On the geometric structure of currents tangent to smooth distributions

Giovanni Alberti, Annalisa Massaccesi, Eugene Stepanov

Abstract. It is well known that a k-dimensional smooth surface in a Euclidean space cannot be tangent to a non-involutive distribution of k-dimensional planes. In this paper we discuss the extension of this statement to weaker notions of surfaces, namely integral and normal currents. We find out that integral currents behave to this regard exactly as smooth surfaces, while the behavior of normal currents is rather multifaceted. This issue is strictly related to a geometric property of the boundary of currents, which is also discussed in details.

Keywords: non-involutive distributions, Frobenius theorem, integral currents, normal currents, geometric property of the boundary.

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1. Introduction

The starting point of this paper is the following implication in Frobenius theorem: if V is a distribution of k-dimensional planes on an open set Ω in $\mathbb{R}^n$, and Σ is a k-dimensional smooth surface which is everywhere tangent to V, then V is involutive at every point of Σ or, equivalently, Σ does not intersect the open set where V is non-involutive. In the following we refer to this statement simply as Frobenius theorem.

In the classical statement it is assumed that both the distribution V and the surface Σ are sufficiently regular. In particular it suffices that V be of class C^1 and Σ be a surface (submanifold) of class C^1, possibly with boundary. In this paper we discuss the generalization of this result to weaker notions of surfaces, though not weakening the regularity assumption on V (see however §1.15).

We first remark that Frobenius theorem does not hold if Σ is just a closed subset of a k-dimensional C^1-surface. More precisely, for every continuous distribution of k-planes V there exists a C^1-surface S such that the set Σ of all points of S where S is tangent to V has positive k-dimensional measure, regardless of the involutivity of V (this result was proved for a special V in [6], Theorem 1.4; the general version can be found in [5]).

On the other hand, Frobenius theorem holds if the boundary of Σ (relative to S) is not too large; for example, it suffices that the $(k-1)$-dimensional Hausdorff measure $\mathscr{H}^{k-1}(\partial\Sigma)$ be finite, see §1.13 for more details.

However the most satisfactory version of this statement is obtained by considering surfaces and boundary in the sense of currents: in Theorem 1.1 we show that Frobenius theorem holds for all integral currents tangent to the distribution V. Thus one is naturally led to wonder what happens for the largest class of currents with

"nice" boundary, namely normal currents. It turns out that this case is much more interesting, and in particular the validity of Frobenius theorem depends also on how "diffuse" the current is (Theorem 1.3).

Notice that our results are local in nature, and therefore, even if stated in the Euclidean space, they actually hold in Riemannian manifolds, and even in Finsler manifolds.

Some of the results in this paper were announced in [3].

Description of the results

Through the rest of this paper V is a C^1-distribution of k-planes on an open set Ω in $\mathbb{R}^n$, and $N(V)$ is the open set of all points where V is non-involutive.[1]

1.1. Theorem. *Let T be an integral current in Ω which is tangent to V.[2] Then the support of T does not intersect the non-involutivity set $N(V)$.*

A version of this statement was first proved in the second author's dissertation ([17], Theorem 2.2.6), following a completely different argument.

The next step is to consider normal currents. We recall here that these currents share many properties with integral currents, including that of having a "nice" boundary, but differ in many regards. In particular integral k-dimensional currents are supported on k-dimensional (rectifiable) sets, while k-dimensional normal currents can be quite "sparse", even absolutely continuous with respect to the Lebesgue measure.

The following example, proposed by M. Zworski in [20], shows that Frobenius theorem does not hold in general for normal currents.

1.2. Example. Consider a simple k-vectorfield $v = v_1 \wedge \cdots \wedge v_k$ of class C^1 on $\mathbb{R}^n$ and let T be the k-current given by $T = v\mathscr{L}^n$. Then T is a normal current on every bounded open set Ω in $\mathbb{R}^n$ (see §2.10 and Remark 2.11(iii)) and it is clearly tangent to the distribution V spanned by $v_1, \ldots, v_k$, regardless of its involutivity.

It turns out that there is a general result behind this example: a normal current T which is tangent to a distribution V must be sufficiently "sparse" on the non-involutivity set $N(V)$, and the degree of "sparseness" depends on how much non-involutive the distribution V is.

A precise statement requires some preparation. We let $\widehat{V}$ be the distribution spanned by the vectorfields tangent to V and their first commutators (see §2.16 for precise definitions), and for every $d = k, \ldots, n$ we set

$$N(V, d) := \left\{ x \in \Omega \colon \dim(\widehat{V}(x)) = d \right\}.$$

(Thus $N(V)$ is the union of all sets $N(V, d)$ with $d = k+1, \ldots, n$.)

We then consider a normal k-current T on Ω, which we write as $T = \tau\mu$ where μ is a finite positive measure and τ is a k-vectorfield which is nonzero μ-a.e. (cf. §2.10

and Remark 2.11(i)); in particular the support of T agrees with the support of μ. Along the same line we write the boundary of T as $\partial T = \tau'\mu'$.

We say that T is tangent to V if $\mathrm{span}(\tau(x)) = V(x)$ for μ-a.e. x.[3]

We define the degree of sparseness of a measure in terms of absolute continuity with respect to the Hausdorff measure $\mathscr{H}^d$ or the integral geometric measures $\mathscr{I}_t^d$ (the higher is d, the sparser is the measure). If T is a normal k-current and we take μ and μ' as above, then μ is absolutely continuous with respect to $\mathscr{I}_t^k$ and therefore also with respect to $\mathscr{H}^k$ (see §2.1), that is, $\mu \ll \mathscr{I}_t^k \ll \mathscr{H}^k$. Similarly $\mu' \ll \mathscr{I}_t^{k-1} \ll \mathscr{H}^{k-1}$.

We can now state the main result for normal currents.

1.3. Theorem. *Take V, $T = \tau\mu$ and $\partial T = \tau'\mu'$ as above, and assume that T be tangent to V. Then*

(i) *the restriction of μ to the set $N(V)$ satisfies $\mu \llcorner N(V) \ll \mu'$;*

(ii) *for $d > k$ there holds $\mu \llcorner N(V, d) \ll \mathscr{I}_t^d \ll \mathscr{H}^d$;*

(iii) *for $d > k$ there holds $\mu' \llcorner N(V, d) \ll \mathscr{H}^{d-1}$.*

Using Theorem 1.3 we can show that Frobenius theorem holds for normal currents that satisfy certain additional conditions:

1.4. Corollary. *Take V, $T = \tau\mu$ and $\partial T = \tau'\mu'$ as above, and assume that T be tangent to V. If any of the following conditions holds then the support of T does not intersect $N(V)$:*

(a) *μ is concentrated on a $\mathscr{I}_t^{k+1}$-null Borel set;[4]*

(b) *μ is concentrated on a μ'-null Borel set;*

(c) *T is a rectifiable current (possibly with non-integral multiplicity);*

(d) *$\partial T = 0$.*

1.5. Remarks. (i) Regarding condition (a) in Corollary 1.4, we recall that the following implications hold for every Borel set E:

$$\mathscr{I}_t^{k+1}(E) = 0 \iff \mathscr{H}^{k+1}(E) = 0 \iff \dim_H(E) < k+1,$$

where $\dim_H(E)$ is the Hausdorff dimension of E.

(ii) Under condition (c), Corollary 1.4 generalizes Theorem 1.1.

(iii) Even though the measures μ and μ' in the representations of T and ∂T are not unique, the statements of Theorem 1.3 and Corollary 1.4 do not depend on the choice of these measures (cf. Remark 2.11(ii)).

As already pointed out in [3], the validity of Frobenius theorem for normal currents is strictly related to the following property of the boundary.

1.6. Geometric property of the boundary. Let $T = \tau\mu$ be a normal k-current on the open set Ω with boundary $\partial T = \tau'\mu'$. We say that T has the *geometric property of the boundary* if, up to a modification of τ in a μ-null set,

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[1] Most of the terminology used in this introduction is properly defined in Section 2; the precise definition of $N(V)$ is given in §2.14.

[2] The precise meaning of "T is tangent to V" is given in §2.12.

[3] The span of a (non necessarily simple) k-vector is defined in §2.3.

[4] We say that μ is concentrated on a Borel set E if $\mu(\Omega \setminus E) = 0$; this implies that the support of μ is contained in the closure of E (but not necessarily in E).

(a) the map $x \mapsto \text{span}(\tau(x))$ is continuous on the support of T;

(b) $\text{span}(\tau'(x)) \subset \text{span}(\tau(x))$ for μ'-a.e. x.

It is easy to check that if T is tangent to the distribution V then these conditions are equivalent to the inclusion

$$\text{span}(\tau'(x)) \subset V(x) \quad \text{for } \mu'\text{-a.e. } x. \tag{1.1}$$

1.7. Theorem. *Take V and $T = \tau\mu$ as above, and assume that T be tangent to V. Then the following assertions are equivalent:*

(i) *T has the geometric property of the boundary, that is, (1.1) holds;*

(ii) *the support of μ does not intersect $N(V)$.*

1.8. Remarks. (i) The current associated to an oriented surface Σ of class C^1 has the geometric property of the boundary; indeed condition (b) in §1.6 reduces to the fact that for every $x \in \partial\Sigma$ the tangent space $T_x(\partial\Sigma)$ is contained in $T_x\Sigma$.

(ii) Example 1.2 and Theorem 1.7 show that there are normal currents T which are tangent to a distribution V of class C^1 and do not have the geometric property of the boundary. In this case one may ask where the inclusion $\text{span}(\tau'(x)) \subset V(x)$ holds and where it does not; a detailed answer is given in Theorem 3.3 and Remark 3.4.

(iii) Theorem 1.7 implies that the geometric property of the boundary holds if T is tangent to a distribution of k-planes of class C^1 and satisfies one of the conditions (a)–(d) in Corollary 1.4, e.g., if T is an integral current. In [2] we give an example of integral current which is tangent to a *continuous* distribution of k-planes and does not have the geometric property of the boundary.

Additional comments

1.9. On the geometric property of the boundary. We collect here further remarks on the property defined in §1.6.

(i) The continuity requirement (a) in §1.6. is needed to make the definition meaningful. Indeed if we drop this requirement then every current T such that ∂T is singular with respect to T (that is, μ' is singular with respect to μ) has the geometric property of the boundary—the point is that the k-vectorfield τ is only determined up to μ-null sets, and therefore it can be arbitrarily modified in a set which is μ'-full.

(ii) The relation between the geometric property of the boundary and Frobenius theorem for currents was first pointed out by the second author in her dissertation [17], where a version of Theorem 1.1 is obtained as a corollary of the geometric property of the boundary of integral currents ([17], Lemma 2.2.1).

(iii) We point out that in [17] and [2] the sentence "the current T is tangent to the distribution V" has a stronger meaning than in this paper. Here it means that the tangent plane $\text{span}(\tau(x))$ is prescribed μ-a.e., while there it means that both the tangent plane $\text{span}(\tau)$ and its orientation are prescribed μ-a.e.

Under this stronger notion of tangency, in [2] it is proved that the geometric property of the boundary holds for integral currents that are tangent to a continuous distribution V of k-planes (while here we need that V be of class C^1, cf. Remark 1.8(iii)).

1.10. An open problem. The statement of Theorem 1.3 depends crucially on the sets $N(V,d)$, which are defined using the distribution $\widehat{V}$ spanned by the vectorfields tangent to V and their first commutators (see §2.16). In this context it is also natural to consider the distribution $\overline{V}$ spanned by the Lie algebra generated by V, that is, by the vectorfields tangent to V and their commutators of all orders. Clearly the distribution $\overline{V}$ contains $\widehat{V}$, and the inclusion may be strict. If this is the case, replacing the sets $N(V,d)$ by

$$\overline{N}(V,d) := \big\{ x \in \Omega \colon \dim(\overline{V}(x)) = d \big\}$$

in statements (ii) and (iii) of Theorem 1.3 yields a stronger results. We believe that such results are true, but cannot be obtained by a modification of the present proof.

1.11. Sobolev sets. Extensions of Frobenius theorem to weaker notions of surfaces have been studied by many authors. For instance, in [16], Theorem 2.1, it is proved that *Sobolev sets* of dimension m in the (sub-Riemannian) Heisenberg group $\mathbb{H}^n$ cannot be horizontal for $m > n$, that is, images of Sobolev maps with derivative of rank m from open subsets of $\mathbb{R}^m$ into $\mathbb{R}^{2n+1} \simeq \mathbb{H}^n$ cannot be tangent to the horizontal distribution.

Using Theorem 1.1 we partially recover this result and extend it to a more general setting:

1.12. Theorem. *Let Ω be an open subset in $\mathbb{R}^n$ and V a C^1-distribution of k-planes on Ω. Let A be an open set in $\mathbb{R}^k$ and let $u : A \to \Omega$ be a continuous map of class $W^{1,p}_{\mathrm{loc}}$ with $p > k$ such that, for a.e. $z \in A$, the image of the differential of u at z is $V(u(z))$. Then $u(A)$ does not intersect the non-involutivity set $N(V)$.*

1.13. Tangency sets. Given a distribution of k-planes V and a k-dimensional surface S of class C^1, we say that a closed subset Σ of S is a *tangency set* of S and V if the tangent space $T_x S$ agrees with $V(x)$ for every $x \in \Sigma$. In this context the statement of Frobenius theorem reduces to

$$\mathscr{H}^k\big(\Sigma \cap N(V)\big) = 0 \,. \tag{1.2}$$

As already pointed out at the beginning of this introduction, if S is of class C^1 then (1.2) does not hold for all tangency sets Σ. However it holds if Σ has finite perimeter relative to S.[5] Note that this condition is implied by (but not equivalent to) any of the following: a) the (topological) boundary of Σ relative to S is $\mathscr{H}^{k-1}$-finite; b) the boundary of the canonical current associated to Σ has finite mass.

On the other hand, if the surface S is of class C^2, or even of class $C^{1,1}$, then (1.2) holds for every tangency set Σ, regardless of the regularity of its boundary (see for instance [7], Theorem 1.3). This result is generalized in [5] by proving that if S is of class $C^{1,\alpha}$ for some $0 < \alpha < 1$ then Frobenius theorem holds for every tangency set Σ whose (distributional) boundary has a suitable *fractional* regularity. This shows that the validity of Frobenius theorem depends on a combination of the regularity of $\partial\Sigma$ and of the regularity of the surface S.

It would be interesting to extend this result to some class of currents.

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[5] This claim follows from two results by S. Delladio: in [9], Corollary 4.1, he proves that $\mathscr{H}^k$-a.e. point x of a finite perimeter set Σ is a *superdensity* point, i.e., $\mathscr{H}^k\big((B(x,r) \cap S)\backslash\Sigma\big) = o(r^{k+1})$, and in [10], Corollary 1.1, he proves that the set of superdensity points of Σ does not intersect $N(V)$, and therefore (1.2) holds.

1.14. Carnot-Carathéodory spaces. The tangency set Σ with $\mathscr{H}^k(\Sigma) > 0$ mentioned in §1.13 is a non-trivial k-rectifiable set in Ω. However, if the distribution V satisfies Hörmander condition and we replace the Euclidean distance in Ω with the Carnot-Carathéodory distance associated to V, then Ω contains only trivial k-rectifiable sets, and in particular Σ is no longer rectifiable (see [15] for the case of Heisenberg groups, and [4] for a more general context).

1.15. Non-smooth distributions. Through this paper we always assume that the distribution V is of class C^1, which is the minimal regularity required to define involutivity in the classical sense (see §2.14). We show in §2.19 that it is possible to define involutivity also if V is less regular than C^1 using a suitable distributional formulation. In order to extend the results stated above to less regular V, a major difficulty seems to be the correct definition of the non-involutivity set $N(V)$.

Structure of the paper. Section 2 contains the notation and some preliminary results. The main result in Section 3 is the key identity (3.4), which allows us to establish a very precise connection between Frobenius theorem for normal currents and the geometric property of the boundary (Theorem 3.3). All statements given in this introduction are more or less straightforward consequences of identity (3.4) and Theorem 3.3; the proofs are collected in Section 4.

2. Notation and preliminary results

We assume that the reader is somewhat familiar with the theory of currents. Therefore in this section we only briefly recall the basic notions of multilinear algebra and of the theory of currents, mainly to fix the notation, and describe in more details only those notions that are of less common use.

Through this paper we tacitly assume that sets and functions are Borel measurable and measures are defined on the Borel σ-algebra, and are real-valued and finite (with the notable exception of Lebesgue, Hausdorff and integral geometric measures).

Here is a list of frequently used notations:

$\mu \llcorner F$ restriction of a measure μ to a Borel set F, i.e., $[\mu \llcorner F](E) := \mu(E \cap F)$ for every Borel set E in X;

$\rho\mu$ measure associated to a measure μ on X and a Borel density ρ, that is, $[\rho\mu](E) := \int_E \rho \, d\mu$ for every Borel set E in X;

$f_{\#}\mu$ pushforward of a measure μ on X according to a Borel map $f : X \to Y$, that is, $[f_{\#}\mu](E) := \mu(f^{-1}(E))$ for every Borel set E in Y;

$f_{\#}T$ pushforward of a current T according to a map f (see, e.g., [13], §7.4.2);

$|\mu|$ variation measure associated to a real- or vector-valued measure μ;

$\mu \ll \lambda$ the measure μ is absolutely continuous with respect to the measure λ;

μ_a, μ_s absolutely continuous and singular part of a measure μ with respect to a given measure λ.

$\mathscr{L}^n, \mathscr{H}^d$ Lebesgue measure on $\mathbb{R}^n$ and d-dimensional Hausdorff measure;

$\mathscr{I}_t^d$ d-dimensional integral geometric measures (§2.1);

$I(n,k)$ set of all multi-indices $\mathbf{i} := (i_1, \ldots, i_k)$ with $1 \le i_1 < \cdots < i_k \le n$;

$\wedge_k(V)$ space of k-vectors in a linear space V; the canonical basis of $\wedge_k(\mathbb{R}^n)$ is formed by the simple k-vectors $e_{\mathbf{i}} := e_{i_1} \wedge \cdots \wedge e_{i_k}$ with $\mathbf{i} \in I(n,k)$, where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$; $\wedge_k(\mathbb{R}^n)$ is endowed with the Euclidean norm $|\cdot|$ associated to this basis;[6]

$\wedge^k(V)$ space of k-covectors on a linear space V; the canonical basis of $\wedge^k(\mathbb{R}^n)$ is formed by the simple k-covectors $dx_{\mathbf{i}} := dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ with $\mathbf{i} \in I(n,k)$, where $\{dx_1, \ldots, dx_n\}$ is the canonical basis of the dual of $\mathbb{R}^n$; $\wedge^k(\mathbb{R}^n)$ is endowed with Euclidean norm $|\cdot|$ associated to this basis;

dx $:= dx_1 \wedge \cdots \wedge dx_n$;

$\wedge$ exterior product of k-vectors, or of h-covectors;

$\lrcorner, \llcorner$ interior products of a k-vector and a h-covector (§2.2);

$\star$ Hodge-type operator on k-vectors and k-covectors (§2.5);

$\mathrm{span}(v)$ span of a k-vector v (§2.3);

d exterior derivative of a k-form (§2.7);

div divergence of a k-vectorfield (§2.7);

$[v, v']$ Lie bracket of vectorfields v and v' (§2.9);

$W(\mu, \cdot)$ decomposability bundle of a measure μ (§2.20);

$N(V)$ non-involutivity set of a distribution of k-planes V (§2.14);

$\widehat{V}$ and $N(V, d)$, see §2.16.

2.1. Integral geometric measure. Given $d = 1, \ldots, n$ and $t \in [1, +\infty]$, we denote by $\mathscr{I}_t^d$ the d-dimensional integral geometric measure of exponent t on $\mathbb{R}^n$ (the precise definition can be found in [11], §2.10.5, or [13], §2.1.4).

The relevant features are that $\mathscr{I}_t^d$ is invariant under isometries of $\mathbb{R}^n$, it agrees with the Hausdorff measure $\mathscr{H}^d$ on regular d-dimensional surfaces of $\mathbb{R}^n$, and in general satisfies $\mathscr{I}_t^d \le \mathscr{H}^d$. Moreover, and this is essential to this paper, a Borel set E is $\mathscr{I}_t^d$-null if and only if

$$\mathscr{H}^d\big(p_V(E)\big) = 0 \quad \text{for a.e. } d\text{-plane } V \text{ in } \mathbb{R}^n,$$

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[6] None of the results in this paper depend on this choice of norm.

where p_V stands for the orthogonal projection on V and "a.e." refers to the Haar measure on the Grassmannian of d-planes in $\mathbb{R}^n$.

Note that the class of $\mathscr{I}_t^d$-null Borel sets is the same for all t and is strictly larger than the class of $\mathscr{H}^d$-null sets (indeed, by the Besicovitch-Federer projection theorem, the first class contains all sets which are $\mathscr{H}^d$-finite and purely d-unrectifiable).

In particular the fact that a measure μ is absolutely continuous with respect to $\mathscr{I}_t^d$ does not depend on the exponent t and implies that μ is absolutely continuous with respect to $\mathscr{H}^d$ (but the converse does not hold).

Multilinear algebra

In this subsection we review the basic notions of multilinear algebra; we consider multivectors and multicovectors in a general linear space V. For a thorough treatise of this topic, we refer the reader to [11], §1.5, from which we borrow the notation, or standard textbooks in Differential Geometry, such as [14], Chapters 11 and 14, and [19], Chapter 2.

2.2. Interior product. Given a k-vector v in V and an h-covector α on V with $h \leq k$, the *interior product* $v \llcorner \alpha$ is the $(k-h)$-vector in V defined by

$$\langle v \llcorner \alpha;\, \beta \rangle := \langle v;\, \alpha \wedge \beta \rangle \quad \text{for every } (k-h)\text{-covector } \beta;$$

if $k \leq h$, the *interior product* $v \lrcorner \alpha$ is the $(h-k)$-covector defined by

$$\langle w;\, v \lrcorner \alpha \rangle := \langle w \wedge v;\, \alpha \rangle \quad \text{for every } (h-k)\text{-vector } w.$$

Note that given a k-vector v, an h-covector α and an h'-covector α' with $h+h' \leq k$, then

$$v \llcorner (\alpha \wedge \alpha') = (v \llcorner \alpha) \llcorner \alpha'.$$

Similarly, given a k-vector v, a k'-vector v' and an h-covector α with $k + k' \leq h$, then

$$(v \wedge v') \lrcorner \alpha = v \lrcorner (v' \lrcorner \alpha).$$

2.3. Span of a k-vector. Given a k-vector v in V, we denote by $\mathrm{span}(v)$ the smallest of all linear subspaces W of V such that v belongs to $\wedge_k(W)$.[7] This definition is well-posed because every k-vector in W is canonically identified with a k-vector in V via the inclusion map $i: W \to V$, and assuming this identification we have $\wedge_k(W) \cap \wedge_k(W') = \wedge_k(W \cap W')$ for every W, W' subspaces of V. We have the following properties (see [1], Proposition 5.9):

(i) if $v = 0$ then $\mathrm{span}(v) = \{0\}$;

(ii) if $v \neq 0$ then $\dim(\mathrm{span}(v)) \geq k$;

(iii) if v is simple and non-trivial, that is, $v = v_1 \wedge \cdots \wedge v_k$ with $v_1, \ldots, v_k$ linearly independent vectors in V, then $\mathrm{span}(v)$ is the subspace of V spanned by $v_1, \ldots, v_k$ and $\dim(\mathrm{span}(v)) = k$;

(iv) conversely, if $\dim(\mathrm{span}(v)) = k$ then v is simple and non-trivial;

(v) $\mathrm{span}(v)$ consists of all vectors of the form $v \llcorner \alpha$ with $\alpha \in \wedge^{k-1}(V)$.

The next lemma will be used later in the proofs.

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[7] For example, given linearly independent vectors $v_1, \ldots, v_4$, then $\mathrm{span}(v_1 \wedge v_2 + v_3 \wedge v_4)$ is the linear subspace spanned by $v_1, \ldots, v_4$.

2.4. Lemma. *Let v be a k-vector in V and let W be a d-dimensional subspace of V. Then $\mathrm{span}(v) \subset W$ if any of the following conditions hold:*

(a) *there exist a subspace W' of W with $d' := \dim(W') \geq k$, and an integer h with $1 \leq h \leq d' - k + 1$ such that $\mathrm{span}(v \wedge w) \subset W$ for every $w \in \wedge^h(W')$;*

(b) *$v \wedge w = 0$ for every $w \in \wedge^{d-k+1}(W)$;*

(c) *$k = 1$ and there exist an integer $1 \leq h \leq d$ and a simple h-vector $w \in \wedge^h(W)$ with $w \neq 0$ such that $\mathrm{span}(v \wedge w) \subset W$;*

Proof. The proof is divided in three steps, one for each condition.

Step 1: if condition (a) holds then $\mathrm{span}(v) \subset W$. We argue by contradiction, and prove that if $\mathrm{span}(v) \not\subset W$ then there exists $w \in \wedge^h(W')$ such that $\mathrm{span}(v \wedge w) \not\subset W$. To this aim we choose vectors $e_1, \ldots, e_n$ in V so that $e_1, \ldots, e_{d'}$ form a basis of W', $e_1, \ldots, e_d$ form a basis of W, and $e_1, \ldots, e_n$ form a basis of V. Then we write v as

$$v = \sum_{\mathbf{i} \in I(n,k)} v_{\mathbf{i}}\, e_{\mathbf{i}}.$$

Since $\mathrm{span}(v)$ is not contained in W there exists a multi-index $\mathbf{j} = (j_1, \ldots, j_k)$ in $I(n,k)$ such that $v_{\mathbf{j}} \neq 0$ and $j_k > d$. This means that at most $k-1$ indices in $\mathbf{j}$ belong to $\{1, \ldots, d'\}$; thus there are at least $d' - k + 1$ indices in $\{1, \ldots, d'\}$ that are not in $\mathbf{j}$, and since $h \leq d' - k + 1$ we can find a multi-index $\mathbf{j}' \in I(d', h)$ whose indices are all different from those of $\mathbf{j}$ (with a slight abuse of notation we write $\mathbf{j}' \cap \mathbf{j} = \varnothing$).

We now set $w := \hat{e}_{\mathbf{j}'}$. Then

$$v \wedge w = \sum_{\mathbf{i}:\, \mathbf{j}' \cap \mathbf{i} = \varnothing} v_{\mathbf{i}}\, \hat{e}_{\mathbf{i}} \wedge \hat{e}_{\mathbf{j}'}.$$

Let $\mathbf{j} \cup \mathbf{j}'$ denote the multi-index in $I(n, k+h)$ that contains the indices in $\mathbf{j}$ and in $\mathbf{j}'$. The formula above shows that the coordinate $(v \wedge w)_{\mathbf{j} \cup \mathbf{j}'}$ is equal to $\pm v_{\mathbf{j}}$ and in particular it does not vanish; since $\mathbf{j} \cup \mathbf{j}'$ contains j_k and $j_k > d$, we deduce that $v \wedge w$ is not a $(k+h)$-vector in W, that is, $\mathrm{span}(v \wedge w) \not\subset W$, as claimed.

Step 2: condition (b) implies condition (a). More precisely, condition (a) holds with $W' := W$ and $h := d' - k + 1 = d - k + 1$.

Step 3: condition (c) implies condition (a). More precisely, condition (a) holds with $W' := \mathrm{span}(w)$: notice indeed that since w is simple then $d' := \dim(W') = h$, and therefore the h-vectors in $\wedge^h(W')$ are just multiples of w. $\square$

2.5. A Hodge-type operator. We consider the operator $\star$ that acts on all vectors and covectors of $\mathbb{R}^n$, and more precisely maps k-vectors into $(n-k)$-covectors and vice versa, and is defined by the following property: for every $v \in \wedge_k(\mathbb{R}^n)$ and every $\alpha \in \wedge^k(\mathbb{R}^n)$ there holds

$$\star v := v \lrcorner dx, \quad \star \alpha := e \llcorner \alpha,$$

where $dx := dx_1 \wedge \cdots \wedge dx_n$ and $e := e_1 \wedge \cdots \wedge e_n$.[8]

Note that the definition of the interior products (see §2.2) yields

$$\langle w;\, \star v \rangle = \langle w \wedge v;\, dx \rangle, \quad \langle \star \alpha;\, \beta \rangle = \langle e;\, \alpha \wedge \beta \rangle,$$

---

[8] This operator is similar to the standard Hodge star operator but not exactly the same; it is defined in [11], §1.5.2, but denoted there with a different symbol.

for every $(n-k)$-vector w and every $(n-k)$-covector β.

Moreover for every $\mathbf{i} \in I(n,k)$ one has

$$\star e_{\mathbf{i}} = \operatorname{sign}(\mathbf{j},\mathbf{i})\,dx_{\mathbf{j}}, \quad \star dx_{\mathbf{i}} = \operatorname{sign}(\mathbf{i},\mathbf{j})\,e_{\mathbf{j}}, \tag{2.1}$$

where $\mathbf{j}$ is the multi-index in $I(n, n-k)$ consisting of all indices which are not in $\mathbf{i}$, and $\operatorname{sign}(\mathbf{j},\mathbf{i})$ is the sign of the permutation that reorders the sequence of indices $j_1, \ldots, j_{n-k}, i_1, \ldots, i_k$. The identities in (2.1) show that $\star$ is an involution, that is, $\star(\star v) = v$ and $\star(\star \alpha) = \alpha$.

Among the many identities relating $\star$ and the various products, we will use the following one: for every k-vector v and every h-covector α with $h \leq k$ one has

$$\star(v \llcorner \alpha) = (\star v) \wedge \alpha. \tag{2.2}$$

Forms, vectorfields, currents

Here we review the basic definitions and results concerning differential forms, vectorfields and currents. These objects will be defined on a general open set Ω in $\mathbb{R}^n$, $n \geq 2$.

Elementary introductions to the theory of currents can be found for instance in [13], [18]; the most complete reference remains [11].

2.6. Forms and vectorfields. A k-form is a map $\omega : \Omega \to \wedge^k(\mathbb{R}^n)$, and we sometime write it in terms of the canonical basis of $\wedge^k(\mathbb{R}^n)$:

$$\omega(x) = \sum_{\mathbf{i} \in I(n,k)} \omega_{\mathbf{i}}(x)\,dx_{\mathbf{i}}.$$

Similarly, a k-vectorfield is a map $v : \Omega \to \wedge_k(\mathbb{R}^n)$, and we sometime write it as

$$v(x) = \sum_{\mathbf{i} \in I(n,k)} v_{\mathbf{i}}(x)\,e_{\mathbf{i}}.$$

2.7. Exterior derivative and divergence. If ω is a k-form of class C^1, the *exterior derivative* $d\omega$ is the $(k+1)$-form defined in coordinates by the usual formula:

$$d\omega(x) := \sum_{\mathbf{i} \in I(n,k)} \sum_{j=1}^{n} \frac{\partial \omega_{\mathbf{i}}}{\partial x_j}(x)\,dx_j \wedge dx_{\mathbf{i}} = \sum_{j=1}^{n} dx_j \wedge \frac{\partial \omega}{\partial x_j}(x). \tag{2.3}$$

If v is a k-vectorfield of class C^1, the *divergence* $\operatorname{div} v$ is the $(k-1)$-vectorfield defined by

$$\operatorname{div} v(x) := \sum_{\mathbf{i} \in I(n,k)} \sum_{j=1}^{n} \frac{\partial v_{\mathbf{i}}}{\partial x_j}(x)\,e_{\mathbf{i}} \llcorner dx_j = \sum_{j=1}^{n} \frac{\partial v}{\partial x_j}(x) \llcorner dx_j. \tag{2.4}$$

Definition (2.4) is not as standard as (2.3): we refer to [11], §4.1.6, for the abstract characterization of the divergence operator and for the following identity, which relates divergence and exterior derivative (it can also be proved using (2.1) and (2.2)):

$$\operatorname{div} v := (-1)^{n-k} \star(d(\star v)). \tag{2.5}$$

For $k = 1$, formula (2.4) reduces to the usual definition of divergence of a vectorfield (recall that $e_i \llcorner dx_j = \langle e_i; dx_j \rangle = \delta_{ij}$).

2.8. Leibniz rules. The exterior derivative satisfies a Leibniz rule with respect to the exterior product: given a k-form ω and a k'-form ω' on Ω, both of class C^1, one has

$$d\left(\omega \wedge \omega'\right) = d\omega \wedge \omega' + (-1)^k\,\omega \wedge d\omega'. \tag{2.6}$$

The divergence satisfies a Leibniz rule with respect to the interior product: given a k-vector v and an h-form ω on Ω, both of class C^1 and with $h \leq k$, one has

$$\operatorname{div}(v \llcorner \omega) = (-1)^h \left((\operatorname{div} v) \llcorner \omega + v \llcorner d\omega\right), \tag{2.7}$$

which follows from (2.6) using (2.2) and (2.5).[9]

2.9. Lie bracket and Cartan's formula. Given two vectorfields v, v' on Ω of class C^1, the Lie bracket $[v, v']$ is the vectorfield on Ω defined by

$$[v, v'](x) := \frac{\partial v}{\partial v'}(x) - \frac{\partial v'}{\partial v}(x) = d_x v\,(v'(x)) - d_x v'\,(v(x)),$$

where $d_x v$ and $d_x v'$ stand for the differentials of v and v' at the point x, viewed as linear maps from $\mathbb{R}^n$ into itself.

Consider now a *simple* k-vectorfield $v = v_1 \wedge \cdots \wedge v_k$ with $k \geq 2$ where each v_i is a vectorfield of class C^1 on Ω. Then the divergence of v can be computed using the following version of Cartan's formula:

$$\begin{aligned}
\operatorname{div} v = {} & \sum_{i=1}^{k} (-1)^{i-1} \operatorname{div} v_i \left(\bigwedge_{j \neq i} v_j \right) \\
& + \sum_{1 \leq i < i' \leq k} (-1)^{i+i'-1} [v_i, v_{i'}] \wedge \left(\bigwedge_{j \neq i, i'} v_j \right).
\end{aligned} \tag{2.8}$$

In particular for $k = 2$ we have

$$\operatorname{div}(v_1 \wedge v_2) = (\operatorname{div} v_1)\,v_2 - (\operatorname{div} v_2)\,v_1 + [v_1, v_2]. \tag{2.9}$$

Formula (2.8) can be found, written in a dual form, in [19], Proposition 2.25(f); we recover the form above using identity (2.5).

2.10. Currents. A k-dimensional current (or simply k-current) T on the open set Ω in $\mathbb{R}^n$ is a continuous linear functional on the space of smooth k-forms with compact support in Ω. The *boundary* of T is the $(k-1)$-current ∂T on Ω defined by $\langle \partial T; \omega \rangle := \langle T; d\omega \rangle$ for every smooth $(k-1)$-form ω with compact support. The *mass* of T, denoted by $\mathbb{M}(T)$, is the supremum of $\langle T; \omega \rangle$ over all k-forms ω such that $|\omega(x)| \leq 1$ for every $x \in \Omega$.

By Riesz theorem, the fact that T has finite mass is equivalent to saying that T can be represented as a finite measure on Ω with values in the space $\wedge_k(\mathbb{R}^n)$, that is, $T = \tau \mu$ where μ is a finite positive measure on Ω and τ is a Borel k-vectorfield in $L^1(\mu)$. Thus

$$\langle T; \omega \rangle = \int_\Omega \langle \tau(x); \omega(x) \rangle\,d\mu(x)$$

for every admissible k-form ω on Ω, and $\mathbb{M}(T) = \int_\Omega |\tau|\,d\mu$.

Finally, a k-current T is said to be:

(a) *normal* if both T and ∂T have finite mass;

---

[9] Formulas relating divergence and exterior product (or exterior derivative and interior product) are more complicated, see §2.9.

(b) *rectifiable* if $T = \tau m \mathscr{H}^k$ where m is a function in $L^1(\mathscr{H}^k)$ such that the set $E := \{x : m(x) \neq 0\}$ is k-rectifiable, and τ is a simple k-vectorfield with $|\tau| = 1$ which spans the approximate tangent space to E at x for $\mathscr{H}^k$-a.e. $x \in E$;

(c) *rectifiable with integer multiplicity* if the multiplicity m is integer-valued;

(d) *integral* if T is rectifiable with integer multiplicity and ∂T has finite mass (if this is the case then also ∂T is rectifiable with integer multiplicity).

2.11. Remarks. (i) When we write a current T in the form $T = \tau\mu$ we always assume that T has finite mass, μ is a (locally) finite positive measure, τ belongs to $L^1(\mu)$ and $\tau(x) \neq 0$ for μ-a.e. x; in particular $\mathrm{spt}(T) = \mathrm{spt}(\mu)$.

(ii) The representation $T = \tau\mu$ is not unique. However, given another representation $T = \hat{\tau}\hat{\mu}$ we have that $\hat{\mu} = \rho\mu$ and $\hat{\tau} = \tau/\rho$ for some strictly positive function ρ. In particular μ and $\hat{\mu}$ are absolutely continuous with respect to each other.

(iii) The boundary operator and the (distributional) divergence operator are related by the formula $\partial T = -\operatorname{div} T$. More precisely, given a current of the form $T = \tau \mathscr{L}^n$, then T is a normal current if and only if the divergence of τ belongs to $L^1(\mathscr{L}^n)$, and in that case $\partial T = -\operatorname{div}\tau \mathscr{L}^n$.

(iv) Given a k-current with finite mass $T = \tau\mu$ and a continuous h-form ω with $h \leq k$, the *interior product* of T and ω is the $(k-h)$-current defined by

$$T \llcorner \omega := (\tau \llcorner \omega)\mu. \qquad (2.10)$$

If T is normal and ω is of class C^1, then the definition of boundary and (2.6) give the following Leibniz rule:

$$\partial(T \llcorner \omega) = (-1)^h \big[(\partial T) \llcorner \omega - T \llcorner d\omega\big]. \qquad (2.11)$$

Distributions of k-planes

In this subsection we consider a distribution of k-planes V defined on the open set Ω, recall the notion of involutivity of V, define the distribution $\widehat{V}$ and the sets $N(V)$ and $N(V, d)$, and give some characterizations that are not widely used.

2.12. Distributions of k-planes. Let $1 \leq k \leq n$. A *distribution of k-planes* on the open set Ω in $\mathbb{R}^n$ is a map V that to every $x \in \Omega$ associates a k-dimensional plane $V(x)$ in $\mathbb{R}^n$, that is, a map from Ω to the Grassmannian $\mathrm{Gr}(k, n)$.

We say that a simple k-vectorfield $v = v_1 \wedge \cdots \wedge v_k$ *spans* V if for every $x \in \Omega$ one has

$$V(x) = \mathrm{span}(v(x)) = \mathrm{span}\{v_1(x), \ldots, v_k(x)\}.$$

Note that a distribution V of class C^r, with $r = 0, 1, \ldots, \infty$, is *locally* spanned by $v = v_1 \wedge \cdots \wedge v_k$, where the vectorfields v_i are of class C^r.

We say that an h-vectorfield w on Ω is *tangent* to V if $\mathrm{span}(w(x)) \subset V(x)$ for every x (simply $w(x) \in V(x)$ when $h = 1$).

We say that an h-current with finite mass $T = \tau\mu$ on Ω is *tangent* to V if $\mathrm{span}(\tau(x)) \subset V(x)$ for μ-a.e. x. Note that this notion does not depend on the choice of τ and μ (recall Remark 2.11(ii)).

2.13. Remarks. (i) If T is a rectifiable k-current and E is the associated rectifiable set (as in §2.10(b)), the fact that T is tangent to V means that $V(x)$ contains the approximate tangent space $T_x E$ for $\mathscr{H}^h$-a.e. $x \in E$.

(ii) If $h = k$ and V is spanned by a simple k-vectorfield v, then a current T with finite mass is tangent to V if and only if it can be written as $T = v\mu$ for some *signed* measure μ.

2.14. Involutivity of V and the set $N(V)$. Let V be a distribution of k-planes of class C^1 on the open set Ω in $\mathbb{R}^n$.

We say that V is *involutive at* $x \in \Omega$ if for every couple of vectorfields w, w' of class C^1 which are tangent to V the commutator $[w, w'](x)$ belongs to $V(x)$. We say that V is involutive if it is involutive at every point of Ω.

The set of all points x where V is not involutive is called the *non-involutivity set* of V and denoted by $N(V)$. Note that this set is open.

2.15. Remark. The involutivity of a distribution V is most often defined in terms of the commutators of a given family of vectorfields $v_1, \ldots, v_k$ that span V; the definition above is equivalent (see Corollary 2.18) and has the slight advantage of being independent of the choice of the vectorfields v_i.

2.16. The distribution $\widehat{V}$ and the sets $N(V, d)$. Given V as in §2.14, for every $x \in \Omega$ we denote by $\widehat{V}(x)$ the subspace of $\mathbb{R}^n$ spanned by all vectors in $V(x)$ and by the commutator (evaluated at x) of every couple of vectorfields w, w' of class C^1 which are tangent to V, that is,

$$\widehat{V} := V + \mathrm{span}\{[w, w'] : w, w' \text{ are tangent to } V\}.$$

For every $d = k, \ldots, n$ we set

$$N(V, d) := \{x \in \Omega : \dim(\widehat{V}(x)) = d\}.$$

Thus $N(V, k)$ is the set of all points where V is involutive and

$$N(V) = \bigcup_{d=k+1}^{n} N(V, d).$$

2.17. Proposition. *Let V and $\widehat{V}$ be as in §2.16, and assume that V be spanned by $v = v_1 \wedge \cdots \wedge v_k$ where each v_i is a vectorfield of class C^1 on Ω. We consider the following distributions of planes on Ω:*

(i) $V_1 := \mathrm{span}\{\operatorname{div}(w \wedge w') : w, w' \text{ are } C^1\text{-vectorfields tangent to } V\}$;

(ii) $V_2 := \mathrm{span}\{[v_i, v_j] : 1 \leq i, j \leq k\}$;

(iii) $V_3 := \mathrm{span}\{\operatorname{div}(v_i \wedge v_j) : 1 \leq i, j \leq k\}$;

(iv) $V_4 := \{\operatorname{div} w : w \text{ is a 2-vectorfield of class } C^1 \text{ tangent to } V\}$.

Then

$$\begin{aligned} \widehat{V} = V + V_1 = V + V_2 &= V + V_3 \\ &= V + V_4 = V + \mathrm{span}(\operatorname{div} v). \end{aligned} \qquad (2.12)$$

2.18. Corollary. *Let V and $v = v_1 \wedge \cdots \wedge v_k$ be as in the previous statement. Then the following assertions are equivalent at every given point $x \in \Omega$:*

(i) *V is involutive at x;*

(ii) $[v_i, v_j] \in V$ or, equivalently, $\mathrm{div}(v_i \wedge v_j) \in V$ for every $1 \le i, j \le k$;

(iii) $\mathrm{span}(\mathrm{div}\, v) \subset V$;

(iv) $v \wedge ((\mathrm{div}\, v) \llcorner \mathrm{d}x_{\mathbf{i}}) = 0$ for every $\mathbf{i} \in I(n, k-2)$.

Proof of Proposition 2.17. The proof of (2.12) is divided in several steps.

Step 1: $V + V_2 = V + V_3$ and $V + V_1 = \widehat{V}$. These equalities follow from the inclusion

$$\mathrm{div}(w \wedge w') - [w, w'] \in V,$$

which holds for every pair of 1-vectorfields w, w' tangent to V, and is a consequences of formula (2.9).

Step 2: $V_4 \subset V + V_3$. Every 2-vectorfield w of class C^1 tangent to V can be written as

$$w = \sum_{1 \le i < j \le k} a_{ij}\, v_i \wedge v_j$$

for suitable C^1-functions a_{ij}. By applying formula (2.7) to the 2-vectorfields $v_i \wedge v_j$ and the 0-forms a_{ij} we obtain

$$\mathrm{div}\, w = \sum_{1 \le i < j \le k} a_{ij}\, \mathrm{div}(v_i \wedge v_j) + (v_i \wedge v_j) \llcorner \mathrm{d}a_{ij};$$

this formula immediately implies the claim.

Step 3: proof the first four equalities in (2.12):

$$
\begin{aligned}
\widehat{V} &= V + V_1 &&\text{by Step 1} \\
&\subset V + V_4 &&\text{because } V_1 \subset V_4 \\
&\subset V + V_3 &&\text{by Step 2} \\
&= V + V_2 &&\text{by Step 1} \\
&\subset \widehat{V} &&\text{because } V_2 \subset \widehat{V}.
\end{aligned}
$$

The last equality in (2.12) follows by the next two steps.

Step 4: $\mathrm{span}(\mathrm{div}\, v) \subset V_4$. Every vector in $\mathrm{span}(\mathrm{div}\, v)$ can be written as $(\mathrm{div}\, v) \llcorner \alpha$ for some $(k-2)$-covector α (see §2.3(v)), and by applying formula (2.7) to v and to the constant form α we obtain

$$(\mathrm{div}\, v) \llcorner \alpha = (-1)^k \mathrm{div}(v \llcorner \alpha),$$

and the right-hand side clearly belongs to V_4.

Step 5: $V_4 \subset V + \mathrm{span}(\mathrm{div}\, v)$. Since V is spanned by the *simple k-vectorfield v*, every 2-vectorfield w tangent to V can be written as $v \llcorner \omega$ for some $(k-2)$-form ω; then formula (2.7) implies that $\mathrm{div}\, w = \mathrm{div}(v \llcorner \omega)$ belongs to $V + \mathrm{span}(\mathrm{div}\, v)$. $\square$

Proof of Corollary 2.18. The equivalence of (i), (ii) and (iii) follows immediately from Proposition 2.17.

Let us prove the implication (iii) $\Rightarrow$ (iv). Assertion (iii) means that $\mathrm{div}\, v$ is a $(k-1)$-vector in V. Thus $(\mathrm{div}\, v) \llcorner \mathrm{d}x_{\mathbf{i}}$ is a 2-vector in V and $v \wedge ((\mathrm{div}\, v) \llcorner \mathrm{d}x_{\mathbf{i}})$ is a $(k+1)$-vector in V, and it must vanish because V has dimension k.

Finally, let us prove the implication (iv) $\Rightarrow$ (iii). Every vector in $\mathrm{span}(\mathrm{div}\, v)$ can be written as $(\mathrm{div}\, v) \llcorner \alpha$ for some $(k-2)$-covector α (see §2.3(v)). Thus (iv) implies

that $v \wedge ((\mathrm{div}\, v) \llcorner \alpha) = 0$, which in turn implies that $(\mathrm{div}\, v) \llcorner \alpha$ belongs to the span of v, which is V (here we use that v is simple and nontrivial). $\square$

2.19. A weak notion of involutivity. Corollary 2.18(iv) shows that the involutivity of a distribution V spanned by a k-vectorfield v is characterized by the equation

$$v \wedge ((\mathrm{div}\, v) \llcorner \mathrm{d}x_{\mathbf{i}}) = 0 \quad \text{for every } \mathbf{i} \in I(n, k-2), \tag{2.13}$$

which for $k = 2$ reduces to $v \wedge \mathrm{div}\, v = 0$.

We point out that equation (2.13) makes sense even if v is less regular than C^1. More precisely, the right-hand side of (2.13) is a well-defined distribution if v and $\mathrm{div}\, v$ belong, locally, to function spaces which are in duality (and are closed under multiplication by functions of class C_c^∞) and therefore one can define involutivity for such classes of vectorfields.

For example, it suffices that v be continuous and $\mathrm{div}\, v$ be a locally finite measure, or that v belong to the Sobolev class H_{loc}^s for some $s \ge 0$ and $\mathrm{div}\, v \in H_{\mathrm{loc}}^{-s}$. In particular it suffices that $v \in H_{\mathrm{loc}}^{1/2}$ (in this case $\mathrm{div}\, v \in H_{\mathrm{loc}}^{-1/2}$ because the divergence is a first-order differential operator).

Decomposability bundle and sparseness of a measure

In this subsection we briefly recall the notion of decomposability bundle of a measure μ, and show that the dimension of this bundle gives a lower bound on the degree of sparseness of μ, expressed in terms of absolute continuity with respect to integral geometric measures $\mathscr{I}_t^d$.

2.20. Decomposability bundle of a measure. Here we briefly sketch the definition of the *decomposability bundle* of a measure, introduced in [1], §2.6. Given a positive finite measure μ on the open set Ω, we denote by $\mathscr{F}(\mu)$ the class of all families $\{F_t : t \in I\}$ parametrized by $I := [0, 1]$ such that:

- each F_t is a 1-dimensional rectifiable set in Ω;
- the measure $\lambda := \int_I (\mathscr{H}^1 \llcorner F_t)\, dt$ satisfies $\lambda \ll \mu$.[10]

The decomposability bundle of μ is a map that to every $x \in \Omega$ associates a (possibly trivial) linear subspace of $\mathbb{R}^n$, denoted in this paper by $W(\mu, x)$, which is uniquely determined up to μ-null sets by the following properties:

(i) for every $\{F_t\} \in \mathscr{F}(\mu)$ there holds $T_x F_t \subset W(\mu, x)$ for $\mathscr{H}^1$-a.e. $x \in F_t$ and a.e. $t \in I$, where $T_x F_t$ is the approximate tangent line to the set F_t at x;

(ii) $W(\mu, \cdot)$ is μ-minimal among all bundles $W(\cdot)$ that satisfy property (i), in the sense that $W(\mu, x) \subset W(x)$ for μ-a.e. x.

Besides some results already contained in [1], we will need the following statement, which is a consequence of a remarkable theorem by G. De Philippis and F. Rindler [8].

2.21. Proposition. *Let μ and $W(\mu, \cdot)$ be as above, d be an integer, and E be a Borel set such that $\dim(W(\mu, x)) \ge d$ for μ-a.e. $x \in E$. Then $\mu \llcorner E \ll \mathscr{I}_t^d \ll \mathscr{H}^d$.*

For the proof we need the following two lemmas.

---

[10] Recall that λ is given by $\lambda(E) := \int_I \mathscr{H}^1(F_t \cap E)\, dt$ for every Borel set $E \subset \Omega$, where dt is the Lebesgue measure; we implicitly require that this integral is well-defined and finite.

2.22. Lemma. *Let μ and $W(\mu, \cdot)$ be as above, let $f : \Omega \to \mathbb{R}^m$ be a map of class C^1, and let $f_\# \mu$ be the pushforward of μ according to f. Then*

$$\mathrm{d}_x f(W(\mu, x)) \subset W(f_\# \mu, f(x)) \quad \text{for } \mu\text{-a.e. } x \in \Omega, \tag{2.14}$$

where $\mathrm{d}_x f : \mathbb{R}^n \to \mathbb{R}^m$ is the differential of f at x.

Sketch of proof. We argue by contradiction and assume that there exists a Borel set F with $\mu(F) > 0$ where the inclusion in (2.14) fails. Then we can find a bounded Borel vectorfield τ on Ω such that

(a) $\mathrm{d}_x f(\tau(x)) \notin W(f_\# \mu, f(x))$ and $|\tau(x)| = 1$ for μ-a.e. $x \in F$;

(b) $\tau(x) \in W(\mu, x)$ for μ-a.e.

Step 1. Using property (b) and Corollary 6.5 in [1] we find a normal 1-current T such that τ is the Radon-Nikodým density of T with respect to μ, that is, $T = \tau\mu + T_s$ with T_s singular with respect to μ. Possibly removing from F a μ-null subset where T_s is concentrated, we can assume that $T \llcorner F = \tau\mu \llcorner F$.

Step 2. Using Theorem 5.5 in [1] we find a family of 1-dimensional rectifiable sets $\{F_t : t \in I\}$ with $I := [0, 1]$ such that for every $t \in I$ and $\mathcal{H}^1$-a.e. $x \in F_t$ the approximate tangent space $T_x F_t$ is spanned by $\tau(x)$, and $\int_I (\mathcal{H}^1 \llcorner F_t)\, dt = \mu \llcorner F$.

Step 3. For every $t \in I$ the set $E_t := f(F_t)$ is rectifiable, the approximate tangent space $T_{f(x)} E_t$ is spanned by $\mathrm{d}_x f(\tau(x))$ for $\mathcal{H}^1$-a.e. $x \in F_t$, the measures $\mathcal{H}^1 \llcorner E_t$ and $f_\#(\mathcal{H}^1 \llcorner F_t)$ are absolutely continuous with respect to each other, and

$$\int_I (\mathcal{H}^1 \llcorner E_t)\, dt \ll \int_I f_\#(\mathcal{H}^1 \llcorner F_t)\, dt = f_\#(\mu \llcorner B) \le f_\# \mu\,.$$

Thus the family $\{E_t : t \in I\}$ belongs to $\mathscr{F}(f_\# \mu)$ and therefore property (i) in §2.20 implies that, for a.e. $t \in I$ and $\mathcal{H}^1$-a.e. $x \in F_t$,

$$\mathrm{d}_x f(\tau(x)) \in T_{f(x)} E_t \subset W(f_\# \mu, f(x))\,,$$

which means that $\mathrm{d}_x f(\tau(x)) \in W(f_\# \mu, f(x))$ for μ-a.e. $x \in F$, in contradiction with property (a) above. $\qquad\square$

2.23. Lemma. *Let μ and $W(\mu, \cdot)$ be as above, and assume that $\dim(W(\mu, x)) \ge d$ for μ-a.e. x. Then $\mu \ll \mathscr{I}_t^d$.*

Proof. We first introduce some notation:

- λ_d is the Haar measure on the Grassmannian $\mathrm{Gr}(d, n)$;
- for every $V \in \mathrm{Gr}(d, n)$, $p_V : \mathbb{R}^n \to V$ is the orthogonal projection onto V and μ_V is the pushforward of the measure μ according to p_V.

Using the characterization of $\mathscr{I}_t^d$-null sets given in §2.1 it is easy to show that the assertion $\mu \ll \mathscr{I}_t^d$ is implied by the assertion $\mu_V \ll \mathscr{H}^d$ for λ_d-a.e. V. The proof of the latter is divided in four steps.

Step 1: if $W \in \mathrm{Gr}(d', n)$ with $d' \ge d$ then $p_V(W) = V$ for λ_d-a.e. V. Possibly replacing W with a subspace, we can assume that W has dimension d. Since $\ker(p_V) = V^\perp$, we have that

$$p_V(W) = V \text{ if and only if } \dim(W \cap V^\perp) = 0\,.$$

Therefore, taking into account that the map $V \mapsto V^\perp$ is a bijection from $\mathrm{Gr}(d, n)$ to $\mathrm{Gr}(n - d, n)$ that preserves the respective Haar measures, we can reformulate the claim as follows:

$$\dim(W \cap Z) = 0 \quad \text{for } \lambda_{n-d}\text{-a.e. } Z \in \mathrm{Gr}(n - d, n).$$

This is equivalent to saying that the set

$$S_k := \{ Z \in \mathrm{Gr}(n - d, n) : \dim(W \cap Z) = k \}$$

is λ_{n-d}-null for every $k > 0$, which is a consequence of the fact that S_k is actually a smooth submanifold of $\mathrm{Gr}(n - d, n)$ with dimension strictly lower than $\mathrm{Gr}(n - d, n)$.

Step 2: for λ_d-a.e. V one has $p_V(W(\mu, x)) = V$ for μ-a.e. $x \in \Omega$. By assumption we have $\dim(W(\mu, x)) \ge d$ for μ-a.e. x, and then it suffices to use Step 1.

Step 3: for λ_d-a.e. V one has

$$W(\mu_V, y) = V \quad \text{for } \mu_V\text{-a.e. } y \in V. \tag{2.15}$$

By applying Lemma 2.22 to the map $f := p_V$ we obtain that, for every d-plane V,

$$W(\mu_V, p_V(x)) \supset p_V(W(\mu, x)) \quad \text{for } \mu\text{-a.e. } x \in \Omega,$$

and recalling Step 2 we obtain that, for λ_d-a.e. V,

$$W(\mu_V, p_V(x)) \supset p_V(W(\mu, x)) = V \quad \text{for } \mu\text{-a.e. } x \in \Omega,$$

which implies

$$W(\mu_V, y) \supset V \quad \text{for } \mu_V\text{-a.e. } y \in V$$

because μ_V is the pushforward of μ through p_V. To obtain (2.15) it is enough to recall that $W(\mu_V, y) \subset V$ for μ_V-a.e. $y \in V$ because μ_V is a measure on V.

Step 4: $\mu_V \ll \mathscr{H}^d$ for λ_d-a.e. V. Identity (2.15) means the following: if we identify the d-plane V with $\mathbb{R}^d$ (isometrically), then μ_V is a measure on $\mathbb{R}^d$ whose decomposability bundle is a.e. equal to $\mathbb{R}^d$, and therefore Corollary 1.12 and Lemma 3.1 in [8] imply that μ_V is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$, that is, the restriction of $\mathscr{H}^d$ to V. $\qquad\square$

Proof of Proposition 2.21. Let $\bar{\mu}$ be the restriction of μ to the set E. By Proposition 2.9(i) in [1] we have that $W(\bar{\mu}, x) = W(\mu, x)$ for $\bar{\mu}$-a.e. x, which implies that $\dim(W(\bar{\mu}, x)) \ge d$ for $\bar{\mu}$-a.e. x. We conclude the proof by applying Lemma 2.23. $\qquad\square$

Sparseness of a normal current and of its boundary

In this subsection we establish a relation between the degree of sparseness of a normal current T and that of its boundary ∂T, both expressed in terms of absolute continuity with respect to Hausdorff measures.

2.24. Proposition. *Let $T = \tau\mu$ be a normal k-current on the open set Ω in $\mathbb{R}^n$ with boundary $\partial T = \tau'\mu'$ such that*

(a) *there exists a real number $\alpha \in [k, n]$ such that $\mu \ll \mathscr{H}^\alpha$.*

(b) *there exists a C^1-vectorfield v on Ω such that $v \wedge \tau = 0$ μ-a.e.*

Let

$$E := \{ x \in \Omega : v(x) \wedge \tau'(x) \ne 0 \}\,.$$

Then $\mu' \llcorner E \ll \mathscr{H}^{\alpha - 1}$.

2.25. Remarks. (i) If τ is simple, the condition $v \wedge \tau = 0$ μ-a.e. in assumption (b) is equivalent to $v(x) \in \mathrm{span}(\tau(x))$ for μ-a.e. x, that is, v is tangent to T.

(ii) If $k > 1$ and τ' is simple, then $E = \{x \colon v(x) \notin \mathrm{span}(\tau'(x))\}$.

(iii) If $k = 1$ then τ' is a real function with $\tau' \neq 0$ μ'-a.e., and therefore E can be equivalently defined as $E = \{x \colon v(x) \neq 0\}$.

Before the proof we present two examples that illustrate the optimality of this statement: the first one shows that the vectorfield v cannot be just continuous, and the second one shows that the Hausdorff measures cannot be replaced by the integral geometric measures.

2.26. Example. For every $a \in [0,1]$ let T_a be the integral 1-current in $\mathbb{R}^2$ associated to the (oriented) curve parametrized by $\gamma_a(t) := (t, at^2)$ with $t \in [0,1]$, and let T be the normal 1-current given by the superposition of all T_a, that is,

$$\langle T; \omega \rangle := \int_0^1 \langle T_a; \omega \rangle \, da \quad \text{for every 1-form } \omega \text{ of class } C_c^\infty.$$

Then

$$T = \left(\frac{1}{x_1^2}, \frac{2x_2}{x_1^3}\right) \mathscr{L}^2 \llcorner F, \quad \partial T = \mathscr{H}^1 \llcorner I - \delta_0,$$

where δ_0 is the Dirac mass at 0, and F, I are the sets in $\mathbb{R}^2$ defined by

$$F := \left\{x \colon 0 \leq x_1 \leq 1, \ 0 \leq x_2 \leq x_1^2\right\},$$

$$I := \left\{x \colon x_1 = 1, \ 0 \leq x_2 \leq 1\right\}.$$

Thus $\mu \ll \mathscr{L}^2 = \mathscr{H}^2$ but $\mu' \not\ll \mathscr{H}^1$.

Let now $v \colon F \to \mathbb{R}^2$ be the vectorfield given by $v(x) := (1, 2x_2/x_1)$ if $x \neq 0$ and $v(0) := (1, 0)$. One can check that v is of class $C^{0,1/2}$ on F and thus can be extended to the entire $\mathbb{R}^2$ with the same regularity. Moreover v is tangent to T and never vanishes on the set F, which contains the support of μ'; therefore the set E contains the support of μ' (see Remark 2.25(iii)) and thus $\mu' \llcorner E = \mu' \not\ll \mathscr{H}^1$, which means that Proposition 2.24 fails for this choice of v. (On the other hand, every vectorfield v of class C^1 tangent to T must vanish at 0, thus for such v the set E does not contain 0 and $\mu' \llcorner E \ll \mathscr{H}^1$, in accordance with Proposition 2.24.)

2.27. Example. Take a Borel function $g \colon [0,1] \to \mathbb{R}$ whose graph Γ is purely unrectifiable and $\mathscr{H}^1$-finite, and for every $r \in \mathbb{R}$ consider the map $f_r \colon [0,1] \to \mathbb{R}^2$ given by $f_r(s) := (s, g(s) + r)$. For every $a \in [0,1]$ we denote by T_a the 1-current in $\mathbb{R}^2$ associated to the (oriented) vertical segment $I_a := [f_0(a), f_1(a)]$, and by T the superposition of all such T_a (defined as in the previous example). Then

$$T = e \, \mathscr{L}^2 \llcorner F, \quad \partial T = \lambda_1 - \lambda_0,$$

where $e := (0,1)$, F is the union of the segments I_a with $0 \leq a \leq 1$, and λ_r is the pushforward of the Lebesgue measure on $[0,1]$ according to the map f_r.

Thus $\mu \ll \mathscr{L}^2 = \mathscr{H}^2 = \mathscr{I}_t^2$. Moreover the constant vectorfield $v(x) := e$ is tangent to T and never vanishes, and then Remark 2.25(iii) yields $\mu' \llcorner E = \mu'$.

On the other hand μ' is supported on the set $\Gamma \cup (\Gamma + e)$, which is $\mathscr{H}^1$-finite and purely unrectifiable, and therefore $\mathscr{I}_t^1$-null, which implies that $\mu' \llcorner E = \mu'$ is singular with respect to $\mathscr{I}_t^1$. (However $\mu' \ll \mathscr{H}^1$, as predicted by Proposition 2.24.)

We now pass to the proof of Proposition 2.24.

Using a localization argument we reduce to the case where $\Omega = \mathbb{R}^n$ and T and v have compact support. Moreover we can assume that $|\tau'| = 1$ μ'-a.e.

In the proof we use the flow associated to the vectorfield v, namely the map $\Phi \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\Phi(0, x) = x, \quad \frac{\partial \Phi}{\partial t}(t, x) = v(\Phi(t, x)) \quad \text{for every } t \in \mathbb{R}, \ x \in \mathbb{R}^n,$$

and we write $\Phi_t(x)$ for $\Phi(t, x)$. Since v is of class C^1 and compactly supported, the map Φ is well-defined and of class C^1, and each map Φ_t is bi-Lipschitz and coincides with the identity out of a compact set which does not depend on t.

2.28. Lemma. *Let T and v be as in Proposition 2.24, and let Φ be as above. Given $t_0, t_1 \in \mathbb{R}$, let $[\![t_0, t_1]\!]$ be the 1-current in $\mathbb{R}$ associated to the oriented interval $[t_0, t_1]$. Then the pushforward of the product current $[\![t_0, t_1]\!] \times \partial T$ on $\mathbb{R} \times \mathbb{R}^n$ according to the map Φ satisfies*

$$\Phi_\#\big([\![t_0, t_1]\!] \times \partial T\big) = (\Phi_{t_1})_\# T - (\Phi_{t_0})_\# T. \tag{2.16}$$

Proof. The homotopy formula (see for instance [13], §7.4.3) states that

$$\partial\big(\Phi_\#([\![t_0, t_1]\!] \times T)\big) = (\Phi_{t_1})_\# T - (\Phi_{t_0})_\# T - \Phi_\#\big([\![t_0, t_1]\!] \times \partial T\big),$$

and then (2.16) is implied by

$$\Phi_\#\big([\![t_0, t_1]\!] \times T\big) = 0. \tag{2.17}$$

The proof of (2.17) is divided in two steps. We denote by $\mathrm{d}_{(t,x)}\Phi$ the differential of Φ at the point (t, x), viewed as a linear map from $\mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}^n$, and let $e_0 := (1, 0) \in \mathbb{R} \times \mathbb{R}^n$. Moreover we tacitly identify $v \in \mathbb{R}^n$ with $(0, v) \in \mathbb{R} \times \mathbb{R}^n$, which yields an identification of k-vectors in $\mathbb{R}^n$ with k-vectors in $\mathbb{R} \times \mathbb{R}^n$.

Step 1: for every t and μ-a.e. x the pushforward of the $(k+1)$-vector $e_0 \wedge \tau(x)$ according to the linear map $\mathrm{d}_{(t,x)}\Phi$ is null, that is

$$\big(\mathrm{d}_{(t,x)}\Phi\big)_\#(e_0 \wedge \tau(x)) = 0. \tag{2.18}$$

Differentiating the semigroup identity $\Phi(t + s, x) = \Phi(t, \Phi(s, x))$ with respect to s at $s = 0$ we get

$$\mathrm{d}_{(t,x)}\Phi(e_0) = \mathrm{d}_{(t,x)}\Phi(v(x)),$$

and then

$$(\mathrm{d}_{(t,x)}\Phi)_\#(e_0 \wedge \tau(x)) = (\mathrm{d}_{(t,x)}\Phi)_\#(v(x) \wedge \tau(x)),$$

and using assumption (b) in Proposition 2.24 we get (2.18).

Step 2: proof of (2.17). Using (2.18), for every test $(k+1)$-form ω on $\mathbb{R} \times \mathbb{R}^n$ we obtain

$$\big\langle \Phi_\#([\![t_0, t_1]\!] \times T); \omega \big\rangle = \big\langle [\![t_0, t_1]\!] \times T; \Phi^\#(\omega) \big\rangle$$

$$= \int_{t_0}^{t_1} \int_{\mathbb{R}^n} \big\langle \omega(\Phi(t, x)); (\mathrm{d}_{(t,x)}\Phi)_\#(e_0 \wedge \tau(x)) \big\rangle \, d\mu(x) \, dt = 0,$$

and (2.17) is proved. $\square$

Proof of Proposition 2.24. We argue by contradiction and assume that there exists a compact set $E' \subset E$ such that

$$\mathscr{H}^{\alpha-1}(E') = 0, \quad \mu'(E') > 0.$$

Next we choose a point $x_0 \in E'$ where the map $v \wedge \tau'$ is approximately continuous and the set E' has density 1 (in both cases the underlying measure is μ'). We fix for the time being $\delta, r > 0$ and consider the sets

$$E'' := E' \cap \overline{B(x_0, r)}, \quad F := [0, \delta] \times E'', \quad G := \Phi(F),$$

and the k-current

$$S := \Phi_{\#}\big([\![0, \delta]\!] \times \partial T\big).$$

We claim that

   (i) $\mathscr{H}^{\alpha}(G) = 0$;

   (ii) S, viewed as a vector-vauled measure, satisfies $S \ll \mathscr{H}^{\alpha}$;

   (iii) $S(G) \neq 0$ for δ and r suitably chosen.

Note that (i) and (ii) imply $S(G) = 0$, which contradicts (iii). To conclude the proof it remains to prove claims (i)–(iii).

Step 1: proof of (i). We have the following chain of implications: $\mathscr{H}^{\alpha-1}(E') = 0$ $\Rightarrow \mathscr{H}^{\alpha-1}(E'') = 0 \Rightarrow \mathscr{H}^{\alpha}(F) = 0 \Rightarrow \mathscr{H}^{\alpha}(G) = 0$ (the second implication follows from [11], Theorem 2.10.45; the third one from the fact that Φ is Lipschitz).

Step 2: proof of (ii). Using the definition of the current S and Lemma 2.28 we obtain $S = (\Phi_{\delta})_{\#}T - T$; to conclude we recall that $T \ll \mu \ll \mathscr{H}^{\alpha}$ by assumption, and observe that $(\Phi_{\delta})_{\#}T \ll (\Phi_{\delta})_{\#}\mu \ll \mathscr{H}^{\alpha}$ because the map Φ_{δ} is bi-Lipschitz.

Step 3: proof of (iii). To prove this claim we set $\rho(r) := \mu'\big(\overline{B(x_0, r)}\big)$ and show that

$$\lim_{r \to 0} \lim_{\delta \to 0} \frac{S(G)}{\delta \rho(r)} = v(x_0) \wedge \tau'(x_0) \neq 0. \tag{2.19}$$

Let λ be the product measure on $\mathbb{R} \times \mathbb{R}^n$ given by $\lambda := (\mathscr{L}^1 \llcorner [0, \delta]) \times \mu'$, and let $g : \mathbb{R} \times \mathbb{R}^n \to \wedge_k(\mathbb{R}^n)$ be the map given by $g(t, x) := (\mathrm{d}_{(t,x)}\Phi)_{\#}(e_0 \wedge \tau'(x))$. Using the definition of pushforward of currents we obtain

$$S(G) = \int_{\Phi^{-1}(G)} g \, d\lambda = A_1 + A_2 + A_3 \tag{2.20}$$

where

$$A_1 := \int_F g(0, x) \, d\lambda(t, x) = \delta \int_{E''} g(0, x) \, d\mu'(x),$$

$$A_2 := \int_F \big(g(t, x) - g(0, x)\big) \, d\lambda(t, x),$$

$$A_3 := \int_{\Phi^{-1}(G) \setminus F} g(t, x) \, d\lambda(t, x).$$

The definitions of g and Φ yield

$$g(0, x) = v(x) \wedge \tau'(x).$$

Using this identity, the definitions of $\rho(r)$ and E'', and the choice of x_0 we obtain that

$$\frac{A_1}{\delta \rho(r)} = \frac{1}{\rho(r)} \int_{B(x_0, r) \cap E'} v \wedge \tau' \, d\mu' \underset{r \to 0}{\longrightarrow} v(x_0) \wedge \tau'(x_0). \tag{2.21}$$

Using the dominated convergence theorem and the fact that g is continuous in t and uniformly bounded we obtain that, for every fixed r,

$$\frac{|A_2|}{\delta} \leq \int_{B(x_0, r)} \Big(\sup_{0 \leq t \leq \delta} |g(t, x) - g(0, x)| \Big) \, d\mu'(x) \underset{\delta \to 0}{\longrightarrow} 0. \tag{2.22}$$

Finally we let E_δ be the projection of $\Phi^{-1}(G) \cap \big([0, \delta] \times \mathbb{R}^n\big)$ onto $\mathbb{R}^n$ and notice that this is a closed set that contains E'' and for every fixed r converges to E'' in the Hausdorff distance as $\delta \to 0$. Since $|g|$ is bounded by some constant m (because so is τ') we obtain that

$$\frac{|A_3|}{\delta} \leq \frac{m}{\delta} \lambda\big(\Phi^{-1}(G) \setminus F\big) \leq m \, \mu'\big(E_\delta \setminus E''\big) \underset{\delta \to 0}{\longrightarrow} 0. \tag{2.23}$$

Putting together (2.20), (2.21), (2.22) and (2.23) we obtain (2.19). $\qquad\square$

3. The key identity

The main result in this section is identity (3.4) in Proposition 3.2. Using this identity we obtain the fundamental relation between the boundary of a normal k-current tangent to a distribution of k-planes V and the distribution $\widehat{V}$ associated to V (Theorem 3.3).

Through this section, k and n are integers that satisfy $2 \leq k < n$, Ω is an open set in $\mathbb{R}^n$, V is a distribution of k-planes V on Ω spanned by vectorfields $v_1, \dots, v_k$ of class C^1 on Ω and $v := v_1 \wedge \cdots \wedge v_k$, as usual.

Moreover T is a normal k-current on Ω which is tangent to V, which we write as $T = v\mu$ where μ is a suitable *signed* measure (see Remark 2.13(ii)). In the sequel it is important to remember that μ is not necessarily positive.

As usual write $\partial T = \tau'\mu'$ where μ' is a positive measure and τ' is a density with values in $(k-1)$-vectors.

Notice that we assume that the distribution V is *globally* spanned by k vectorfields, and not just locally (cf. §2.12). There is however no loss of generality, because all statements we are interested in are local in nature.

3.1. Lemma. *Take v and V as above and consider the $(k-1)$-form*

$$\alpha := \star(w \wedge u) \tag{3.1}$$

where $u = u_1 \wedge \cdots \wedge u_{n-k-1}$ is a simple $(n-k-1)$-vector and $w = w_1 \wedge w_2$ is a simple 2-vectorfield on Ω with w_1, w_2 vectorfields of class C^1 tangent to V. Then

   (i) $v \llcorner \alpha = 0$ *on Ω;*

   (ii) $\langle v; \mathrm{d}\alpha \rangle = \langle v \wedge \mathrm{div}\, w \wedge u; \mathrm{d}x \rangle$ *on Ω;*

   (iii) $\langle v; \mathrm{d}\alpha \rangle \neq 0$ *at every point of Ω where $v \wedge \mathrm{div}\, w \wedge u \neq 0$.*

Proof. To prove (i) we show that $\langle v \llcorner \alpha; \beta \rangle = 0$ for every 1-covector β. Indeed

$$\langle v \llcorner \alpha; \beta \rangle = \langle v; \alpha \wedge \beta \rangle = (-1)^{k-1} \langle v; \beta \wedge \alpha \rangle$$
$$= (-1)^{k-1} \langle v \llcorner \beta; \alpha \rangle$$
$$= (-1)^{k-1} \langle v \llcorner \beta; \star(w \wedge u) \rangle$$
$$= (-1)^{k-1} \langle (v \llcorner \beta) \wedge w \wedge u; \mathrm{d}x \rangle = 0 \,,$$

where the last equality holds because $(v \llcorner \beta) \wedge w$ is a $(k+1)$-vectorfield tangent to the distribution of k-planes V, and therefore it is everywhere null.

Let us prove (ii). Using (2.5) we get

$$\langle v; \mathrm{d}\alpha \rangle = \langle v; \mathrm{d}(\star(w \wedge u)) \rangle = \langle v; \star(\mathrm{div}(w \wedge u)) \rangle$$
$$= \langle v \wedge (\mathrm{div}(w \wedge u)); \mathrm{d}x \rangle \,. \tag{3.2}$$

Since both w and u are simple we can use formula (2.8) to compute the divergence of $w \wedge u$, obtaining

$$\mathrm{div}(w \wedge u) = [w_1, w_2] \wedge u + w' \,, \tag{3.3}$$

where

$$w' = (\mathrm{div}\, w_1)\, w_2 \wedge u - (\mathrm{div}\, w_2)\, w_1 \wedge u$$
$$+ \sum_{i=1}^{n-k-1} (-1)^i \left([w_1, u_i] \wedge w_2 - [w_2, u_i] \wedge w_1\right) \wedge \left(\bigwedge_{j \neq i} u_j\right) \,.$$

Now, each w_i belongs to $V = \mathrm{span}(v)$ by assumption, hence $v \wedge w_i = 0$, which implies that $v \wedge w' = 0$. Therefore using (3.3) and (2.9) we get

$$v \wedge \mathrm{div}(w \wedge u) = v \wedge [w_1, w_2] \wedge u = v \wedge \mathrm{div}\, w \wedge u \,.$$

Plugging this formula into (3.2) proves (ii).

Finally, (iii) is an immediate consequence of (ii). □

3.2. Proposition. *Take v, V, $T = v\mu$ and $\partial T = \tau'\mu'$ as above, and let w be a 2-vectorfield of class C^1 on Ω which is tangent to V. Then the following identity of measures (with values in $(k+1)$-vectors) holds:*

$$(\tau' \wedge w)\, \mu' = (v \wedge \mathrm{div}\, w)\, \mu \,. \tag{3.4}$$

Proof. The proof is divided in two cases.

Case 1: $w = w_1 \wedge w_2$ with w_1, w_2 vectorfields of class C^1 tangent to V. Fix a simple $(n-k-1)$-vector u and let α be the $(k-1)$-form defined in (3.1). Recalling definition (2.10) and Lemma 3.1(i) we obtain that $T \llcorner \alpha = (v \llcorner \alpha)\mu = 0$. Therefore formula (2.11) yields

$$0 = (-1)^{k-1}\partial(T \llcorner \alpha) = \partial T \llcorner \alpha - T \llcorner \mathrm{d}\alpha$$
$$= (\tau' \llcorner \alpha)\, \mu' - (v \llcorner \mathrm{d}\alpha)\, \mu$$
$$= \langle \tau' \wedge w \wedge u; \mathrm{d}x \rangle\, \mu' - \langle v \wedge \mathrm{div}\, w \wedge u; \mathrm{d}x \rangle\, \mu$$

(in the last equality we used Lemma 3.1(ii)). Hence

$$(\tau' \wedge w \wedge u)\, \mu' = (v \wedge \mathrm{div}\, w \wedge u)\, \mu \,,$$

---

which implies (3.4) by the arbitrariness of u.

Case 2: w is arbitrary. Then w can be written in the form

$$w = \sum_{1 \leq i < j \leq k} a_{ij}\, v_i \wedge v_j$$

for suitable functions a_{ij} of class C^1. Then identity (3.4) holds for each addendum $w_{ij} := a_{ij}\, v_i \wedge v_j$ as just proved, and therefore it holds for w, too. □

Using formula (3.4) we can easily establish the following key relation between the boundary of T and the distribution $\widehat{V}$ defined in §2.16.

3.3. Theorem. *Take v, V, $T = v\mu$ and $\partial T = \tau'\mu'$ as above, and let $\mu' = \mu'_a + \mu'_s$ be the Lebesgue decomposition of the measure μ' with respect to μ. Then*
   (i) $\mathrm{span}(\tau'(x)) \subset V(x)$ *for μ'_s-a.e. x;*
   (ii) $\mathrm{span}(\tau'(x)) + V(x) = \widehat{V}(x)$ *for μ'_a-a.e. x.*

Proof. We denote by $X(V)$ the space of all 2-vectorfield of class C^1 on Ω tangent to V and write $\mu'_a = \rho\mu$ for a suitable density ρ.

Let $w \in X(V)$. Then equation (3.4) can be rewritten as

$$\tau' \wedge w = 0 \qquad \text{for } \mu'_s\text{-a.e. } x, \tag{3.5}$$
$$\tau' \wedge w = \tfrac{1}{\rho} v \wedge \mathrm{div}\, w \quad \text{for } \mu'_a\text{-a.e. } x. \tag{3.6}$$

The proof is now divided in three steps. The first one contains the proof of statement (i), while the others give statement (ii).

Step 1: proof of statement (i). Equation (3.5) implies that for every $w \in X(V)$ there exists a μ'_s-null set N_w such that $\tau'(x) \wedge w(x) = 0$ for every $x \notin N_w$. Take now a countable dense family $X' \subset X(V)$, and let N be the union of N_w over all $w \in X'$. Then N is μ'_s-null and it is easy to check that for every $x \notin N$ and every $w \in X(V)$ there holds $\tau'(x) \wedge w(x) = 0$, which means that

$$\tau'(x) \wedge w = 0 \quad \text{for every 2-vector } w \text{ in } V(x),$$

and therefore $\mathrm{span}(\tau'(x)) \subset V(x)$ (apply Lemma 2.4 with assumption (b)).

Step 2: $\mathrm{span}(\tau'(x)) \subset \widehat{V}(x)$ for μ'_a-a.e. x. Let $w \in X(V)$. Using equation (3.6) and the inclusion $\mathrm{span}(v \wedge \mathrm{div}\, w) \subset \widehat{V}$ (Proposition 2.17) we obtain

$$\mathrm{span}(\tau'(x) \wedge w(x)) \subset \widehat{V}(x) \text{ for } \mu'_a\text{-a.e. } x,$$

and proceeding as in Step 1 we find a μ'_a-null set N such that, for every $x \notin N$,

$$\mathrm{span}(\tau'(x) \wedge w) \subset \widehat{V}(x) \quad \text{for every 2-vector } w \text{ in } V(x),$$

and then $\mathrm{span}(\tau'(x)) \subset \widehat{V}(x)$ (apply Lemma 2.4 with assumption (a)).

Step 3: $\widehat{V}(x) \subset V(x) + \mathrm{span}(\tau'(x))$ for μ'_a-a.e. x. Let $w \in X(V)$. Using equation (3.6) and the inclusion $\mathrm{span}(\tau' \wedge w) \subset \mathrm{span}(\tau') + V$, we obtain that

$$\mathrm{span}(v(x) \wedge \mathrm{div}\, w(x)) \subset \mathrm{span}(\tau'(x)) + V(x) \text{ for } \mu'_a\text{-a.e. } x.$$

Proceeding as in Step 1 we find a μ'_a-null set N such that, for every $x \notin N$ and every $w \in X(V)$,

$$\mathrm{span}(v(x) \wedge \mathrm{div}\, w(x)) \subset V(x) + \mathrm{span}(\tau'(x)),$$

and using Lemma 2.4 with assumption (c) we obtain that, for every $x \notin N$,

$$\operatorname{div} w(x) \in V(x) + \operatorname{span}(\tau'(x)).$$

Then the claim follows using Proposition 2.17. □

3.4. Remark. Recall that $\widehat{V}(x)$ agrees with $V(x)$ for every x in the involutivity set $N(V,k) = \Omega \setminus N(V)$, and strictly contains $V(x)$ for every x in the non-involutivity set $N(V)$. Then Theorem 3.3 implies that the inclusion that defines the geometric property of the boundary for T, namely

$$\operatorname{span}(\tau'(x)) \subset V(x),$$

holds for μ'_s-a.e. $x \in \Omega$ and for μ'_a-a.e. $x \in \Omega \setminus N(V)$, and does not hold for μ'_a-a.e. $x \in N(V)$.

3.5. Remark. The case $k = 2$ and $n = 3$ of Proposition 3.2 is especially significant. In this case a form α with properties (i)-(iii) in Lemma 3.1 is simply given by $\alpha := \star v$, and equation (3.4) in Proposition 3.2 reduces to

$$(\tau' \wedge v)\,\mu' = (v \wedge \operatorname{div} v)\,\mu.$$

4. Proofs of the results in Section 1

In this section we prove Theorem 1.3, Corollary 1.4, and Theorems 1.1, 1.7 and 1.12 (in this order).

We follow the notation of Section 3. In particular V is spanned by $v_1, \ldots, v_k$, $v := v_1 \wedge \cdots \wedge v_k$, and $T = \tau\mu$ where μ is a suitable *signed* measure. Note that it is sufficient to prove the statements above for this measure μ, despite the fact that it may be not positive (cf. Remark 2.11(ii)).

Proof of Theorem 1.3. We denote by $X(V)$ the space of all 2-vectorfield of class C^1 on Ω tangent to V. The proof is divided in several steps.

Step 1: proof of statement (i). Let $w \in X(V)$ and let $\mu = \mu_a + \mu_s$ be the Lebesgue decomposition of μ with respect to μ'. Then identity (3.4) yields $(v \wedge \operatorname{div} w)\,\mu_s = 0$, and therefore we can find a $|\mu_s|$-null set N_w such that, for every $x \notin N_w$,

$$v(x) \wedge \operatorname{div} w(x) = 0.$$

Proceeding as in Step 1 of the proof of Theorem 3.3 we find a $|\mu_s|$-null set N such that the previous equation holds for every $x \notin N$ and every $w \in X(V)$, and applying Lemma 2.4 with assumption (c) we obtain

$$\operatorname{div} w(x) \in V(x).$$

By Proposition 2.17 this means $V(x) = \widehat{V}(x)$ at every $x \notin N$, that is, V is involutive at every $x \notin N$ or, in other words, the non-involutivity set $N(V)$ is $|\mu_s|$-null, which finally implies $\mu \llcorner N(V) \ll \mu_a \ll \mu'$.

For the next step we denote by $\bar{\mu}$ the restriction of $|\mu|$ to $N(V)$. Recall that $W(\bar{\mu}, \cdot)$ is the decomposability bundle of $\bar{\mu}$ (see §2.20).

Step 2: $W(\bar{\mu}, x) \supset \widehat{V}(x)$ for $\bar{\mu}$-a.e. x. Indeed, since $T = v\mu$ and $\partial T = \tau'\mu'$ are normal currents, Theorem 5.10 in [1] implies that the decomposability bundles of

the measures $|\mu|$ and μ' contain the span of τ and τ' respectively, that is,

$$W(|\mu|, x) \supset \operatorname{span}(v(x)) = V(x) \quad \text{for } |\mu|\text{-a.e. } x, \tag{4.1}$$

$$W(\mu', x) \supset \operatorname{span}(\tau'(x)) \qquad \text{for } \mu'\text{-a.e. } x. \tag{4.2}$$

On the other hand $\bar{\mu}$ is absolutely continuous with respect to $|\mu|$ and also with respect to μ' (by statement (i)) and therefore Proposition 2.9(i) in [1] yields

$$W(\bar{\mu}, x) = W(|\mu|, x) = W(\mu', x) \quad \text{for } \bar{\mu}\text{-a.e. } x. \tag{4.3}$$

Putting together (4.1), (4.2) and (4.3) we obtain

$$W(\bar{\mu}, x) \supset V(x) + \operatorname{span}(\tau'(x)) \quad \text{for } \bar{\mu}\text{-a.e. } x,$$

and we conclude the proof of the claim recalling that $V + \operatorname{span}(\tau') = \widehat{V}$ by Proposition 2.17.

Step 3: proof of statement (ii). For every $d = k+1, \ldots, n$ consider the measure $\mu_d := |\mu| \llcorner N(V, d)$. Since μ_d is absolutely continuous with respect to $\bar{\mu}$, using Proposition 2.9(i) in [1] and Step 2 we obtain that

$$W(\mu_d, x) = W(\bar{\mu}, x) \supset \widehat{V}(x) \quad \text{for } \mu_d\text{-a.e. } x,$$

and in particular $\dim(W(\mu_d, x)) \geq d$ for μ_d-a.e. x. We now conclude using Proposition 2.21 or Lemma 2.23.

For the rest of the proof we fix $d = k+1, \ldots, n$ and consider the following sets:

$$\Omega_d := \left\{ x \in \Omega : \dim(\widehat{V}(x)) \geq d \right\} = N(V, d) \cup \cdots \cup N(V, n),$$

$$F := \left\{ x \in \Omega_d : V(x) \subset \operatorname{span}(\tau'(x)) \right\},$$

$$E_i := \left\{ x \in \Omega_d : v_i(x) \notin \operatorname{span}(\tau'(x)) \right\} \quad \text{with } i = 1, \ldots, k.$$

Step 4: $\mu' \llcorner F \ll \mathscr{H}^d \ll \mathscr{H}^{d-1}$. Using the identity $\widehat{V} = V + \operatorname{span}(\tau')$ (Proposition 2.17), we have that for $x \in F$ the linear space $\operatorname{span}(\tau'(x))$ contains $\widehat{V}(x)$ and therefore has dimension at least d. Thus (4.2) implies that $W(\mu', x)$ has dimension at least d for μ'-a.e. $x \in F$, and the claim follows from Proposition 2.21.

Step 5: $\mu' \llcorner E_i \ll \mathscr{H}^{d-1}$ for $i = 1, \ldots, k$. We prove this claim by applying Proposition 2.24 to the open set Ω_d, the current $T = v\mu = \tau|\mu|$ and the vectorfield v_i. To this end we notice that

- $|\mu| \llcorner \Omega_d \ll \mathscr{H}^d$ by statement (ii) and the definition of Ω_d;
- $v_i \wedge v = 0$ everywhere in Ω and then also in Ω_d;
- $v_i \wedge \tau' \neq 0$ on E_i by the definition of E_i.

Step 6: $\mu' \llcorner \Omega_d \ll \mathscr{H}^{d-1}$, which implies statement (iii). To prove the first part of the claim we use that $\Omega_d = F \cup E_1 \cup \cdots \cup E_k$ and Steps 4 and 5. To prove statement (iii) use that $N(V, d) \subset \Omega_d$. □

Proof of Corollary 1.4. Since the set $N(V)$ is open, proving that the support of μ does not intersect $N(V)$ is equivalent to showing that $\mu \llcorner N(V) = 0$.

If condition (a) holds, then μ is singular with respect to $\mathscr{I}_t^d$ for every $d \geq k+1$, and this fact and Theorem 1.3(ii) imply $\mu \llcorner N(V, d) = 0$, and since $N(V)$ is the union of all $N(V, d)$ with $d \geq k+1$, we obtain $\mu \llcorner N(V) = 0$, as desired.

If condition (b) holds then Theorem 1.3(i) yields $\mu \llcorner N(V) = 0$.

To conclude the proof we notice that condition (c) implies condition (a), and condition (d) implies condition (b). □

Proof of Theorem 1.1. Apply Corollary 1.4 with condition (c). □

Proof of Theorem 1.7. We begin with the implication (i) ⇒ (ii). The geometric property of the boundary for T, namely inclusion (1.1), and Theorem 3.3(ii) imply that $V = \widehat{V}$ μ'_a-a.e., where μ'_a is the absolutely continuous part of μ' with respect to μ.

Since $V(x) \neq \widehat{V}(x)$ at every $x \in N(V)$ we infer that $\mu'_a \llcorner N(V) = 0$ and using Theorem 1.3(i) we deduce that $\mu \llcorner N(V) = 0$. Since $N(V)$ is open, this means that the support of μ does not intersect $N(V)$.

We now prove (ii) ⇒ (i). The assumption $\widehat{V}(x) = V(x)$ for $|\mu|$-a.e. x and Theorem 3.3(ii) imply that $\mathrm{span}(\tau'(x)) \subset V(x)$ for μ'_a-a.e. x. On the other hand this inclusion holds also for μ'_s-a.e. x by Theorem 3.3(i), and therefore T has the geometric property of the boundary. □

Sketch of proof of Theorem 1.12. Assume by contradiction that there exists $z \in A$ such that $u(z) \in N(V)$. Since $N(V)$ is open and u is continuous and of class $W^{1,p}_{\mathrm{loc}}$ we can find a ball U centered at z such that

- $u(U)$ is contained in $N(V)$;
- the restriction of u to ∂U belongs to $W^{1,p}(\partial U)$.

Then the graph of the restriction of u to U, denoted by Γ, is a k-dimensional rectifiable set with $\mathscr{H}^k(\Gamma) < +\infty$. Moreover it is proved in [12], §2.5, Theorem 1, that the rectifiable current canonically associated to Γ, denoted by $[\![\Gamma]\!]$, has boundary with finite mass, and therefore is an integral current in $\mathbb{R}^k \times \mathbb{R}^n$ (for this step we need the assumption $p > k$).

Since ∇u has maximal rank, possibly replacing U with a suitable open subset, we have that the pushforward of $[\![\Gamma]\!]$ through the projection $p : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$ is a non-trivial integral k-current tangent to V. But the support of such current is contained in $N(V)$, thus violating Theorem 1.1. □

References

[1] G. Alberti and A. Marchese, *On the differentiability of Lipschitz functions with respect to measures in the Euclidean space*, Geom. Funct. Anal., 26 (2016), pp. 1–66.

[2] G. Alberti and A. Massaccesi, *On a geometric property of the boundaries of currents, and related questions*. Paper in preparation.

[3] ———, *On some geometric properties of currents and Frobenius theorem*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl., 28 (2017), pp. 861–869.

[4] G. Alberti, A. Massaccesi, and A. Merlo, *About non-rectifiability in Carnot-Carathéodory spaces*. Paper in preparation.

[5] ———, *On tangency sets of non-involutive distributions*. Paper in preparation.

[6] Z. M. Balogh, *Size of characteristic sets and functions with prescribed gradient*, J. Reine Angew. Math., 564 (2003), pp. 63–83.

[7] Z. M. Balogh, C. Pintea, and H. Rohner, *Size of tangencies to non-involutive distributions*, Indiana Univ. Math. J., 60 (2011), pp. 2061–2092.

[8] G. De Philippis and F. Rindler, *On the structure of $\mathscr{A}$-free measures and applications*, Ann. of Math. (2), 184 (2016), pp. 1017–1039.

[9] S. Delladio, *Functions of class C^1 subject to a Legendre condition in an enhanced density set*, Rev. Mat. Iberoam., 28 (2012), pp. 127–140.

[10] ———, *The tangency of a C^1 smooth submanifold with respect to a non-involutive C^1 distribution has no superdensity points*, Indiana Univ. Math. J., 68 (2019), pp. 393–412.

[11] H. Federer, *Geometric measure theory*, vol. 153 of Die Grundlehren der mathematischen Wissenschaften, Springer-Verlag, Berlin Heidelberg New York, 1969. Reprinted in the series Classic in Mathematics, Springer-Verlag, Berlin Heidelberg, 1996.

[12] M. Giaquinta, G. Modica, and J. Souček, *Cartesian currents in the calculus of variations. I. Cartesian currents*, vol. 37 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, Springer-Verlag, Berlin, 1998.

[13] S. G. Krantz and H. R. Parks, *Geometric integration theory*, Cornerstones, Birkhäuser, Boston, MA, 2008.

[14] J. M. Lee, *Introduction to smooth manifolds. 2nd revised ed.*, vol. 218 of Graduate Texts in Mathematics, Springer-Verlag, New York, 2nd revised ed ed., 2013.

[15] V. Magnani, *Unrectifiability and rigidity in stratified groups*, Arch. Math., 83 (2004), pp. 568–576.

[16] V. Magnani, J. Malý, and S. Mongodi, *A low rank property and nonexistence of higher-dimensional horizontal Sobolev sets*, J. Geom. Anal., 25 (2015), pp. 1444–1458.

[17] A. Massaccesi, *Currents with coefficients in groups, applications and other problems in Geometric Measure Theory*, PhD thesis, Scuola Normale Superiore, Pisa, 2014.

[18] L. Simon, *Lectures on geometric measure theory*, vol. 3 of Proceedings of the Centre for Mathematical Analysis, Australian National University, Canberra, 1983.

[19] F. W. Warner, *Foundations of differentiable manifolds and Lie groups*, vol. 94 of Graduate Texts in Mathematics, Springer-Verlag, New York-Berlin, 1983. Corrected reprint of the 1971 edition.

[20] M. Zworski, *Decomposition of normal currents*, Proc. Amer. Math. Soc., 102 (1988), pp. 831–839.

G.A.

Dipartimento di Matematica, Università di Pisa
largo Pontecorvo 5, 56127 Pisa, Italy
e-mail: `giovanni.alberti@unipi.it`

A.M.

Dipartimento di Tecnica e Gestione dei Sistemi Industriali (DTG), Università di Padova
stradella S. Nicola 3, 36100 Vicenza, Italy
e-mail: `annalisa.massaccesi@unipd.it`

E.S.

St. Petersburg Branch of the Steklov Institute of Mathematics of the Russian Academy of Sciences
Fontanka 27, 191023 St. Petersburg, Russia

and

Faculty of Mathematics, Higher School of Economics,
Usacheva 6, 119048 Moscow, Russia
e-mail: `stepanov.eugene@gmail.com`