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# Differentiability of Lipschitz functions, structure of null sets, and other problems

Giovanni Alberti, Marianna Csörnyei, David Preiss

**Abstract.** The research presented here developed from rather mysterious observations, originally made by the authors independently and in different circumstances, that Lebesgue null sets may have uniquely defined tangent directions that are still seen even if the set is much enlarged (but still kept Lebesgue null). This phenomenon appeared, for example, in the rank-one property of derivatives of BV functions and, perhaps in its most striking form, in attempts to decide whether Rademacher's theorem on differentiability of Lipschitz functions may be strengthened or not.

We describe the non-differentiability sets of Lipschitz functions on $\mathbb{R}^n$ and use this description to explain the development of the ideas and various approaches to the definition of the tangent fields to null sets. We also indicate connections to other current results, including results related to the study of structure of sets of small measure, and present some of the main remaining open problems.

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## 1. Differentiability of Lipschitz functions

One of the important results of Lebesgue tells us that Lipschitz functions on the real line are differentiable almost everywhere. It is also well-known that the converse is true: for every Lebesgue null set E on the real line there is a real-valued Lipschitz function which is non-differentiable at any point of $E$. That is:

**Theorem 1.1.** *For a given set $E \subset \mathbb{R}$ there is a Lipschitz function $f\colon \mathbb{R} \to \mathbb{R}$ which is not differentiable at any point $x \in E$ if and only if $E$ is Lebesgue null.*

One of our aims is to generalise Theorem 1.1, and also its more precise variants that will be described in Theorem 1.13, to Lipschitz functions $f\colon \mathbb{R}^n \to \mathbb{R}^m$.

Since a Lipschitz function on $\mathbb{R}$ is differentiable almost everywhere, Fubini Theorem implies immediately that the *directional (or partial) derivative*

$$f'(x;u) := \lim_{t\to 0} \frac{f(x+tu) - f(x)}{t}$$

of a Lipschitz function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ exists for each direction $u$ at a.e. $x$.

Although differentiability is not the same as the existence of sufficiently many partial derivatives, the set of points at which these two notions differ is relatively easy to control. First recall the following definition:

**Definition 1.2.** A set $E \subset \mathbb{R}^n$ is *porous at a point* $x \in E$ if there is a $c > 0$ and there is a sequence $y_n \to 0$ such that the balls $B(x + y_n, c|y_n|)$ are disjoint from $E$. The set $E$ is *porous* if it is porous at each of its points, and it is called *$\sigma$-porous* if it is a countable union of porous sets.

**Theorem 1.3** ([3])**.** *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz function. Then the set of those points at which $f$ is not differentiable but it is differentiable in $n$ linearly independent directions is $\sigma$-porous.*

It follows from Lebesgue's density theorem that $\sigma$-porous sets have Lebesgue measure zero. Therefore as an immediate corollary we obtain:

**Theorem 1.4** (Rademacher)**.** *Every Lipschitz function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is differentiable almost everywhere.*

The converse direction, i.e. the description of those sets $E \subset \mathbb{R}^n$ for which there is a non-differentiable Lipschitz function, is much harder. D. Preiss proved that the converse of Rademacher's theorem is false, already in dimension 2:

**Theorem 1.5** ([9])**.** *There is a Lebesgue null set $E \subset \mathbb{R}^2$ such that every Lipschitz function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is differentiable in at least one point of $E$.*

Unlike in the classical Lebesgue and Rademacher theorem, Preiss's result is not an 'almost everywhere' result, he does not show that the function is differentiable at 'most' of the points $x \in E$. Indeed this is not possible. We prove the following theorem:

**Theorem 1.6.** *For every Lebesgue null set $E \subset \mathbb{R}^2$ there is a Lipschitz function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ which is not differentiable at any point $x \in E$.*

This theorem says that for every Lebesgue null set there are two real-valued Lipschitz functions, namely, the coordinate functions of $f$, such that at each $x \in E$ at least one of the two functions are non-differentiable.

**Remark.** In [9] the result is proved not only in $\mathbb{R}^2$, but in every Banach space with a smooth norm. Preiss's set $E$ is dense. In a recent paper [5], M. Doré and O. Maleva constructed a closed (and hence nowhere dense) null set with the same property: in every Banach space $X$ with separable dual there exists a closed bounded set of Hausdorff dimension one containing a Fréchet-differentiability point of every Lipschitz function $f\colon X \to \mathbb{R}$.

Let $E \subset \mathbb{R}^n$. It is immediate from the definition that a set $E$ is porous at $x \in E$ if and only if the Lipschitz function $f(x) = \operatorname{dist}(x, E)$ is non-differentiable at $x$. Of course $\sigma$-porous sets cannot fully describe non-differentiability sets of Lipschitz functions (not even in $\mathbb{R}$, since not all Lebesgue null sets of $\mathbb{R}$ are $\sigma$-porous). But by Theorem 1.3, in order to find all Lebesgue null sets for which there is a non-differentiable Lipschitz function, it is enough to consider functions not having enough many directional derivatives.

From the point of view of differentiability problems, the sets that are the most negligible are the sets of points at which a Lipschitz function may be differentiable in no direction. We show that these sets form a $\sigma$-ideal. We call them *uniformly purely unrectifiable*. Notice that uniformly purely unrectifiable sets are purely unrectifiable, i.e. they are null on every rectifiable curve, since a Lipschitz function is differentiable in the tangent direction at a.e. point of a curve. We will see later that uniformly purely unrectifiable sets have the (possibly only formally) stronger property that they can be covered by an open set which is small on many curves simultaneously.



For simplicity, consider just Lipschitz functions $f\colon \mathbb{R}^2 \to \mathbb{R}^m$. We will show that if $f$ is not differentiable at the points of $E \subset \mathbb{R}^2$, then at each point $x \in E$ except for a uniformly purely unrectifiable set, there is a *unique* differentiability direction $\tau(x)$ of $f$. Moreover, this direction is determined by the geometry of the set $E$, it is independent of the function $f$: for any other Lipschitz function $g$, the direction constructed using $f$ and $g$ agree at each point of $E$ except for a uniformly purely unrectifiable set. Indeed, if $E$ is contained in the non-differentiability set of both $f\colon \mathbb{R}^2 \to \mathbb{R}^{m_1}$ and $g\colon \mathbb{R}^2 \to \mathbb{R}^{m_2}$, then the direction $\tau$ defined by the function $h = (f, g)\colon \mathbb{R}^2 \to \mathbb{R}^{m_1+m_2}$ must coincide with the directions defined by $f$ or $g$, whenever $f$, $g$ and $h$ have a unique direction of differentiability.

Using also Theorem 1.6, we obtain:

**Corollary 1.7.** *For every planar Lebesgue null set $E$, at each point $x \in E$ there is a direction $\tau(x)$ with the following property: every Lipschitz function $f\colon \mathbb{R}^2 \to \mathbb{R}^m$ is differentiable in the direction $\tau(x)$ at every $x \in E$, except at a uniformly purely unrectifiable set of points. This direction is determined uniquely, except for a uniformly purely unrectifiable set.*

**Remark.** There are null sets which are very far from being purely unrectifiable. For instance, R. O. Davies showed in [4] that every Borel set $B \subset \mathbb{R}^2$ can be covered by infinite straight lines without increasing its Lebesgue measure. One can even put continuum many lines through each of the points of $B$ so that the union of these lines has the same measure as $B$. Now if $B = B_0$ is, say, a point, applying Davies's theorem iteratively, we can find $B_0 \subset B_1 \subset B_2 \subset \dots$ such that each $B_k$ has continuum many lines through the points of $B_{k-1}$, and the sets $B_k$ are Lebesgue null. Then $\bigcup B_k$ is also Lebesgue null, and it has continuum many lines through each of its points. What could be $\tau$ on $\bigcup B_k$? Since Lipschitz functions are differentiable along lines, at each line of the construction, $\tau$ must agree with the direction of the line at a.e. of its points. But there are continuum many lines at each point, how can we choose only one of these, so that along any given line at a.e. point we choose the direction of the given line and not one of the others?

Now, consider Lipschitz functions on $\mathbb{R}^n$.

**Notation.** We denote by $\mathcal{N}_{n,k}$ the $\sigma$-ideal of subsets of $\mathbb{R}^n$ generated by sets for which there is a Lipschitz function $f\colon \mathbb{R}^n \to \mathbb{R}$ differentiable in at most $k$ linearly independent directions.

So $\mathcal{N}_{n,0}$ are exactly the uniformly purely unrectifiable sets, while $\mathcal{N}_{n,n-1}$ are the non-differentiability sets we are mainly interested in.

Since a Lipschitz function is differentiable in the tangent directions of any $k$-rectifiable set at $\mathcal{H}^k$-almost all of its points, therefore $\mathcal{N}_{n,k-1}$ sets are $k$-purely unrectifiable, i.e. they meet every $k$-rectifiable set in an $\mathcal{H}^k$-null set.

As a refinement of the above observations on directions of differentiability in the plane, we will show that whenever $E \in \mathcal{N}_{n,k}$, there is $\tau\colon E \to G(n,k)$ such that for all $x \in E$ except those belonging to an $\mathcal{N}_{n,k-1}$ set, every Lipschitz function $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable in the direction $\tau(x)$.

**Definition 1.8.** $\tau: E \to G(n,k)$ is called a *$k$-dimensional tangent field* of a set $E$ if every Lipschitz function $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable in the direction $\tau(x)$ at all $x \in E$ except those belonging to an $\mathcal{N}_{n,k-1}$ set.

**Theorem 1.9.** *Every set $E \in \mathcal{N}_{n,k}$ has a $k$-dimensional tangent field. Moreover, the tangent field is unique up to an $\mathcal{N}_{n,k-1}$ set.*

It is easy to see that:

**Proposition 1.10.** *The set of (directional) non-differentiability of a Lipschitz function $f: \mathbb{R}^n \to \mathbb{R}$ can be written as a countable union of sets $E$, for each of which we may find a direction $u$ and numbers $a < b$ such that*

$$\liminf_{t\to 0} \frac{f(x+tu)-f(x)}{t} < a < b < \limsup_{t\to 0} \frac{f(x+tu)-f(x)}{t}.$$

Since our $f$ is Lipschitz, such set is null not only on every line in direction $u$, but also on every curve $\gamma: \mathbb{R} \to \mathbb{R}^n$ provided that $|\gamma' - u|$ is small enough.

We can do slightly better: if $\delta > 0$ is small enough, for every $\varepsilon > 0$ there is an open set $G \supset E$ such that the length of $G \cap \gamma$ is less than $\varepsilon$ for every curve $\gamma: \mathbb{R} \to \mathbb{R}^n$ with $|\gamma' - u| < \delta$.

This observation motivates the following definition. Given a convex cone $C$, we may define the $C$-width of an open set $G$ as the supremum of the lengths of $\gamma \cap G$ where the supremum is taken over all Lipschitz curves $\gamma$ that 'go in the direction of $C$', i.e. for which $\gamma'(t) \in C$ for a.e. $t$. Then we define the $C$-width for general sets as the infimum of the $C$-widths of open sets containing it. In fact, our definition of the width is slightly more complicated: instead of the length we use a technically more convenient measure (that also depends on a vector $e \in \mathrm{int}(C)$) of the part of the curve that lies in the set $G$. (See later, Definition 1.14.)

Using this notion of width, an equivalent description of the tangent field of a set can be obtained without referring to non-differentiability sets and non-differentiability directions of Lipschitz functions:

**Definition 1.11.** If $E \subset \mathbb{R}^n$, we say that the mapping $\tau: E \to G(n,k)$ is a *$k$-dimensional tangent field* of $E$ if for every cone $C$, the set of those points $x \in E$ for which $\tau(x) \cap C = \{0\}$ has $C$-width zero.

This defines the same tangent field as Definition 1.8: one can show that the family of those subsets of $\mathbb{R}^n$ that admit a $k$-dimensional tangent field according to Definition 1.11 coincides with the $\sigma$-ideal $\mathcal{N}_{n,k}$, and also that the two tangent fields coincide.

According to Proposition 1.10 (and paragraphs preceding it), the set where $f$ is not differentiable can be covered by countably many sets, each of which has width zero with respect to some cone.

We do not know whether this is a full description, i.e. we do not know whether the non-differentiability sets of Lipschitz functions (i.e. those sets that admit an $(n-1)$-tangent field) are exactly described by the property that they can be covered by countably many sets, each of which has width zero with respect to some cone.



It is not very hard to show, using Definition 1.11, that the existence of an $(n-1)$-tangent field of a set is equivalent to the property that for every $\varepsilon > 0$ the set can be covered by a finite number of sets each of which has width zero with respect to some cone that is only $\varepsilon$-far from a halfspace.

Our results include:

**Theorem 1.12.** *For every set $E \subset \mathbb{R}^n$, the following are equivalent:*

(i) *There is a Lipschitz function $f: \mathbb{R}^n \to \mathbb{R}^n$ that is non-differentiable at any point of $E$.*
(ii) *There is a sequence (possibly infinite) of Lipschitz functions $f_j: \mathbb{R}^n \to \mathbb{R}$ such that at every point of $E$ at least one of the $f_j$ is non-differentiable.*
(iii) *The set $E$ is in $\mathcal{N}_{n,n-1}$.*
(iv) *The set $E$ has an $(n-1)$-tangent field.*
(v) *If $n \le 2$: $E$ has Lebesgue measure zero.*

We do not know whether every Lebesgue null set is in $\mathcal{N}_{n,n-1}$ for $n > 2$. And we do not know whether it is true that for every $m < n$ there is a null set $E \in \mathbb{R}^n$ such that every Lipschitz function $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at some point of $E$. Preiss proved in [9] that the answer is 'yes' for $1 = m < n$. Doré and Maleva in [6], building heavily on methods due to Lindenstrauss, Preiss and Tišer in [8] in their study of differentiability problems in infinite dimensional Banach spaces, proved that the answer is also yes for $2 = m < n$. But their current methods do not work for $m \ge 3$.

So far we didn't say anything about how we can construct a non-differentiable function for a given (small) set $E$. This is much harder than the other direction, i.e. showing that the set of points of non-differentiability must be small. In dimension 1 it is easy, and one may try to use the 1-dimensional proof as a guidance. One could even consider generalising the more precise description of the sets of non-differentiability of Lipschitz functions $f: \mathbb{R} \to \mathbb{R}$ due (with slightly worse constants) to Zahorski [11]. (See [7] for a more recent proof.)

**Theorem 1.13** (Zahorski)**.** *For any $G_\delta$ set $E \subset \mathbb{R}$ of Lebesgue measure zero there is a Lipschitz function $f: \mathbb{R} \to \mathbb{R}$ with* $\mathrm{Lip}(f) \le 1$ *which is differentiable at every point $x \notin E$ and*

$$\liminf_{t\to 0} \frac{f(x+t)-f(x)}{t} = -1 < 1 = \limsup_{t\to 0} \frac{f(x+t)-f(x)}{t}$$

*for every $x \in E$.*

Recall that a set is $G_\delta$ if it is an intersection of countably many open sets. Recall also that, by adding together suitable multiples of functions obtained by this theorem for $G_\delta$ sets $E_i$, Zahorski showed that $E \subset \mathbb{R}$ is the set of points of non-differentiability of some Lipschitz function $f: \mathbb{R} \to \mathbb{R}$ if and only if $E$ is of Lebesgue measure zero and of type $G_{\delta\sigma}$ (a union of countably many $G_\delta$ sets).

So let us see how one can construct a Lipschitz function $f: \mathbb{R} \to \mathbb{R}$ non-differentiable at the points of the given Lebesgue null set $E$. We recursively find open sets $G_1 \supset G_2 \supset \cdots \supset E$ so small that $G_k$ is small in every component of $G_{k-1}$. (For example, $|G_k \cap C| < 2^{-k}|b-a|$ for any component $C = (a,b)$ of $G_{k-1}$.) Let $f_k(x)$ denote the measure of $(-\infty, x) \cap G_k$. Then $f_k'(x) = 1$ at each point $x \in G_k$, but the slope $(f_k(b) - f_k(a)/(b-a)$ is close to 0. Using this it is easy to check that $f(x) = \sum(-1)^k f_k(x)$ is not differentiable at any point of $\bigcap G_k$. If $E$ is $G_\delta$ and $\varepsilon > 0$, it is not difficult to choose the $G_k$ so that, defining $f(x) = \sum \lambda_k f_k(x)$ where $|\lambda_k| < \varepsilon$ and the partial sums of the $\lambda_k$ oscillate between $\pm 1$, we get a function that almost satisfies the statement of Theorem 1.13. However, at the points of $\mathbb{R} \setminus E$ we would only get that the upper and lower derivatives of $f$ differ by no more than $2\varepsilon$, not that $f$ is differentiable. We are in fact able to find a higher dimensional analogue of this construction. Recall however that Theorem 1.13 is proved in a different way, and that the weaker statement that we have just indicated is not sufficient for showing the full description of non-differentiability sets mentioned above.

As a higher dimensional analogue of the functions $f_k$, for an open set $G \subset \mathbb{R}^n$ of (small) $C$-width $w$ and unit vector $e$ from the interior of $C$, we construct a function $\omega: \mathbb{R}^n \to \mathbb{R}$ such that $\mathrm{Lip}(\omega)$ is bounded by a constant depending on $C$ and $e$, $\omega(y) \geq \omega(x)$ if $y - x \in C$, $\omega(x + te) = \omega(x) + t$ if the segment $[x, x + te]$ lies in $G$, and $0 \leq \omega(x) \leq w$ for all $x \in \mathbb{R}^n$.

The function $\omega$ can be used to construct non-differentiable functions, in a similar way as the functions $f_k$ were used in dimension 1. Indeed, $\omega$ has directional derivative 1 in the direction $e$ at each $x \in G$, but from the more global point of view $\omega$ looks like having derivative zero.

The technical details of the construction are quite complicated. They may be somewhat simplified in the case of sets $E \in \mathcal{N}_{n,0}$. Given any vector $e$, we choose an open set $G \supset E$ with small $C$-width where $C$ is close to the halfspace $\{x : \langle x, e\rangle \geq 0\}$. The function $\langle x, e\rangle - \omega(x)$ sees, from every point of $G$, some points in the direction $e$ with slope almost one, but has local Lipschitz constant close to zero on $G$. This allows us to iterate the construction locally. Moving also the vectors $e$ through a dense subset of the unit sphere, we get a function which is non-differentiable at any point of $E$ in any direction. More precisely, here is our definition and the results we prove:

**Definition 1.14.** Let $C$ be a convex cone and let $e$ be a unit vector in $C$.

(i) We define $M = M_{C,e}: \mathbb{R}^n \to \mathbb{R}$ by
$$M(x) = \sup\{\lambda \in \mathbb{R} : x - \lambda e \in C\}\,.$$

(ii) The *$C$-width* $w(G) = w_{C,e}(G)$ of an open set $G \subset \mathbb{R}^n$ is defined as the supremum of the numbers
$$\int_{\{t:\gamma(t)\in G\}} M(\gamma'(t))\,dt$$
among all Lipschitz curves $\gamma: \mathbb{R} \to \mathbb{R}^n$ which go in the direction of $C$.

(iii) For a general set $E \subset \mathbb{R}^n$, $w(E)$ is the infimum of $w(G)$ among all open sets $G$ which contain $E$.

(iv) Let $G \subset \mathbb{R}^n$ be an open set of finite width. For every point $x \in \mathbb{R}^n$ we set $\omega(x) = \omega_{G,C,e}(x)$ as the supremum of the numbers
$$-\lambda + \int_{t\in[a,b],\gamma(t)\in G} M(\gamma'(t))\,dt$$
among all $a, b \in \mathbb{R}$, $\lambda \geq 0$ and $\gamma : [a, b] \to \mathbb{R}^n$ such that $\gamma(b) - x = \lambda e$ and $\gamma$ goes in the direction of $C$.



We use this function $\omega$ to prove:

**Theorem 1.15.** *For every $\tilde\varepsilon > 0$ and for every set $E$ which is $G_\delta$ and uniformly purely unrectifiable there is a function $f: \mathbb{R}^n \to \mathbb{R}$ such that*

(i) $\mathrm{Lip}(f) = 1$;

(ii) *$f$ is $\tilde\varepsilon$-differentiable on $\mathbb{R}^n \setminus E$, that is, for every $x \in \mathbb{R}^n \setminus E$ there is $r > 0$ and a vector $u$ such that*
$$|f(x) - f(y) - \langle u, y - x\rangle| \leq \tilde\varepsilon|y - x| \quad \text{for all } y \in B(x, r),$$

(iii) *for every $x \in E$, $\eta \in B(0, 1) \subset \mathbb{R}^n$ and $\varepsilon > 0$ there is an $r < \varepsilon$ such that*
$$|f(y) - f(x) - \langle \eta, y - x\rangle| \leq \varepsilon r \quad \text{for all } y \in B(x, r).$$

*In particular, $f$ is not differentiable at the points of $E$, it is not even $\varepsilon$-differentiable for any $\varepsilon < 1$.*

Since every uniformly purely unrectifiable set is contained in a $G_\delta$ uniformly purely unrectifiable set, this indeed shows that for every $\mathcal{N}_{n,0}$ set there is a Lipschitz function that is non-differentiable in any direction. However this result does not provide Zahorski-type exact description of sets of non-differentiability in any direction (which, by analogy, one would conjecture to be $\mathcal{N}_{n,0}$ sets of type $G_{\delta\sigma}$), since we do not know (in dimension $n > 1$) whether (ii) of Theorem 1.15 can be replaced by the condition that $f$ is differentiable on $\mathbb{R}^n \setminus E$.

By a rather delicate induction with respect to $k$ (which is where we need the condition (ii) of Theorem 1.15) we show that the sets of points of $k$-dimensional differentiability can be characterised as follows:

**Theorem 1.16.** (i) *Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz function, and for each $x \in \mathbb{R}^n$ choose $\tau(x)$ to be a maximal dimensional subspace such that the restriction of $f$ to $x + \tau(x)$ is differentiable at $x$. For each $0 \leq k \leq n-1$, let $E_k$ denote the set of those points at which $\dim \tau(x) = k$. Then $E_k \in \mathcal{N}_{n,k}$.*

(ii) *Let $E_k \subset \mathbb{R}^n$ be an $\mathcal{N}_{n,k}$ set for some $0 \leq k \leq n-1$. Then there is a Lipschitz function $f: \mathbb{R}^n \to \mathbb{R}^{k+1}$ and a $k$-tangent field $\tau$ of $E_k$ such that $f$ is not differentiable at any $x \in E_k$ in any direction $e$ that is orthogonal to $\tau(x)$.*

We can make (ii) of Theorem 1.16 more quantitative. Again, this is a weaker analogy of Theorem 1.13, which is needed for induction and to which the same remarks as to the case $k = 0$ apply.

**Theorem 1.17.** *For each $0 \leq k < n$ there is a constant $c_{n,k} > 0$ such that, whenever $l > k$, $\varepsilon > 0$ and $E$ is a $G_\delta$, $\mathcal{N}_{n,k}$ subset of $\mathbb{R}^n$, then there is a function $f: \mathbb{R}^n \to \mathbb{R}^l$ with $\mathrm{Lip}(f) \leq 1$ which is $\varepsilon$-directionally differentiable at every point of $\mathbb{R}^n \setminus E$ and has the property that for every $x \in E$ there are $k$-dimensional linear subspaces $V, W$ of $\mathbb{R}^n, \mathbb{R}^l$, respectively, so that for any unit vectors $v \in V^\perp$ and $w \in W^\perp$,*
$$\limsup_{t\searrow 0} \frac{\langle f(x + tv) - f(x), w\rangle}{t} - \liminf_{t\searrow 0} \frac{\langle f(x + tv) - f(x), w\rangle}{t} \geq c_{n,k}\,.$$

According to (iii) of Theorem 1.15, $c_{n,0} = 2$. We do not know whether $c_{n,k} = 2$ for $k > 0$.

We finish this section by showing that for differentiability with respect to a measure (instead of at every point of a given set) it is sufficient to consider real-valued functions:

**Theorem 1.18.** *Let $\mu$ be a $\sigma$-finite Borel measure on $\mathbb{R}^n$.*

(i) *Every real-valued Lipschitz function $f\colon \mathbb{R}^n \to \mathbb{R}$ is differentiable $\mu$-almost everywhere, if and only if every set in $\mathcal{N}_{n,n-1}$ is $\mu$-null.*
(ii) *On the other hand, if an $\mathcal{N}_{n,n-1}$-set has positive $\mu$-measure, then there is a Lipschitz function $f\colon \mathbb{R}^n \to \mathbb{R}$ which is non-differentiable $\mu$-almost everywhere on this set.*

*In particular, for every singular probability measure $\mu$ in the plane there is a Lipschitz function $f\colon \mathbb{R}^2 \to \mathbb{R}$ which is non-differentiable $\mu$-almost everywhere.*

This nicely complements the result of Preiss mentioned before, according to which there is a null set $E \subset \mathbb{R}^2$ such that every Lipschitz function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is differentiable in at least one point of $E$.

As we have already pointed out, the proof of Theorem 1.16 is rather involved. However, Theorem 1.18 may be proved in a simpler way, closer to the argument that we indicated for Theorem 1.15. Recall that the key point of this argument was that for an open set $G$ of small $C$-width and $e \in C$ we constructed a function $\omega$ with directional derivative 1 in the direction $e$ at each $x \in G$, but looking like having derivative zero from the global point of view. To prove Theorem 1.15, we needed only one such $G$ (as it contained the whole set $E$) while to prove Theorem 1.16 we need several of them which may overlap and so constructions that we need to do cannot be independent. However, to show Theorem 1.18, we may throw away sets of small measure, and so achieve that the sets $G$ in which we have to construct the function $\omega$ are in positive distance from each other. These constructions may still be handled independently, resulting in a reasonably accessible proof.

## 2. Structure of null sets and other problems

In this section we list various results that can be proved using similar techniques and ideas as the ones we use for the characterisation of non-differentiability of Lipschitz functions.

2.1. **Tangent of null sets.** In the planar case, we know that the $\sigma$-ideal $\mathcal{N}_{2,1}$ and the $\sigma$-ideal of Lebesgue null sets coincide, i.e. every planar Lebesgue null set admits a 1-tangent field. We do not know if the same is true in higher dimension. However, there is another, weaker notion of tangent fields that can be defined for any Lebesgue null set in $\mathbb{R}^n$:

**Definition 2.1.** Given a set $E \subset \mathbb{R}^n$, we say that a Borel measurable map $\tau\colon E \to G(n,k)$ defines a *weak $k$-tangent field* to $E$ if for every $k$-rectifiable set $S$, $\mathrm{Tan}(S,x) = \tau(x)$ for $\mathcal{H}^k$-a.e. $x \in S \cap E$.

Notice that in this definition we had to include a measurability assumption. It was not needed in Definition 1.8 since the tangent field defined there is automatically Borel measurable (after a modification on an $\mathcal{N}_{n,k-1}$ set). However, under the continuum hypothesis one can define a non-measurable weak $k$-tangent field by ordering $k$ dimensional $C^1$ surfaces in $\mathbb{R}^n$ into $S_\alpha$, $\alpha < \omega_1$ and defining $\tau(x)$ as the tangent space of $S_\alpha$ at $x$ where $\alpha$ is the first ordinal for which $x \in S_\alpha$.



It follows from the definition that, given a set $E \subset \mathbb{R}^n$, the weak $k$-tangent field, provided that it exists, is uniquely defined up to $k$-purely unrectifiable subsets of $E$ (recall that the $k$-tangent field is uniquely defined up to an $\mathcal{N}_{n,k-1}$ set). Also, if a set admits a $k$-tangent field then it is also a weak $k$-tangent field. We do not know (even in the planar case for $k = 1$) whether the $\sigma$-ideal $\mathcal{N}_{n,k-1}$ coincides with the $\sigma$-ideal generated by $G_\delta$ (or Borel, or analytic) $k$-purely unrectifiable sets, and we do not know in dimensions $n > 2$ whether every set admitting a weak $k$-tangent field also admits a $k$-tangent field. However, we can prove that:

**Theorem 2.2.** *Any set $E \subset \mathbb{R}^n$ of Lebesgue measure zero admits a weak $(n-1)$-tangent field.*

This result can be understood as saying the rather mysterious fact that one can prescribe in which direction an $(n-1)$-surface meets a null set $E$, without knowing the surface itself. The mystery would deepen if, for example, one had a purely 1-unrectifiable set in $\mathcal{N}_{n,1} \setminus \mathcal{N}_{n,0}$: this set would have uniquely prescribed directions that would not be possible to describe by meeting with curves.

2.2. **Covering by Lipschitz slabs and intersecting by curves.** The notion '$C$-width' can be defined in the following, equivalent way. Given a cone $C$ and a vector $e \in \mathrm{int}(C)$, if $E$ is a '$C$-Lipschitz set', i.e. $E \cap (x + C) = \{0\}$ and $E$ meets each line of direction $e$ in exactly one point, then we call the set between $E$ and its shifted copy $E + we$ $(w > 0)$ a *$C$-Lipschitz slab of width $w$*. If $K \subset \mathbb{R}^n$ is compact, we may define its *$C$-width* as the infimum of the total width of families of $C$-Lipschitz slabs covering it. If $G \subset \mathbb{R}^n$ is open, then we define its $C$-width as the supremum of $C$-widths of compact sets contained in it, and finally if $E \subset \mathbb{R}^n$ is arbitrary, then its $C$-width is defined as the infimum of the $C$-widths of open sets containing it.

In our original definition of $C$-width, we measured the part of the curve $\gamma$ that lies in the set $G$ (i.e. we chose the function $M(x)$ in (i) of Definition 1.14) in such a way that we obtain *exactly* the same width as the one defined using $C$-Lipschitz slabs.

So a compact set has $C$-width zero if it can be covered by $C$-Lipschitz slabs of arbitrary small total width. In particular, in $\mathbb{R}^2$, every compact Lebesgue null set is in $\mathcal{N}_{2,1}$, therefore it can be covered by Lipschitz slabs of arbitrary small total width. In fact, in the plane one can cover any null set, and it is enough to use the coordinate directions and Lipschitz graphs with Lipschitz constant one. We show the following:

**Theorem 2.3.** *Every set $E \subset [0,1]^2$ of measure $0 \le m < ab$ is the union of two sets $E = A \cup B$, where $A$ has $C$-width less than $a$ for $C = \{(x,y) : |x| > |y|\}$ and $B$ has $C$-width less than $b$ for $C = \{(x,y) : |y| > |x|\}$.*

*That is, there are Lipschitz functions $f_i\colon \mathbb{R} \to \mathbb{R}$, $g_j\colon \mathbb{R} \to \mathbb{R}$ with Lipschitz constant 1 and $w_i, w_j > 0$ with $\sum_i w_i < a$, $\sum_j w_j < b$, such that*

$$A \subset \bigcup_i \{(x,y) : f_i(x) \le y \le f_i(x) + w_i\} \quad B \subset \bigcup_j \{(x,y) : g_j(y) \le x \le g_j(y) + w_j\}.$$

This can be used e.g. to show that there is a 1-Lipschitz function $f\colon \mathbb{R} \to \mathbb{R}$ whose graph $(x, f(x))$ or $(f(x), x)$ meets $E$ in length at least $m^{1/2}$. The analogous result is also true in higher dimension:

**Theorem 2.4.** *For every set $E \subset [0,1]^n$ of measure $m$ there is a Lipschitz curve (with a fixed Lipschitz constant that depends only on the dimension $n$) that meets $E$ in length at least $c_n m^{1/n}$.*

Here a curve means the graph of a map from one of the coordinate axis into its orthogonal complement. We do not know whether there is a $k$-dimensional Lipschitz surface (where surfaces are understood similarly) that meets $E$ in $\mathcal{H}^k$-measure $c_{k,n} m^{k/n}$.

2.3. **Mappings onto balls and weak derivatives.** Among the problems exploring the geometric structure of sets with positive Lebesgue measure, the following one, proposed by M. Laczkovich, is particularly interesting:

**Problem 2.5.** *Given a set $E \subset \mathbb{R}^n$ of positive Lebesgue measure, is there a Lipschitz function $f\colon \mathbb{R}^n \to \mathbb{R}^n$ which maps $E$ onto a set with non-empty interior (or, equivalently, that maps $E$ onto a ball)?*

Without loss of generality we can assume that $E$ is compact. In dimension $n=1$, $f(x) = |(-\infty, x) \cap E|$ maps $E$ onto an interval and $\mathbb{R} \setminus E$ onto a countable set.

P. Jones called our attention to a result of N.X. Uy in [10]:

**Theorem 2.6** ([10])**.** *For every compact set $E \subset \mathbb{R}^2$ of positive Lebesgue measure there is a non-constant complex-valued Lipschitz function that is holomorphic everywhere outside $E$ (including infinity).*

If we identify $\mathbb{C}$ and $\mathbb{R}^2$, we obtain a mapping $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ that is orientation preserving and open on the complement of $E$; using degree theory it follows that $f(E) \supset f(\mathbb{R}^2 \setminus E) \supset$ a ball. This gives a positive answer to Problem 2.5 in dimension $n=2$.

In dimension $n=2$ a completely different construction can also be obtained using our function $\omega$ from (iv) Definition 1.14 (more precisely, the function $u(x) = x - \omega(x)e$, whose distance from the identity is small). Instead of constructing an open mapping on $\mathbb{R}^2 \setminus E$, we show that close to a density point of $E$ a Lipschitz perturbation of the identity can be found which maps $\mathbb{R}^2 \setminus E$ onto a 1-rectifiable set (and consequently, it maps $E$ onto a set of non-empty interior):

**Theorem 2.7.** *For $n = 1, 2$ and for every $E \subset \mathbb{R}^n$ of positive Lebesgue measure there is an orientation-preserving Lipschitz mapping $f\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $f(E) = [0,1]^n$ and $f(\mathbb{R}^2 \setminus E)$ is $(n-1)$-rectifiable.*

Unfortunately none of these methods are powerful enough to construct such a mapping in higher dimension; the question in dimensions $n \geq 3$ remains open. It may be true that, in any dimension, there is a Lipschitz perturbation of the identity that maps $\mathbb{R}^n \setminus E$ onto an $(n-1)$-rectifiable set.

Another use of $\omega$ is the following. Let $\mu$ be a measure such that $\mu(S) > 0$ for some $S \in \mathcal{N}_{n,n-1}$, and let $E$ be a subset of $S$ with $\mu(E) > 0$ of $C$-width zero for some cone $C$. Let $\omega_j$ denote the function $\omega$ for $w = 1/j$. Then the functions $\omega_j\colon \mathbb{R}^n \to \mathbb{R}$ have uniformly bounded Lipschitz constants, they converge to constant 0 as $j$ tends to infinity, and $\omega_j'(x;e) \geq 0$ everywhere and $\omega_j'(x;e) = 1$ for $x \in E$.



A moment's reflection shows that $\omega_j'$ cannot converge to $0 = \omega'$ in any weak sense with respect to $\mu$ (and a straightforward smoothing argument can make them $C^1$). Therefore, for a measure $\mu$ in $\mathbb{R}^n$, weak derivatives of Lipschitz functions may be defined iff $\mu$ is absolutely continuous with respect to $\mathcal{N}_{n,n-1}$, hence iff every Lipschitz function is differentiable $\mu$-almost everywhere. For $n=2$ we know that the above holds iff $\mu$ is absolutely continuous with respect to the Lebesgue measure. This answers a problem due to G. Mokobodzki.

2.4. **Tangents of measures.** Alberti proved in [1] the so-called 'rank-one property' of $BV$ functions:

**Theorem 2.8** ([1])**.** *Let $u$ and $v$ be $BV$ functions on $\mathbb{R}^n$. Then the direction of the gradients of $u$, $v$ agree $\mu$-a.e. whenever the measure $\mu$ is singular, and absolutely continuous with respect to the variation of the gradients of both $u$ and $v$.*

This result can be understood as saying that certain class of $\mathbb{R}^n$-valued measures in $\mathbb{R}^n$, namely those that arise as singular parts of derivatives of $BV$ functions, have a.e. uniquely defined normal directions and so also 'tangent' hyperplanes. The question naturally arises: for what measures is our $(n-1)$-dimensional tangent field uniquely defined almost everywhere? Is it the same as the hyperplane defined via derivatives of $BV$ functions?

The measure has to be concentrated on $\mathcal{N}_{n,n-1}$ and it has to be absolutely continuous with respect to $\mathcal{N}_{n,n-2}$. Since sets from $\mathcal{N}_{n,n-2}$ are purely $(n-1)$-unrectifiable, for the later requirement it suffices that the measure is absolutely continuous with respect to purely $(n-1)$-unrectifiable sets. The former requirement would be equivalent to singularity if $\mathcal{N}_{n,n-1}$ coincided with Lebesgue null sets, which we do not know. But the methods used to prove it when $n=2$ are powerful enough to show that every Lebesgue null set in $\mathbb{R}^n$ is a union of a set from $\mathcal{N}_{n,n-1}$ and a purely $(n-1)$-unrectifiable set. So it suffices to assume that the measure is singular, and absolutely continuous with respect to purely $(n-1)$-unrectifiable sets.

**Definition 2.9.** A measure on $\mathbb{R}^n$ is called *$k$-rectifiable* if it is absolutely continuous with respect to $\mathcal{H}^k|_E$, where $E \subset \mathbb{R}^n$ is a $k$-rectifiable set. Measures which can be represented as integral combinations $\mu = \int \mu_t \, dP(t)$ of $k$-rectifiable measures $\mu_t$ are called *$k$-rectifiably representable.*

**Theorem 2.10.** *A measure $\mu$ is $k$-rectifiably representable if and only if $\mu(E) = 0$ for every $k$-purely unrectifiable set $E$.*

**Definition 2.11.** Given a $k$-rectifiably representable measure $\mu$ on $\mathbb{R}^n$, $\tau\colon \mathbb{R}^n \to G(n,k)$ defines a *$k$-tangent field of $\mu$*, if for every representation $\mu = \int \mu_t \, dP(t)$ where $\mu_t$ is supported on a $k$-rectifiable set $E_t$, there holds $\mathrm{Tan}(E_t, x) = \tau(x)$ for $\mu_t$-a.e. $x$ and $P$-a.e. $t$.

The $k$-tangent field, if it exists, is uniquely determined up to $\mu$-negligible sets. We show that:

**Theorem 2.12.** *An $(n-1)$-rectifiably representable measure admits an $(n-1)$-tangent field if and only if it is singular.*

Singular parts of derivatives of $BV$ functions are $(n-1)$-rectifiably representable, and indeed, the hyperplane orthogonal to the gradient and the $(n-1)$-tangent field of these measures coincide.

We finish this section by saying that, applying a version of Radon-Nýkodim Theorem, we show that

**Theorem 2.13.** *Every measure $\mu$ on $\mathbb{R}^n$ can be uniquely decomposed as*

$$\mu = \mu_n + \mu_{n-1} + \cdots + \mu_0 \,,$$

*where each $\mu_k$ is a $k$-rectifiably representable measure supported on a $(k+1)$-purely unrectifiable set.*

We do not know whether the measure $\mu_k$ admits a $k$-tangent field for $k < n-1$.

## 3. Combinatorial connections

Combinatorial connections of our results were first noted by Matoušek. He observed that a part of our proof of Laczkovich's problem is similar to the proof of the Erdős-Szekeres Theorem, and he recognised that this part may be replaced by its corollary: for any planar set $M$ having $m^2$ points there is a function $\psi\colon \mathbb{R} \to \mathbb{R}$ with $\mathrm{Lip}(\psi) \le 1$ such that one of the sets

$$\{(x,y) \in M : y = \psi(x)\} \quad \text{or} \quad \{(x,y) \in M : x = \psi(y)\}$$

has at least $m$ points.

Some of the results on which our proofs are based exploited this connection and may be considered as a continuous analogy of the Erdős-Szekeres or Dilworth Theorems. For example, if a set $E$ admits a $k$-tangent field then for every decomposition $G(n,k) = \bigcup A_j$ there corresponds a partition $E \subset \bigcup E_j$, where $E_j$ contains those $x \in E$ for which $\tau(x) \in A_j$. By definition, the set $E_j$ has width 0 with respect to any proper closed convex cone $C$ for which $C \cap S = \{0\}$ for all $S \in A_j$. That is, we can decompose $E$ into parts that can be covered by Lipschitz slabs of arbitrary small total width. Discrete analogue of this statement says that, in the plane, a finite set of points can always be covered by a small number of Lipschitz curves of given directions (and then of course one of them must contain many points).

The relation to the combinatorial results becomes even more apparent if we consider weak $k$-tangent fields. Look at only the special case $k = n-1$, and suppose that $E \subset [0,1]^n$ is Lebesgue null and compact. Then we can approximate $E$ by a grid: it intersects $o(N^n)$ out of $N^n$ subcubes of $[0,1]^n$. Let $C$ be a convex cone, and consider the partial order on $\mathbb{R}^n$ defined by $x_1 \prec x_2 \iff x_2 - x_1 \in C$. By Dilworth Theorem, the set of the centres of the cubes intersecting $E$ can be covered by $o(N^{n-1})$ chains and $o(N)$ antichains. Chains are curves going in the direction of $C$ and antichains are $C$-Lipschitz surfaces. Since $E$ lies in a $O(1/N)$ neighbourhood of the set of the centres of the cubes, it is covered by $o(N^{n-1})$ 'tubes' going in the direction $C$ and by $o(N)$ Lipschitz slabs of width $O(1/N)$, i.e. by tubes of arbitrary small total cross-sectional volume and by slabs of arbitrary small total width. The set covered by tubes meets $C$-Lipschitz surfaces in a set of small $\mathcal{H}^{n-1}$ measure, and the set covered by slabs meets curves going in the direction $C$ in small length.



This decomposition leads to a weak $(n-1)$-tangent field as we let the angle of $C$ tend to a halfspace and its direction run through a dense set of directions. For other results we would need to cover by slabs only, and we do not know if this is always possible, except for the 2-dimensional case where there is no difference between tubes and slabs.

Many results presented here are connected to a possibility of decomposing certain small sets, or perhaps even all Lebesgue null sets, in a way reminiscent of the decompositions of finite sets in combinatorial results. As we have seen above, the existence of a weak $(n-1)$-tangent field is a direct corollary of Dilworth Theorem. For other problems we need a much finer, continuous version of the combinatorial results whose proofs also use techniques that are not available in the discrete world, they are purely analytic.

There are also problems that could be solved using discrete decomposition results, but we do not know if the discrete versions are true. Matoušek conjectured a higher dimensional variant of the Erdős-Szekeres Theorem that would fully solve Laczkovich's problem. This conjecture was disproved by Tardos. One can however modify his conjecture so that it would imply a positive answer to our main problem (all Lebesgue null sets would belong to $\mathcal{N}_{n,n-1}$):

**Conjecture 3.1.** For any set $M \subset \mathbb{R}^n$ having $m^n$ points there is a function $\psi\colon \mathbb{R}^{n-1} \to \mathbb{R}$ with $\mathrm{Lip}(\psi) \le C_n$ and *an orthonormal system of coordinates* such that the set

$$\{(x_1,\dots,x_n) \in M : x_n = \psi(x_1,\dots,x_{n-1})\}$$

has at least $c_n m^{n-1}$ points.

This problem is open. We only show that, unlike in the plane, the coordinate systems cannot be restricted to permutations of the standard coordinate system, not even in $\mathbb{R}^3$:

**Theorem 3.2.** *For every Lipschitz constant $L$ and for every $\varepsilon > 0$ there exists a finite set $M \subset \mathbb{R}^3$ of $m^3$ points, such that for every $\phi\colon \mathbb{R}^2 \to \mathbb{R}$ with $\mathrm{Lip}(\phi) < L$, in the standard coordinate system in $\mathbb{R}^3$, all the three graphs $x = \phi(y,z)$, $y = \phi(x,z)$ and $z = \phi(x,y)$ contain less than $\varepsilon m^2$ points of $M$.*

However, the *dyadic* analogue of Conjecture 3.1 is true in any dimension, even in the standard coordinate-system for Lipschitz mappings with constant 1.

Consider the unit cube $Q = [0,1]^n \subset \mathbb{R}^n$. A cube in $Q$ is called a *dyadic cube* of size $1/2^k$, if it is obtained by dividing $Q$ to $2^{kn}$ subcubes of equal sizes in the obvious manner. Let $Q_0$ be the set of points that are not on the boundary of any dyadic cube. The *dyadic distance* of two points $x, y \in Q_0$ is the size of the smallest dyadic cube that contains both $x$ and $y$. This defines a metric on $Q_0$. In a current work M. Csörnyei and P. Jones showed that:

**Theorem 3.3.** (i) *For any set $M \subset Q_0 \subset \mathbb{R}^n$ having $m^n$ points there is a function $\psi\colon \mathbb{R}^{n-1} \to \mathbb{R}$ with dyadic Lipschitz constant 1 and there is a coordinate-direction $x_k$ $(k = 1,2,\dots,n)$ such that the set*

$$\{(x_1,\dots,x_n) \in M : x_k = \psi(x_1,\dots,x_{k-1},x_{k+1},\dots,x_n)\}$$

*has at least $m^{n-1}$ points.*

(ii) *Every set $E \subset Q_0 \subset \mathbb{R}^n$ of Lebesgue measure $m$ can be covered by dyadic Lipschitz slabs of total width at most $m^{1/n}$.*

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G.A.
Dipartimento di Matematica, Università di Pisa
largo Pontecorvo 5, 56127 Pisa, Italy
e-mail: `galberti1@dm.unipi.it`

M.C.
Department of Mathematics, University College London
Gower Street, London, WC1E 6BT, United Kingdom
e-mail: `mari@math.ucl.ac.uk`

D.P.
Mathematics Institute, University of Warwick
Coventry, CV4 7AL, United Kingdom
e-mail: `d.preiss@warwick.ac.uk`