The Topology of Two-Dimensional Real Algebraic Varieties (*).

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Sunto. – *È noto che ogni spazio analitico reale è localmente omeomorfo al cono su un poliedro con caratteristica di Eulero-Poincaré pari. Si dimostra che questa condizione è anche sufficiente affinchè un poliedro (compatto) di dimensione due P sia omeomorfo ad una varietà algebrica reale affine $\hat{P}$. Segue inoltre dalla costruzione che la $\hat{P}$ ottenuta ha, in un certo senso, un insieme di singolarità algebriche minimale, compatibilmente con la topologia di P.*

Introduction.

The topological resolution of singularities is often a suitable tool for studying different kind of questions (see [5] or [4] for an application to the representation of homology classes). In [1] it is given a complete topological characterization of real algebraic affine varieties with isolated singularities, by means of both algebraic approximations of differentiable objects and the construction of a good resolution of singularities (see also [3]).

It seems natural that one can generalize this technique. It is known that every real analytic space is locally homeomorphic to the cone over a polyhedron with even Euler characteristic (see [7]; we shall call this property condition (E)). In this paper we show that every two-dimensional (compact) stratified space P is homeomorphic to a real algebraic affine variety $\hat{P}$ if and only if P satisfies (E).

The main tool is again the construction of a good topological resolution of the singularities (similar, in some sense, to the algebraic one), whose existence is essentially equivalent to condition (E). Using the one point compactification, we give at the end a complete topological characterisation of two-dimensional real algebraic varieties.

Many proofs are elementary; moreover the details of the constructions allow us to get precise informations about the algebraic singularities of $\hat{P}$: we thus obtain a subset $\{A - B\}$ of the set of spaces satisfying (E), such that any $P \in \{A - B\}$ is homeomorphic to a $\hat{P}$ whose algebraic and topological singularities are the same. This is not possible in general: however, we give a standard way to add a « minimal » (with respect to the topology of P) set of singularities in $\hat{P}$ (see 2.11 b) for the precise statement).

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The definition of $\{A - B\}$ and the proofs in this case seem to be more immediately generalizable to higher dimensional spaces. We have just learned that a similar result is announced in a later version of [1].

1. – Preliminaries.

We shall first make some remarks. By the word «smooth» we shall always mean differentiable of class C^∞; due to the low dimension of the spaces considered, many constructions are clear: thus, for example, for the sake of simplicity, all topological constructions are meant up to smoothing, or else we shall assume some notions like attaching a handle to a manifold.

We shall work in the category of two-dimensional compact stratified spaces (see THOM [8], [9] and MATHER [6]), eventually with (not empty collared) boundary. We recall here some known facts.

Let P be such a space, where we assume that every 0-dimensional stratum is exactly one point; if X_0 and X_1 are strata of P, $X_0 < X_1$ means that $X_0 \subset \bar{X}_1$; $\dot{P}$ is the boundary of P. We can assume that P is realized in an euclidean space $\boldsymbol{R}^N$, where N is big enough.

For each $x \in P$, there exists a fundamental system of neighbourhoods of the kind xQ_i (that is, the cone on Q_i with vertex x), where Q_i is a 1-dimensional stratified space isomorphic to Q, $\forall i$; Q is called the link of x in P, and we write $Q = \text{lk}\,(x, P)$.

1.1 REMARK. – If $x = X_0$ is a stratum of P, then $\text{lk}\,(x, P)$ is isomorphic to the boundary of a tubular neighbourhood of X_0 in P (see THOM [8], [9] and MATHER [6]).

Let p be the greatest integer such that $\text{lk}\,(x, P)$ is homeomorphic to $S^p * T$, where S^p is the unit sphere in $\boldsymbol{R}^{p+1}$ ($p \geqslant \dim X_0$, if x belongs to the stratum X_0) and $*$ is the join operation defined by

$$X * Y = X \times Y \times [0, 1]/(x, y, 0) \sim (x, y', 0);\; (x, y, 1) \sim (x', y, 1)\,.$$

1.2 DEFINITION. – The intrinsic codimension of x in P is

$$\text{CI}\,(x, P) = \dim T = 2 - (p + 1)\,.$$

1.3 DEFINITION. – Let i be the length of a maximal chain of strata $x \in X_0 < < X_1 < \ldots < X_i$. The coheight of x in P is

$$\text{CA}\,(x, P) = i\,.$$

1.4 DEFINITION. – A stratified space P is good if

$$\text{CI}\,(x, P) \geqslant \text{CA}\,(x, P) \quad \text{for each } x \in P\,.$$

1.5 DEFINITION. – $\Sigma P = \{x \in P\colon \mathrm{CI}(x, P) + \mathrm{CA}(x, P) \neq 0\}$; $x \in P$ is regular if $\mathrm{CI}(x, P) = \mathrm{CA}(x, P) = 0$.

1.6 REMARK. – Let P be a good stratified space; this means that its stratification describes exactly the topological regularity of a point in P and in ΣP. In particular,

$$\Sigma P = \{x \in P\colon \mathrm{lk}(x, P) \text{ is not homeomorphic to } S^1\}.$$

As in dimension two there are no smoothing problems, we shall only consider, without loss of generality, good stratified spaces.

1.7 DEFINITION. – Let P be a good stratified space. We define

$$\Sigma_0 P = \{x \in P\colon \mathrm{CA}(x, P) = 2\} \quad \text{and} \quad \Sigma_1 P = \{x \in P\colon \mathrm{CA}(x, P) = 1\}.$$

1.8 REMARK. – *a*) $\Sigma P \neq \Sigma_0 P \cup \Sigma_1 P$; if $x \in \Sigma P \setminus (\Sigma_0 P \cup \Sigma_1 P)$, then or x is an isolated point, or it belongs to a 1-dimensional stratum which is not incident to any 2-dimensional stratum;

b) $\Sigma_0 P$ consists of a finite number of points (as P is compact and, if $x \in \Sigma_0 P$, then x is a stratum);

c) $\Sigma(\Sigma P) = \Sigma_0 P \cup \{\text{isolated points of } P\}$.

Using the tubular neighbourhoods of P (see THOM [8], [9] and MATHER [6]), we can find a closed neighbourhood N of ΣP in P such that:

a) the boundary $\dot{N}$ of N is a closed manifold;

b) there exists a (piecewise smooth) projection $p\colon N \to \Sigma P$ which is a deformation retraction;

c) (N, p) is unique, up to isotopy;

d) N is the mapping cylinder of $\dot{p} = p|_{\dot{N}}\colon \dot{N} \to \Sigma P$.

In the following we shall refer to (N, p) as the regular neighbourhood of ΣP in P.

1.9 DEFINITION. – Let f, g be two loops in X, $f(0) = f(1) = g(0) = g(1)$. We say that f and g are specially homotopic if they only differ for constant intervals; that is, if there exists a finite number of loops $f = f_1, f_2, \ldots, f_n = g$, such that f_{i+1} can be obtained from f_i (or vice-versa) in the following way:

$$f_{i+1}(t) = \begin{cases} f_i\big((x_0/t_0)\cdot t\big), & 0 \leqslant t \leqslant t_0 \\ f_i(x_0), & t_0 \leqslant t \leqslant t_1 \\ f_i\big(((1 - x_0)/(1 - t_1))\cdot(t - t_1) + x_0\big), & t_1 \leqslant t \leqslant 1 \end{cases}$$

with $0 \leqslant t_0 \leqslant t_1 \leqslant 1$ and $0 \leqslant x_0 \leqslant 1$.

1.10 REMARK. – *a*) if f and g are specially homotopic, then their mapping cylinders are homeomorphic;

b) $\dot{N}$ consists of a finite number of circles embedded in $P \setminus \Sigma P$ and the homeomorphism type of N depends only on the class of special homotopy of $\dot{p}: \dot{N} \to \Sigma P$; it is thus possible to change the map $\dot{p}$, up to special homotopy, without changing the homeomorphism class of the stratified space P;

c) moreover, it is clear that, if we change $\dot{p}$ in p', specially homotopic to $\dot{p}$ and piecewise smooth (according to the strata of P), then we don't change the isomorphism class of P, as a stratified space.

Let P be a good stratified space. We call (N_1, p_1) the regular neighbourhood of ΣP in P. The regular neighbourhood of $\Sigma(\Sigma P)$ in ΣP is the union of a neighbourhood N_{01} of $\Sigma_0 P$ in ΣP and a finite number of isolated points in P. N_{01}, with the natural projection $p_{01}: N_{01} \to \Sigma_0 P$, will be called the regular neighbourhood of $\Sigma_0 P$ in ΣP. Moreover, we can choose a tubular neighbourhood (N_0, p_0) of $\Sigma_0 P$ in P such that:

a) if $\Sigma_0 P = \{x_1, \ldots, x_n\}$, then N_0 is isomorphic to the disjoint union $\coprod_{i=1,\ldots,n} x_i \operatorname{lk}(x_i, P)$;

b) $N_{01} = N_0 \cap \Sigma P$;

c) $p_0|_{N_1 \cap N_0} = p_{01} \circ p_1|_{N_1 \cap N_0}$;

d) $\dot{p}_1^{-1}(N_{01}) = \dot{N}_0 \cap \dot{N}_1$.

From now on, if no statement is made to the contrary, all stratified spaces will be without boundary.

1.11 DEFINITION. – Let P be a good stratified space. We say that P satisfies condition (A) if $\forall x \in \Sigma_1 P$ such that $\operatorname{lk}(x, P) = S^0 * M$, M consists of an even number of points.

Let $\dot{N}_{01} = \dot{N}_0 \cap \Sigma P = \{r_1, \ldots, r_s\}$ and $\operatorname{lk}(r_i, P) = S^0 * M_i$, with $M_i = \operatorname{lk}(r_i, \dot{N}_0) = \{n_i \text{ points}\}$, $n_i \geq 0$.

1.12 DEFINITION. – Let P be a good stratified space. We say that P satisfies condition (B) if, $\forall n \in \mathbf{N}$, $\forall x \in \Sigma_0 P$, $\# \{i: n_i = n \text{ and } \dot{p}_{01}(r_i) = x\}$ is even.

1.13 REMARK. – *a*) If P satisfies (A) and (B), then it also satisfies Sullivan's condition (E): $\chi(\operatorname{lk}(x, P))$ is even, for each $x \in P$. To see this, note that if $x \in \Sigma_0 P$ (otherwise the statement is obvious), then $\operatorname{lk}(x, P)$ is a graph Γ with $2k$ vertices $r_1, \ldots, r_{2k}$ (for (B)) and $(n_1 + \ldots + n_{2k})/2$ edges; from (A) it follows that each n_i is even, and from (B) that they are equal in pairs; therefore $\chi(\Gamma)$ is even;

b) the contrary of *a*) is no longer true: for example the suspension of the wedge of three circles $P = S^1 \vee S^1 \vee S^1$ satisfies (E) and doesn't satisfy (B).

2. – Topological resolution of singularities.

We shall give polynomial equations for a (compact) stratified space satisfying (E) by means of a topological resolution of singularities of a special kind, whose existence we shall prove in this paragraph.

We shall first give the construction for a good stratified space P satisfying (A) and (B), and then we generalize it to a space P satisfying (E); we do this for many reasons: first, the $(A-B)$ case is much simpler, and it is easier then to understand the modifications which must be given in the (E) case; the existence of the $(A-B)$ special resolution of singularities characterizes the spaces satisfying conditions (A) and (B) and it seems easier to generalize this construction to higher dimensional spaces (see remark 2.7); the $(A-B)$ case is the most general one such that we can make the construction without changing the stratification of P: in the (E) case it will be necessary to add some 1-dimensional strata to the topological singularities of P (thus P, in particular, will no more be a good stratified space).

We first give a construction which will be useful later:

2.1 REMARK. – Let M_n be a two-dimensional compact orientable manifold of genus n and with boundary $\partial M_n = S_1 \cup \ldots \cup S_{n+2}$. We shall give a standard way to find a family $\{\gamma_i\}$ of circles embedded in M_n, in general position and such that $M_n \setminus \{\gamma_i\}$ is a collar of ∂M_n in M_n. We say also that M_n is a normal neighbourhood of $\bigcup_i \gamma_i$.

The proof is by induction on n; M_0 is the cylinder $S^1 \times [0,1]$ and M_{k+1} can be obtained from M_k by attaching a «handle with a hole».

On M_0, the family is the only circle $\gamma = S^1 \times \{\frac{1}{2}\}$; suppose now we have given the family $\{\gamma_i\}$ on M_k and consider

$$M_{k+1} = \overline{M_k \setminus (S^0 \times D^2)} \bigcup_{S^0 \times S^1} \overline{(([0,1] \times S^1) \setminus D)} .$$

D is a 2-disk embedded in $[0,1] \times S^1$ and we can suppose there exists $x_0 \in S^1$ such that $D \subset]\frac{1}{4}, \frac{3}{4}[\times (S^1 \setminus \{x_0\})$.

$S^0 \times D^2$ are two disks D_1 and D_2 embedded in M_k, which we choose to be in different connected components V_1 and V_2 of $M_k \setminus \{\gamma_i\}$, such that $\overline{V}_1 \cap \overline{V}_2$ is a circle γ_{i_0} of the given family.

Let $x_1 = (x_0, 0) \in \partial D_1$ and $x_2 = (x_0, 1) \in \partial D_2$; there exists a path α in M_k, with endpoints x_1 and x_2, which intersects γ_{i_0} transversally in one point and doesn't intersect any other circle of the family $\{\gamma_i\}$. Put

$$\tilde{\gamma}_1 = \alpha \bigcup_{\{x_1, x_2\}} ([0,1] \times \{x_0\}); \qquad \tilde{\gamma}_2 = S^1 \times \{\tfrac{1}{4}\}; \qquad \tilde{\gamma}_3 = S^1 \times \{\tfrac{3}{4}\} .$$

Then the required family on M_{k+1} is $\{\gamma_i, \tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3\}$: to see this, it is enough to note that the connected component of $M_{k+1} \setminus ((\cup \gamma_{i_0}) \cup (\cup \tilde{\gamma}_j))$ containing ∂D is homeomorphic to a cylinder, while the others, when different from a connected component of $M_k \setminus (\cup \gamma_i)$, may be obtained from one of these by adding a hole and a cutting from the boundary of the hole to the previous boundary (and thus they are still homeomorphic to a cylinder).

2.2 Remark. - $M_n \setminus (\cup \gamma_i)$ has $n+2$ connected components $V_1, \ldots, V_{n+2}$, where we denote by V_i the one containing $S_i \subset \partial M_n$. In the following, we shall need that the closure of one of these components, say V_1, intersects each $\overline{V}_j$, $j = 2, \ldots, n+2$, in a circle of the family $\{\gamma_i\}$. To achieve this, it is enough to choose one of the two disks D_1 and D_2 of the last remark to be always in the connected component of $M_k \setminus (\cup \gamma_i)$ containing S_1. Note that, in this case, we can choose paths α_j, with endpoints a point of S_1 and a point of S_j $(j = 2, \ldots, n+2)$, such that each α_j intersects in exactly one point and transversally just one circle of the family $\{\gamma_i\}$, and different paths intersect different circles (see fig. 1).

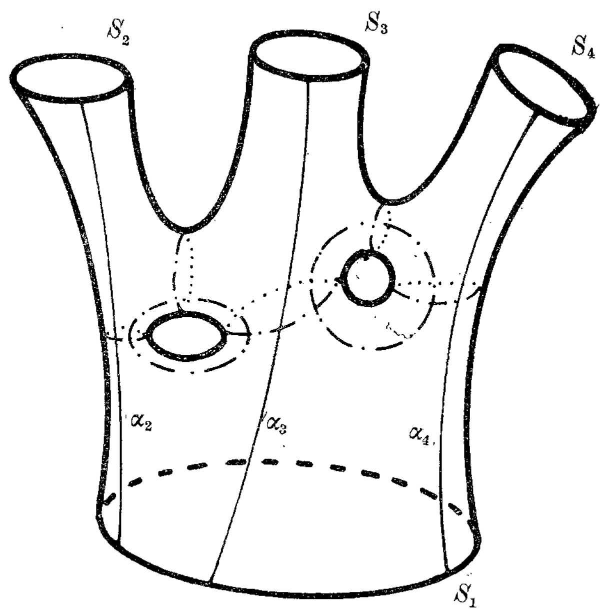


Figure 1

We want to prove the following theorem, whose statement makes clear what we mean by a special resolution of singularities.

2.3 Theorem. - Let P be a good stratified space, satisfying (A) and (B); then there exists an $(A-B)$ special resolution of the singularities of P, that is a chain $P'' \xrightarrow{f''} P' \xrightarrow{f'} P$ such that:

1) P' is a good stratified space, satisfying (A) and (B) and such that $\Sigma_0 P' = \emptyset$;

2) $f'^{-1}(\Sigma_0 P) = \mathcal{F} = \bigcup_r F_r$, where

a) for each r, F_r is or a circle or a wedge of circles (where we agree that a wedge of 0 circles is a single point);

b) if F_r is a circle, then F_r is embedded in $P' \setminus \Sigma P'$;

c) if F_r is a wedge of m_r circles, then the center x_r of F_r is a point of $\Sigma P'$ such that $\mathrm{lk}(x_r, P') = S^0 * (2m_r \text{ points})$; $x_r = F_r \cap \Sigma P'$;

d) the F_r's intersect transversally in $P' \setminus \Sigma P'$;

3) f' is a continuous epimorphism such that $f'|_{P' \setminus \mathcal{F}}: P' \setminus \mathcal{F} \to P \setminus \Sigma_0 P$ is an isomorphism and $f'|_{P' \setminus f'^{-1}(N_0)}: P' \setminus f'^{-1}(N_0) \to P \setminus N_0$ is the identity;

4) P'' is a good stratified space such that $\Sigma_0 P'' = \Sigma_1 P'' = \emptyset$; that is, P'' is a manifold, maybe not equidimensional;

5) $f''^{-1}(\Sigma_1 P') = \mathcal{F}' = \bigcup_k F'_k$ is a family of circles in general position embedded in P'';

6) f'' is a continuous epimorphism such that $f''|_{P'' \setminus \mathcal{F}'}: P'' \setminus \mathcal{F}' \to P' \setminus \Sigma_1 P'$ is an isomorphism and $f''|_{P'' \setminus f''^{-1}(N'_1)}: P'' \setminus f''^{-1}(N'_1) \to P' \setminus N'_1$ is the identity;

7) $f''^{-1}(\mathcal{F}) = \mathcal{F}'' = \bigcup_h F''_h$ is a family of circles embedded in P'' and such that $\mathcal{F}' \cup \mathcal{F}''$ is a family in general position;

8) putting $f = f' \circ f''$, we have that $f|: P'' \setminus (\mathcal{F}' \cup \mathcal{F}'') \to P \setminus \Sigma P$ is an isomorphism; moreover, for each $F''_i \in \mathcal{F}''$, $f|_{F''_i}$ is the constant map on a point of $\Sigma_0 P$, while, for each $F'_j \in \mathcal{F}'$, $f''|_{F'_j}$ or is the constant map on a point of $\Sigma P'$, or it is an n-covering of a circle of $\Sigma P'$.

We shall prove this theorem in three steps:

2.4 *Step* 1: *construction of* P'. – Let P be a good stratified space satisfying (A) and (B) and assume first $\Sigma_0 P = \{x_0\}$. $\Gamma = \mathrm{lk}(x_0, P)$ is a graph with an even number of vertices $r_1, \ldots, r_{2k} = \dot{N}_{01}$ and such that, for each $i = 1, \ldots, 2k$, a neighbourhood U_i of r_i in Γ is a cone with vertex r_i on an even number of points P_{ij} $(j = 1, \ldots, 2n_i;\ n_i \geq 0)$; as P satisfies (B), we may assume also $n_1 = n_2, \ldots, n_{2k-1} = n_{2k}$.

One can prove easily (by induction on k) that there exist s circles $\Gamma_1, \ldots, \Gamma_s$ in Γ such that Γ is the quotient of the disjoint union $\left(\coprod_{t=1,\ldots,s} \Gamma_t\right) \coprod \left(\coprod_{i=1,\ldots,2k} r_i\right)$ by an equivalence relation such that:

a) if r_i is isolated in Γ, then $[r_i] = \{r_i\}$;

b) if r_i is not isolated in Γ, then $[r_i] = \{r_i, P_1, \ldots, P_{n_i}\}$, where $P_j \in \Gamma_j$ and $j \neq h \Rightarrow \Gamma_j \neq \Gamma_h$;

c) if $p \notin [r_i]$ for some i, then $[p] = \{p\}$.

Let us choose once for all $\Gamma_1, \ldots, \Gamma_s$; reorder then the points $P_{ij} \in \partial U_i$ so that $P_{i,2j-1}$ and $P_{i,2j}$ belong to the same circle Γ_h, for any $j = 1, \ldots, 2n_i$ (this construction is clearly empty if $n_i = 0$).

Attach now $n_1 + \ldots + n_k$ edges to $\coprod_{i=1,\ldots,2k} U_i$ so that the boundary of each edge are two points of the kind $P_{i,2j-1}$ and $P_{i,2j}$, and let $\tilde{\Gamma}$ be the resulting graph.

$\tilde{\Gamma}$ is the disjoint union of $2k$ wedges of circles, with centers in r_i, such that each circle meets the boundary of U_i in two points belonging to the same Γ_h.

Put

$$\tilde{P} = \overline{P \setminus N_0} \bigcup_{U_i \times \{0\}} U_i \times [0, 1] \bigcup_{U_i \times \{1\}} \tilde{\Gamma}$$

and consider one of the circles Γ_h, say Γ_1; let $r_1, \ldots, r_t$ be the vertices of Γ belonging to Γ_1; for each $i = 1, \ldots, t$, choose in the wedge of circles with center in $(r_i, 1) \in \tilde{\Gamma}$ that one containing the two points $(P_{ij}, 1)$ such that $P_{ij} \in \Gamma_1$; we thus get t circles $S_1, \ldots, S_t$.

Consider now the manifold $M^{(1)}_{t-1}$ and identify $\partial M^{(1)}_{t-1}$ with $\Gamma_1 \cup S_1 \cup \ldots \cup S_t$; choose a family $\{\gamma_i\}$ of circles in general position embedded in $M^{(1)}_{t-1}$ as in 2.1 and 2.2, where Γ_1 is now the circle playing the role of S_1 in 2.2. Choose then t paths $\alpha_1, \ldots, \alpha_t$ as in 2.2, such that the endpoints of α_i are $(r_i, 0) \in \Gamma_1$ and $(r_1, 1) \in S_i$; let N_i be a tubular neighbourhood of α_i in $M^{(1)}_{t-1}$ such that

$$N_i \cap \Gamma_1 = (U_i \times \{0\}) \cap \Gamma_1 \quad \text{and} \quad N_i \cap S_j = (U_i \times \{1\}) \cap S_j .$$

We can then attach $M^{(1)}_{t-1}$ to $\tilde{P}$ by identifying N_i with $(U_i \cap \Gamma_1) \times [0, 1]$ in the natural way (in particular, α_i is identified with $\{r_i\} \times [0, 1]$).

Do this for each circle $\Gamma_1, \ldots, \Gamma_s$ and call $\hat{P}$ the resulting space; $\hat{P}$ is a good stratified space, with boundary $\tilde{\Gamma}$:

$$\hat{P} = \tilde{P} \bigcup_{\{N_i\}} (M^{(1)} \cup \ldots \cup M^{(s)}) = \overline{P \setminus N_0} \bigcup_{\dot{N}_0} (\underbrace{M^{(1)} \cup \ldots \cup M^{(s)}}_{\{\alpha_i\}}) .$$

We denote by C_i the wedge of n_{2i} circles; let P' be the quotient of the disjoint union $\hat{P} \coprod ((C_1 \cup \ldots \cup C_k) \times [0, 1])$ by the identification of $C_i \times \{0\}$ with the wedge of $n_{2i-1} = n_{2i}$ circles with center in $(r_{2i-1}, 1) \in \tilde{\Gamma}$ and $C_i \times \{1\}$ with the wedge of n_{2i} circles with center in $(r_{2i}, 1) \in \tilde{\Gamma}$, for each $i = 1, \ldots, k$.

P', with the natural stratification, is a good stratified space and

$$\Sigma P' = \overline{\Sigma P \setminus N_{01}} \coprod \Big(\coprod_{i=1,\ldots,k} \{r_{2i}\} \times [0, 1] \Big) / \sim ,$$

where the equivalence relation $\sim$ is defined by

$$(r_{2i}, 0) \sim r_{2i-1} \in \dot{N}_{01} \quad \text{and} \quad (r_{2i}, 1) \sim r_{2i} \in \dot{N}_{01} .$$

Thus $\Sigma_0 P' = \emptyset$ and condition (B) is empty with respect to P'; moreover, for each $x \in \Sigma P' \setminus (\Sigma P \cap \Sigma P')$, $\mathrm{lk}\,(x, P') = S^0 * \{2n_i$ points$\}$ for some i: therefore P' satisfies condition (A).

Observe finally that, if $\Sigma_0 P$ consists of more than one point, we can make the same construction on disjoint neighbourhoods of the points belonging to $\Sigma_0 P$. Therefore we have constructed a space P' satisfying property 1) of 2.3.

2.5 *Step* 2: *construction of* f'. – We always assume, for the sake of simplicity, $\Sigma_0 P = \{x_0\}$.

$$P' = \overline{P \setminus N_0} \bigcup_{\dot N_0} \tilde Q;\ \tilde Q = \tilde Q_1 \coprod \tilde Q_2/\sim;\ \tilde Q_1 = (M^{(1)} \cup \ldots \cup M^{(s)})/\sim;\ \tilde Q_2 = (C_1 \cup \ldots \cup C_k) \times [0, 1]\,,$$

where $\sim$ denotes the identifications previously described.

N_0 is a cone with vertex x_0 on $\dot N_0$; thus, in order to define a map $f'\colon P' \to P$ satisfying properties 2) and 3) of theorem 2.3, it is enough to find a family $\mathcal{F} = \bigcup_r F_r$, satisfying properties a), b), c), d) of 2) and such that $\tilde Q \setminus \bigcup_r F_r$ is a collar on $\dot N_0 = \partial \tilde Q$.

First of all, for each $i = 1, \ldots, s$, we choose a family of circles $\{\gamma_h^{(i)}\}$ embedded in the manifold $M^{(i)}$, as in the remark 2.2 with respect to Γ_i. Note that, for each $i = 1, \ldots, s$, there are t_i paths $\alpha_1^{(i)}, \ldots, \alpha_{t_i}^{(i)}$ in $M^{(i)}$ (the ones where we make the identifications to get $\tilde Q_1$), such that the endpoints of $\alpha_j^{(i)}$ are the two points $(r_j, 0) \in \Gamma_i$ and $(r_j, 1) \in S_j^{(i)}$; as we saw, we can choose the circles $\{\gamma_h^{(i)}\}$ so that one and only one (which we call $\gamma_j^{(i)}$) meets the path $\alpha_j^{(i)}$ transversally.

Let us fix then a point $x_j \in \alpha_j^{(i)}$, for example $x_j = (r_j, \frac12)$, and choose $\gamma_j^{(i)}$ so that $x_j \in \gamma_j^{(i)}$. Note also that $\gamma_j^{(i)}$ is the only circle of the family $\{\gamma_h^{(i)}\}$ which is contained in the closures of the two connected components of $M^{(i)} \setminus \{\gamma_h^{(i)}\}$ meeting $S_j^{(i)}$ and Γ_i respectively.

Therefore, if we choose the families $\{\gamma_h^{(i)}\}$ as described, after the identifications we shall get a family $\mathcal{F}' = \{F_s'\}$ satisfying properties a), b), c), d), and such that $\tilde Q_1 \setminus \mathcal{F}'$ is a collar on $\dot N_0 \cup \tilde\Gamma$.

Let us now attach to $\tilde Q_1$ a «handle» $C_p \times [0, 1]$; if C_p is *a* wedge of $n_{2p} = 0$ circles, that is the single point r_{2p}, it is enough to add to the family $\mathcal{F}'$ the point $(r_{2p}, \frac12)$.

Suppose then $n_{2p} > 0$ and let S be a circle belonging to the wedge C_p: we shall describe how to make some modifications to the family $\mathcal{F}'$ in order to get a family satisfying the same properties with respect to the space $\tilde Q_1 \cup (S \times [0, 1])$.

$S \times \{0\}$ is identified with a circle $S_j^{(p)} \subset \partial M^{(p)}$ and $S \times \{1\}$ is identified with a circle $S_h^{(q)} \subset \partial M^{(q)}$ (maybe $p = q$).

As we saw, we can associate to $S_j^{(p)}$ (resp. $S_h^{(q)}$) a well-defined circle $\gamma_j^{(p)}$ (resp. $\gamma_h^{(q)}$); we shall use the simpler notation $S' = S_j^{(p)}$, $S'' = S_h^{(q)}$, $\gamma' = \gamma_j^{(p)}$, $\gamma'' = \gamma_h^{(q)}$. Let V' (resp. $V^{(p)}$) be the connected component of $M^{(p)} \setminus \bigcup_s \gamma_s^{(p)}$ which meets S' (resp. Γ_p); V'' and $V^{(q)}$ are defined similarly.

There exists a projection $p'\colon M^{(p)} \to \bigcup_s \gamma_s^{(p)}$ (resp. $p''\colon M^{(q)} \to \bigcup_t \gamma_t^{(q)}$) such that V'

is the mapping cylinder of $\dot{p}' = p'|_{S'}$ and $\gamma' \subset p'(V')$; moreover, as $\gamma' \subset \overline{V^{(p)}}$ and $V^{(p)} \neq V'$, there exists an arc $\beta' \subset \gamma'$, with endpoints x_1' and x_2', such that $x_j' \in \beta'$, $p'^{-1}(\beta')$ is homeomorphic to a disk and $\dot{p}'^{-1}(\beta')$ is an arc $\sigma' \subset S'$ with endpoints y_1' and y_2'; in a similar way we choose β'', with endpoints x_1'' and x_2'', and σ'', with endpoints y_1'' and y_2''. Choose now two arcs ϱ_1 and ϱ_2 in $S \times [0, 1]$ such that: i) $S \times \times [0, 1] \setminus (\varrho_1 \cup \varrho_2)$ is the disjoint union of two disks D_1 and D_2; ii) $\partial D_1 = \sigma' \cup \varrho_1 \cup \cup (S'' \setminus \sigma'') \cup \varrho_2$; iii) $\partial D_2 = \sigma'' \cup \varrho_1 \cup (S' \setminus \sigma') \cup \varrho_2$; iv) $(r_{2p}, \frac{1}{2}) \in \varrho_1$.

Consider the circle

$$\gamma = \overline{(\gamma' \setminus \beta')} \bigcup_{x_1', x_2'} p'^{-1}\{x_1', x_2'\} \bigcup_{y_1', y_2'} (\varrho_1 \cup \varrho_2) \bigcup_{y_1'', y_2''} p''^{-1}\{x_1'', x_2''\} \bigcup_{x_1'', x_2''} \overline{(\gamma'' \setminus \beta'')}$$

and the family of circles $\tilde{\mathcal{F}}' = \mathcal{F}' \setminus \{\gamma', \gamma''\} \cup \{\gamma\}$ (see fig. 2).

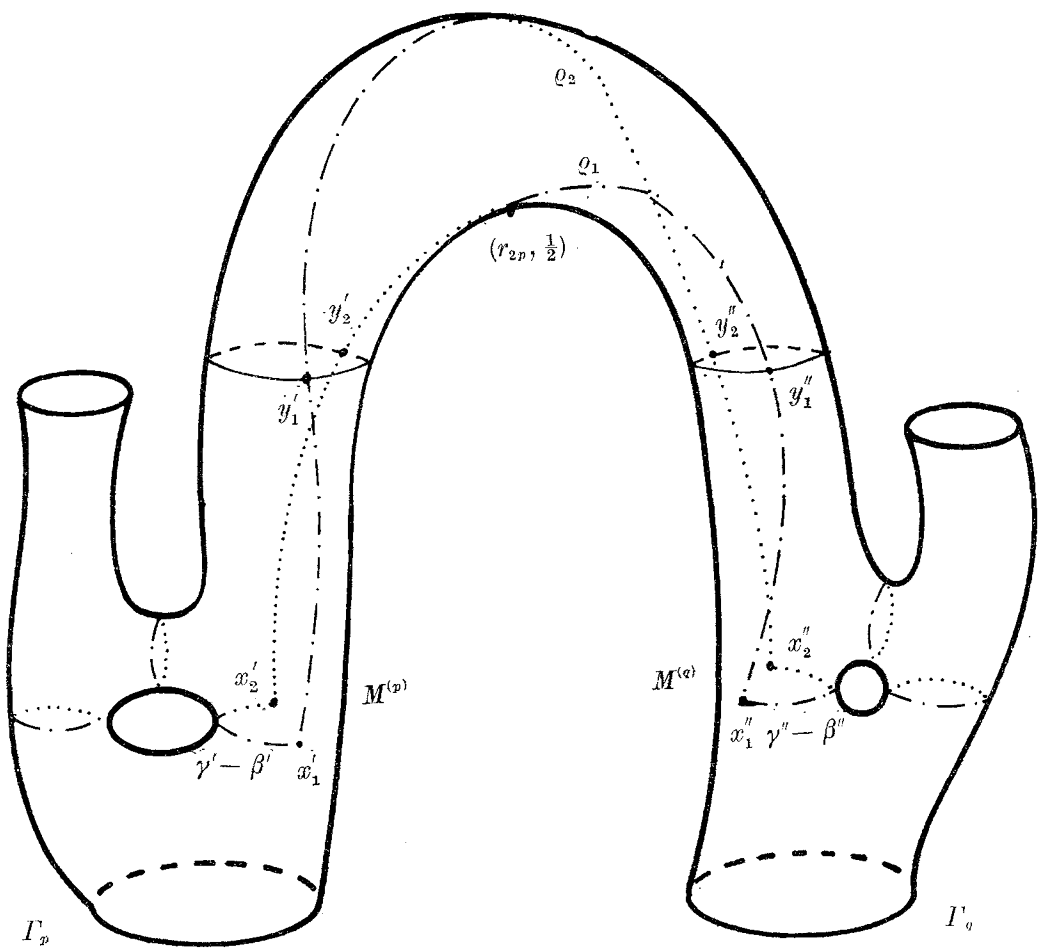


Figure 2

This family satisfies the required properties with respect to the space $\tilde{Q}_1 \cup (S \times \times [0,1])$; observe that the only connected components of $\tilde{Q}_1 \cup (S \times [0,1]) - \tilde{\mathcal{F}}'$ which are not connected components of $\tilde{Q}_1 \setminus \mathcal{F}'$ are the union of $V^{(p)}$ (resp. $V^{(q)}$) and three disks:

$$p'^{-1}(\beta') \qquad (\text{resp. } p''^{-1}(\beta''))\,,$$

D_1 (resp. D_2) and

$$p''^{-1}(p''(S''\setminus\sigma'')) \qquad (\text{resp. } p'^{-1}p'(S'\setminus\sigma'))$$

such that

$$V^{(p)} \cap p'^{-1}(\beta') = \beta'\,, \qquad \Big[V^{(p)} \bigcup_{\beta'} p'^{-1}(\beta')\Big] \cap D_1 = \sigma'\,,$$
$$\Big[V^{(p)} \bigcup_{\beta'} p'^{-1}(\beta') \bigcup_{\sigma'} D_1\Big] \cap p''^{-1}(p''(S''\setminus\sigma'')) = S''\setminus\sigma''$$

and similarly for $V^{(q)}$; thus the resulting connected component is homeomorphic to $V^{(p)}$ (resp. $V^{(q)}$). The same holds if $p = q$, as the intersections of $V^{(p)} = V^{(q)}$ with the first two disks are two disjoint arcs $\beta' \subset \gamma'$ and $\beta'' \subset \gamma'' \neq \gamma'$.

Note finally that the only properties of the family $\mathcal{F}'$ which we used are:

1) $\mathcal{F}'$ satisfies the required conditions with respect to $\tilde{Q}_1$;
2) to any circle $S_j^{(i)} \subset \Gamma = \partial\tilde{Q}_1 \setminus \dot{N}_0$ we can associate a circle $\gamma_j^{(i)} \in \mathcal{F}'$ with the described properties.

As these two properties hold for the new family $\tilde{\mathcal{F}}'$ with respect to the space $\tilde{Q}_1 \cup S \times [0,1]$, we can repeat the same construction until there are no handles left. We shall get at the end the required family $\mathcal{F} = \{F_r\}$: note that the property iv) of the arcs ϱ_i ensures that $\mathcal{F}$ satisfies condition c).

As before, we can remove the first assumption $\Sigma_0 P = \{x_0\}$ by working in disjoint neighbourhoods of the points belonging to $\Sigma_0 P$, so that we have proved the existence of a map $f'\colon P' \to P$ satisfying conditions 2) and 3) of the theorem 2.3.

2.6 *Step* 3: *construction of* P'' *and* f''. – Consider the stratified space P', satisfying (A) and (B) and such that $\Sigma_0 P' = \emptyset$; $\Sigma P'$ is the disjoint union of a finite number of circles $S_1, \ldots, S_r$ and a finite number of points $x_1, \ldots, x_s$. Without loss of generality we can assume $\Sigma P' = \Sigma_1 P'$ (as $\Sigma P' \setminus \Sigma_1 P'$ consists of connected components of P').

Let (N', p') be the regular neighbourhood of $\Sigma P'$ in P', $N_i' = p'^{-1}(x_i)$,

$$\dot{N}_i' = \dot{p}'^{-1}(x_i) \qquad (\text{for each } i = 1, \ldots, s)\,, \qquad N_j = p'^{-1}(S_j)$$

and

$$\dot{N}_j = \dot{p}'^{-1}(S_j) \qquad (\text{for each } j = 1, \ldots, r)\,.$$

$\dot{N}_i'$ and $\dot{N}_j$ are both disjoint unions of circles embedded in $P' \setminus \Sigma P'$; note that we could obviously get a space P'' as required by putting $P'' = \overline{P' \setminus N'} \bigcup_{\dot{N}'}$ (disjoint union of a finite number of disks); however, we shall give a different construction of P'', which makes clear the existence of the map $f'': P'' \to P'$.

As P' satisfies condition (A), if $\dot{N}_j = S_{j_1} \cup \ldots \cup S_{j_m}$, there are exactly an even number of indices k such that the map $\dot{p}'|: S_{j_k} \to S_j$ has odd degree; as $\Sigma_0 P' = \emptyset$, we can also assume that, if $\dot{p}'|: S_{j_k} \to S_j$ has degree n, then it is in fact an n-covering.

These remarks show that it is enough to prove the theorem in the three following particular cases:

a) $\Sigma P' = x$;

b) $\Sigma P' = S$, $\dot{p}'^{-1}(S) = \dot{N}' = S_1$ and $\dot{p}'|: S_1 \to S$ is a $2m$-covering;

c) $\Sigma P' = S$, $\dot{p}'^{-1}(S) = \dot{N}' = S_1 \cup S_2$, $\dot{p}'|: S_1 \to S$ is a $2m+1$-covering and $\dot{p}'|: S_2 \to S$ is a $2n+1$-covering.

Case a): let $\dot{N}' = \dot{p}'^{-1}(x) = S_1 \cup \ldots \cup S_k$ ($k \geqslant 2$, as P' is good).

Consider the manifold M_{k-2} described in 2.1, and the family $\{\gamma_i\}$ of circles embedded in M_{k-2} in general position; there is a natural map $\varphi: M_{k-2} \to N'$ such that $\varphi^{-1}(x) = \bigcup_i \gamma_i$ and $\varphi_1: \partial M_{k-2} \to \dot{N}' = S_1 \cup \ldots \cup S_k$ is a homeomorphism, according to the mapping cylinder structure of M_{k-2} and the cone structure of N'.

Put then $P'' = \overline{P' \setminus N'} \bigcup_{\dot{N}'} M_{k-2}$ and $f'': P'' \to P'$ defined by extending φ with the identity on $\overline{P' \setminus N'}$. It is clear that P'' is a manifold and f'' satisfies conditions 5), 6) and 8); condition 7) is empty.

Case b): let M be a Moebius band and $\gamma \subset M$ a circle such that M is the mapping cylinder of a 2-covering $\pi: \partial M \to \gamma$.

Put $P'' = \overline{P' \setminus N'} \cup M$, identifying $\dot{N}' = S_1$ with ∂M.

There exists a map $\varphi: \gamma \to S$ (which is an m-covering) such that $\varphi \circ \pi: \partial M \to S$ is the same map as $\dot{p}'|: S_1 \to S$, up to the given identification. As

$$M = \partial M \times [0, 1]/ \ (x, 1) \sim (x', 1) \text{ iff } \pi(x) = \pi(x')$$

and

$$N' = S_1 \times [0, 1]/ \quad (y, 1) \sim (y', 1) \text{ iff } \dot{p}'(y) = \dot{p}'(y'),$$

we can define $f'': M \to N'$ by extending the given identification between ∂M and S, according to the mapping cylinder structures.

It is clear that P'' is a manifold and f'' satisfies conditions 5), 6) and 8) $(f''^{-1}(\Sigma P') = \gamma)$. As for property 7), let $F_i \in \mathcal{F}$; if F_i is a circle, there is nothing to check, because $F_i \cap N' = \emptyset$, so that $f''^{-1}(F_i) \cap \gamma = F_i \cap \gamma = \emptyset$. If F_i is a wedge of m circles, with center $x \in S$, $f''^{-1}(x) = \{x_1, \ldots, x_m\} \in \gamma$ and, if $U = F_i \cap N' =$ cone with vertex x on $2m$ points, $f''^{-1}(U)$ consists of m arcs embedded in M, meeting γ transversally in the points $x_1, \ldots, x_m$. Thus $\{f''^{-1}(F_i)\} \cup \gamma$ is a family of circles in general position.

Case c): if $m = n$, put $V = S\times[0,1]$, $\gamma = S\times\{\frac{1}{2}\}$ and do the same construction as in the previous case.

Suppose now $m < n$. Let V be the union of a manifold M_1 (as described in 2.1) and a Moebius band M, where ∂M is identified with

$$S_3'\subset \partial M_1 = S_1'\cup S_2'\cup S_3' .$$

Put $P'' = \overline{P'\setminus N'}\cup V$, where the union is made by an identification of $\dot{N}' = S_1\cup S_2$ and $\partial V = S_1'\cup S_2'$.

In order to define f'', consider the circles $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ embedded in M_1 (as in the remark 2.2) and let $\alpha_1, \alpha_2, \alpha_3$ be the arcs of γ_4 such that, if $\pi\colon M_1\to\gamma_4\cup\gamma_1\cup\gamma_2\cup\gamma_3$ is the retraction,

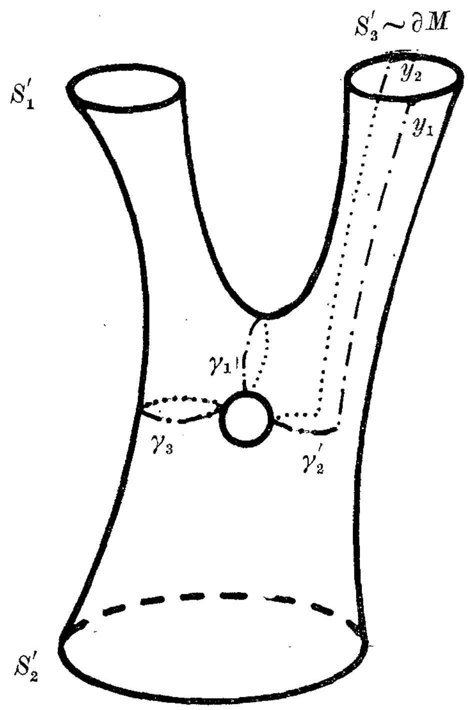
$$\pi(S_1') = \gamma_3\alpha_2\gamma_1\alpha_2^{-1}, \qquad \pi(S_2') = \gamma_3\alpha_1\gamma_2\alpha_1^{-1}, \qquad \pi(S_3') = \gamma_2\alpha_3\gamma_1\alpha_3^{-1} .$$

Change now γ_2 into a new circle γ_2' as follows: let $\beta\subset\gamma_2$ be an arc, with endpoints x_1 and x_2, such that $\beta\cap\gamma_4 = \emptyset$; $\sigma = \pi^{-1}(\beta)\cap\partial M$ is an arc with endpoints y_1 and y_2.

Let $\varrho\subset M$ be a path with endponts y_1 and y_2 such that $M\setminus\varrho$ is connected. Define

$$\gamma_2' = \overline{\gamma_2\setminus\beta}\bigcup_{x_1,x_2}(\pi^{-1}\{x_1x_2\}\cap V_M)\bigcup_{\gamma_1,\gamma_2}\varrho$$

where V_M is the connected component of $M_1\setminus\{\gamma_i\}$ such that $\partial M\subset V_M$ (see fig. 3).



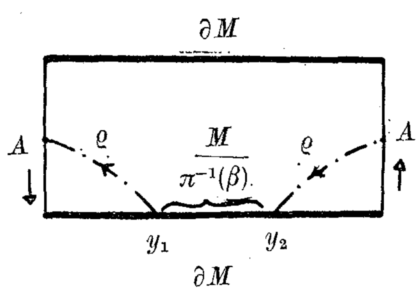


Figure 3

Then V is a regular neighbourhood of $\gamma_1 \cup \gamma_2' \cup \gamma_3 \cup \gamma_4$ and, if $q\colon V \to \gamma_1 \cup \gamma_2' \cup \cup \gamma_3 \cup \gamma_4$ is the retraction such that V is the mapping cylinder of $\dot{q} = q|_{\partial V}$, then

$$q(S_1') = \gamma_3\alpha_2\gamma_1\alpha_2^{-1} \quad \text{and} \quad q(S_2') = \gamma_3\alpha_1\gamma_2'\alpha_3\gamma_1\alpha_3^{-1}\gamma_2'\alpha_1^{-1}.$$

Define now $\varphi\colon \gamma_1 \cup \gamma_2' \cup \gamma_3 \cup \gamma_4 \to S$ such that

$\varphi|_{\gamma_1 \cup \gamma_4}$ is the constant map on a point $x_0 \in S$,

$\varphi|_{\gamma_3}$ is a $(2m+1)$-covering,

$\varphi|_{\gamma_2'}$ is a $(n-m)$-covering.

It follows that the map $\varphi \circ q\colon \partial V \to S$ is specially homotopic to a $(2m+1)$-covering when restricted to S_1', while it is specially homotopic to a $(2n+1)$-covering when restricted to S_2'. Therefore, one can assume that $\varphi \circ q$ is the same map as $\dot{p}'\colon S_1 \cup S_2 \to S$, up to the given identification between $S_1 \cup S_2$ and $S_1' \cup S_2'$, and we can define $f''\colon P'' \to P'$ as in the previous case.

As before, properties 5), 6) and 8) are obvious from the constructions; property 7) is proved similarly to the previous case: the only difference is that we must take care to choose the point $x_0 \in S$ such that $f''^{-1}(x_0) = \gamma_1 \cup \gamma_4$ so that it is not the center of a wedge of circles $F_i \in \mathcal{F}$.

It is clear how the general case follows from these three particular cases, so that theorem 2.3 is now completely proved.

2.7 Remarks and examples.

1) It is easy to check that, if P is a good stratified space which has an $(A-B)$ special resolution of singularities, in the sense of theorem 2.3, then P satisfies conditions (A) and (B).

2) (A) is equivalent to the following condition:

(K): the (smooth unoriented) bordism class $[\dot{p}\colon \dot{N} \to \Sigma P]$ is zero which is a necessary and sufficient condition to the existence of a blow-up $f\colon \tilde{P} \to P$ of ΣP in P (in the sense of KATO [5]).

3) (K) does not imply the existence of an $(A-B)$ special resolution: for example, let $(S^1 \vee S^1)_i$, $i = 1, 2, 3$, be three copies of $S^1 \vee S^1$ and x_i be the center of the wedge $(S^1 \vee S^1)_i$. Put

$$K_i = S(S^1 \vee S^1)_i = (S^1 \vee S^1)_i \times [0,1]/(x,0) \sim (x_i, 0) \quad \text{for each } x$$

$$(x,1) \sim (x_i, 1) \quad \text{for each } x$$

and

$$P = K_1 \cup K_2 \cup K_3/(x_1', 0) \sim (x_1, 1) \sim (x_2, 0)$$
$$(x_2, 1) \sim (x_3, 0) \sim (x_3, 1)$$

with the natural good stratification.

P satisfies (K) and does not satisfy (B).

4) The stratified space P of the last example does not even satisfy the following condition:

(K'): the (smooth unoriented) bordism class $[\dot{p}_{01}: \dot{N}_{01} \to \Sigma_0 P]$ is zero which is a necessary and sufficient condition to the existence of a blow-up of $\Sigma_0 P$ in ΣP. In fact, (K') is strictly weaker than (B) and not even (K) and (K') together imply the existence of an $(A - B)$ special resolution of singularities, as we can see from the following example.

5) Let $K_1 = S(S^1 \vee S^1)$ and $K_2 = S(S^1 \vee S^1 \vee S^1)$ and (x_i, j) be defined as in example 3) $(i = 1, 2; j = 0, 1)$.

Put

$$P = K_1 \cup K_2/(x_1, 0) \sim (x_2, 0); \quad (x_1, 1) \sim (x_2, 1)$$

with the natural good stratification. P satisfies (K) and (K') and does not satisfy (B). P does not even satisfy condition (E).

6) Let $P = S(S^1 \vee S^1 \vee S^1)$, which is a stratified space satisfying (E) and not (B) (see remark 1.13 b)). P is the example of a space which can't be homeomorphic to a real algebraic affine variety whose algebraic and topological singularities are the same.

7) It is clear enough how the notion of an $(A - B)$ special resolution can be generalized to higher dimensional stratified spaces.

We have seen that conditions (A) and (B) are strictly stronger than condition (E); we want now to give a construction, similar to the $(A - B)$ special resolution, for spaces satisfying only (E). More precisely, we want to prove the following theorem, which is the analogue of 2.3:

2.8 THEOREM. - Let P be a good stratified space satisfying (E). Then there exists an (E) special resolution of the singularities of P, that is a chain $P'' \xrightarrow{f''} P' \xrightarrow{f'} P$ such that:

1) P' is a good stratified space, satisfying (E) and such that $\Sigma_0 P' = \{x_1, \ldots, x_n\}$ and, for each $i = 1, \ldots, n$, $\operatorname{lk}(x_i, P')$ or is a wedge of an odd number of circles, or

is a graph with two vertices y_i and z_i, $4n_i$ edges with endpoints y_i and z_i and a wedge of $2m_i$ circles with center z_i $(m_i > 0)$;

2) $f'^{-1}(\Sigma_0 P) = \mathcal{F} = \bigcup_r F_r$, where $\mathcal{F}$ satisfies conditions *a*), *b*) and *d*) of theorem 2.3, 2) and the following

c') if F_r is a wedge of m_r circles with center x_r, then or $x_r \in \Sigma_0 P'$, or $x_r \in \Sigma_1 P'$ and $\operatorname{lk}(x_r, P') = S^0 * (2m_r \text{ points})$; $x_r = F_r \cap \Sigma P'$;

3) as in 2.3;

4) as in 2.3;

5) we can define a (not good) stratification of P' by adding some 1-dimensional strata so that, if $\tilde{\Sigma} P' = \Sigma P' \cup \{\text{new strata}\}$, then $f''^{-1}(\tilde{\Sigma} P') = \mathcal{F}' = \bigcup_k F'_k$ is a family of circles in general position embedded in P'';

6) as in 2.3, putting $\tilde{\Sigma} P'$ instead of $\Sigma P'$;

7) as in 2.3;

8) let $f = f' \circ f''$; then $f|: P'' \setminus (\mathcal{F}' \cup \mathcal{F}'') \to P \setminus \Sigma P$ is an isomorphism; for each $F''_i \in \mathcal{F}''$, $f|_{F''_i}$ is the constant map; for each $F'_j \in \mathcal{F}'$, $f''|_{F'_j}$ or is the constant map. or it is an n-covering of a circle of $\tilde{\Sigma} P'$, or it is a double covering, branched in two points, of an arc of $\tilde{\Sigma} P'$.

As for 2.3, we shall prove this theorem in three steps.

2.9 *Step* 1: *construction of* P'. – Let us always suppose, for the sake of simplicity, $\Sigma_0 P = \{x_0\}$.

We can first make the same construction as in 2.4, until we get the stratified space $\hat{P} = \overline{P \setminus N_0} \cup [M^{(1)} \cup \ldots \cup M^{(s)}]/\sim$ with boundary $\hat{\Gamma}$. $\hat{\Gamma}$ is the disjoint union $\hat{\Gamma} = C_{n_1} \cup \ldots \cup C_{n_r}$, where C_{n_i} is the wedge of n_i circles. It is no longer true that the n_i's are equal in pairs; however, as P satisfies (E) $\big($so that $\chi(\hat{\Gamma}) = \chi(\Gamma) = \chi(\operatorname{lk}(x_0, P))$ is even$\big)$, there are exactly an even number of indices i such that n_i is even. Make then the following constructions:

1) if there exist n_i and n_j such that $n_i = n_j = k$, we attach to $\hat{P}$ the « handle » $C_k \times [0, 1]$, identifying $C_k \times \{0\}$ with C_{n_i} and $C_k \times \{1\}$ with C_{n_j} (as in 2.4). Do this until there are no pairs of equal n_i's left;

2) if n_i is odd, consider the space $B_{n_i} = D^2/\sim$, where $\sim$ is the equivalence relation which identifies n_i distinct points of ∂D^2 to a single point (which we call the vertex of B_{n_i}). Then attach B_{n_i} to $\hat{P}$, identifying $\dot{B}_{n_i}$ with C_{n_i};

3) if n_i is even, then there exists another wedge C_{n_j} left, with n_j even; let $n_i = 2h$, $n_j = 2k$ and $k > h$.

Consider the space

$$T_{n_i,n_j} = (C_{2h}\times[0,1]) \cup (C_{2(k-h)}\times[0,\tfrac{1}{2}]) \cup B_{2(k-h)}/\sim$$

where $\sim$ is the equivalence relation which identifies $\dot{B}_{2(k-h)}$ with $C_{2(k-h)}\times\{\frac{1}{2}\}$ and $\{x\}\times[0,\frac{1}{2}]$ with $\{y\}\times[0,\frac{1}{2}]$ (where x is the center of C_{2h} and y is the center of $C_{2(k-h)}$). We can then attach T_{n_i,n_j} to $\hat{P}$, identifying its boundary with $C_{n_i}\cup C_{n_j}$.

After all these constructions, we shall get a stratified space P' which satisfies condition 1) of 2.8; to see this, note that $\Sigma_0 P'$ consists exactly of the vertices z_h of the spaces B_h: in the case 2), $\mathrm{lk}(z_h, P')$ is the wedge of an odd number of circles, while, in the case 3), it is a graph of the required kind.

2.10 *Step* 2: *construction of* f'.

$$P' = \overline{P\setminus N_0} \cup \Big(\bigcup_{i=1,\dots,s} M^{(i)}\Big) \cup \Big(\bigcup_{k} C_k\times[0,1]\Big) \cup \Big(\bigcup_{n_i} B_{n_i}\Big) \cup \big(\bigcup T_{n_i,n_j}\big)/\sim\; = \overline{P\setminus N_0}\cup\tilde{Q};$$

we have to find a family $\mathcal{F}=\{F_r\}$ in $\tilde{Q}$ which satisfies properties a), b), c'), d) and such that $\tilde{Q}\setminus\bigcup_r F_r$ is a collar on $\dot{N}_0$.

We first consider the families $\{\gamma_h^{(i)}\}$ in $M^{\,i)}$ and the family $\mathcal{F}'$ obtained from these by the given identifications; proceed then as in 2.5 whenever we add a handle $S\times[0,1]$ with $S\subset\tilde{\Gamma}$ and $S\times[0,1]\subset C_k\times[0,1]$ or $S\times[0,1]\subset T_{n_i,n_j}$.

We thus get a family $\mathcal{F}^* = \{F_r^*\}$ satisfying conditions a), b), c), d) and such that $Q'\setminus\bigcup F_r^*$ is a collar of $\partial Q'$, where

$$Q' = \tilde{Q}\setminus\big(\bigcup B_{n_i}\big)\setminus\big(\bigcup B_{n_i,n_j}\big) \quad \text{and} \quad B_{n_i,n_j} = (C_{n_j-n_i}\times[0,\tfrac{1}{2}]) \cup B_{n_j-n_i}/\sim\; \subset T_{n_i,n_j}.$$

As we have done in 2.5 with respect to the handles, we shall give now a standard way to change the family $\mathcal{F}^*$, whenever we add a space B_{n_i} (or B_{n_i,n_j}, which will be the same).

Let $B_n = D/z_1\sim\dots\sim z_n$ with $z_1,\dots,z_n\in\partial D$ (n is odd if $B_n = B_{n_i}$ and it is even if $B_{n_i,n_j} = B_n\cup\dot{B}_n\times[0,\frac{1}{2}]$).

Let $S_1,\dots,S_n$ be the circles of $\tilde{\Gamma}$ belonging to the wedge which is identified with $C_n = \dot{B}_n$ and $\gamma_1,\dots,\gamma_n$ be the circles associated to $S_1,\dots,S_n$ as in 2.5. Note that $\gamma_i\in\mathcal{F}^*$, as these circles have never been changed by the previous modifications.

Let π_j be the retraction $\pi_j\colon V_j\to\bigcup_h \gamma_h^{(p_j)}$, where $S_j\subset\partial M^{(p_j)}$ and V_j is the connected component of $M^{(p_j)}\setminus\bigcup_h\gamma_h^{(p_j)}$ containing S_j.

Choose, as in 2.5, an arc $\beta_j\subset\gamma_j$ (for each $j=1,\dots,n$) intersecting ΣP and such that $\pi_j^{-1}(\beta_j)$ is homeomorphic to a disk.

Let $\sigma_j = \dot{\pi}^{-1}(\beta_j)\subset S_j$ and $\tau_j = \overline{S_j\setminus\sigma_j}$.

We make now the following modifications of the family $\mathcal{F}^*$:

A) take off the circles $\gamma_1, \ldots, \gamma_n$;

B) add s circles $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_s$, $s = [n/2] - 1$; for each $k \leqslant s$, $\tilde{\gamma}_k$ is the quotient of an arc in D with endpoints z_1 and z_{2k+1} and $\tilde{\gamma}_j \cap \tilde{\gamma}_k = \{\text{vertex of } B_n\}$ if $j \neq k$;

C) add $s+1$ circles $\hat{\gamma}_1, \ldots, \hat{\gamma}_{s+1}$ obtained by « connecting » two or three of

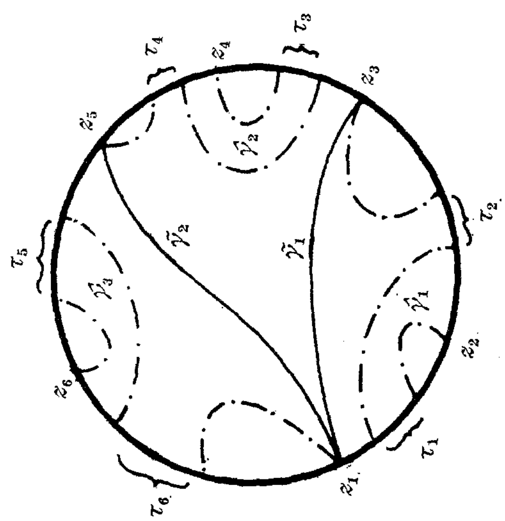


Figure 4

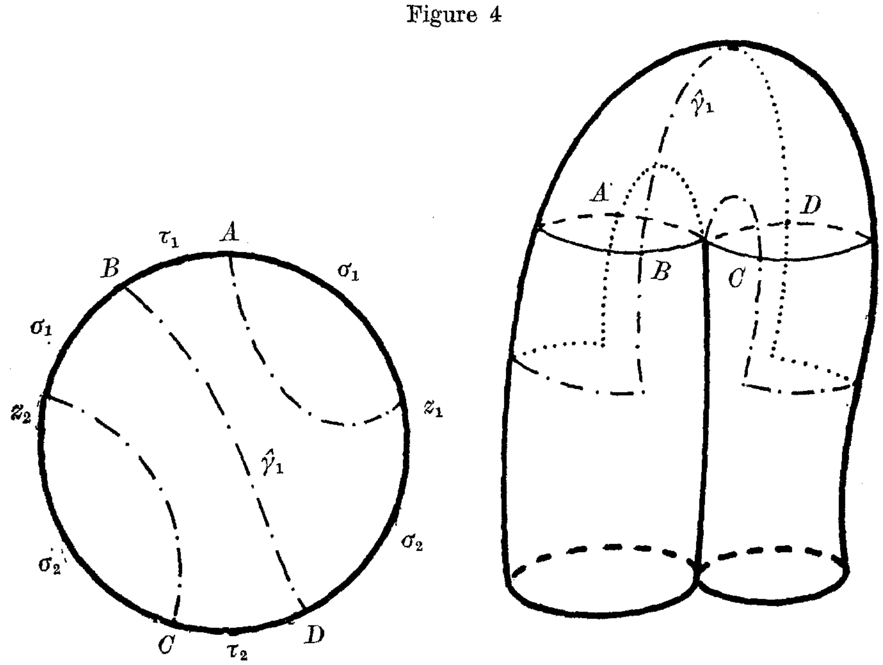


Figure 5

the circles γ_i's. More precisely, $\hat{\gamma}_1$ is obtained by « connecting » γ_1 and γ_2, $\hat{\gamma}_2$ by « connecting » γ_3 and γ_4 and finally $\hat{\gamma}_{s+1}$ is obtained by « connecting » γ_n, γ_{n-1} (and γ_{n-2}, if n is odd) (see fig. 4). The connecting operation is made as in 2.5: for example, in order to construct $\hat{\gamma}_1$, we choose two arcs in the connected component of $D\setminus(\bigcup \tilde{\gamma}_i)$ containing $S_1 \cup S_2$ so as to divide it in (four) disks such that, if the boundary of one of these disks contains τ_1, then its intersection with τ_2 is empty (see fig. 5). We can make a similar construction in the case of three circles, as we can see in fig. 6.

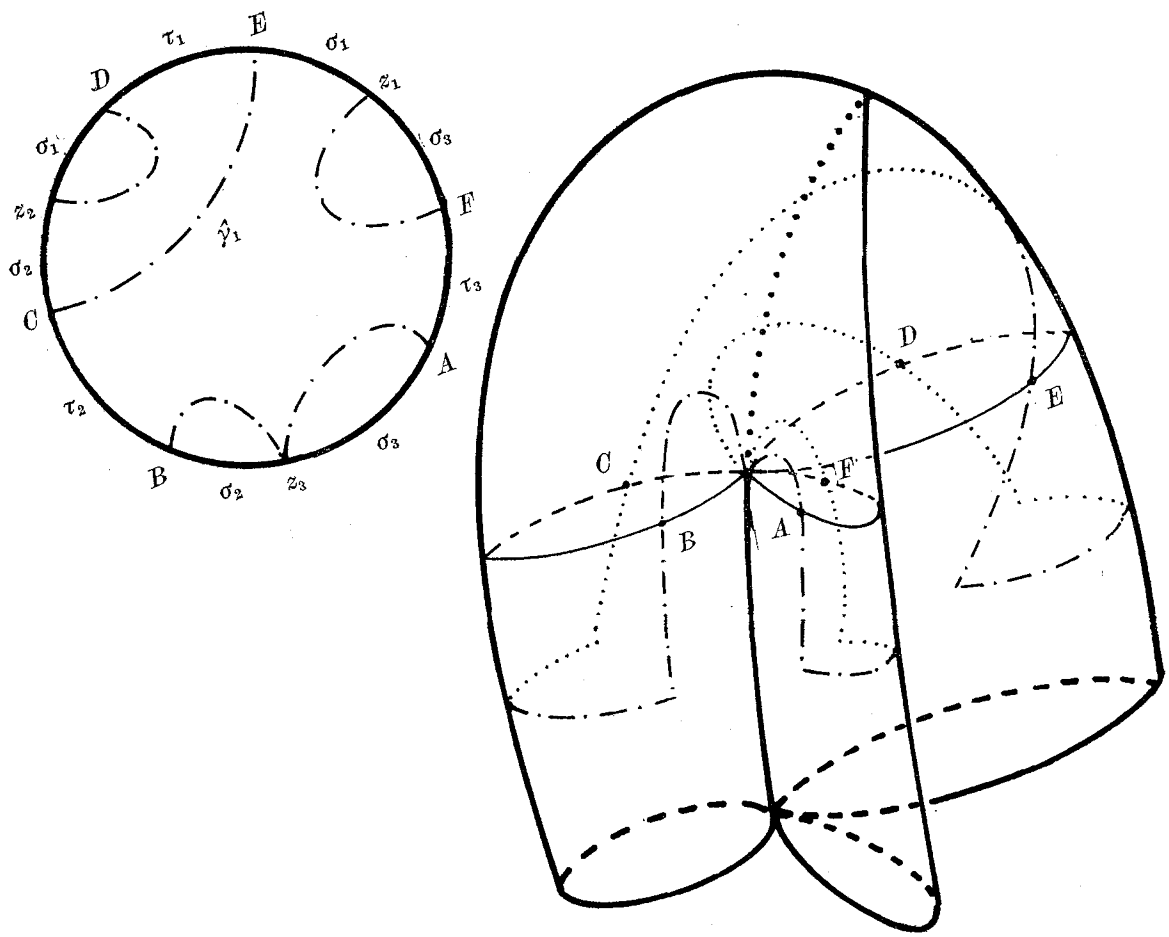


Figure 6

Note that, if z is the vertex of B_n, then $z \in \hat{\gamma}_k$, for each $k = 1, \dots, s+1$, as one of the arcs chosen in D necessarily contains z_i for some i. After all the operations just described, we shall get a new family $\tilde{\mathcal{F}}^*$; with the same kind of arguments as in 2.5, one can easily prove that the family $\tilde{\mathcal{F}}^*$ satisfies the required conditions with respect to the space $Q' \cup B_n$.

Moreover, we didn't change the γ_i's associated to the circles of $\tilde{\Gamma}$ different from $S_1, \dots, S_n$; this ensures that we can repeat the construction until there are no more B_n or B_{n_i,n_j} left. We thus get the required family $\mathcal{F} = \{F_r\}$ in $\tilde{Q}$ and, as a consequence, the map $f'\colon P' \to P$ satisfying properties 2) and 3) of 2.8.

Note finally that, if z is the vertex of B_n, the number of circles F_r such that $z \in F_r$ is exactly $s + (s+1)$, that is $n-1$ if n is even and $n-2$ if n is odd.

2.11 *Step* 3: *construction of* P'' *and* f''. – Consider now the stratified space P'; as P' satisfies property 1) of 2.8, its singularities may be of the following three kinds:

a) circles or isolated points in $\Sigma P'$, which do not intersect $\Sigma_0 P'$;

b) arcs whose endpoints are two points $x_1, x_2 \in \Sigma_0 P'$, such that $\mathrm{lk}(x_1, P') = \mathrm{lk}(x_2, P') =$ wedge of an odd number of circles;

c) circles intersecting $\Sigma_0 P'$ in points whose links are graphs with two vertices of the kind described in 1).

The case *a*) is dealt with exactly as in 2.6.

b) assume $\Sigma P' = \alpha$, where α is an arc with endpoints x_1 and x_2 and $\mathrm{lk}(x_1, P') = \mathrm{lk}(x_2, P') =$ wedge of $(2n+1)$ circles.

Let (N', p') be a regular neighbourhood of $\Sigma P'$ in P'; N'_i $(i = 1, 2)$ a regular neighbourhood of x_i in P'; $y_1 \in \dot{N}' \cap \dot{N}'_1 = \{(4n+2) \text{ points}\}$ and α' be the unique arc of $\dot{N}' \setminus (\dot{N}' \cap (N'_1 \cup N'_2))$ containing y_1.

Let y_2 be the other endpoint of α', $y_2 \in \dot{N}' \cap \dot{N}'_2$ and $\tilde{\alpha}$ be the arc with endpoints x_1 and x_2 obtained by extending α' according to the cone structures of N'_1 and N'_2.

Put $\tilde{\Sigma} P' = \{\alpha \cup \tilde{\alpha}\}$ and let $(\tilde{N}, \tilde{p})$ be a regular neighbourhood of $\tilde{\Sigma} P'$ in P' (remark: we shall define $\tilde{\Sigma} P$ as $f'(\tilde{\Sigma} P')$). $\dot{\tilde{N}}$ is the disjoint union of a finite number of circles $S_1, \dots, S_k$.

Notation: let us fix an orientation of S and $\tilde{\Sigma} P'$; by $S = \alpha_{i_1}^{\pm 1} \dots \alpha_{i_k}^{\pm 1}$ we shall mean that we can divide S into k arcs such that the map $\tilde{p}|\colon S \to \tilde{\Sigma} P'$, when restricted to the j-th arc, is an homeomorphism with the arc $\alpha_{i_j} \subset \tilde{\Sigma} P'$, conserving (resp. inverting) the orientation if the exponent is 1 (resp. -1).

By construction we have then, up to a permutation of the circles S_i,

$$S_1 = \alpha\tilde{\alpha}\,, \qquad S_2 = \alpha\tilde{\alpha}(\alpha\alpha^{-1})^{h_2}$$

$$S_i = (\alpha\alpha^{-1})^{h_i}\,, \qquad \text{for each } i = 3, \dots, k$$

and $h_2 + h_3 + \dots + h_k = \big((4n+2) - 2\big)/2 = 2n$.

For each $i = 1, \dots, k$, consider a manifold $M^{(i)} = M_j^{(i)}$, where

$j = 0$, if $i = 1$ or $i > 2$ and h_i is even;

$j = 1$, if $i = 2$ and h_2 is even or if $i > 2$ and h_i is odd;

$j = 2$, if $i = 2$ and h_2 is odd.

Let $\partial M_j^{(i)} = S_i' \cup S_1^{(i)} \cup \dots \cup S_{j+1}^{(i)}$ and identify S_i' with S_i; let $\tilde{\mathcal{F}} = \{\gamma_l^{(i)}\}$ be the family of circles in general position in $M_j^{(i)}$ constructed as in 2.2 with respect to S_i'; $\tilde{V} = \bigcup_i M_j^{(i)}$ and $\tilde{q}: \tilde{V} \to \tilde{\mathcal{F}}$ be the retraction such that $\tilde{V}$ is the mapping cylinder of $\dot{\tilde{q}} = \tilde{q}|_{\partial\tilde{V}}$.

It is clear then that, following 2.6, we can define a map $\tilde{\varphi}: \tilde{\mathcal{F}} \to \tilde{\Sigma}P'$ such that:

a) $\tilde{\varphi}\circ\dot{\tilde{q}}|_{S_i'}$ is specially homotopic to $\tilde{p}|_{S_i}$, up to the given identification;

b) $\tilde{\varphi}\circ\dot{\tilde{q}}|_{S_1^{(i)}} = \begin{cases} (\alpha\alpha^{-1})^{h_i} & \text{if } h_i \text{ is even, } i \geqslant 2 \\ (\alpha\alpha^{-1})^{h_i - 1} & \text{if } h_i \text{ is odd, } i \geqslant 2; \end{cases}$

c) $\tilde{\varphi}\circ\dot{\tilde{q}}|_{S_2^{(i)}} = \alpha\alpha^{-1}$ if h_i is odd, $i \geqslant 2$;

d) $\tilde{\varphi}\circ\dot{\tilde{q}}|_{S_1^{(1)}} = \tilde{\varphi}\circ\dot{\tilde{q}}|_{S_r^{(2)}} = \alpha\tilde{\alpha}$, where $r = 2$ if h_2 is even and $r = 3$ if h_2 is odd.

Note that, as $h_2 + \dots + h_k$ is even, the number of circles $S_j^{(i)}$ such that $\tilde{\varphi}\circ\dot{\tilde{q}}|_{S_j^{(i)}} = \alpha\alpha^{-1}$ is even.

We shall outline now the modifications to make on $\tilde{V}$, $\tilde{\mathcal{F}}$, $\tilde{\varphi}$ to get a manifold V, a family $\mathcal{F}'$ of circles in general position in V and a map $\varphi: \mathcal{F}' \to \tilde{\Sigma}P'$ satisfying the required properties:

1) if $\tilde{\varphi}\circ\dot{\tilde{q}}|_{S_j^{(i)}} = (\alpha\alpha^{-1})^{2s}$, add a Moebius band M to V, identifying its boundary with $S^{)i)}$; change then the circle $\tilde{\gamma} \in \mathcal{F}$ «associated» to $S_j^{i)}$ to get a new circle γ as in 2.6, case *b)*, and define $\varphi|_\gamma = (\alpha\alpha^{-1})^s$;

2) if $\tilde{\varphi}\circ\dot{\tilde{q}}|_{S_j^{(i)}} = \tilde{\varphi}\circ\dot{\tilde{q}}|_{S_m^{(l)}} = \alpha\alpha^{-1}$, add a cylinder $S \times [0, 1]$ to $\tilde{V}$ identifying its boundary with $S_j^{(i)} \cup S_m^{(l)}$; take off then the two circles «associated» to $S_j^{(i)}$ and $S_m^{(l)}$, add a new circle γ as in 2.5 and define $\varphi|_\gamma = \alpha\alpha^{-1}$;

3) do the same as in 2) for $S_1^{(i)}$ and $S_2^{(r)}$, putting $\varphi|_\gamma = \alpha\tilde{\alpha}$.

We thus get a manifold V with $\partial V = S_1' \cup \dots \cup S_k'$, a family $\mathcal{F}'$ of circles in general position and a map $\varphi: \mathcal{F}' \to \tilde{\Sigma}P'$ such that, if $q: V \to \mathcal{F}'$ is the usual retraction, then $\varphi\circ\dot{q}|_{S_i'}$ is specially homotopic to $\dot{p}|_{S_i}$, up to the given identification (see fig. 7).

Put $P'' = \overline{P' \setminus \tilde{N}} \bigcup_{\tilde{N}} V$ and define $f'': P'' \to P'$ according to the mapping cylinder structures, as in 2.6. It is clear that P'' and f'' satisfy properties 4), 5), 6) and 8) of 2.8.

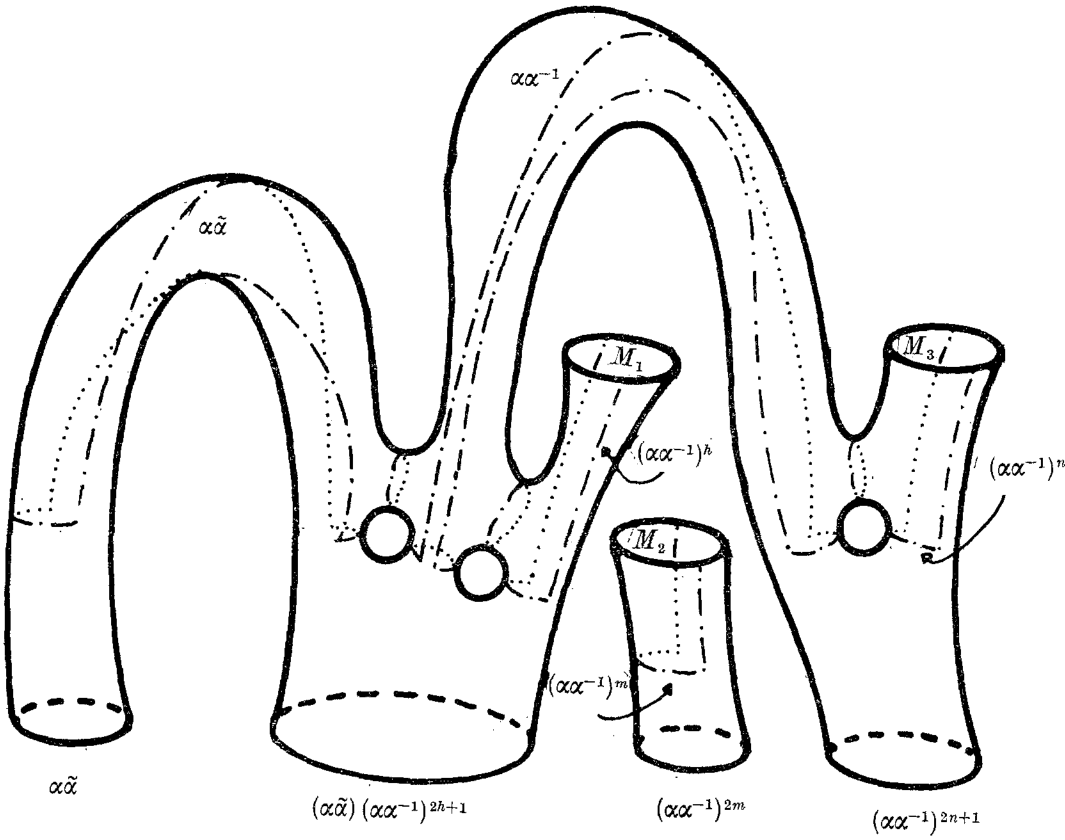


Figure 7

As for property 7), let us consider a wedge of circles $F \in \mathcal{F}$ with center in $x_1 \in \Sigma_0 P'$: $f''^{-1}(x_1)$ is the disjoint union of a finite number of points belonging to the circles $\gamma_1, \ldots, \gamma_{k-1}, \gamma'_k$ and of a finite number of the circles γ_j $(j \neq 1, \ldots, k-1)$.

Let $U = F \cap \mathring{N}$; $\dot{U}$ consists of an even number of points belonging to $S'_1, \ldots, S'_k$: It is clear then that (eventually changing V or φ, and with the same kind of arguments as those up to now used), we can achieve that $f''^{-1}(\mathcal{F}) \cup \{\gamma_j\}$ is a family of circles in general position.

c) assume now $\Sigma P' = S$, and $S \cap \Sigma_0 P' = \{x_1, \ldots, x_m\}$.

Let α_i $(i < m)$ be the arc of S with endpoints x_i and x_{i+1} and α_m the arc with endpoints x_m and x_1; let n_i be such that $y_i \in \alpha_i \Rightarrow \operatorname{lk}(y_i, P') = S^0 * \{4n_i \text{ points}\}$ (note that $n_i \neq n_{i+1}$).

Let (N', p') be a regular neighbourhood of $\Sigma P'$ in P' and $\dot{N}' = S_1 \cup \ldots \cup S_k$: We saw that $\operatorname{lk}(x_i, P')$ is a graph with two vertices y_{i-1} and y_i, $4n_{i-1}$ edges with endpoints y_{i-1} and y_i and a wedge of $2(n_i - n_{i-1})$ circles with center y_i. This ensures that the map $\dot{p}'$: $S_1 \cup \ldots \cup S_k \to S$ is described as follows (using the same notation as before):

$$S_i = S^{h_{i1}}(\alpha_1' \alpha_1'^{-1})^{h_{i2}} \dots (\alpha_{s-1}', \alpha_{s-1}'^{-1})^{h_{is}}$$

($i = 1, \dots, k$), where α_i' denotes a suitable arc of S (not necessarily $\alpha_i' = \alpha_i$) and, for each $j = 1, \dots, s$, $h_{1j} + h_{2j} + \dots + h_{kj}$ is an even number (because it is equal to the sum, or the difference, of some of the $4n_i$'s).

For each $i = 1, \dots, k$, consider a manifold $M_j^{(i)}$, where

$$j = \# \{r\colon h_{ir} \neq 0\} + \# \{r\colon h_{ir} \text{ is odd}\} - 1$$

and proceed then exactly as in the previous case.

Note only that, if $n_i = 0$ (as in this case we can't suppose, as in the $(A - B)$ case, that $\Sigma P' = \Sigma_1 P'$) V is the disjoint union of a circle $\tilde{S}$ and a two-dimensional manifold V' and the map $f''|_{\tilde{S}}$ is a homeomorphism between $\tilde{S}$ and S (otherwise f'' wouldn't be an epimorphism).

2.12 REMARKS AND EXAMPLES.

a) it is clear that in special cases one can give a simpler construction than the general one here described:

b) we finish this paragraph with some figures explaining all the steps of the constructions of 2.3 and 2.8:

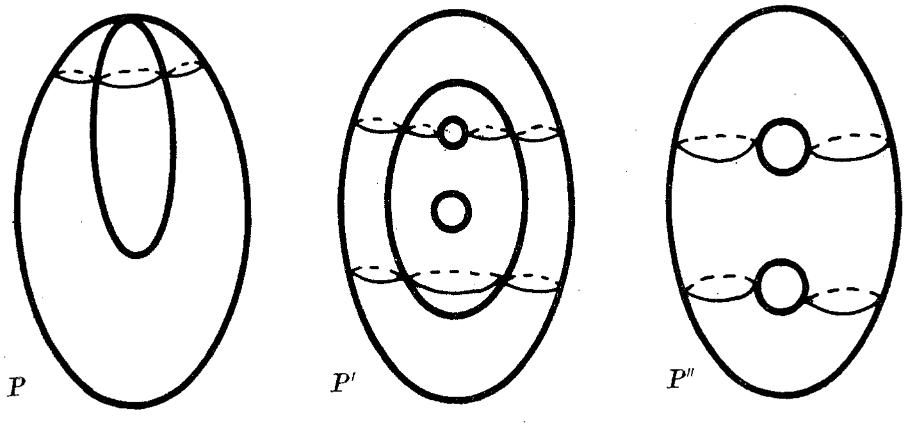


Figure 8

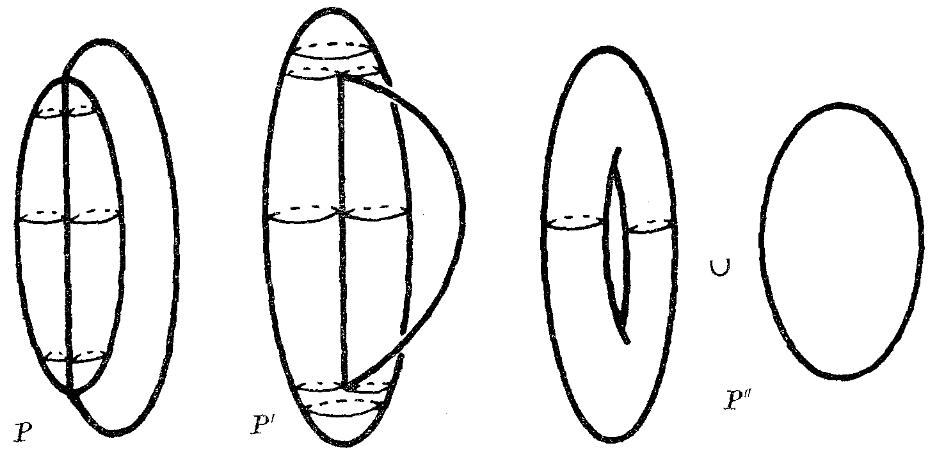


Figure 9

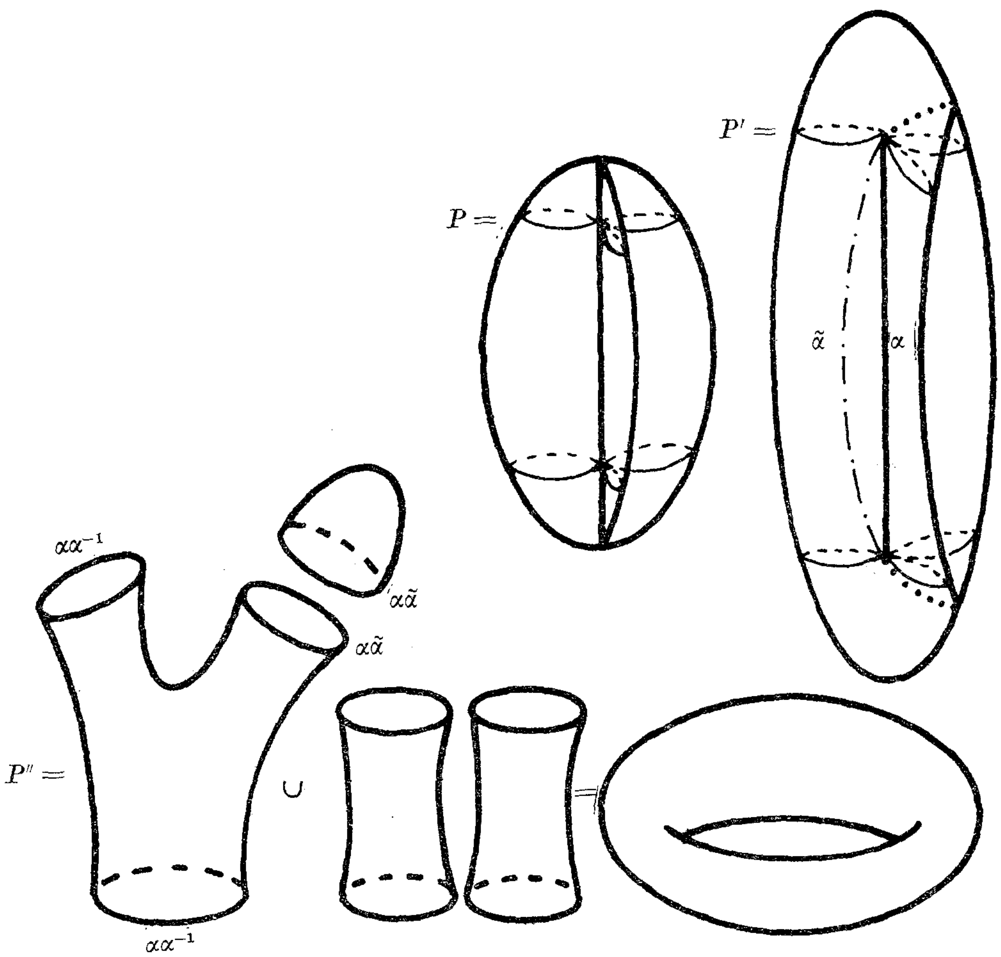


Figure 10

3. – Polynomial equations.

We first recall some facts (see [2] and [3] for precise definitions). We say that V is an algebraic variety iff

$$V = \{x \in \boldsymbol{R}^n\colon P_1(x) = P_2(x) = \ldots = P_k(x) = 0,\ P_i \in \boldsymbol{R}[x_1, \ldots, x_n],\ i = 1, \ldots, k\}$$

$I(V) \subset \boldsymbol{R}[x_1, \ldots, x_n]$ is the ideal of polynomials vanishing on V.

Let $W \subset \boldsymbol{R}^m$ be an other algebraic variety; a map $\varphi\colon V \to W$ is regular iff it is locally given (in the Zariski topology) by non singular rational functions. It is known that for such a φ there exists a regular extension $\Phi\colon \boldsymbol{R}^n \to \boldsymbol{R}^m$.

V is regular iff for every $x \in V$ there exist $Q_1, \ldots, Q_q \in I(V)$ such that $q = n - \dim V$ and $dQ_1, \ldots, dQ_q$ are linearly indipendent.

Let M be a smooth compact submanifold of $\boldsymbol{R}^n$, d the usual metric on $\boldsymbol{R}^n$, $G_{n,r}$ the Grassmann manifold of r-linear spaces in $\boldsymbol{R}^n$ and d' a metric on $G_{n,r}$ which induces the usual topology. Then, for every $\varepsilon > 0$, a submanifold M' of $\boldsymbol{R}^n$ is an ε-approximation of M in $\boldsymbol{R}^n$ iff there exists a diffeomorphism $h\colon M \to M'$ such that:

(i) $d(x, h(x)) < \varepsilon$;

(ii) $d'(TM_x, TM'_{h(x)}) < \varepsilon$,

where TM_x and $TM'_{h(x)}$ are the linear tangent varieties to M in x and to M' in $h(x)$ respectively. If ε is small enough, there exists a (small) isotopy $H_t\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ such that $H_0 = id$, $H_1|_M = h$ and H_t is the identity outside a fixed compact neighbourhood K of M, $K \supset M'$ (see [2]). The set of differentiable maps between manifolds is endowed with the Whitney topology.

Let $X \subset V$ and Y be topological spaces and $\varphi\colon X \to Y$ be a map. We denote by $Q(V, \varphi, Y)$ the quotient space $V \coprod Y/\sim$, where $x \sim y$ iff: (i) $x = y$; (ii) $x, y \in X$ and $\varphi(x) = \varphi(y)$; (iii) $Y \ni y = \varphi(x)$.

We can now state the main results of this paper:

3.1 THEOREM. – Let P be a good compact two-dimensional stratified space satisfying (A) and (B). Then there exists a homeomorphism $g\colon P \to \hat{P}$, where $\hat{P}$ is real algebraic and the real algebraic singularities of $\hat{P}$ are equal to $g(\Sigma P)$.

3.2 THEOREM. – Let P be as before, satisfying (E) instead of (A) and (B). Then there exists a homeomorphism $g\colon P \to \hat{P}$, where $\hat{P}$ is real algebraic and the algebraic singularities of $\hat{P}$ are equal to $g(\tilde{\Sigma} P)$ (see 2.8, 5)).

PROOFS OF 3.1 AND 3.2. – We can suppose there are no isolated points in P (actually, the theorems hold for P if and only if they hold for $P \setminus \{\text{isolated points}\}$).

Fix an $(A - B)$ (or (E)) special resolution of the singularities of P: $P'' \xrightarrow{f''} P' \xrightarrow{f'} P$;

we shall always write $\tilde{\Sigma}P, \tilde{\Sigma}P'$, meaning that in the first case $\tilde{\Sigma}P = \Sigma P$ and $\tilde{\Sigma}P' = \Sigma P'$.

Let $\mathcal{F} = \{F_r\} = f'^{-1}(\Sigma_0 P)$; then $P = f'(P')$ is naturally homeomorphic to $Q(P', f'|_{\mathcal{F}}, \Sigma_0 P)$ and $\tilde{\Sigma}P$ to $Q(\tilde{\Sigma}P', f'|_{\mathcal{F}\cap\tilde{\Sigma}P'}, \Sigma_0 P)$.

P'' consists of the disjoint union of one- and two-dimensional manifolds $N = \{N_1, ..., N_r\}$ and $M = \{M_1, ..., M_s\}$ respectively.

$$f''^{-1}(\tilde{\Sigma}P') = \big(f''^{-1}(\tilde{\Sigma}P') \cap M\big) \cup N = \{S_p\} \cup N\,,$$

where $S = \{S_p\}$ is a finite family of circles in general position in M.

$$f''^{-1}(\mathcal{F}) = \big(f''^{-1}(\mathcal{F}) \cap M\big) \cup \big(f''^{-1}(\mathcal{F}) \cap N\big) = \{\tilde{S}_q\} \cup \{q_j\}\,,$$

where $\tilde{S} = \{\tilde{S}_q\}$ is a finite family of circles in M such that $S \cup \tilde{S}$ is in general position and $Q = \{q_j\}$ consists of a finite number of points.

There exists a natural relative homeomorphism between

$$(P', \mathcal{F}) = \big(f''(P''), f''(\tilde{S} \cup Q)\big)$$

and

$$\Big(Q(P'', f''|_{S\cup N}, \tilde{\Sigma}P'),\ Q\big(\tilde{S} \cup Q, f''|_{(S\cup N)\cap(\tilde{S}\cup Q)}, \mathcal{F} \cap \tilde{\Sigma}P'\big)\Big)\,.$$

We can assume that P'', P' and P are realized in three copies of an $\boldsymbol{R}^A$, for a big A. $\tilde{\Sigma}P'$ consists of a finite number of smooth circles of $\boldsymbol{R}^A$: $\tilde{\Sigma}P' = C = \{C_1, ..., C_k\}$; $C \cap \mathcal{F}$ is a finite number of points.

Let us approximate every C_i with a regular algebraic curve C_i' such that, if $h_i\colon C_i \to C_i'$ is the related diffeomorphism, then: $h_i|_{C_i\cap\mathcal{F}} = id$; there exist isotopies H_t^i of $\boldsymbol{R}^A$ such that $H_0^i = id$, $H_1^i|_{C_i} = h_i$, and $H_t^i|_{C_i\cap\mathcal{F}} = id$, for each t; every H_t^i has a compact support K_i which is a neighbourhood of C_i and $K_i \cap K_j = \emptyset$ if $i \neq j$ (see [3]). Then the H_t^i's define a global isotopy H_t of $\boldsymbol{R}^A$.

We can construct a special resolution $P'' \xrightarrow{g''} \tilde{P}' \xrightarrow{g'} P$ of P such that $\tilde{P}' = H_1(P')$; $\tilde{\Sigma}\tilde{P}' = C' = \{C_i'\}$;

$$g'^{-1}(\Sigma_0 P) \cap C' = f'^{-1}(\Sigma_0 P) \cap C; \qquad g'' = H_1 \circ f''; \qquad g' = f' \circ H_1^{-1}\,.$$

Let $\tilde{\mathcal{F}} = \{\tilde{F}_r\} = \{H_1(F_r)\}$.

Clearly $(\tilde{P}', \tilde{\mathcal{F}}) = \big(g''(P''), g''(\tilde{S} \cup Q)\big)$ is homeomorphic to $(P', \mathcal{F})$ and to $\Big(Q(P'', g''|_{S\cup N}, \tilde{\Sigma}\tilde{P}'), Q\big(\tilde{S} \cup Q, g''|_{(S\cup N)\cap(S\cup Q)}, \tilde{\mathcal{F}} \cap \tilde{\Sigma}\tilde{P}'\big)\Big)$; P is homeomorphic to $Q(\tilde{P}', g'|_{\tilde{\mathcal{F}}}, \Sigma_0 P)$.

3.3 REMARK. – The unoriented smooth bordism of C', $\eta_*(C')$ is generated by algebraic elements ($\eta_j(C') = 0$ if $j \neq 0, 1$; $\eta_0(C')$ is generated by the classes [point $\to$ $\to C'$]; $\eta_1(C')$ is generated by the classes $[C_i' \hookrightarrow C']$).

If A is big enough, the following facts hold:

1) There exist approximations $h_M: M \to M'$, $h_N: N \to N'$ of M and N in $\mathbf{R}^A$ such that:

   (*a*) M' and N' are regular algebraic varieties;

   (*b*) for each p, $h_p = h_M|_{S_p}: S_p \to Z_p$ is an approximation of S_p in $\mathbf{R}^A$, where Z_p is regular algebraic and $h_p|_{S_p \cap \tilde{S}} = id$; $h_p|_{S_p \cap \{S_q, q \neq p\}} = id$. Let $Z = \{Z_p\}$.

   (*c*) For each q, $\tilde{h}_q = h_M|_{\tilde{S}_q}: \tilde{S}_q \to \tilde{Z}_q$ is an approximation of $\tilde{S}_q$ in $\mathbf{R}^A$, where $\tilde{Z}_q$ is regular algebraic and $\tilde{h}_q|_{\tilde{S}_q \cap S} = id$, $\tilde{h}_q|_{\tilde{S}_r \cap \{\tilde{S}r, r \neq q\}} = id$. Let $\tilde{Z} = \{\tilde{Z}_i\}$.

   (*d*) For each i, $k_i = h_N|_{N_i}: N_i \to N'_i$ is an approximation of N_i in $\mathbf{R}^A$, where N'_i is regular algebraic and $k_i|_{N_i \cap Q} = id$.

2) There exists a regular map $\Phi: Z \cup N' \to C'$ such that:

   (*a*) for each p, if $\varphi_p = \Phi|_{Z_p}$, then: (i) $\varphi_p \circ h_p$ approaches $g''|_{S_p}$; (ii) $\varphi_p|_{S_p \cap \tilde{S}} = g''|$; $\varphi_p|_{S_p \cap \{Z_q, q \neq p\}} = g''|$; (iii) $\varphi_p \circ h_p$ is specially homotopic to $g''|_{S_p}$; (iv) if $g''|_{S_p}$ is the constant map, then $\varphi_p = g''|_{S_p}$ is the constant map;

   (*b*) for each i, if $g_i = \Phi|_{N'_i}$, then: (i) $g_i \circ k_i$ approaches $g''|_{N_i}$; (ii) $g_i|_{N_i \cap Q} = g''$; (iii) $g_i \circ k_i$ is specially homotopic to $g''|_{N_i}$.

By means of remark 3.3, all these claims follow almost immediately from theorem 3 and proposition 1 (with its related lemma) of [3] (the same results and the same proofs are again in paragraph *e*) of [10]).

The only one which is not immediate to prove is the following: if $g''|_{S_p}: S_p \to C'_j$ is a 2-covering of an arc $A \subset C'_j$ with y_0 and y_1 as endpoints, which is branched on y_0 and y_1 in $x_0, x_1 \in S_p$, then $\varphi_p: Z_p \to C'_j$ has the same property (as x_0 and $x_1 \in S_p \cap \cap (\tilde{S} \cup S_p) \cap \{S_q, q \neq p\}$, they also belong to Z_p; therefore, the above statement is equivalent to the property 2 (iii) for $\varphi_p \circ h_p$).

We give here a sketch of the proof (using standard arguments: see [2], [3] and [11]).

Let U_i, V_i be neighbourhoods of x_i and y_i in $\mathbf{R}^A$ ($i = 0, 1$) such that $U_0 \cap U_1 = V_0 \cap V_1 = \emptyset$ and there exist diffeomorphisms $b_i: U_i \to B_i$, $d_i: V_i \to D_i$, where B_i and D_i are balls in $\mathbf{R}^A$ with the origin as center and:

$$b_i(S_p \cap U_i) = \{B_i \cap \{x_2 = \ldots = x_A = 0\}\};$$

$$d_i(C_j \cap V_i) = \{D_i \cap \{x_2 = \ldots = x_A = 0\}\}; \quad b_i(x_i) = 0 = d_i(y_i)\,.$$

We can assume that:

(1) $d_i \circ g'' \circ b_i^{-1}|_{b_i(S_p \cap U_i)} = (x_1^2, 0, \ldots, 0)$;

(2) $g''|_{S_p}$ is the restriction to S_p of a differentiable map $G = (G_1, \ldots, G_A)$ defined on a neighbourhood W of S_p in $\mathbf{R}^A$ such that $dG_j(x_i) = 0$, $i = 0, 1$, $j = 1, \ldots, A$.

By (1), every map near to $g''|_{S_p}$ has only two critical points lying in two fixed disjoint neighbourhoods W_i of x_i ($i = 0, 1$); then it is a 2-covering of another arc A' of C'_j, branched at the endpoints.

Let $P = (P_1, \dots, P_A)$ be polynomials of $\boldsymbol{R}[x_1, \dots, x_A]$ such that $P_j(x_i) = G_j(x_i)$ and $dP_j(x_i) = 0$, for each i and j. Then, by (2), for each j, if $H_j(x) = G_j(x) - P_j(x)$, then $H_j(x_i) = 0$ and $dH_j(x_i) = 0$.

Fix a compact neighbourhood K of S_p, $K \subset W$; there exists a finite open covering $\{U_s\}$ of K and polynomials $Q_1, \dots, Q_t$ such that $Q_r(x_i) = 0$, $dQ_r(x_i) = 0$ for each r $H_j(x) = \sum_{r=1}^{t} F_r^s(x) Q_r(x)$, $x \in U_s$, where every F_r^s is smooth on U_s.

Using a partition of unity (as in [11]) and the usual Weierstrass approximation theorem, we can approximate $H = (H_1, \dots, H_A)$ with $\tilde{P} = (\tilde{P}_1, \dots, \tilde{P}_A)$ where every $\tilde{P}_j$ is a polynomial and $\tilde{P}_j(x_i) = 0$, $d\tilde{P}_j(x_i) = 0$, for each i and j.

Then $\bar{P} = (\tilde{P}_1 + P_1, \dots, \tilde{P}_A + P_A)$ approaches G and $\bar{P}_j(x_i) = G_j(x_i)$, $d\bar{P}_j(x_i) = 0$. Now (see [3], lemma 2) Z_p (and φ_p) is obtained by generic projection on $\boldsymbol{R}^A$ of a regular algebraic copy $\bar{Z}_p$ of S_p in

$$\{(x, y) \in K \times \boldsymbol{R}^A : y = \pi(\bar{P}(x)) - \bar{P}(x)\},$$

where π is the projection of a tubular neighbourhood of C'_j in $\boldsymbol{R}^A$; $(x_i, 0) \in Z_p$, $i = 0, 1$; φ_p is given by the restriction to Γ_p of $\bar{P}(x) + y$, which is defined on $K \times \boldsymbol{R}^A$.

By means of our choice of $\bar{P}$, it is clear that the only critical points of $\varphi_p \circ h_p$ are x_0 and x_1, which is what we had to prove.

Let us return now to M', satisfying properties 1) and 2); it is not hard to prove that there exists a natural relative homeomorphism ϱ between $(\tilde{P}', \tilde{\mathcal{F}})$ and $(\hat{P}', \hat{\mathcal{F}}) = (Q(M' \cup N', \Phi, \tilde{\Sigma}\tilde{P}'), Q(\tilde{Z} \cup Q, \Phi|_{(Z \cup N') \cap (\tilde{Z} \cup Q)} = g''|_{(S \cup N) \cap (\tilde{S} \cup Q)}, \tilde{\mathcal{F}} \cap \tilde{\Sigma}\tilde{P}'))$ and there exists a homeomorphism between $g'(\tilde{P}') = P$ and $\hat{P} = Q(\hat{P}', g' \circ \varrho^{-1}|_{\hat{\mathcal{F}}}, \Sigma_0 P)$; therefore, the theorems 3.1 and 3.2 are proved by using twice the following proposition.

3.4 PROPOSITION. – Let $Z, X \subset V \subset \boldsymbol{R}^n$, $T \subset Y \subset \boldsymbol{R}^m$ be algebraic varieties and $\varphi: X \to Y$ a regular map such that $\varphi(Z \cap X) \subset T$. Suppose that V is compact. Then there exist an algebraic variety $W \subset \boldsymbol{R}^k \times \boldsymbol{R}^m$ and a regular map $\Phi: V \to W$ such that:

1) $W = \Phi(V) \cup \{0 \times Y\} = \Phi(V) \cup \hat{Y}$;

2) $\Phi(V) \cap \hat{Y} = \Phi(X)$;

3) $\Phi|: V \setminus X \to \Phi(V) \setminus \hat{Y}$ is an algebraic isomorphism;

4) $\Phi|_X = (0, \varphi)$;

5) $W' = \Phi(Z) \cup \{0 \times T\} = \Phi(Z) \cup \hat{T}$ is an algebraic subvariety of W.

PROOF. – Let $s: S^n \setminus p \to \boldsymbol{R}^n$ be the stereographic projection from the north pole and let i be its inverse. Put $\tilde{V} = i(V)$, $\tilde{Z} = i(Z)$, $\tilde{X} = i(X)$ and $\tilde{\varphi} = \varphi \circ s|_{\tilde{X}}$. Let $\Theta\colon V \to \boldsymbol{R}^m$ be a regular map which extends φ and put $\tilde{\Theta} = \Theta \circ s|_{\tilde{V}}$.

Choose a set of generators $\psi_1, \ldots, \psi_r$ of $I(X)$ and let

$$\tilde{\Psi} = (\psi_1, \ldots, \psi_r) \circ s|_{\tilde{V}}\colon \tilde{V} \to \boldsymbol{R}^r .$$

$$\Gamma_{\tilde{V}} = \{(x, y, z) \in S^n \times \boldsymbol{R}^r \times \boldsymbol{R}^m\colon x \in \tilde{V},\ y = \tilde{\Psi}(x),\ z = \tilde{\Theta}(x)\}$$

is isomorphic to $\tilde{V}$ and contains the subvarieties $\Gamma_{\tilde{Z}}$ and $\Gamma_{\tilde{X}}$ which are isomorphic to $\tilde{Z}$ and $\tilde{X}$ respectively. Clearly, it is enough to prove the proposition for $\Gamma_{\tilde{V}}$, $\Gamma_{\tilde{Z}}$, $\Gamma_{\tilde{X}}$, $\pi|\Gamma_{\tilde{X}}$ where $\pi\colon S^n \times \boldsymbol{R}^r \times \boldsymbol{R}^m \to \boldsymbol{R}^m$ is the natural projection.

Let $\overline{Y} = \{0 \times Y\} \subset \boldsymbol{R}^r \times \boldsymbol{R}^m$, $\overline{T} = \{0 \times T\} \subset \overline{Y}$ and $R(y, z)$, $S(y, z)$ be polynomials such that $\overline{Y} = \{R = 0\}$, $\overline{T} = \{S = 0\}$.

Let $F\colon \boldsymbol{R}^{n+1} \times \boldsymbol{R}^r \times \boldsymbol{R}^m \to \boldsymbol{R}^{n+1} \times \boldsymbol{R}^r \times \boldsymbol{R}^m$ be defined by $F(x, y, z) = \big(R(y, z)\, x, y, z\big)$.

Claim: $W = F(\Gamma_{\tilde{V}}) \cup \hat{Y}$, $W' = F(\Gamma_{\tilde{Z}}) \cup \hat{T}$ and $\Phi = F|\Gamma_{\tilde{V}}$ satisfy the required conditions:

1) $F|_{\tilde{X}} = (0, 0, \varphi)\colon \Gamma_{\tilde{X}} \ni x_0 = \big(x, 0, \tilde{\varphi}(x)\big)$; therefore $R\big(0, \tilde{\varphi}(x)\big) = 0$ and $F(x_0) = \big(0, 0, \tilde{\varphi}(x)\big)$.

2) If $x_0 \in \Gamma_{\tilde{V}}$ and $F(x_0) \in \hat{Y}$, then $x_0 \in \Gamma_{\tilde{X}}$: $\tilde{\Psi}(x_0) = 0$ implies $x_0 \in \Gamma_{\tilde{X}}$.

3) From 2) it follows that $\Gamma_{\tilde{V}} \setminus \Gamma_{\tilde{X}} = \Gamma_{\tilde{V}} \setminus \{(x, y, z)\colon R(y, z) = 0\}$ and the inverse of the isomorphism $F\colon \Gamma_{\tilde{V}} \setminus \Gamma_{\tilde{X}} \to W \setminus \hat{Y}$ is $(x', y', z') \to \big(x'/R(y', z'), y', z'\big)$.

It is now enough to prove that W is algebraic.

Since $\Gamma_{\tilde{V}} \subset S^n \times \boldsymbol{R}^r \times \boldsymbol{R}^m$, we can suppose that $\Gamma_{\tilde{V}} = \{P(x, y, z) = 0\}$ where

$$P(x, y, z) = \big(|x|^2 - 1\big)^t + \sum_{i=1,\ldots,s} P_i(x, y, z)$$

and each P_i is an homogeneous polynomial of degree $t_i < 2t$ with respect to x.

Using the isomorphism of 3), it is easy to see that

$$W \setminus \hat{Y} = \{\tilde{P} = 0,\ R(y, z) \neq 0\},$$

where

$$\tilde{P}(x, y, z) = \big(|x|^2 - R(y, z)^2\big)^t + \sum_i R(y, z)^{2t - t_i} P_i(x, y, z) .$$

But, if $\tilde{P}(x, y, z) = 0$ and $R(y, z) = 0$, then $x = 0$ and therefore $W = \{\tilde{P} = 0\}$. We can do the same for W', using $S(y, z)$, thus proving the proposition.

3.5 REMARK. – It is known that a topological space V has an algebraic structure if and only if its one point compactification, $\tilde{V} = V \cup \{\infty\}$, has one; moreover, for every structure on V, it is possible to get $\tilde{V}$ such that V and $\tilde{V}\setminus\{\infty\}$ are isomorphic. It follows that the previous proposition is true with the weaker hypothesis that only X is compact (see [1], [3]).

Furthermore, it is easy to prove the following

3.6 COROLLARY. – A two-dimensional (non compact) stratified space P is homeomorphic to an algebraic variety if and only if it is homeomorphic to $\bar{P}\setminus x\mathrm{lk}(x, \bar{P})$, where $\bar{P}$ is a compact two-dimensional stratified space satisfying (E).

PROOF. – It follows from theorems 3.1 and 3.2 and remark 3.5.

3.7 FINAL REMARKS.

a) The remark 3.3, which is obvious in this case, has been essentially used to apply the results of [3]. In order to generalize these results to higher dimensional spaces, we may need a theorem saying, roughly speaking, that every compact closed smooth manifold M in $\boldsymbol{R}^N$ (where N is big enough) can be approximated by a regular algebraic variety M' such that $\eta_*(M')$ has algebraic generators.

b) Every compact two-dimensional real analytic space is homeomorphic to an algebraic variety.

c) From the details of the proofs of the theorems 3.1 and 3.2 we can get informations about the irreducible algebraic components of $\hat{P}$ (besides informations about the singularities of $\hat{P}$); moreover, this points out that, « up to little modifications », every two-dimensional (compact) stratified space is homeomorphic to an algebraic variety.

d) It seems interesting to study the relations between the topological properties $(A-B)$ and the (coherence of the) analytic structure of P.

e) Let Q be an algebraic variety and $\mathrm{Sing}\, Q$ be the algebraic singularities of Q; if $\mathrm{Sing}^{(i)} Q$ is defined inductively by $\mathrm{Sing}^{(i)} Q = \mathrm{Sing}\,\mathrm{Sing}^{(i-1)} Q$, then the algebraic structure $\hat{P}$ has the property that the index j such that $\mathrm{Sing}^{(j)} \hat{P} = \emptyset$ is the least possible with respect to the topology of P.

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