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Optimal reinforcing networks for elastic membranes

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Abstract. In this paper we study the optimal reinforcement of an elastic membrane, fixed at its boundary, by means of a network (connected one-dimensional structure), that has to be found in a suitable admissible class. We show the existence of an optimal network, and observe that such network carries a multiplicity that in principle can be strictly larger than one. Some numerical simulations are shown to confirm this issue and to illustrate the complexity of the optimal network when the total length becomes large.

Keywords: Optimal networks, elastic membranes, reinforcement, relaxed solution, Golab's semicontinuity theorem.

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1. Introduction

In the present paper we consider the vertical displacement of an elastic membrane under the action of an exterior load f and fixed at its boundary; this amounts to solve the variational problem

$$\min\left\{\frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega fu\,dx \;:\; u\in H^1_0(\Omega)\right\}, \tag{1.1}$$

or equivalently the elliptic PDE

$$-\Delta u = f \text{ in } \Omega, \qquad u\in H^1_0(\Omega).$$

Here Ω is a bounded Lipschitz domain of $\mathbb{R}^2$, $f\in L^2(\Omega)$, and $H^1_0(\Omega)$ is the usual Sobolev space of functions with zero trace on the boundary $\partial\Omega$.

Our goal is to rigidify the membrane by adding a one-dimensional reinforcement in the most efficient way; the reinforcement is described by a one-dimensional set $S\subset\Omega$ which varies in a suitable class of admissible choices. The effect of S on the membrane is described by the energy

$$\mathcal{E}_f(S) := \inf\left\{\frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{m}{2}\int_S |\nabla u|^2\,d\mathscr{H}^1 - \int_\Omega fu\,dx \;:\; u\in C^\infty_c(\Omega)\right\} \tag{1.2}$$

that has to be maximized in the class of admissible choices for S.

Here $m>0$ is a fixed parameter that represents the stiffness coefficient of the one-dimensional reinforcement, $\mathscr{H}^1$ denotes the 1-dimensional Hausdorff measure (that is, the length measure), while $C^\infty_c(\Omega)$ denotes the class of smooth functions with compact support in Ω.

The optimization problem we deal with consists in finding the "best" reinforcement S among all networks with total length bounded by a prescribed L, that is, all S in the class

$$\mathscr{A}_L := \Big\{S \text{ closed connected subset of } \Omega \text{ with } \mathscr{H}^1(S)\le L\Big\},$$

We then consider the maximization of the energy functional $\mathcal{E}_f(S)$ in (1.2) over this class, that is,

$$\max\Big\{\mathcal{E}_f(S)\colon S\in\mathscr{A}_L\Big\}\,. \tag{1.3}$$

1.1. Gradient versus tangential gradient. From the modeling point of view, it is natural to ask whether the gradient ∇u that appears in the line integral

$$\frac{m}{2}\int_S|\nabla u|^2\,d\mathscr{H}^1$$

in (1.2) should be replaced by the tangential gradient $\nabla_\tau u$. It turns out that the question is irrelevant, at least if we strictly follow a variational approach, because the value of $\mathcal{E}_f(S)$ is not affected by this change (Theorem 2.5).

Indeed, if S is a compact curve of class C^1 contained in Ω, it is well-known (see for instance [4]) that the relaxation of the integral

$$F(u):=\int_S|\nabla u|^2\,d\mathscr{H}^1\,,\quad u\in C_c^\infty(\Omega)\,,$$

is given by

$$F^*(u):=\int_S|\nabla_\tau u|^2\,d\mathscr{H}^1\,,\quad u\in H^1(S)\,.$$

This relaxation result holds also when S is a compact connected set with finite length, provided that $H^1(S)$ and ∇_τ are properly defined (this statement is implicitly contained in Proposition 2.11). However, we warn the reader that this relaxation result does not holds if S is an arbitrary compact subset of a curve of class C^1 with positive length; in particular, if S is totally disconnected then the relaxation of F is equal to 0 for every u.

1.2. Concentrated loads. Besides the case of *distributed loads*, which consists in assuming that f belongs to some Lebesgue class $L^p(\Omega)$, we may also consider the case of *concentrated loads*, in which f may have a more singular behavior. More precisely, we may assume that f is a signed measure on Ω. (In this case the linear term $\int_\Omega fu\,dx$ in (1.2) should be written as $\int_\Omega u\,df$.)

We recall that a measure f does not necessarily belong to the dual of the Sobolev space $H_0^1(\Omega)$, and therefore $\mathcal{E}_f(S)=-\infty$ for some choices of S. Clearly, if $\mathcal{E}_f(S)$ is finite for at least one S problem (1.3) still makes sense, and we may discard all S such that $\mathcal{E}_f(S)=-\infty$. However, it may happen that $\mathcal{E}_f(S)=-\infty$ for every S in $\mathscr{A}_L$ and in that case problem (1.3) does not make sense (see Example 2.8).

1.3. An optimization problem in a model for traffic congestion. Another optimization problem requiring a similar analytical approach arises in a model for the reduction of traffic congestion in a given geographic area. Here the minimum problem is

$$\min\left\{\int_\Omega H(\sigma)\,dx\;:\,-\operatorname{div}\sigma=f\ \text{in}\ \Omega,\ \sigma\cdot n=0\ \text{on}\ \partial\Omega\right\}, \tag{1.4}$$

where $f=f^+-f^-$ and in the region Ω the function f^+ represents the density of residents while f^- is the density of working places. The vector σ is the traffic flux and the function H describes the transportation cost; the case $H(s):=|s|$ gives the classical Monge's problem, while we talk of *congested transport* if the function H is super-linear at infinity, that is,

$$\lim_{|s|\to\infty}\frac{H(s)}{|s|}=+\infty\,.$$

We refer to [3, 5, 6, 18] and to the references therein for a detailed description of this model. In the case $H(s) := |s|^2/2$, the minimization problem (1.4) reduces, via a duality argument, to a problem of the form (1.1).

The optimization problem arises when a new road, or network of roads, S has to be built to reduce the congestion; the total length L is prescribed and on the new road the congestion function is strictly lower than $|s|^2/2$, for example $\alpha|s|^2/2$ with $\alpha < 1$. The problem then consists in finding the optimal one-dimensional set S, and we end up, via a duality argument, with a problem similar to (1.3), with $m := 1/\alpha$.

1.4. Relaxed formulation of the optimization problem. The optimization problem (1.3) is solved, *in a suitable relaxed form*, in Section 2, to which we refer for precise statements and definitions.

We explain first the need for a relaxed formulation. Consider a maximizing sequence (S_n) for problem (1.3): since these sets are closed, connected, and satisfy $\mathscr{H}^1(S_n) \le L$, they converge, up to subsequence and in Hausdorff distance, to some connected compact set S_∞ with $\mathscr{H}^1(S_\infty) \le L$ contained in the closure $\overline{\Omega}$. The problem is that the functional $\mathcal{E}_f(S)$ is not upper semicontinuous in S with respect to Hausdorff convergence, and therefore S_∞ may be not a solution of problem (1.3).

However it turns out that $\mathcal{E}_f(S)$ is upper semicontinuous if we identify the sets S with the measures $\mathscr{H}^1 \llcorner S$, namely the restrictions of the Hausdorff measure $\mathscr{H}^1$ to S, and consider the weak* convergence of measures instead of the Hausdorff convergence of sets. More precisely, we extend the energy functional (1.2) to general positive measures μ on $\overline{\Omega}$ by setting

$$\mathcal{E}_f(\mu) := \inf\left\{\frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{m}{2}\int_\Omega |\nabla u|^2\,d\mu - \int_\Omega u\,df : u \in C_c^\infty(\Omega)\right\}$$

(here we assume that the load f is a signed measure). Notice that this new functional extends the previous one in the sense that $\mathcal{E}_f(\mu) = \mathcal{E}_f(S)$ when $\mu = \mathscr{H}^1 \llcorner S$, and it is upper semicontinuous with respect to the weak* convergence of μ (Proposition 3.1).

The problem now is that weak* limits of measures of the form $\mathscr{H}^1 \llcorner S$ are not necessarily measures of the same form, and in particular the limit μ_∞ of the measures $\mathscr{H}^1 \llcorner S_n$ is a measure supported on the set S_∞, but may be not the measure $\mathscr{H}^1 \llcorner S_\infty$; if it is, then S_∞ is a solution of the optimization problem (1.3), but otherwise it is not.

These considerations lead to the following relaxed version of problem (1.3):

$$\max\left\{\mathcal{E}_f(\mu) : \mu \in \mathscr{M}_L\right\}, \tag{1.5}$$

where $\mathscr{M}_L$ is the class of all weak* limits of measures of the form $\mathscr{H}^1 \llcorner S$ with S admissible network, that is, $S \in \mathscr{A}_L$.

The class $\mathscr{M}_L$ is completely described in Proposition 2.1, and the existence of a solution of the relaxed optimization problem (1.5) is proved in Theorem 2.2. In Theorem 2.6 we show that there is always a solution of the form $\mu = \theta\,\mathscr{H}^1 \llcorner S$ where S is a compact, connected set with finite length contained in $\overline{\Omega}$, and θ is a real-valued multiplicity function which satisfies $\theta(x) \ge 1$ for $\mathscr{H}^1$-a.e. $x \in S$.

If $\theta(x) = 1$ for $\mathscr{H}^1$-a.e. $x \in S$, then S is a solution of the original optimization problem (1.3). If not, then problem (1.3) may have no solution.

In Section 5 we present some numerical simulations which show unexpected behaviors of the optimal measures μ; in particular we have evidence that in some situations (and perhaps most situations) the multiplicity θ may be strictly larger than 1 in a subset of positive length of S.

1.5. Final remarks. (i) We do not know if problem (1.5) is "the" relaxation of problem (1.3), and in particular we cannot exclude that some kind of Lavrentiev phenomenon occurs (for more details see Problem 6.2 and Remark 6.3).

(ii) The connectedness assumption on S is crucial: indeed, removing this constraint allows a sequence of maximizing sets S_n to spread all over Ω and leads to a relaxed problem of the form

$$\max\Big\{\mathcal{E}_f(\mu)\colon \mu\in\mathscr{M}^+(\overline{\Omega})\,,\ \mu(\overline{\Omega})\le L\Big\}\,,$$

where $\mathscr{M}^+(\overline{\Omega})$ is the class of all positive measures on $\overline{\Omega}$. This optimization problem has been studied in [11] and in [7], where it is shown that the optimal measure μ actually belongs to $L^p(\Omega)$ and the exponent p depends on the summability of the right-hand side f. Similar problems, in the extreme case when in the reinforcing region a Dirichlet condition is imposed, have been considered in [9, 10].

(iii) In the definitions of $\mathcal{E}_f(S)$ and $\mathcal{E}_f(\mu)$ we required that u belongs to $C_c^\infty(\Omega)$ to ensure that all integrals make sense. Clearly it would be equivalent to consider continuous functions in $H_0^1(\Omega)$ that are of class C^1 in a neighborhood of S or of the support of μ. One can go further, and take u in a suitably defined Sobolev space, so that the infimum in the definition of $\mathcal{E}_f(\mu)$ is a minimum (Proposition 2.11).

(iv) In our model the stiffener S is a one-dimensional set and its contribution to the total energy is described by the line integral

$$\frac{m}{2}\int_S |\nabla_\tau u|^2\,d\mathscr{H}^1\,.$$

This choice is consistent with the fact that the integral above is the variational limit as $\varepsilon\to 0$ of the integrals

$$\frac{m}{2\varepsilon}\int_{S_\varepsilon}|\nabla u|^2\,dx\,,$$

where S_ε is the thin strip $S_\varepsilon := \big\{x\in\mathbb{R}^2\colon \operatorname{dist}(x,S)<\varepsilon/2\big\}$. In other words, the one-dimensional stiffener S can be seen as the limit structure of two-dimensional thin strips of thickness ε and elastic constants m/ε (see for instance [17] and [2]).

Structure of the paper. In Section 2 we give a precise formulation of the relaxed optimization problem, and state the main existence results (Theorems 2.2 and 2.6). In Section 3 we prove the results stated in Section 2. In Section 4 we give some additional properties that solutions of the relaxed optimization problem must satisfy. Section 5 is devoted to the numerical approximation of the relaxed problem. Section 6 contains additional remarks and open problems.

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2. Existence of solutions of the relaxed problem

Let us fix/recall the basic notation. Unless we specify otherwise, for the rest of this paper Ω is a bounded Lipschitz domain in $\mathbb{R}^2$, the load f is a signed measure on

Ω, the class $\mathscr{A}_L$ of admissible reinforcements consists of all closed connected subsets of Ω with $\mathscr{H}^1(S) \leq L$.

For every $S \in \mathscr{A}_L$ the functional $\mathcal{E}_f(S)$ is defined by formula (1.2), with the linear term $\int_\Omega fu\,dx$ written as $\int_\Omega u\,df$ because f is now a measure. The optimization problem (1.3) consists in finding the maximum of $\mathcal{E}_f(S)$ among all $S \in \mathscr{A}_L$, and it makes sense provided that $\mathcal{E}_f$ is not identically $-\infty$ (see Subsection 1.2).

In the following we assume that $\mathcal{E}_f(S)$ is finite for some set $S \in \mathscr{A}_L$.

As explained in Subsection 1.4, we denote by $\mathscr{M}^+(\overline{\Omega})$ the class of all positive finite measures on $\overline{\Omega}$, and extend $\mathcal{E}_f$ to all μ in $\mathscr{M}^+(\overline{\Omega})$ by

$$\mathcal{E}_f(\mu) := \inf\Big\{E_f(\mu,u) \colon u \in C_c^\infty(\Omega)\Big\}, \tag{2.1}$$

where

$$E_f(\mu,u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{m}{2}\int_\Omega |\nabla u|^2\,d\mu - \int_\Omega u\,df\,. \tag{2.2}$$

Then we denote by $\mathscr{M}_L$ the weak* closure in $\mathscr{M}^+(\overline{\Omega})$ of the class of all measures of the form $\mathscr{H}^1 \llcorner S$ with $S \in \mathscr{A}_L$, in short,

$$\mathscr{M}_L := \overline{\big\{\mathscr{H}^1 \llcorner S \colon S \in \mathscr{A}_L\big\}}^{\text{weak*}},$$

and the relaxed optimization problem (1.5) consists in finding the maximum of $\mathcal{E}_f(\mu)$ among all $\mu \in \mathscr{M}_L$.

Proposition 2.1. *The class $\mathscr{M}_L$ consist of all $\mu \in \mathscr{M}^+(\overline{\Omega})$ such that*

(a) *$\mu(\overline{\Omega}) \leq L$;*
(b) *the support S of μ is a connected, compact set in $\overline{\Omega}$ with $\mathscr{H}^1(S) \leq L$;*
(c) *$\mathscr{H}^1 \llcorner S \leq \mu$.*

We can now state our first existence result:

Theorem 2.2. *The optimization problem (1.5) admits a solution $\mu \in \mathscr{M}_L$.*

Let μ be a measure in $\mathscr{M}_L$ with support S. We want to show that the value of $\mathcal{E}_f(\mu)$ is not affected if we replace the full gradient ∇u in the integral $\int_\Omega |\nabla u|^2 d\mu$ in (2.2) with the tangential gradient $\nabla_\tau u$, and if we remove from the measure μ the part which is singular with respect to $\mathscr{H}^1 \llcorner S$.

To this purpose we need to recall some well-known properties of connected sets S with finite length (for more details we refer to standard references, such as [13]).

2.3. Connected sets with finite length. Let S be a compact connected set in $\mathbb{R}^d$ with finite length. Then S is rectifiable, which means that it can be covered (up to an $\mathscr{H}^1$-negligible subset) by countably many embedded curves of class C^1, and indeed it can be parametrized (although not bijectively) by a single Lipschitz path.

Moreover S admits a tangent line $\tau(x)$ at $\mathscr{H}^1$-a.e. $x \in S$. Thus, for every such x and every $u : \mathbb{R}^d \to \mathbb{R}$ of class C^1 we can define the tangential gradient $\nabla_\tau u(x)$.

2.4. The functionals E_f^* and $\mathcal{E}_f^*$. Let μ be a measure in $\mathscr{M}^+(\overline{\Omega})$ of the form

$$\mu = \theta\,\mathscr{H}^1 \llcorner S\,,$$

where S is a compact connected set in $\overline{\Omega}$ with finite length. We set

$$\mathcal{E}_f^*(\mu) := \inf\Big\{E_f^*(\mu,u) \colon u \in C_c^\infty(\Omega)\Big\}, \tag{2.3}$$

where

$$E_f^*(\mu,u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{m}{2}\int_S |\nabla_\tau u|^2\,\theta\,d\mathscr{H}^1 - \int_\Omega u\,df\,.$$

Theorem 2.5. *Let μ be a measure in $\mathscr{M}_L$ with support S, and let μ^a be the absolutely continuous part of μ with respect to $\mathscr{H}^1 \llcorner S$. Then*

$$\mathcal{E}_f(\mu) = \mathcal{E}_f(\mu^a) = \mathcal{E}^*_f(\mu^a) .$$

This statement shows that the maximum problem (1.5) can be reformulated as the maximum of $\mathcal{E}_f(\mu)$, or equivalently of $\mathcal{E}^*_f(\mu)$, on the class

$$\mathscr{M}^a_L := \Big\{ \mu \in \mathscr{M}_L \colon \mu \text{ is absolutely continuous w.r.t. } \mathscr{H}^1 \llcorner \operatorname{spt}(\mu) \Big\} . \tag{2.4}$$

In view of Proposition 2.1, $\mathscr{M}^a_L$ agrees with the class of all measures of the form $\mu = \theta\, \mathscr{H}^1 \llcorner S$ where

(a) S is a compact connected set in $\overline{\Omega}$ with $\mathscr{H}^1(S) = L$,
(b) θ is a real-valued multiplicity function with $\theta(x) \geq 1$ for $\mathscr{H}^1$-a.e. $x \in S$,
(c) $\mu(\overline{\Omega}) = \int_S \theta \, d\mathscr{H}^1 \leq L$.

The following improvement of Theorem 2.2 holds:

Theorem 2.6. *Problem (1.5) admits a solution μ in the class $\mathscr{M}^a_L$ with $\mu(\overline{\Omega}) = L$, which is therefore a solution of*

$$\max \Big\{ \mathcal{E}^*_f(\mu) \colon \mu \in \mathscr{M}^a_L \, , \ \mu(\overline{\Omega}) = L \Big\} .$$

If in addition f belongs to $L^p(\Omega)$ for some $p > 1$ and the support of f is $\overline{\Omega}$, then every *solution of problem (1.5) belongs to $\mathscr{M}^a_L$ and satisfies $\mu(\overline{\Omega}) = L$.*

Remark 2.7. (i) The first part of Theorem 2.6 is little more than a corollary of Theorem 2.5. The second part, however, requires a more delicate argument.

(ii) The assumption that the support of f is $\overline{\Omega}$ in the second part of Theorem 2.6 can probably be weakened, but not entirely removed. Indeed if $f = 0$ then $\mathcal{E}_f(\mu) = 0$ for every $\mu \in \mathscr{M}^+(\overline{\Omega})$, and in particular every $\mu \in \mathscr{M}_L$ is a solution of problem (1.5).

(iii) As already pointed out in Subsection 1.4, if the solution $\mu = \theta \mathscr{H}^1 \llcorner S$ given by Theorem 2.6 verifies $\theta = 1$ a.e., that is, $\mu = \mathscr{H}^1 \llcorner S$, then S is a solution of the original optimization problem (1.3). However, the following example suggests that this is not always the case.

Example 2.8. Let A, B be two points in Ω such that the closed segment $[A, B]$ is contained in Ω, and let $f := \delta_A - \delta_B$. Regarding the maximization problem (1.5) three possibilities may occur, depending on the choice of L:

- If $L < |A - B|$ then we have $\mathcal{E}_f(S) = -\infty$ for every S in the class $\mathscr{A}_L$. Indeed, since a connected set S of length $L < |A - B|$ cannot contain both A and B, and since the capacity of a point in the plane is zero, we may construct a sequence of function $u_n \in C^\infty_c(\Omega)$ which vanish on S, tend to 0 in $H^1(\Omega)$, and satisfy $u_n(A) - u_n(B) \to +\infty$.
- If $L = |A - B|$ then the unique set S for which the energy is not $-\infty$ is the segment $[A, B]$, which is then the unique solution of the maximization problem (1.3), while $\mu := \mathscr{H}^1 \llcorner S$ is the unique solution of problem (1.5).
- If $L > |A - B|$ and $\mu = \theta\, \mathscr{H}^1 \llcorner S$ is a solution of problem (1.5) as in Theorem 2.6, then the numerical simulations in Section 5 give a strong indication that $\theta > 1$ on a subset of S with positive length.

We conclude this section by defining the Sobolev space $H^1_0(\Omega) \cap H^1(S)$, and showing that the infimum in formula (2.3) is actually a minimum. This fact will be used in the proof of the second part of Theorem 2.6.

2.9. The Sobolev space $H^1(S)$. Let S be a compact connected set in $\mathbb{R}^d$ with finite length $\ell := \mathscr{H}^1(S)$. Using [1, Theorem 4.4] we obtain a closed Lipschitz path $\gamma : [0,1] \to S$ which parametrizes S, and more precisely

- γ has multiplicity 2 at $\mathscr{H}^1$-a.e. $x \in S$, that is, $\#(\gamma^{-1}(x)) = 2$;
- $|\dot\gamma(t)| = 2\ell$ for a.e. $t \in [0,1]$.

We then define the Sobolev space $H^1(S)$ as the space of all continuous functions $u : S \to \mathbb{R}$ such that the composition $u \circ \gamma$ belongs to $H^1(I)$, where $I := (0,1)$.[1]

Note that for a.e. $t \in [0,1]$ the vector $\dot\gamma(t)$ spans $\tau(x)$, that is, the tangent line to S at $x := \gamma(t)$. Therefore a function $u \in H^1(S)$ is differentiable along $\tau(x)$ for $\mathscr{H}^1$-a.e. $x \in S$ and the tangential derivative satisfies

$$|\nabla_\tau u(x)| = \frac{1}{2\ell}\big|(u\circ\gamma)'(t)\big|\,.$$

Thus the area formula for Lipschitz maps yields

$$\big\|u\big\|_2^2 := \int_S |u|^2\, d\mathscr{H}^1 = \ell \int_0^1 \big|(u\circ\gamma)'(t)\big|^2\, dt\,.$$

and

$$\big\|\nabla_\tau u\big\|_2^2 := \int_S |\nabla_\tau u|^2\, d\mathscr{H}^1 = \frac{1}{4\ell}\int_0^1 \big|(u\circ\gamma)'(t)\big|^2\, dt\,.$$

We endow $H^1(S)$ with the Hilbert norm

$$\|u\|^2_{H^1(S)} := \|u\|_2^2 + \|\nabla_\tau u\|_2^2.$$

2.10. The Sobolev space $H^1_0(\Omega) \cap H^1(S)$. Let S be a connected compact set in $\overline{\Omega}$ with finite length $\ell := \mathscr{H}^1(S)$. Since S is rectifiable, we can find a strictly positive (Borel) density function $m : S \to \mathbb{R}$ such that the trace operator

$$T_S : H^1(\mathbb{R}^2) \to L^2(m\,\mathscr{H}^1 \llcorner S)$$

is well defined and bounded.[2]

Then we define $H^1_0(\Omega)\cap H^1(S)$ as the space of all $u \in H^1_0(\Omega)$ such that $T_S u$ agrees a.e. with a function in $H^1(S)$. In the following we tacitly assume that u agrees on S with the representative of the trace $T_S u$ in $H^1(S)$, and in particular u is continuous on S. We endow $H^1_0(\Omega) \cap H^1(S)$ with the Hilbert norm

$$\big\|u\big\|^2_{H^1_0(\Omega)\cap H^1(S)} := \big\|\nabla u\big\|^2_{L^2(\Omega)} + \big\|\nabla_\tau u\big\|^2_{L^2(S)}\,.$$

(Completeness can be proved using the continuity of the trace operator T_S.) Using the fact that functions of class $H^{1/2}$ on intervals do not admit discontinuities of jump type, one can prove that for every u in $H^1_0(\Omega) \cap H^1(S)$, there holds

$$u(x) = 0 \quad \text{for every } x \in S \cap \partial\Omega.$$

[1] It can be proved that given a function u in $H^1(S)$ and a Lipschitz function $\varphi : [0,1] \to S$ such that $|\dot\varphi(t)| \geq \delta$ for some given positive δ and for a.e. t, then $u \circ \varphi$ belongs to $H^1(I)$. In particular the definition of $H^1(S)$ does not depend on the choice of the parametrization γ.

[2] Take for instance countably many compact curves S_n of class C^1 in $\mathbb{R}^2$ that cover $\mathscr{H}^1$-almost all of S, let m_n be the norm of the trace operators $T_n : H^1(\mathbb{R}^2) \to L^2(S_n)$, and set

$$m(x) := \sum_n \frac{1}{2^n m_n^2} 1_{S_n}(x)\,.$$

Proposition 2.11. *Let μ be a measure of the form $\mu = \theta\,\mathscr{H}^1 \llcorner S$ where S is a compact connected set with finite length in $\overline{\Omega}$ and the multiplicity θ is larger than some positive constant. If f belongs to $L^p(\Omega)$ for some $p > 1$ then $\mathcal{E}_f(\mu) = \mathcal{E}^*_f(\mu)$ is finite and*

$$\mathcal{E}_f(\mu) = \mathcal{E}^*_f(\mu) = \min\Big\{E^*_f(\mu, u) \colon u \in H^1_0(\Omega) \cap H^1(S)\Big\}. \tag{2.5}$$

3. Proofs of the results in Section 2

Through this section, we denote by $|M|$ the operator norm of a matrix M.

Proposition 3.1. *The functional $\mathcal{E}_f$ defined in (2.1) is weakly* upper semicontinuous on $\mathscr{M}^+(\overline{\Omega})$.*

Proof. Just notice that $\mathcal{E}_f(\mu)$ is defined as the infimum of $E_f(\mu, u)$ over all u in $C^\infty_c(\Omega)$ and $E_f(\mu, u)$ is clearly weakly* continuous in μ for every such u, cf. (2.2). □

Proof of Theorem 2.2. This statement follows from Proposition 3.1 and the weak* compactness of the class $\mathscr{M}_L$ (an immediate consequence of its definition). □

To prove Proposition 2.1 we need the following results, which we state in $\mathbb{R}^d$ even if we need only the case $d = 2$.

Proposition 3.2. *Let (S_n) be a sequence of compact connected sets in $\mathbb{R}^d$ which converge to a set S in the Hausdorff distance, let (μ_n) be a sequence of positive finite measures on $\mathbb{R}^d$ which converge to a measure μ in the weak* sense, and assume that*

- *μ_n is supported on S_n and $\mathscr{H}^1 \llcorner S_n \le \mu_n$;*
- *$\mathscr{H}^1(S_n) \le L$ for some finite constant L.*

Then

- *μ is supported on S and $\mathscr{H}^1 \llcorner S \le \mu$;*
- *$\mathscr{H}^1(S) \le \liminf \mathscr{H}^1(S_n) \le L$.*

Proof. The fact that μ is supported on S follows easily from the weak* convergence of μ_n to μ and the convergence of S_n to S in the Hausdorff distance.

The inequality $\mathscr{H}^1(S) \le \liminf \mathscr{H}^1(S_n)$ is Gołąb's semicontinuity theorem (see [14, Section 3], or [1, Theorem 2.9]).

The inequality $\mathscr{H}^1 \llcorner S \le \mu$ can be viewed as a localized version of Gołąb's theorem, and the proof is slightly more complicated. Using [1, Theorem 4.4], we obtain that each S_n can be parametrized by a closed path $\gamma_n : [0, 1] \to S_n$ such that

- γ_n has multiplicity 2 at $\mathscr{H}^1$-a.e. $x \in S_n$, that is, $\#(\gamma_n^{-1}(x)) = 2$;
- γ_n has degree 0 at $\mathscr{H}^1$-a.e. $x \in S_n$ (for a precise definition see [1, §4.1]);
- each γ_n has Lipschitz constant $\mathrm{Lip}(\gamma_n) \le 2L$.

Passing to a subsequence we can assume that the paths γ_n converge uniformly to some $\gamma : [0, 1] \to S$ with $\mathrm{Lip}(\gamma) \le 2L$, and one easily checks that γ parametrizes S. Moreover [1, Proposition 4.3] shows that γ has degree 0 and in particular has multiplicity at least 2 at $\mathscr{H}^1$-a.e. $x \in S$.

Therefore, for every *positive* test function $\varphi \in C_c(\mathbb{R}^d)$ there holds

$$\begin{aligned}\int_{\mathbb{R}^d} \varphi\, d\mu = \lim_{n\to\infty} \int_{\mathbb{R}^d} \varphi\, d\mu_n &\geq \liminf_{n\to\infty} \int_{S_n} \varphi\, d\mathscr{H}^1 \\ &= \liminf_{n\to\infty} \frac{1}{2}\int_0^1 \varphi(\gamma_n)\, |\dot\gamma_n|\, dt \\ &\geq \frac{1}{2}\int_0^1 \varphi(\gamma)\, |\dot\gamma|\, dt \geq \int_S \varphi\, d\mathscr{H}^1\,.\end{aligned}$$

The first inequality follows from the assumption $\mu_n \geq \mathscr{H}^1 \llcorner S_n$ and the fact that φ is positive, the second equality and the third inequality follow from the area formula for Lipschitz maps, and finally the second inequality follows by a standard semicontinuity argument.

We have thus proved that

$$\int_{\mathbb{R}^d} \varphi\, d\mu \geq \int_S \varphi\, d\mathscr{H}^1$$

for every test function $\varphi \geq 0$, which yields the desired inequality $\mu \geq \mathscr{H}^1 \llcorner S$. □

Lemma 3.3. *Let S be a connected compact set with finite length in $\mathbb{R}^d$, and let μ be a positive finite measure supported on S such that $\mu \geq \mathscr{H}^1 \llcorner S$. Then there exists a sequence of connected compacts sets S_n in $\mathbb{R}^d$ such that*

- *$\mathscr{H}^1(S_n) \leq \mu(\mathbb{R}^d)$ for every n,*
- *the sets S_n converge to S in the Hausdorff distance;*
- *the measures $\mathscr{H}^1 \llcorner S_n$ converge weakly* to μ.*

Proof. Choose a unit vector $e \in \mathbb{R}^d$ which is not in the approximate tangent line to S at x for $\mathscr{H}^1$-a.e. $x \in S$. Consider then the measure $\lambda := \mu - \mathscr{H}^1 \llcorner S$. Since λ is positive and supported on S, it can be approximated by a sequence of positive discrete measures

$$\lambda_n := \sum_j a_{nj} \delta_{x_{nj}}$$

where

- the points x_{nj} belong to S,
- the coefficients a_{nj} converge uniformly to 0 as $n \to \infty$,
- $\lambda_n(\mathbb{R}^d) \leq \lambda(\mathbb{R}^d)$ for every n, that is, $\sum_j a_{nj} \leq \mu(\mathbb{R}^d) - \mathscr{H}^1(S)$.

Thanks to the choice of e we can further require that

- $x_{nj} - x_{ni}$ is not parallel to e for every n and every $i \neq j$.

Finally we set

$$S_n := S \cup \Big(\bigcup_j I_{nj}\Big)$$

where I_{nj} is the closed segment with endpoints x_{nj} and $x_{nj} + a_{nj}e$. By the choice of the points x_{ni} we have that the segments I_{nj} are pairwise disjoint (for fixed n) and have negligible intersection with S. Now it is easy to check that the sets S_n have the required properties. □

Proof of Proposition 2.1. We first prove that every measure $\mu \in \mathscr{M}_L$ satisfies properties (a)–(c) in Proposition 2.1. Take indeed a sequence of sets $S_n \in \mathscr{A}_L$ such that $\mathscr{H}^1 \llcorner S_n$ weakly* converge to μ. Possibly passing to a subsequence we can assume that the sets S_n converge in Hausdorff distance to some compact connected set S

contained in $\overline{\Omega}$. Then Proposition 3.2 implies that S is the support of μ and μ belongs to $\mathscr{M}_L$.

The converse implication, namely that every positive measure that satisfies properties (a)-(c) belongs to $\mathscr{M}_L$, follows from Lemma 3.3. $\square$

The proof of Theorem 2.5 is split in two parts (Propositions 3.9 and 3.12), the proofs of which require several lemmas. Some of these lemmas are stated in general dimension d even if we only need the case $d = 2$.

Lemma 3.4. *Let $K \subset \mathbb{R}$ be a compact set with $|K| = 0$. For every $\varepsilon > 0$ there exists a function $f : \mathbb{R} \to \mathbb{R}$ of class $C^\infty(\mathbb{R})$ such that*

- *$|f(x) - x| \le \varepsilon$ for all $x \in \mathbb{R}$;*
- *$f' = 0$ in a neighborhood of K;*
- *$0 \le f'(x) \le 1$ for all $x \in \mathbb{R}$;*
- *the open set $A := \{x \colon f'(x) \ne 1\}$ satisfies $|A| \le \varepsilon$.*

Proof. For every $\delta > 0$ let

- K_δ be the open δ-neighborhood of K;
- ρ_δ be a smooth regularizing kernel with support contained in $[-\delta, \delta]$,
- $g_\delta := \rho_\delta * 1_{K_{2\delta}}$ and f_δ be a primitive of $1 - g_\delta$ such that $f_\delta(0) = 0$.

One easily checks that g_δ and f_δ are smooth functions and satisfy the following properties:

- $0 \le g_\delta(x) \le 1$ for every $x \in \mathbb{R}$, which implies $0 \le f'_\delta(x) \le 1$;
- if $x \in K_\delta$ then $[x - \delta, x + \delta] \subset K_{2\delta}$, hence $g_\delta(x) = 1$, and $f'_\delta(x) = 0$;
- if $x \notin K_{3\delta}$ then $[x - \delta, x + \delta] \cap K_{2\delta} = \varnothing$, hence $g_\delta(x) = 0$ and $f'_\delta(x) = 1$;
- for every $x \in \mathbb{R}$, $|f_\delta(x) - x| \le \|g_\delta\|_1 = |K_{2\delta}|$.

Notice now that since K is compact then $|K_\delta|$ converges to $|K| = 0$ as $\delta \to 0$, and therefore we can find $\bar\delta$ such that $|K_{2\bar\delta}| \le |K_{3\bar\delta}| \le \varepsilon$. We conclude the proof by setting $f := f_{\bar\delta}$. $\square$

Lemma 3.5. *Let K be a compact set in $\mathbb{R}^d$, $d \ge 2$, with $\mathscr{H}^1(K) = 0$. For every $\varepsilon > 0$ there exists a map $\phi : \mathbb{R}^d \to \mathbb{R}^d$ of class C^∞ such that*

(i) *$|\phi(x) - x| \le \varepsilon$ for all $x \in \mathbb{R}^d$;*
(ii) *$\nabla\phi = 0$ in a neighborhood of K;*
(iii) *$|\nabla\phi(x)| \le 1$ for all $x \in \mathbb{R}^d$.*

Moreover, having fixed $r > 0$, we can further require that $\nabla\phi(x) = I$, where I is the $d \times d$-identity matrix, out of an open set A with $|A \cap (-r, r)^d| \le \varepsilon$.

Proof. Fix for the time being $\varepsilon' > 0$, to be properly chosen later. For $i = 1, \dots, d$ let K_i be the projection of K on the i-th coordinate axis, and let $f_i : \mathbb{R} \to \mathbb{R}$ be the function obtained by applying Lemma 3.4 with K_i and ε' in place of K and ε, and let A_i be the set where $f'_i \ne 1$. Let

$$\phi(x_1, \dots, x_d) := \big(f_1(x_1), \dots, f_d(x_d)\big) .$$

It is easy to check that ϕ has properties (i)-(iii) for ε' small enough. Moreover the set A where $\nabla\phi \ne I$ is contained in the set of all x such that $x_i \in A_i$ for some $i = 1, \dots, d$, and therefore $|A \cap (-r, r)^d| \le d\varepsilon'(2r)^{d-1}$, which is less that ε for ε' small enough. $\square$

Lemma 3.6. *Let μ, μ' be measures in $\mathscr{M}^+(\overline{\Omega})$ and assume that $\mu = \mu' + \lambda$ where λ is a positive measure supported on a Borel set E with $\mathscr{H}^1(E) = 0$. Then for every*

$u \in C_c^\infty(\Omega)$ and every $\delta > 0$ there exist $v \in C_c^\infty(\Omega)$ such that $\|v - u\|_\infty \le \delta$ and

$$\int_\Omega |\nabla v|^2 d\mu \le \int_\Omega |\nabla u|^2 d\mu' + \delta\,. \tag{3.1}$$

Proof. We fix for the time being $\varepsilon > 0$, to be chosen properly through the proof. Then we choose a compact set $K \subset E$ such that $\lambda(\overline{\Omega} \setminus K) \le \varepsilon$, we let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be the map constructed in Lemma 3.5 for the set K, and we set

$$v(x) := u(\phi(x)) \quad \text{for every } x \in \mathbb{R}^2.$$

The function v is clearly smooth and compactly supported on $\mathbb{R}^2$, and the support is contained in Ω for ε sufficiently small.

The rest of the proof is divided in three steps. In the following we use the letter C to denote any constant that may depend on μ and u but not on ε; the value of C may change at every occurrence.

Step 1. Estimate of $|u - v|$. Using property (i) in Lemma 3.5 we obtain

$$|v - u| = |u(\phi) - u| \le \mathrm{Lip}(u)\,|\phi(x) - x| \le C\varepsilon\,,$$

which implies $\|v - u\|_\infty \le \delta$ if we choose ε small enough.

Step 2. Estimates of $|\nabla v|$. Using property (iii) in Lemma 3.5 we obtain

$$\begin{aligned} |\nabla v| = |\nabla u(\phi)|\,|\nabla \phi| \le |\nabla u(\phi)| &\le |\nabla u| + \big|\nabla u(\phi) - \nabla u\big| \\ &\le |\nabla u| + \mathrm{Lip}(\nabla u)\,\varepsilon = |\nabla u| + C\varepsilon\,. \end{aligned}$$

Step 3. Proof of estimate (3.1). Property (ii) in Lemma 3.5 implies that $\nabla v = 0$ on K. Using this fact and the estimate in Step 2, and recalling the choice of K, we obtain

$$\begin{aligned} \int_\Omega |\nabla v|^2 d\mu = \int_{\Omega\setminus K} |\nabla v|^2 d\mu &\le \int_{\Omega\setminus K} \big(|\nabla u| + C\varepsilon\big)^2 \, d\mu \\ &= \int_{\Omega\setminus K} |\nabla u|^2 + C\varepsilon \, d\mu \\ &\le \int_\Omega |\nabla u|^2 \, d\mu' + \int_{\Omega\setminus K} |\nabla u|^2 \, d\lambda + C\varepsilon \\ &\le \int_\Omega |\nabla u|^2 \, d\mu' + C\varepsilon\,, \end{aligned}$$

which implies the desired estimate if we choose ε small enough. □

Lemma 3.7. *Let μ, μ' be as in Lemma 3.6. Then for every $u \in C_c^\infty(\Omega)$ and every $\delta > 0$ there exist $v \in C_c^\infty(\Omega)$ such that $\|v - u\|_\infty \le \delta$ and*

$$E_f(\mu, v) \le E_f(\mu', u) + \delta\,. \tag{3.2}$$

Proof. Consider the measures $\bar\mu := dx + m\mu$ and $\bar\mu' := dx + m\mu'$, where dx is the Lebesgue measure on Ω, and fix $\bar\delta \in (0, \delta]$, to be chosen later.

Let now v be the function obtained by applying Lemma 3.6 with $\bar\mu$, $\bar\mu'$, $\bar\delta$ in place of μ, μ', δ. Then $\|v - u\|_\infty \le \bar\delta \le \delta$ and

$$\begin{aligned} E_f(\mu, v) &= \frac{1}{2}\int_\Omega |\nabla v|^2 d\bar\mu + \int_\Omega v\, df \\ &\le \frac{1}{2}\left[\int_\Omega |\nabla u|^2 d\bar\mu' + \bar\delta\right] + \left[\int_\Omega u\, df + \|u - v\|_\infty \|f\|\right] \\ &= E_f(\mu', u) + \left(\frac{1}{2} + \|f\|\right)\bar\delta\,, \end{aligned}$$

where $\|f\|$ is the total mass of the signed measure f. Thus estimate (3.2) follows by choosing δ sufficiently small. □

Lemma 3.8. *Given $\mu, \mu' \in \mathscr{M}^+(\overline{\Omega})$ such that $\mu \geq \mu'$, then $\mathcal{E}_f(\mu) \geq \mathcal{E}_f(\mu')$.*

Proof. This statement follows by the fact that $E_f(\mu, u) \geq E_f(\mu', u)$ for every u in $C_c^\infty(\Omega)$, cf. (2.1) and (2.2). □

Proposition 3.9. *Let μ, μ' be measures in $\mathscr{M}^+(\overline{\Omega})$ and assume that $\mu = \mu' + \lambda$ where λ is a positive measure supported on a Borel set E with $\mathscr{H}^1(E) = 0$. Then*

$$\mathcal{E}_f(\mu) = \mathcal{E}_f(\mu') \,.$$

Proof. The inequality $\mathcal{E}_f(\mu) \geq \mathcal{E}_f(\mu')$ is contained in Lemma 3.8, the opposite inequality follows from Lemma 3.7. □

Lemma 3.10. *Let μ be a finite positive measure on $\mathbb{R}^d$ of the form $\mu = \theta\,\mathscr{H}^1 \llcorner \Sigma$ where Σ is a rectifiable set, and let A be an open set that contains Σ. For a.e. $x \in \Sigma$ let $\tau(x)$ be the tangent line to Σ at x and let $P(x)$ be the matrix associated to the orthogonal projection of $\mathbb{R}^d$ onto $\tau(x)$. Then for every $\varepsilon > 0$ there exist a smooth map $\phi : \mathbb{R}^d \to \mathbb{R}^d$ and a compact set $K \subset \Sigma$ such that*

(i) *$|\phi(x) - x| \leq \varepsilon$ for all x, and $\phi(x) = x$ for $x \in \mathbb{R}^d \setminus A$;*
(ii) *$|\nabla\phi(x)| \leq 10$ for all x, and $\nabla\phi(x) = I$ for $x \in \mathbb{R}^d \setminus A$;*
(iii) *$|\nabla\phi(x) - P(x)| \leq \varepsilon$ for all $x \in K$ and $\mu(\mathbb{R}^d \setminus K) \leq \varepsilon$.*

Proof. The proof is divided in several steps. We fix for the time being $\varepsilon' \in (0, 1]$, to be chosen at the end of the proof.

Step 1. Construction of ϕ. Using the fact that Σ is rectifiable and μ is supported on Σ we can find compact sets $K_i \subset \Sigma$ and curves Γ_i of class C^1, with $i = 1, \dots, n$, such that:

- the sets K_i are disjoint and contained in Γ_i;
- $\mu(\mathbb{R}^d \setminus K) \leq \varepsilon$ where $K := K_1 \cup \dots \cup K_n$;
- up to a rotation R_i, Γ_i agrees with the graph of a C^1 map $\gamma_i : \mathbb{R} \to \mathbb{R}^{d-1}$ (we identify $\mathbb{R}^d$ with $\mathbb{R} \times \mathbb{R}^{d-1}$) and $|\dot\gamma_i| \leq \varepsilon'$ everywhere.

Then we find $\delta \in (0, \varepsilon/3]$ such that

- the open δ-neighborhoods K_i^δ are disjoint and contained in A.

Next we find smooth curves G_i and smooth functions $\sigma_i : \mathbb{R}^d \to [0, 1]$ such that

- $\sigma_i = 1$ on a neighborhood of K_i and $\sigma_i = 0$ out of K_i^δ;
- $|\nabla\sigma_i| \leq 2/\delta$ everywhere;
- G_i agrees, up to the rotation R_i, with the graph of a C^∞ map $g_i : \mathbb{R} \to \mathbb{R}^{d-1}$ such that $|\gamma_i - g_i| \leq \delta$ and $|\dot g_i| \leq \varepsilon'$ everywhere.

For every i we let p_i be the projection of $\mathbb{R}^n$ onto G_i defined by

$$p_i(x) = p_i(x_1, \dots, x_d) := \big(x_1, g_i(x_1)\big)$$

(modulo the rotation R_i). Finally we set $\sigma_0 := 1 - (\sigma_1 + \dots + \sigma_n)$, so that the functions $\sigma_0, \dots, \sigma_n$ form a partition of unity, and define

$$\phi(x) := \sigma_0(x)\, x + \sum_{i=1}^n \sigma_i(x)\, p_i(x) = x + \sum_{i=1}^n \sigma_i(x)\, \big(p_i(x) - x\big) \,. \tag{3.3}$$

In the rest of the proof we assume for simplicity that the rotations R_i are the identity, and we denote by C any constant that does not depend on ε' and δ; the value of C may vary at every occurrence.

Step 2. $|x-p_i(x)| \le 3\delta$ for every $x \in K_i^\delta$. Write $x = (x_1, x')$ with $x' \in \mathbb{R}^{d-1}$. Since Γ_i is the graph of γ_i and $|\dot\gamma_i| \le 1$, the fact that $\operatorname{dist}(x, \Gamma_i) \le \delta$ implies $|x' - \gamma_i(x_1)| \le 2\delta$. Then the assumption $|\gamma_i - g_i| \le \delta$ yields $|x' - g_i(x_1)| \le 3\delta$, which is the desired estimate.

Step 3. Proof of statement (i). Given $x \in \mathbb{R}^n$, for every i such that $\sigma_i(x) \ne 0$ there holds $x \in K_i^\delta$, and then $|x - p_i(x)| \le 3\delta$ by Step 2. Hence (3.3) gives

$$|\phi(x) - x| \le \sum_{i=1}^n \sigma_i(x) \, \big|p_i(x) - x\big| \le 3\delta \le \varepsilon \, .$$

On the other hand, if x does not belong to A then it does not belong to any K_i^δ, which means $\sigma_i(x) = 0$, and therefore (3.3) yields $\phi(x) = x$.

Step 4. Proof of statement (ii). Formula (3.3) gives

$$\nabla\phi(x) = I + \sum_{i=1}^n \sigma_i(x)\big(\nabla p_i(x) - I\big) + \sum_{i=1}^n \big(p_i(x) - x\big) \otimes \nabla\sigma_i(x) \, . \tag{3.4}$$

If x does not belong to A then x does not belong to the support of σ_i for every i, and formula (3.4) reduces to $\nabla\phi(x) = I$.

Consider now x arbitrary. From formula (3.4) we obtain

$$\begin{aligned} |\nabla\phi(x)| &\le 1 + \sum_{i=1}^n \sigma_i(x) \, \big(1 + |\nabla p_i(x)|\big) + \sum_{i=1}^n \big|x - p_i(x)\big| \cdot \big|\nabla\sigma_i(x)\big| \\ &\le 1 + 3 + 3\delta \cdot \frac{2}{\delta} = 10 \, . \end{aligned}$$

For the second inequality we used that $\sigma_i(x) = 0$ and $\nabla\sigma_i(x) = 0$ for all i except at most one, and the following estimates: $|\nabla p_i(x)| \le 2$ (use the definition of p_i and the bound $|\dot g_i| \le \varepsilon' \le 1$), $|\nabla\sigma_i| \le 2/\delta$ (by the choice of σ_i), and $|x - p_i(x)| \le 3\delta$ (Step 2).

Step 5. $|\nabla p_i(x) - P(x)| \le C\varepsilon'$ for $x \in K_i$. Let P be the $d \times d$ matrix associated to the projection of $\mathbb{R}^d$ onto the line $\mathbb{R} \times \{0\}$, that is, the matrix with all entries equal to 0 except $P_{11} = 1$. The definition of p_i and the assumption $|\dot g_i| \le \varepsilon'$ imply $|\nabla p_i(x) - P| \le C\varepsilon'$, while the assumption $|\dot\gamma_i| \le \varepsilon'$ implies $|P(x) - P| \le C\varepsilon'$.

Step 6. Proof of statement (iii). We already know that $\mu(\mathbb{R}^d \setminus K) \le \varepsilon$. Moreover, if x belongs to K then it belongs to K_i for some i and since $\sigma_i = 1$ in a neighborhood of K_i, formula (3.4) reduce to $\nabla\phi(x) = \nabla p_i(x)$. We conclude the proof using the estimate in Step 5 and choosing ε' small enough. □

Lemma 3.11. *Let μ be a measure in $\mathscr{M}^+(\overline{\Omega})$ of the form $\mu = \theta \, \mathscr{H}^1 \llcorner \Sigma$ where Σ is a rectifiable set. Then for every $u \in C_c^\infty(\Omega)$ and every $\delta > 0$ there exist $v \in C_c^\infty(\Omega)$ such that $\|v - u\|_\infty \le \delta$ and*

$$E_f(\mu, v) \le E_f^*(\mu, u) + \delta \, . \tag{3.5}$$

Proof. We fix $\varepsilon > 0$, to be chosen later. We take an open set $A \supset \Sigma$ such that $|A| \le \varepsilon$, and we then let ϕ and K be the map and the compact set given by Lemma 3.10. We now set

$$v(x) := u(\phi(x)) \quad \text{for every } x \in \mathbb{R}^2.$$

The function v is smooth and its support is contained in Ω for ε sufficiently small, and the desired properties of v follow by the properties of ϕ stated in Lemma 3.10. As usual, the letter C denotes any constant which does not depend on ε.

Using property (i) in Lemma 3.10, for every $x \in \mathbb{R}^2$ we obtain

$$|v - u| = |u(\phi) - u| \le \operatorname{Lip}(u) \, |\phi(x) - x| \le C\varepsilon \, ,$$

which implies $\|v-u\|_\infty \le \delta$ for ε small enough, and

$$\left|\int_\Omega v\,df - \int_\Omega u\,df\right| \le C\varepsilon\,. \tag{3.6}$$

Using property (ii) in Lemma 3.10 we obtain

$$|\nabla v| = |\nabla u(\phi)|\cdot|\nabla\phi| \le C\,, \tag{3.7}$$

while properties (i) and (ii) imply that $\nabla v = \nabla u$ for every $x\in\Omega\setminus A$. Therefore

$$\int_\Omega |\nabla v|^2\,dx \le \int_{\Omega\setminus A} |\nabla u|^2\,dx + C|A| \le \int_\Omega |\nabla u|^2\,dx + C\varepsilon\,. \tag{3.8}$$

For every $x\in K$ we write ∇v as follows, where $P(x)$ is the matrix associated to the projection on the approximate tangent line to Σ at x:

$$\nabla v = \nabla u(\phi)\,\nabla\phi = \big(\nabla u(\phi)-\nabla u\big)\,\nabla\phi + \nabla u\,\big(\nabla\phi - P\big) + \nabla u\,P\,.$$

Then, recalling that $\nabla u\,P = \nabla_\tau u$ and using properties (i)-(iii), we obtain:

$$\begin{aligned}|\nabla v| &\le \big|\nabla u(\phi)-\nabla u\big|\cdot|\nabla\phi| + |\nabla u|\cdot|\nabla\phi - P| + |\nabla_\tau u|\\ &\le \mathrm{Lip}(\nabla u)\,|\phi(x)-x|\,C + C\varepsilon + |\nabla_\tau u| = |\nabla_\tau u| + C\varepsilon\,.\end{aligned}$$

Using this estimate and (3.7), and the fact that $\mu(\mathbb{R}^2\setminus K)\le\varepsilon$, we obtain

$$\int_\Sigma |\nabla v|^2\,d\mu = \int_K |\nabla_\tau u|^2 + C\varepsilon\,d\mu + \int_{\Sigma\setminus K} C\,d\mu \le \int_\Sigma |\nabla_\tau u|^2\,d\mu + C\varepsilon\,. \tag{3.9}$$

Finally (3.5) follows from (3.6), (3.8) and (3.9) by choosing ε small enough. □

Proposition 3.12. *Let μ be a measure in $\mathscr{M}^+(\overline{\Omega})$ of the form $\mu = \theta\,\mathscr{H}^1\llcorner\Sigma$ where Σ is a rectifiable set. Then*

$$\mathcal{E}_f(\mu) = \mathcal{E}^*_f(\mu)\,.$$

Proof. The trivial inequality $E_f(\mu,u)\ge E^*_f(\mu,u)$ implies $\mathcal{E}_f(\mu)\ge\mathcal{E}^*_f(\mu)$; the opposite inequality follows from Lemma 3.11. □

Proof of Theorem 2.5. Combine Propositions 3.9 and 3.12. □

Proof of Proposition 2.11. If f belongs to $L^p(\Omega)$ for some $p>1$ then it belongs also to the dual of $H^1_0(\Omega)$, and therefore the functional $E^*_f(\mu,\cdot)$ is coercive, which implies that the infimum $\mathcal{E}^*_f(\mu)$ is not $-\infty$. Clearly the same holds for $\mathcal{E}_f(\mu)$.

Next we notice that the minimum of $E^*_f(\mu,u)$ over all $u\in H^1_0(\Omega)\cap H^1(S)$ is attained because this functional is coercive and weakly lower-semicontinuous.

The first equality in (2.5) is proved in Theorem 2.5.

To prove the second equality in (2.5) it is enough to show that $C^\infty_c(\Omega)$ is dense in norm in $H^1_0(\Omega)\cap H^1(S)$. The proof of this density result is a bit delicate, but ultimately standard, and we simply list the key steps:

- $C^\infty_c(\Omega)$ is dense in $\mathrm{Lip}_c(\Omega)$;
- $\mathrm{Lip}_c(\Omega)$ is dense in $\mathrm{Lip}_0(\Omega)$;
- $\mathrm{Lip}_0(\Omega)$ is dense in the subspace X of all $u\in H^1_0(\Omega)\cap H^1(S)$ which are constant on some open set A (depending on u) such that $S\setminus A$ can be covered by finitely many disjoint compact curves of class C^1;
- X is dense in $H^1_0(\Omega)\cap H^1(S)$.

In all these steps "dense" refers to the norm of $H^1_0(\Omega)\cap H^1(S)$; the last step is the most delicate, and can be proved arguing as in the proof of Lemma 3.6. □

Lemma 3.13. *Assume that $f \in L^p(\Omega)$ for some $p > 1$ and that the support of f is Ω, and let μ be a measure in $\mathscr{M}_L$ such that $\mu(\overline{\Omega}) < L$. Then there exists μ' in $\mathscr{M}_L$ such that $\mu' \geq \mu$ and $\mathcal{E}_f(\mu') > \mathcal{E}_f(\mu)$.*

In particular every solution μ of problem (1.5) satisfies $\mu(\overline{\Omega}) = L$.

Proof. Let S be the support of μ. By Proposition 2.11, $\mathcal{E}_f(\mu) = E^*_f(\mu, u)$ where $u \in H^1_0(\Omega) \cap H^1(S)$ is a minimizer of $E^*_f(\mu, \cdot)$. Then u solves the equation $\Delta u = -f$ in $\Omega' := \Omega \setminus S$, which implies that u is of class C^1 on Ω' and the set of all $x \in \Omega'$ such that $\nabla u(x) = 0$ has empty interior.

In particular we can find a point $x \in \Omega'$ such that $\nabla u(x) \neq 0$ and $\operatorname{dist}(x, S) < \ell$ where $\ell := L - \mu(\overline{\Omega})$. We then choose a segment S' which connects x to S, has length $\mathscr{H}^1(S') \leq \ell$, and is not orthogonal to $\nabla u(x)$.

We set $\mu' := \mu + \mathscr{H}^1 \llcorner S$. Clearly $\mu' \geq \mu$ and the support of μ' is $S \cup S'$, and one easily checks that μ' belongs to $\mathscr{M}_L$. Since $\mu' \geq \mu$ then $\mathcal{E}_f(\mu') \geq \mathcal{E}_f(\mu)$ (cf. Lemma 3.8), and we claim that this inequality is strict.

Assume by contradiction that $\mathcal{E}_f(\mu') = \mathcal{E}_f(\mu)$, and let $u' \in H^1_0(\Omega) \cap H^1(S \cup S')$ be a minimizer of $E^*_f(\mu', \cdot)$. Then u' is also a minimizer of $E^*_f(\mu, \cdot)$, and since this functional is strictly convex we have that u and u' agree as elements of the space $H^1_0(\Omega) \cap H^1(S)$. This means that

$$E^*_f(\mu, u') = \mathcal{E}_f(\mu) = \mathcal{E}_f(\mu') = E^*_f(\mu', u')\,.$$

On the other hand u is of class C^1 on $\Omega \setminus S$, and in particular is continuous, and therefore u agrees with u' on S', which implies that $\nabla_\tau u' = \nabla_\tau u$ a.e. on S', and by the choice of S' we have that $\nabla_\tau u$ is not identically null on S'. This yields the contradiction $E^*_f(\mu, u') < E^*_f(\mu', u')$. ☐

Proof of Theorem 2.6. Let us prove the first part of the statement. Let $\bar\mu \in \mathscr{M}_L$ be an arbitrary solution of problem (1.5) (which exists by Theorem 2.2), let S be the support of $\bar\mu$ and let $\bar\mu^a = \bar\theta\, \mathscr{H}^1 \llcorner S$ be the absolutely continuous part of $\bar\mu$ with respect to $\mathscr{H}^1 \llcorner S$. Then $\bar\mu^a$ is also a solution of problem (1.5) by Theorem 2.5.

If $L = \bar\mu^a(\overline{\Omega})$ we set $\mu := \bar\mu^a = \bar\theta\, \mathscr{H}^1 \llcorner S$.

If $L > \bar\mu^a(\overline{\Omega}) = \int_S \bar\theta \, d\mathscr{H}^1$ we set $\mu := \theta\, \mathscr{H}^1 \llcorner S$ where θ is a any function such that $\theta \geq \bar\theta$ and $L = \int_S \theta \, d\mathscr{H}^1 = \mu(\overline{\Omega})$.

Let us now prove the second part of the statement. Since $\bar\mu^a$ is a solution of problem (1.5), Lemma 3.13 implies that $\bar\mu^a(\overline{\Omega}) = L$. On the other hand $\bar\mu(\overline{\Omega}) \leq L$ because of the definition of $\mathscr{M}_L$, and therefore we must have $\bar\mu(\overline{\Omega}) = L$ and $\bar\mu = \bar\mu^a$, which concludes the proof. ☐

4. Some necessary conditions of optimality

In this section we assume that the load f belongs to $L^2(\Omega)$, and we consider a measure $\mu = \theta \mathscr{H}^1 \llcorner S$ in $\mathscr{M}^a_L$ (see (2.4)) and the function $u \in H^1_0(\Omega) \cap H^1(S)$ that solves problem (2.5), that is, the unique minimizer of $E^*_f(\mu, \cdot)$.

In Proposition 4.1 we derive some necessary conditions that μ and u must satisfy if μ solves the maximum problem (1.5).

In Proposition 4.2 we derive the Euler-Lagrange equations for u in strong form (assuming some regularity on S and u).

Proposition 4.1. *Assume that μ solves of the optimization problem (1.5) and that the set $S_+ := \{x \in S \colon \theta(x) > 1\}$ has positive length. Then there exists a constant $c \in \mathbb{R}$ such that*

(i) $|\nabla_\tau u| = c$ *a.e. on* S_+;
(ii) $|\nabla_\tau u| \le c$ *a.e. on* $S \setminus S_+$.

Proof. The proof is divided in several steps; the key inequality is (4.4), which is obtained from (4.2).

We consider variations of μ of the form $\mu_\varepsilon := (\theta + \varepsilon\eta)\,\mathscr{H}^1 \llcorner S$, with $\varepsilon > 0$ and $\eta \in L^\infty(S)$ (in particular we keep the set S fixed). In order that μ_ε be admissible, that is, $\mu_\varepsilon \in \mathscr{M}_L$ for $0 \le \varepsilon \le 1$, we assume that

$$\int_S \eta \, d\mathscr{H}^1 = 0 \text{ and } \eta \ge 1 - \theta \text{ a.e.} \tag{4.1}$$

Step 1. Let u_ε *be the minimizer of* $E_f^*(\mu_\varepsilon, \cdot)$*: then for every* η *that satisfies* (4.1) *there holds*

$$\int_S |\nabla_\tau u_\varepsilon|^2 \, \eta \, d\mathscr{H}^1 \le 0\,. \tag{4.2}$$

By the choice of u and u_ε we have that $\mathcal{E}_f(\mu) = E_f^*(\mu, u)$ and $\mathcal{E}_f(\mu_\varepsilon) = E_f^*(\mu_\varepsilon, u_\varepsilon)$. Therefore, the optimality of μ yields

$$E_f^*(\mu, u_\varepsilon) \ge E_f^*(\mu, u) \ge E_f^*(\mu_\varepsilon, u_\varepsilon) = E_f^*(\mu, u_\varepsilon) + \frac{m\varepsilon}{2} \int_S |\nabla_\tau u_\varepsilon|^2 \, \eta \, d\mathscr{H}^1\,, \tag{4.3}$$

and the comparison of the first and last terms of (4.3) gives (4.2).

In the next four steps we prove that the functions u_ε converge strongly to u, which will imply that (4.2) holds with u in place of u_ε.

Step 2. The functions u_ε *are uniformly bounded in* $H_0^1(\Omega) \cap H^1(S)$. Note indeed that for ε small enough there holds $1/2 \le \theta + \varepsilon\eta$ and therefore

$$\begin{aligned} \frac{1}{2} \int_\Omega |\nabla u_\varepsilon|^2 \, dx + \frac{m}{4} \int_S |\nabla_\tau u_\varepsilon|^2 \, d\mathscr{H}^1 - \int_\Omega f u_\varepsilon \, dx \\ \le E_f^*(\mu_\varepsilon, u_\varepsilon) \le E_f^*(\mu_\varepsilon, 0) = 0\,, \end{aligned}$$

and the functional in the first line is clearly coercive on $H_0^1(\Omega) \cap H^1(S)$.

Step 3. $E_f^*(\mu, u_\varepsilon)$ *converge to* $E_f^*(\mu, u)$ *as* $\varepsilon \to 0$. From (4.3) we obtain

$$0 \le E_f^*(\mu, u_\varepsilon) - E_f^*(\mu, u) \le -\frac{m\varepsilon}{2} \int_S |\nabla_\tau u_\varepsilon|^2 \, \eta \, d\mathscr{H}^1\,,$$

and the last term tends to 0 as $\varepsilon \to 0$ by Step 2.

Step 4. The functions u_ε *converge to* u *weakly in* $H_0^1(\Omega) \cap H^1(S)$ *as* $\varepsilon \to 0$. By Step 3 and the weak lower-semicontinuity of $E_f^*(\mu, \cdot)$, every weak* limit of the sequence u_ε is a minimizer of $E_f^*(\mu, \cdot)$ and therefore it must be u because this functional is strictly convex.

Step 5. The functions u_ε *converge to* u *strongly in* $H_0^1(\Omega) \cap H^1(S)$ *as* $\varepsilon \to 0$*, and for every* η *that satisfies* (4.1) *there holds*

$$\int_S |\nabla_\tau u|^2 \, \eta \, d\mathscr{H}^1 \le 0\,. \tag{4.4}$$

Since the linear term in $E_f^*(\mu, \cdot)$ is weakly continuous, the convergence of the energies in Step 3 implies the convergence of the energies without linear term, that is $E_0^*(\mu, u_\varepsilon) \to E_0^*(\mu, u)$. Notice now that

$$\Phi(\cdot) := \big(E_0^*(\mu, \cdot)\big)^{1/2}$$

is an equivalent Hilbert norm on $H_0^1(\Omega) \cap H^1(S)$, and recall that in Hilbert spaces weak convergence plus convergence of the norms implies strong convergence.

Inequality (4.4) follows from (4.2) and the fact that the functions $|\nabla_\tau u_\varepsilon|^2$ converge to $|\nabla_\tau u|^2$ in $L^1(S)$.

Step 6. Conclusion of the proof. Set $S_\delta := \{x \in S \colon \theta(x) \geq 1 + \delta\}$. Note that inequality (4.4) holds for all η which vanish on $S \setminus S_\delta$, satisfy $|\eta| \leq \delta$ on S_δ, and have integral 0 on S_δ. Since this class of functions is closed by change of sign, the inequality is actually an equality, which can be written as

$$\int_{S_\delta} |\nabla_\tau u|^2 \, \eta \, d\mathscr{H}^1 = 0 \, ,$$

and implies that $|\nabla_\tau u|$ is equal to some constant c a.e. on S_δ. Since this holds for every $\delta > 0$, we have proved statement (i).

Using statement (i) and recalling that for every admissible η there holds

$$\int_{S \setminus S_+} \eta \, d\mathscr{H}^1 = - \int_{S_+} \eta \, d\mathscr{H}^1 \, ,$$

we rewrite (4.4) as

$$0 \geq \int_{S \setminus S_+} |\nabla_\tau u|^2 \, \eta \, d\mathscr{H}^1 + c^2 \int_{S_+} \eta \, d\mathscr{H}^1 = \int_{S \setminus S_+} \big(|\nabla_\tau u|^2 - c^2 \big) \, \eta \, d\mathscr{H}^1 \, ,$$

and since the restriction of η to $S \setminus S_+$ can be an arbitrary positive bounded function with integral less than $\int_S (\theta - 1) \, d\mathscr{H}^1$, this inequality implies that $|\nabla_\tau u|^2 - c^2 \leq 0$ a.e. on $S \setminus S_+$, which is statement (ii). □

For the next result we need some assumptions on S, θ and u.

We assume that θ is a continuous function and that S is a network of class C^1, that is, it can be written as a finite union of simple curves S_i of class C^1 contained in $\overline{\Omega}$ that intersect each other and $\partial\Omega$ only at the endpoints. We denote by $S^\#$ the set of all endpoints of the curves S_i, and we say that $x \in S^\#$ is

- a boundary point if $x \in \partial\Omega$;
- a terminal point if $x \in \Omega$ and x belongs to only one curve S_i;
- a branching point if $x \in \Omega$ and x belongs to more than one curve S_i.

We choose an orientation τ of S,[3] we denote by ν the associated normal, that is, the rotation of τ by 90° counterclockwise, and write ∂_τ for the tangential derivative, $\partial_\nu^\pm$ for the normal derivatives on the two sides of S.

Finally we assume that u is of class C^1 on $\Omega \setminus S$, and that the normal derivatives $\partial_\nu^\pm u$ exist at every point of $S \setminus S^\#$ and belong to $L^1(S)$. We write

$$\big[\partial_\nu u \big] := \partial_\nu^+ u - \partial_\nu^+ u \, .$$

(Note that this quantity does not depend on the choice of the normal ν.)

Proposition 4.2. *Under the assumptions on S, θ and u stated above, we have that*

- *u solves $-\Delta u = f$ on $\Omega \setminus S$ with boundary condition $u = 0$ on $\partial\Omega$;*
- *u solves $-m \, \partial_\tau \big(\theta \, \partial_\tau u \big) = \big[\partial_\nu u \big]$ on each curve S_i minus the endpoints;*
- *u is of class C^1 on each curve S_i, including the endpoints.*

In particular the values of $\partial_\tau u$ at the endpoints of S_i, denoted by $(\partial_\tau u)_i$ are well-defined, and for every $x \in S^\#$ we set

$$\big[\partial_\tau u(x) \big] := \sum (\partial_\tau u(x))_i \, ,$$

where the sum is taken over all i such that x is an endpoint of S_i. Then

[3] This means that τ agrees on each curve S_i (except the endpoints) with is a continuous unit tangent field to S_i; we do not require that τ is continuous at branching points.

- *if x is a boundary point, the Dirichlet condition $u(x) = 0$ holds;*
- *if x is a terminal point, the Neumann condition $\partial_\tau u(x) = 0$ holds;*
- *if x is a branching point, the Kirchhoff condition $\big[\partial_\tau u(x)\big] = 0$ holds.*

Proof. The full Euler-Lagrange equation for u in the weak form is

$$\int_\Omega \nabla u \cdot \nabla \phi \, dx + m \int_S \nabla_\tau u \cdot \nabla_\tau \phi \, \theta \, d\mathscr{H}^1 - \int_\Omega f \phi \, dx = 0 \quad \forall \phi \in C_c^\infty(\Omega) . \tag{4.5}$$

Thus u satisfies the equation $\Delta u = f$ on $\Omega \backslash S$ in the weak sense, and hence it belongs to $H^2_{\rm loc}(\Omega \setminus S)$.

Integrating by parts the first integral in (4.5) we obtain

$$\int_\Omega \nabla u \cdot \nabla \phi \, dx = \int_{\Omega \backslash S} \nabla u \cdot \nabla \phi \, dx = \int_{\Omega \backslash S} f \phi \, dx - \int_S [\partial_\nu u] \, \phi \, d\mathscr{H}^1 ,$$

and therefore (4.5) becomes

$$m \int_S \partial_\tau u \cdot \partial_\tau \phi \, \theta \, d\mathscr{H}^1 - \int_S [\partial_\nu u] \, \phi \, d\mathscr{H}^1 = 0 \quad \forall \phi \in C_c^\infty(\Omega) . \tag{4.6}$$

Thus u solves the equation $-m \, \partial_\tau(\theta \, \partial_\tau u) = \big[\partial_\nu u\big]$ in the weak sense on each curve S_i, which implies that $\theta \, \partial_\tau u$ belongs $W^{1,1}(S_i)$, and then also to $C^0(S_i)$, which in turn implies that u belongs to $C^1(S_i)$.

Finally we integrate by parts the first integral in (4.6) and obtain

$$\sum_{x \in S^\#} \theta(x) \, \big[\partial_\tau u(x)\big] \, \phi(x) = 0 \quad \forall \phi \in C_c^\infty(\Omega) ,$$

This implies that $\big[\partial_\tau u(x)\big] = 0$ for every $x \in S^\#$ which is not a boundary point; if x is a terminal point this means $\partial_\tau u(x) = 0$. □

5. Numerical approximation of optimal reinforcing networks

In this section we introduce a numerical strategy to approximate the solutions of the relaxed reinforcement problem (1.5). Through this section we assume that Ω is a bounded *convex* domain, and that the load f belongs to $L^2(\Omega)$.

Thanks to Theorem 2.6 we can rewrite this optimization problem as

$$\max_{S,\theta} \min_u \left[\frac{1}{2} \int_\Omega |\nabla u|^2 \, dx + \frac{m}{2} \int_S |\nabla u|^2 \, \theta \, d\mathscr{H}^1 - \int_\Omega f u \, dx \right] , \tag{5.1}$$

where the minimum is taken over all function $u \in H^1_0(\Omega) \cap H^1(S)$ and the maximum is taken over all S and θ such that S is a compact, connected set with finite length contained in $\overline{\Omega}$, θ is a function on S with $\theta \geq 1$ a.e. and $\int_S \theta \, d\mathscr{H}^1 = L$.

Since we expect problem (5.1) to have many local maxima, we focus on stochastic optimization algorithms which only require cost function evaluations to proceed.

5.1. Spanning tree parametrization and discrete functional. To discretize problem (5.1), we consider a mesh $\mathcal{T}$ associated to the domain Ω made of n_p points and n_t triangles. We denote by K and M respectively the stiffness and mass matrices of dimensions $n_p \times n_p$ associated to the finite elements $P1$ on $\mathcal{T}$. Moreover, we define K_x and K_y to be the differentiation matrices of $P1$ functions. More precisely, K_x and K_y are matrices of dimensions $n_t \times n_p$ which evaluate the operators ∂_x and ∂_y on piecewise linear continuous functions on the mesh $\mathcal{T}$. Observe that due to the linearity of $P1$ elements, ∂_x and ∂_y are constant on every triangle of the mesh.

Algorithm 1 Projection on weighted length and bound constraints.

Input: L, $SP(P_1,\dots,P_{n_d})$, θ_{weights}, h_s.

step 1: Compute the length $\overline{L}$ of $SP(P_1,\dots,P_{n_d})$ and the center of mass $\overline{C}$ of the points $P_1,\dots,P_{n_d}$.

step 2: Define $(\overline{P_1},\dots,\overline{P_{n_d}})$ to be the image of $SP(P_1,\dots,P_{n_d})$ by the homothetic transformation with center $\overline{C}$ and ratio $h_s L/\overline{L}$.

step 3: Project the weight vector θ_{weights} on the convex set which is the intersection of the linear constraint (5.2) with respect to $SP(\overline{P_1},\dots,\overline{P_{n_d}})$ and the bound constraints $\theta_{\text{weights}}\geq 1$. The projected vector is denoted by $\overline{\theta_{\text{weights}}}$.

Output: $SP(\overline{P_1},\dots,\overline{P_{n_d}})$, $\overline{\theta_{\text{weights}}}$.

Denoting by V_{areas} the column vector of size $n_t\times 1$ containing the area measures of every triangle, we recall the simple identity

$$K = K_x^T V_{\text{areas}} K_x + K_y^T V_{\text{areas}} K_y\,.$$

We denote by the letter U a real vector of n_p node values representing an element of $P1\cap H_0^1(\Omega)$.

Problem (5.1) involves both a connected set and an associated weight function. In order to parametrize connected one dimensional structures, we follow the strategy developed in [8]. Take $n_d=1,2,\dots$ and consider a set of n_d points $P_1,\dots,P_{n_d}\in\Omega$. We associate to such a set its canonical spanning tree $SP(P_1,\dots,P_{n_d})$, which is the polygonal set of minimal length connecting these points without introducing new branching points. Let us point out that, generically, $SP(P_1,\dots,P_{n_d})$ is the union of n_d-1 arcs.

It is straightforward to establish that the family of all such spanning trees (with n_d varying among all integers) is dense with respect to Hausdorff distance among compact connected subsets of $\overline{\Omega}$. To describe an L^1 element of $SP(P_1,\dots,P_{n_d})$, we simply consider a vector θ_{weights} of n_d-1 values greater than 1 which represents a piecewise constant function on every arc of the tree.

Let $V_{\text{lengths}}(P_1,\dots,P_{n_d},\theta_{\text{weights}})$ be the vector of size $n_t\times 1$ which contains the weighted lengths of $SP(P_1,\dots,P_{n_d})$ intersected with every triangle of the mesh $\mathcal{T}$. With previous notations, we can now introduce a discrete approximation of problem (5.1):

$$\max\,\min\left[\frac{1}{2}U^TKU+\frac{m}{2}U^T\left(K_x^TV_{\text{lengths}}K_x+K_y^TV_{\text{lengths}}K_y\right)U-MF\right]$$

where F is the linear interpolation of the function f at the vertices of the mesh $\mathcal{T}$, the minimum is taken over all $U\in P1\cap H_0^1(\Omega)$, the maximum is taken over all pairs $(SP(P_1,\dots,P_{n_d}),\theta_{\text{weights}})$, that satisfy the constraints that every value of θ_{weights} is greater than 1 and the following measure equality holds:

$$\sum_{i=1}^{n_d-1}\mathscr{H}^1(S_i)\,\theta_{\text{weights}}(i)=L\,, \tag{5.2}$$

where the $(S_i)_{1\leq i\leq n_d}$ are the n_d-1 edges of $SP(P_1,\dots,P_{n_d})$.

Since the minimization problem is a strictly convex quadratic problem, it reduces to solve the linear system

$$\left[K+\frac{m}{2}\left(K_x^TV_{\text{lengths}}K_x+K_y^TV_{\text{lengths}}K_y\right)\right]U=MF\,. \tag{5.3}$$

Algorithm 2 Summary of one cost evaluation.

Input: m, l, $SP(P_1,\dots,P_{n_d})$, θ_{weights}, h_s.
step 1: Project $(SP(P_1,\dots,P_{n_d}),\ \theta_{\text{weights}})$ with Algorithm 1 to obtain an admissible couple $(SP(\overline{P_1},\dots,\overline{P_{n_d}}),\ \overline{\theta_{\text{weights}}})$.
step 2: Locate points $\overline{P_1},\dots,\overline{P_{n_d}}$ in the mesh $\mathcal{T}$.
step 3: Compute the intersection of every arc of $SP(\overline{P_1},\dots,\overline{P_{n_d}})$ with every triangle of $\mathcal{T}$ to evaluate $V_{\text{lengths}}(\overline{P_1},\dots,\overline{P_{n_d}},\ \overline{\theta_{\text{weights}}})$.
step 4: Assemble matrix $K_x^T V_{\text{lengths}} K_x + K_y^T V_{\text{lengths}} K_y$ and solve linear system (5.3) to compute its solution $\overline{U}$.
Return: $\frac{1}{2}\overline{U}^T K \overline{U} + \frac{m}{2}\overline{U}^T\big(K_x^T V_{\text{lengths}} K_x + K_y^T V_{\text{lengths}} K_y\big)\,\overline{U} - MF$

5.2. Parametrization of the constraints. As explained in the previous sections, we need the couple $(SP(P_1,\dots,P_{n_d}),\theta_{\text{weights}})$ to have weights greater than one and satisfies equality constraint (5.3). To parametrize such admissible couples we introduce a last scale parameter denoted by $h_s \in (0,1)$. We introduce in Algorithm 1 a three steps procedure to produce an admissible pair $\big(SP(\overline{P_1},\dots,\overline{P_{n_d}}),\overline{\theta_{\text{weights}}}\big)$ for a given triplet of parameters $\big(SP(P_1,\dots,P_{n_d}),\theta_{\text{weights}},h_s\big)$.

5.3. Technical details and complexity. We summarize in Algorithm 2 the different steps required to compute the cost associated to a given set of parameters, that we choose as $\big(SP(P_1,\dots,P_{n_d}),\theta_{\text{weights}},h_s\big)$.

We give below some technical details and underline the computational complexity of every step.

In the first phase of projection, only the final step of the procedure is not computationally trivial. Whereas the projection of a point onto an hyperplane can be analytically described, the projection on an hyperplane intersected with a box requires a specific attention. In all our experiments, we used Dai and Fletcher algorithm [12] to obtain a fast and precise approximation of this projection.

Observe that the spanning tree $SP(\overline{P_1},\dots,\overline{P_{n_d}})$ is precisely by construction of length $h_s L \le L$ which implies that constraints (5.2) and $\theta_{\text{weights}} \ge 1$ are compatible. In our situation, an order of only n_d iterations was required to reach a relative error of 10^{-6} on first order optimality conditions with respect to the infinity norm which reduces to a complexity of order n_d^2.

The second and third steps have been carried out using an hash structure representation of the mesh $\mathcal{T}$ combined with a Quad-tree associated to its vertices. Using those precomputed information, these operations required in practice an order of $(n_d+n_p)\log(n_d+n_p)$ operations.

Finally, assembling and solving the linear system has been performed by a standard Cholesky decomposition which concentrated the main part of the computational effort in our experiments where the number of parameters $3n_d$ was negligible with respect to n_p which was of order 10^4.

5.4. Numerical experiments. Based on previous discretization, we approximate optimal triplet solutions (S,θ,u) of problem (5.1) using a stochastic algorithm. We focus our study on the homogeneous load case corresponding to f constantly equal to 1 and on the sum of two Dirac masses $f = \delta_{(-1/2,0)} - \delta_{(1/2,0)}$.

In all our experiments, we used the NLopt library (see [15]) and its implementation of ISRES algorithm with its default parameters which combine local and global stochastic optimization.

We carried out optimization runs limited to five hours of computation leading to an order of 2×10^6 cost function evaluations based on algorithm 2 on a standard computer for a mesh made of 10^4 triangles.

In Figures 1 and 3 (at the end of this paper) we describe the optimal configurations we obtained for $L = 1$ to $L = 6$ with $n_d = 20$. Observe that the resulting number of parameters in the triplet is exactly $3n_d$. Moreover, in order to obtain a fine and stable description of optimal structures, we performed a local optimization step of the obtained structure increasing the number of points to $n_d = 50$. We used the NLopt implementation of the BOBYQA algorithm for this final step which does not require gradient base information.

Finally, we give in Table 1 several numerical estimates obtained on a fine mesh with 10^5 elements of our computed sets and also of natural networks which could be guess to be optimal. As illustrated by these numerical values, neither the radius (for $L = 1$), a diameter (for $L = 2$), a triple junction (for $L = 3$) or a cross for ($L = 4$) seem to be optimal.

We recover the fact, described in Proposition 4.1, that, for optimal structures, the tangential gradient of u is almost constant where $\theta > 1$ whereas we can observe drastic changes of magnitude where $\theta = 1$ (see Figures 1, 3 and 2 at the end of this paper).

Length constraint	Theoretical guesses	Computed optimal networks
1	-0.179471 (radius)	-0.178873
2	-0.165095 (diameter)	-0.161944
3	-0.152676 (star)	-0.149601
4	-0.141969 (cross)	-0.138076
5	-	-0.127661
6	-	-0.117140

TABLE 1. Reinforcement values computed on a fine mesh of 10^6 elements for classical and computed connected sets for $m = 0.5$

6. Remarks and open questions

There are several remarks and open problems related to the optimization problem (1.3) and the relaxed optimization problem (1.5); we list below those we deem more interesting.

Remark 6.1. In general the functional $\mathcal{E}_f(\cdot)$ is not weakly* continuous on $\mathscr{M}_L$. We prove this claim by an explicitly example. We let $S \subset \Omega$ be a closed segment with length 2δ, which we identify with the interval $[-\delta, \delta]$, and let f be a signed measure of the form $f := \rho \mathscr{H}^1 \llcorner S$ where ρ is a function on S with integral 0.

We then consider the measures $\mu_n := \theta_n \mathscr{H}^1 \llcorner S$ where $\theta_n(s) := g(ns/\delta)$ and g is the 2-periodic function on $\mathbb{R}$ defined by $g = 1$ on $[-1, 0)$ and $g = 2$ on $[0, 1)$. Thus μ_n converge to $\mu := \frac{3}{2} \mathscr{H}^1 \llcorner S$. However, the functionals

$$F(\mu_n, u) := \frac{1}{2} \int_S |\nabla_\tau u|^2 d\mu_n - \int_S u\, df = \int_{-\delta}^{\delta} \frac{\theta_n}{2} |\dot{u}|^2 - \rho u\, ds$$

Gamma-converge (on $H^1(S)$ endowed with the weak topology) to

$$F(u) := \int_{-\delta}^{\delta} \frac{2}{3} |\dot{u}|^2 - \rho u\, ds\,,$$

and $F(u) < F(\mu, u)$ for every non constant u (because $2/3$ is strictly less than $3/4$, which is the density of μ divided by 2). In particular if f is not a.e. equal to 0 then

$$\lim_{n\to\infty} \min_u F(\mu_n, u) = \min_u F(u) < \min_u F(\mu, u)$$

(all minima are taken over $u \in H^1(S)$). Using the strict inequality we can prove that if the constant m that appears in (2.2) is sufficiently large, then

$$\limsup_{n\to\infty} \mathcal{E}_{mf}(\mu_n) < \mathcal{E}_{mf}(\mu)\,.$$

Problem 6.2. We do not know if problem (1.5) is the relaxation of problem (1.3). In other words, we do not know if the following approximation property holds: *for every $\mu \in \mathscr{M}_L$ there exists a sequence of sets $S_n \in \mathscr{A}_L$ such that*

$$\mathscr{H}^1 \llcorner S_n \to \mu \quad \textit{and} \quad \mathcal{E}_f(S_n) \to \mathcal{E}_f(\mu)\,. \tag{6.1}$$

Indeed, by the definition of $\mathscr{M}_L$ every μ in this class is the limit of $\mathscr{H}^1 \llcorner S_n$ for some sequence of sets $S_n \in \mathscr{A}_L$, but since $\mathcal{E}_f$ is not continuous (Remark 6.1), the second limit in (6.1) does not necessarily hold.

Remark 6.3. If the approximation in energy (6.1) does not hold, then some kind of Lavrentiev phenomenon may occur. This means that

- the value of the maximum/supremum in the original optimization problem (1.3) could be strictly smaller than the value of the maximum in the relaxed optimization problem (1.5);
- given a maximizing sequence (S_n) for problem (1.3), the associated measures $\mathscr{H}^1 \llcorner S_n$ may not converge to a solution of the relaxed problem (1.5).

Remark 6.4. Assume that f belongs to $L^p(\Omega)$ for some $p > 1$ and that μ is a measure in $\mathscr{M}_L$ with support S, and let μ^a be the absolutely continuous part of μ with respect to $\mathscr{H}^1 \llcorner S$. Using Lemma 3.7, Lemma 3.11 and Proposition 2.11 we easily obtain the following: *the relaxation of $E_f(\mu, u)$ with $u \in C_c^\infty(\Omega)$ is the functional $E_f^*(\mu^a, u)$ with $u \in H_0^1(\Omega) \cap H^1(S)$.*

Notice that for $f = 0$ we can rewrite $E_0(\mu, u)$ as

$$F(u) := \frac{1}{2} \int |\nabla u|^2 d\lambda$$

where $\lambda := dx + m\mu$, dx is the Lebesgue measure on Ω, and m is the number that appears in (2.2). Functionals of this type has been studied in detail in [4], where it is proved that the relaxation of $F(u)$ with $u \in C_c^\infty(\Omega)$ is

$$F^*(u) := \frac{1}{2} \int |\nabla_\lambda u|^2 d\lambda\,, \quad u \in H^1(\lambda)\,,$$

where the space H^1_λ and the operator ∇_λ are defined in a suitable abstract sense.

Thus the relaxation result stated above can be rephrased as follows: the space H^1_λ agrees with $H_0^1(\Omega) \cap H^1(S)$ and the operator ∇_λ agrees with the full gradient ∇ for Lebesgue-a.e. x, with the tangential gradient ∇_τ for $\mathscr{H}^1$-a.e. $x \in S$, and with the null-operator for μ^s-a.e. x, where μ^s is the singular part of μ w.r.t. $\mathscr{H}^1 \llcorner S$.

Problem 6.5. We denote by $\mu = \theta\mathscr{H}^1 \llcorner S$ a solution of problem (1.5) given in Theorem 2.6, and by u the unique minimizer of $E_f^*(\mu, \cdot)$. Here are some open questions concerning μ and u.

(a) Intuition tells that it is never convenient to use part of S to reinforce the boundary the membrane, because it is already reinforced by the Dirichlet boundary condition inscribed in the problem. On the other hand, the requirement that S be connected might force part of it to lie on the boundary of Ω, even if this part does not contribute to reinforcing the membrane. Here are two plausible statements that would be interesting to investigate:

- for some non-convex domain Ω the set $S \cap \partial\Omega$ may have positive length, but S cannot be entirely contained in $\partial\Omega$;
- if Ω is strictly convex then the set $S \cap \partial\Omega$ has zero-length, and perhaps it is even finite.

Note that using the second part of Theorem 2.6 (and in particular assuming that the support of f is Ω) we can prove the following: if Ω is strictly convex then $S\cap\partial\Omega$ does not contain any arc.

(b) In principle the density θ belongs to $L^1(S)$. It would be interesting to investigate if θ is bounded and, possibly refining the assumptions on the data, prove further regularity properties.

(c) According to the numerical simulations we made, the set S never contains closed curves; it would be interesting to show this fact under general assumptions.

(d) Numerical simulations also show that S may present branching points at least for values of L large enough. However, the regularity of the set S seems a difficult issue: is it true that, under suitable assumptions on the data, the set S is smooth except a finite number of branching points? And if a branching occurs, what are the necessary condition of optimality for the related angles?

(e) When the support of f is Ω and the total length L tends to $+\infty$, then the optimal set S tends to fill the entire Ω. Can we say more on the asymptotic behavior of S in this regime? This question is reminiscent of a Γ-convergence result for the irrigation problem proved in [16].

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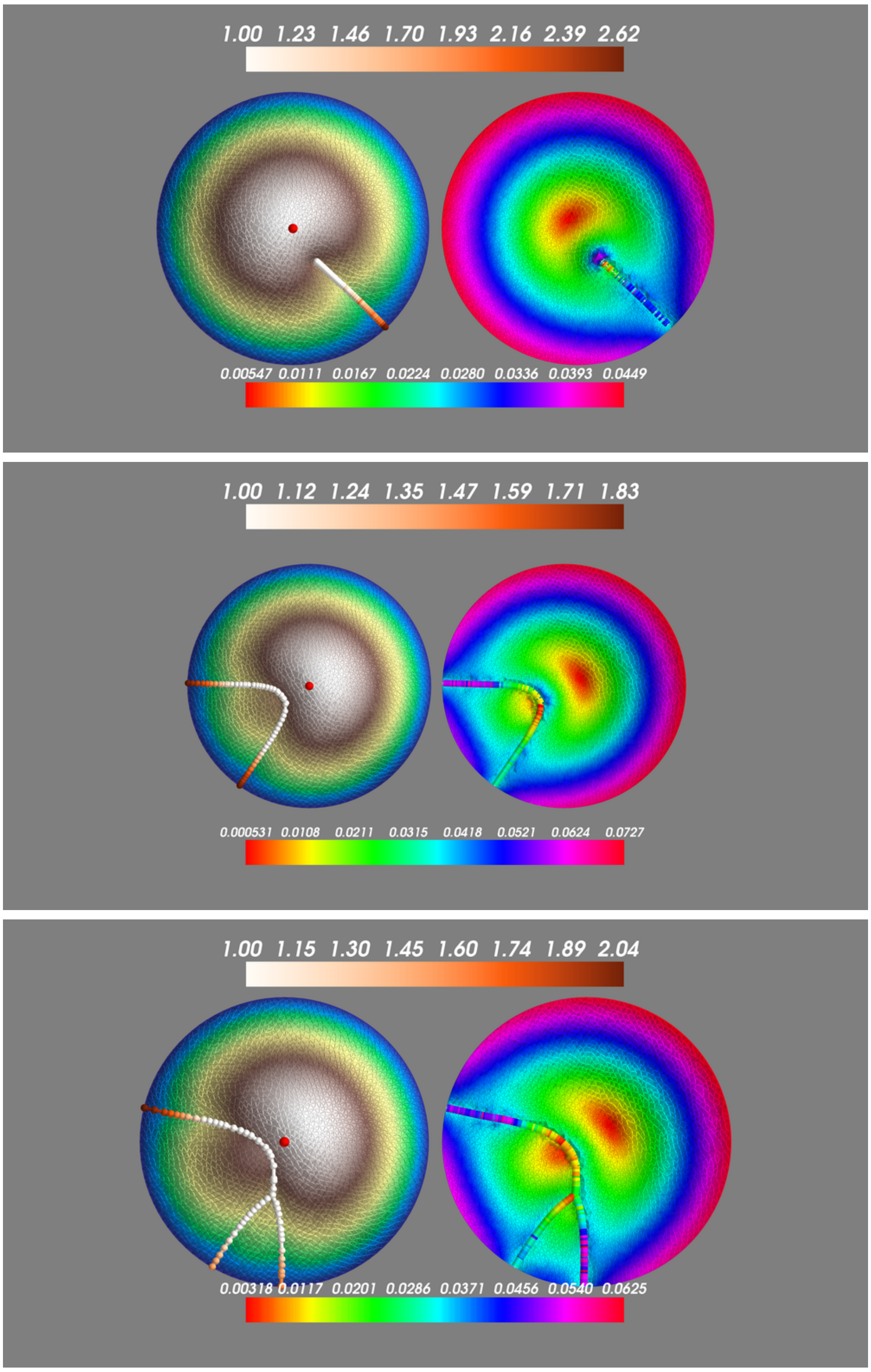

Figure 1. Approximation of globally optimal reinforcement structures for $m = 0.5$, $L = 1$, 2 and 3. The upper colorbar is related to the weights θ which colors the optimal reinforcement set on the left, whereas the lower colorbar stands for the tangential gradient plotted on the connected set on the right picture

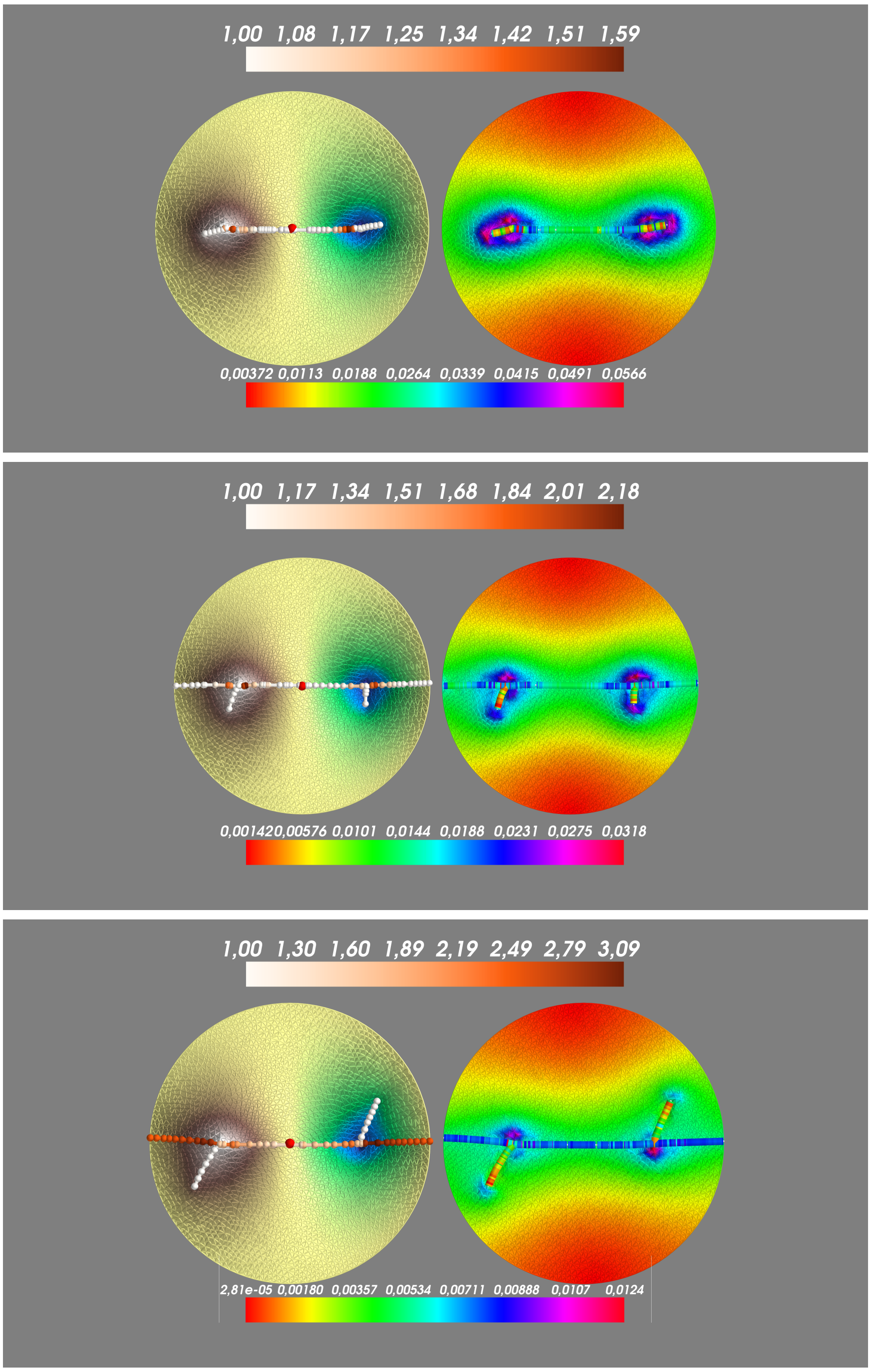

Figure 2. Approximation of globaly optimal reinforcement strucutres for $m = 0.5$, $L = 1.5$, 2.5 and 5 for a source consisting of two dirac masses. The upper colorbar is related to the weights θ which colors the optimal reinforcement set on the left, whereas the lower colorbar stands for the tangential gradient plotted on the connected set on the right picture

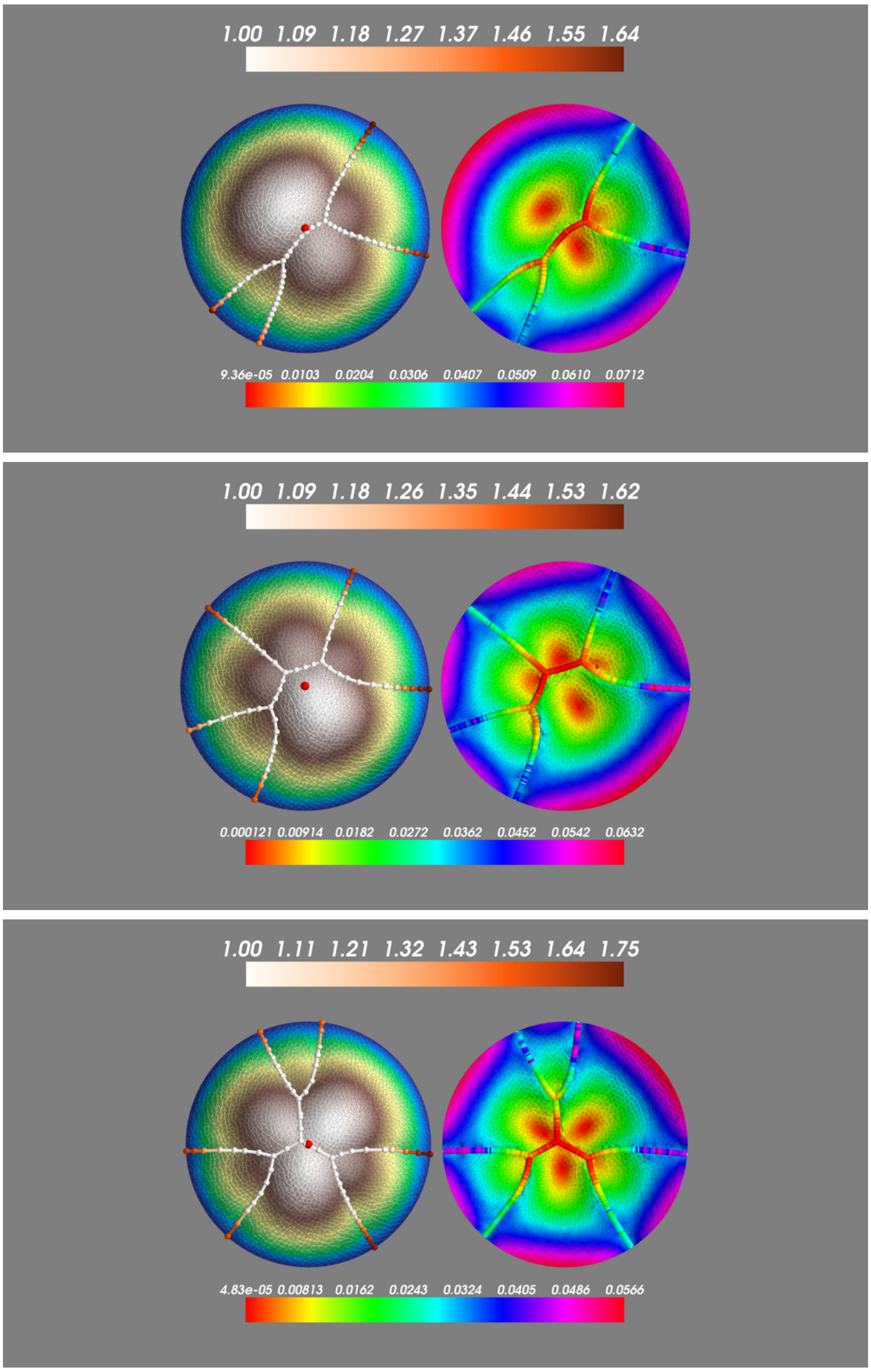

FIGURE 3. Approximation of globaly optimal reinforcement structures for $m = 0.5$, $L = 4$, 5 and 6. The upper colorbar is related to the weights θ which colors the optimal reinforcement set on the left, whereas the lower colorbar stands for the tangential gradient plotted on the connected set on the right picture