Lecture 4

4. Rigorous existence results

Take $\mathcal{E}$ and $\mathcal{D}$ as in the previous lecture
Fix a volume V and an initial stable configuration E_0. (possibly time dependent)

We can prove that under suitable assumptions on the container Ω and the potential $\varphi(t,x)$ there exists an energetic solution $t \mapsto E(t)$ with initial condition E_0 and volume V.

4.1. Construction by time-discretization

Fix a time step $\delta > 0$.

Define $E_\delta(0) := E_0$ and construct $E_\delta(n\delta)$ by induction on n by

$$E_\delta(n\delta) \in \operatorname{argmin} \left\{ \begin{array}{c} \mathcal{E}(n\delta, E) + \mathcal{D}(E, E_\delta(n\delta - \delta)) \\ \text{with } \mathrm{vol}(E) = v \end{array} \right\}$$

Then, for every t, we set $t_\delta := \sup\{n\delta : n\delta \leq t\}$ and

$$E_\delta(t) := E_\delta(t_\delta)$$

This is the construction of discretized solutions explained in the first lecture!

The solution $E(t)$ is then obtained by taking the limit of $E_\delta(t)$ as $\delta \to 0$.

There are however many difficulties.

The main one is due to the fact that "our" dissipation potential is very <u>degenerate</u>: there is no friction associated to the movement of the free surface, and therefore we cannot control the "oscillations" (variation) of $t \to E_\delta(t)$ but only that of $t \to \Sigma_\delta^c(t)$.

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Instead, I will explain an analytical tool (finite perimeter sets) used to prove existence for the minimum problem that appears in the constructions of discretized solutions $E^\delta(t)$, and more generally to prove the existence of minimizers of

$$\mathcal{E}(E) := \sigma_{LV}\left(|\Sigma^\ell| - \int_{\Sigma^c} \varphi\right) + V$$

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among all $E \subset \Omega$ with $\mathrm{vol}(E) = v$.

Here $\varphi : \partial\Omega \to \mathbb{R}$ satisfies the wetting condition $|\varphi| \leq 1$ and, as usual,

$$V(E) := \int_E \rho(x)\, dx .$$

4.2. The direct method of Calculus of Variations

The general approach we follow is to prove existence of minimizers by the so-called direct method (semicontinuity and compactness): the basic idea is that a lower semicontinuous function on a compact metric space has always a minimum point.

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The general approach we follow is to prove existence of minimizers by the so-called direct method (semicontinuity and compactness): the basic idea is that a lower semicontinuous function on a compact metric space has always a minimum point.

Classical example: the Dirichlet Energy

$$\mathcal{E} = \frac{1}{2}\int_{\Omega} |\nabla u|^2$$

admits a minimum on the class of all Sobolev functions $u \in H^1(\Omega)$ with prescribed boundary values $u = u_0$ on $\partial\Omega$.

Note that the direct method would not work with $\mathcal{C}^1$ functions in place of Sobolev ones.

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What do we need for our problem?

A class $\mathcal{F}$ of sets in $\mathbb{R}^d$ with good compactness properties (w.r.t. a suitable distance) such that smooth sets are included in $\mathcal{F}$ and dense.

Moreover we need to extend the notion of "area of the boundary" to sets in $\mathcal{F}$ in a lower semicontinuous fashion.

4.3. Finite perimeter sets

Key observation: if $E = [a,b]$ then the distributional derivative of 1_E (on $\mathbb{R}$) is $D1_E = \delta_a - \delta_b$.

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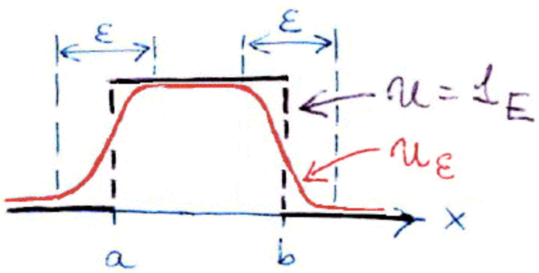

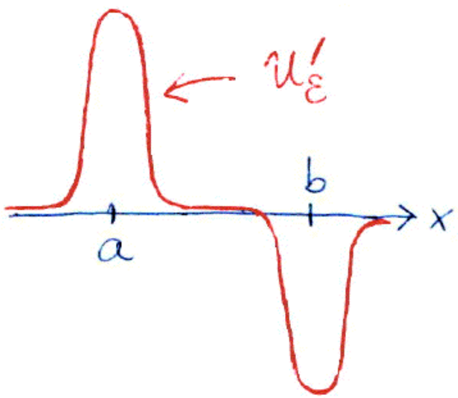

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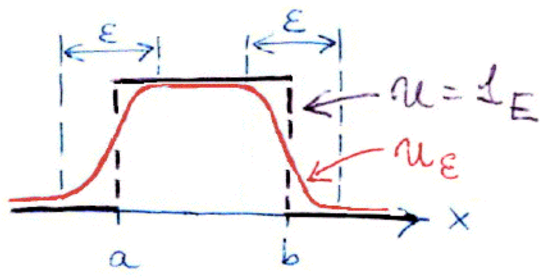

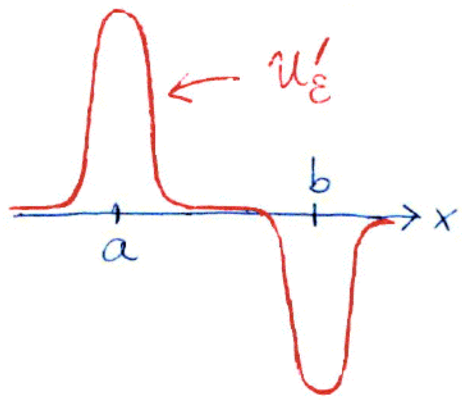

So $D1_E$ is related to ∂E...

4.4. Definition of Finite Perimeter sets.

We say that $E \subset \mathbb{R}^d$ has finite perimeter if $D\mathbb{1}_E$ is a (vector-valued) measure. That is, there exists $\mu = (\mu_1, \dots, \mu_d)$ s.t.

$$\int_E \frac{\partial \varphi}{\partial x_i} \, \underset{\substack{\uparrow \\ \text{Lebesgue} \\ \text{measure}}}{dx} = -\int_{\mathbb{R}^d} \varphi \, d\mu_i \quad \forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$$

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Or equivalently

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we write the vector meas. μ as unit vectorfield η times the positive measure $|\mu|$.

We then define the perimeter of E:

$$\mathrm{Per}(E) := \text{mass of } |\mu| := |\mu|(\mathbb{R}^d)$$

$$= \sup_{|\phi|\leq 1} \int_{\mathbb{R}^d} \phi \cdot \eta \, d|\mu|$$

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\end{aligned}$$

We endow the class $\mathcal{F}$ of finite perim. sets in $\mathbb{R}^d$ with the L^1-distance

$$d(E,E') := \| 1_E - 1_{E'} \|_{L^1} = |E \Delta E'| \leftarrow$$

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Identity (1) above express Per(E) as a supremum of continuous (linear) functions of E, showing that Per(E) is lower semicontinuous in E.

4.5. Basic example

Let E be a smooth set in $\mathbb{R}^d$. Then

E has finite perimeter

$|\mu|$ = surface measure σ on ∂E

η = inner (unit) normal to ∂E

$\mathrm{Per}(E)$ = "area" of $\partial E = \mathcal{H}^{d-1}(\partial E)$

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Proof By the divergence theorem,

$$\int_E \operatorname{div}\phi \, dx = \int_{\partial E} \phi \cdot \eta \, d\sigma \quad \forall \phi \in \mathcal{C}_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$$

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This implies $\mu = \eta \cdot \sigma$.

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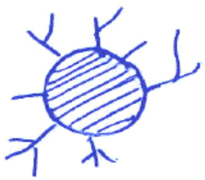
b)  is the same as 

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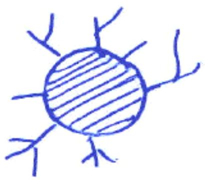
b)  is the same as 

In general there holds $\mathrm{Per}(E) \leq \mathcal{H}^{d-1}(\partial E)$, and both examples above show that $=$ may not occur when E is not smooth.

4.6. Approximation by smooth sets.

For every $E \subset \mathcal{Y}$ there exists a sequence of smooth sets E_n s.t. $E_n \to E$ (in the distance d) and $\mathrm{Per}(E_n) \to \mathrm{Per}(E)$.

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Therefore, if $\mathrm{Per}(E) < \mathcal{H}^{d-1}(\partial E)$, then

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Thus the quantity $\mathcal{H}^{d-1}(\partial E)$ is not lower semicontinuous (and not even well-defined, if you think about it).

4.7. Compactness and lower semicontinuity

Let E_n be a sequence of sets in $\mathcal{F}$ s.t.

$E_n \subset \Omega$ bounded domain;

$Per(E_n) \leq C < +\infty$.

Then, passing to subsequence, $E_n \to E \in \mathcal{F}$ and

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Proof (idea of): embed $\mathcal{F}$ in the space $BV(\mathbb{R}^d)$; use Sobolev (compact) embedding $BV(\mathbb{R}^d) \hookrightarrow L^1_{loc}(\mathbb{R}^d)$; use the fact that BV is a (weak* closed subspace of a) dual and Banach-Alaoglu for compactness. Semicontinuity we have already seen.

4.8. A simple warm-up problem

Let Ω be a bounded domain in $\mathbb{R}^d$, $\rho : \Omega \to \mathbb{R}$ a bounded function, $v > 0$, and $\mathcal{F}_v := \{ E \in \mathcal{F} \text{ s.t. } E \subset \Omega, \, vol(E) = v \}$.
Then

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admits a minimizer in $\mathcal{F}_v$.

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Conclude that $\overline{E}$ is minimizer.

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But there is a catch or two.

The method does not provide any algorithm to find minimizers. And minimizers can be in principle crazy objects.

The last problem is solved by proving regularity results, that is, by showing that a minimizer is actually better than a generic element of $\mathcal{F}$. In fact, much better. But this is really hard.

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On the bright side, our minimum problem is (easily) approximated by discretized ones.

A first step towards regularity is the Rectifiability Theorem by De Giorgi and Federer:

Let E be a finite perimeter set.
Let $\partial_* E$ be the set of $x \in \mathbb{R}^d$ s.t. E has density $\frac{1}{2}$ at x (the measure theoretic bdry of E).

Then: $|\mu|$ = restriction of $\mathcal{H}^{d-1}$ to $\partial_* E$;

$\partial^* E$ is <u>rectifiable</u> (that is, ...);

η is a suitably defined inner normal to $\partial_* E$.

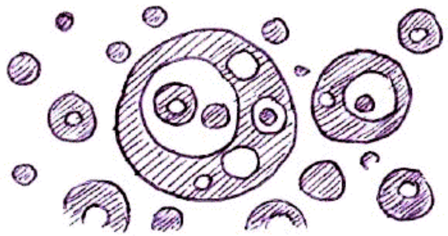
The relevance of this result is better understood in view of the following

Example (of a "bad" finite perimeter set).

Take (x_n) dense in $\mathbb{R}^d$.
Take r_n s.t. $r_n > 0$, $r_n \neq |x_n - x_m|$ $\forall m$,
$r_n < \big| r_m - |x_n - x_m| \big|$ $\forall m < n$ $(*)$, $\sum r_n^{d-1} < +\infty$.

Yes! you can...

Let B_n be the open ball with center x_n and radius r_n.
Assumption $(*)$ implies that either $\overline{B_n} \subset E_{n-1}$ or $\overline{B_n} \cap \overline{E_{n-1}} = \emptyset$.
Set: $E_1 := B_1$; $E_2 := E_1 \Delta B_2$; $E_3 := E_2 \Delta B_3 \dots$



$\leftarrow$ Typical picture of E_n

Let E be the limit of E_n (the sequence converge in $\mathcal{F}$ because...)
For this set E, the support of $|\mu|$ is $\mathbb{R}^d$!

4.9. Minimizing the capillary energy

Let be given:

Ω bounded domain in $\mathbb{R}^3$ (container);

$\rho : \Omega \to \mathbb{R}$ bounded (volume energy density);

$\varphi : \partial\Omega \to \mathbb{R}$ ($\sim$ boundary surface tension coeff.);

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Then $\mathcal{E}$ admits a minimizer on $\mathcal{F}_v$.

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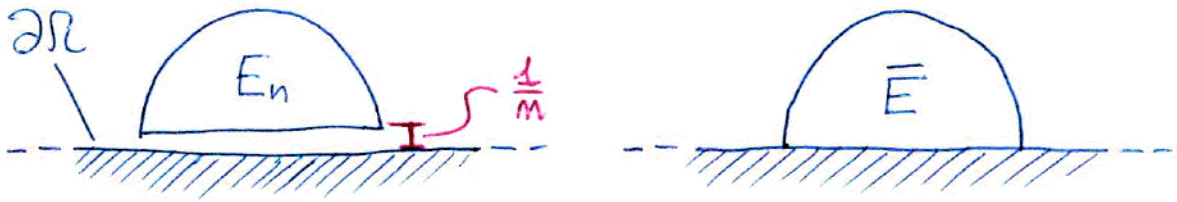
The only problem is that $E_n \to \bar{E}$ does not imply that

$$\liminf_{n\to+\infty} |\Sigma_n^f| \geq |\bar{\Sigma}^f| \quad \& \quad \liminf_{n\to+\infty} |\Sigma_n^c| \geq |\bar{\Sigma}^c|$$



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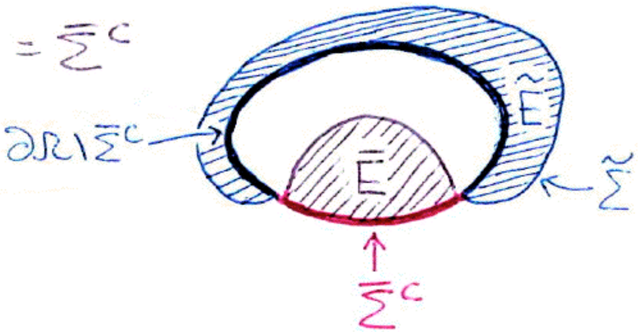
Construct $\tilde{E} \subset \mathbb{R}^3 \setminus \Omega$ s.t. $\partial_* \tilde{E} \cap \partial\Omega = \overline{\Sigma}^c$

(tricky but possible!)

and set $\tilde{\Sigma} := \partial_* \tilde{E} \setminus \partial\Omega$.



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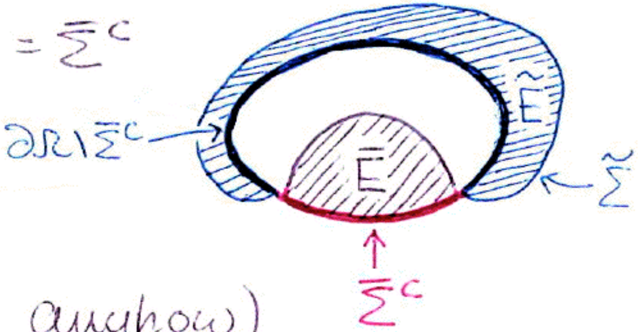

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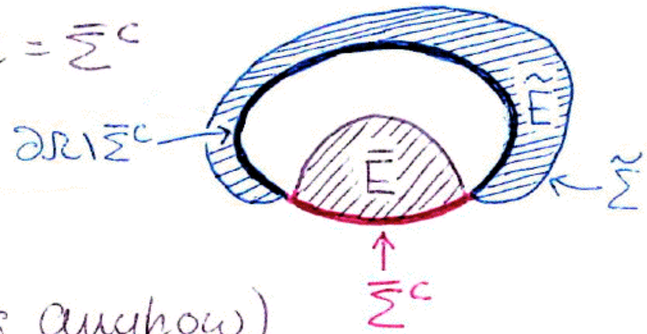

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$$\mathcal{E}(E_n) - \mathcal{E}(E) = |\Sigma_n^f| - |\Sigma^f| + \int_{\Sigma_n^c \setminus \bar{\Sigma}^c} \varphi - \int_{\bar{\Sigma}^c \setminus \Sigma_n^c} \varphi$$

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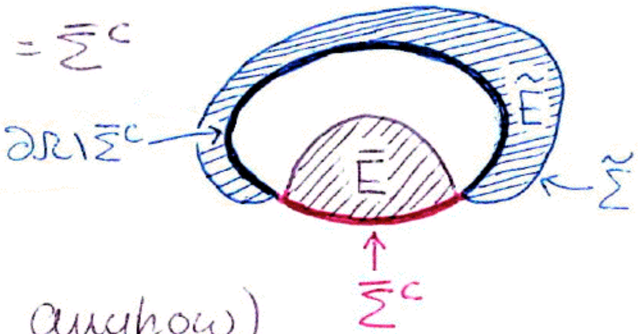

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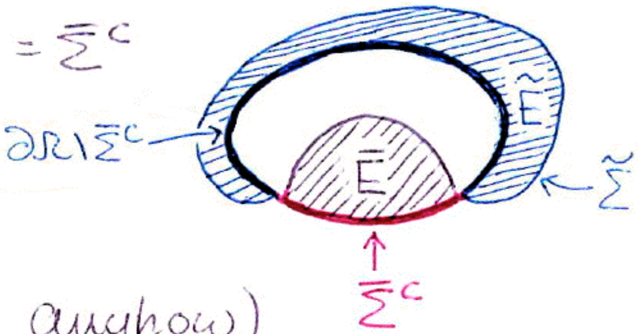

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$$= \mathrm{Per}(E_n \cup \tilde{E}) - \mathrm{Per}(E \cup \tilde{E})$$

(a picture will convince you of the last identity).

Then

$$\liminf_{n\to+\infty} \mathcal{E}(E_n) - \mathcal{E}(\bar{E})$$

$$\geq \liminf_{n\to+\infty} \mathrm{Per}(E_n \cup \tilde{E}) - \mathrm{Per}(\bar{E} \cup \tilde{E})$$

Then

$$\liminf_{n\to+\infty} \mathcal{E}(E_n) - \mathcal{E}(\bar{E})$$

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Hence $\mathcal{E}$ is lower semicontinuous and the rest of the proof works as before.

Concluding remarks

1. We can talk (with some care) of "boundary" of a finite perimeter set, and even of "interior" and "exterior" points.

But there are objects that cannot be defined in the framework of finite perimeter sets, not even in a weak sense.

One is the mean curvature.

Others (in the context of capillarity) are the contact line and the contact angle.

So the equilibrium conditions cannot even be written...

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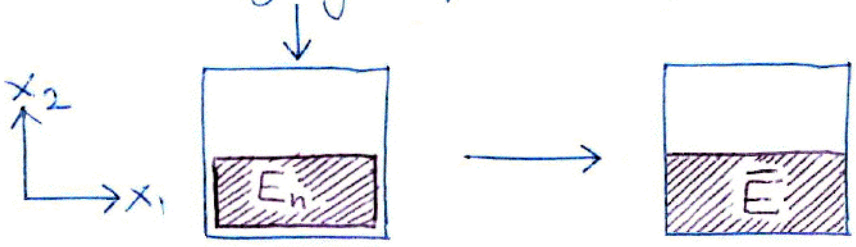
Example: $\varphi = 2$; $\Omega = \square \updownarrow e$; $\rho = k x_2$ with $k \gg \frac{\sigma_{LV}}{e^2}$ (strong gravity directed downward).

Minimizing sequence E_n



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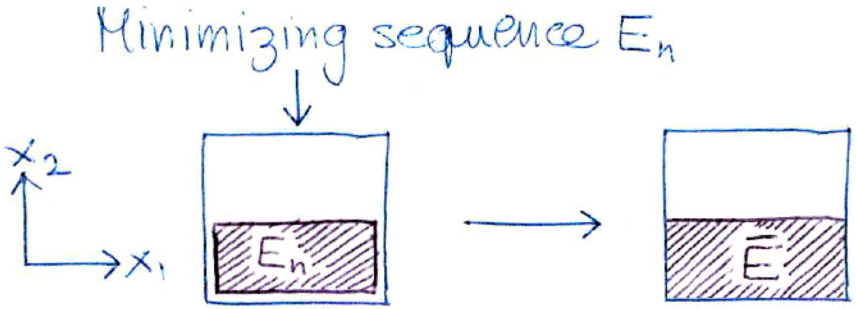

But E is not a minimizer (and there exists no minimizer).

Proving existence results is also a way of checking that there are no hidden traps in the model...