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SEARCHING AROUND $H_*^{alg}(-)$

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Dedicated to the memory of Gus Efroymson

Let X be a compact connected smooth manifold without boundary. The aim of this short communication is to illustrate a geometric explanation of the isomorphism β: $T_q(X) \cong \sum_{i+j=q} H_i(X) \otimes T_j$, where $T_q(X)$ denotes the q-unoriented differentiable bordism group of X and $T_q = T_q$ (point). That is, for each $m \geqq 0$ fix a basis $B_m = \{|M_h^m|; h = 1, \ldots, d_m = \dim T_m\}$ of T_m and characteristic classes $s_q^m \in H^m(-, \mathbf{Z}_2)$ (that is, polynomials in the Stiefel-Whitney classes) such that the matrix of characteristic numbers $(c_{hk}) = (s_h^m(M_k^m), M_k^m)$ is the unit matrix. Let $z \in T_q(X)$ and fix any representative $P \rightarrow_f X$ of z. Define $\beta_h^m \in H_{q-m}(X)$ by $\beta_h^m = f_* D(s_h^m(P))$ (where D: $H^* \rightarrow H_*$ is the Poincaré duality).

Proposition. *The isomorphism β acts as follows*: a) $\beta(z) = \sum_{m,h} \beta_h^m \otimes |M_h^m|$; b) *if $P_h^m \rightarrow^{g(m,h)} X$ is any representative of β_k^m such that P_h^m is a smooth manifold and N_h^m is any representative of $|M_h^m|$, then*: $z = \sum_{m,h} |P_h^m \times N_h^m \rightarrow P_h^m \rightarrow^{g(m,h)} X|$.

Assuming furthermore that X is a non singular affine real algebraic variety, the above proposition, maybe, could help in detecting when such a z is algebraic, that is, admits a real algebraic representative. This problem, or more generally to count the algebraic bordism elements (and the algebraic homology cycles), even up to diffeomorphism, would be interesting from several points of view (see the papers in this volume by Akbulut, Bochnak, King, Tognoli). A trivial consequence is the following.

Corollary. $z = |P \rightarrow X|$ *is algebraic iff $f_*D(s)$ is an algebraic cycle for every characteristic class s of the tangent bundle of P.*

Assuming that $P \rightarrow X$ as before smoothly fibres over X, one can prove the following.

Proposition. *Let $w^k(P)$ be the k-th-Stiefel-Whitney class of the tangent bundle of P. Then $f_*D(w^k(P)) = \chi(F)D(w^{k-h}(X))$, where F is the fibre and $h = \dim F$.*

PROBLEM. Let s be as in the corollary and (P, f) be as in the last proposition. Is $f_* D(s) = D(p(c_1, \ldots, c_k))$, where p is a polynomial and each c_j is a Stiefel-Whitney class over X (not necessarily of its tangent bundle)? If yes, one would have that every bordism class which fibres; over X is algebraic over the same suitable algebraic variety X' diffeomorphic to X.

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