ARTICLE IN PRESS

Available online at www.sciencedirect.com

SciVerse ScienceDirect

Expo. Math. ▮ (▮▮▮▮) ▮▮▮–▮▮▮

EXPOSITIONES MATHEMATICAE

www.elsevier.de/exmath

The topology of Helmholtz domains

R. Benedetti[a], R. Frigerio[a,*], R. Ghiloni[b]

[a] *Dipartimento di Matematica, Università di Pisa, Largo B. Pontecorvo 5, 56127 Pisa, Italy*
[b] *Dipartimento di Matematica, Università di Trento, Via Sommarive 14, 38123 Povo, Italy*

Received 2 March 2010; received in revised form 2 August 2012

Abstract

The use of cuts along surfaces for the study of domains in Euclidean 3-space widely occurs in the theoretical and applied literature about *electromagnetism*, *fluid dynamics* and *elasticity*. This paper is aimed at discussing techniques and results of 3-dimensional topology that provide an appropriate theoretical background to the method of cuts along surfaces. We consider two classes of bounded domains that become "simple" after a finite number of cuts along disjoint properly embedded surfaces. The difference between the two classes arises from the different meanings that the word "simple" may assume, when referred to spatial domains. In the definition of *Helmholtz* domain, we require that the domain may be cut along disjoint surfaces into pieces such that any curl-free smooth vector field defined on a piece admits a potential. On the contrary, in the definition of *weakly Helmholtz* domain we only require that a potential exists for the restriction to every piece of any curl-free smooth vector field defined on the whole initial domain. We use classical and rather elementary facts of 3-dimensional geometric and algebraic topology to give an exhaustive description of Helmholtz domains, proving that their topology is forced to be quite elementary: in particular, Helmholtz domains with connected boundary are just possibly knotted handlebodies, and the complement of any nontrivial link is not Helmholtz. The discussion about weakly Helmholtz domains is more advanced, and their classification is a more demanding task. Nevertheless, we provide interesting characterizations and examples of weakly Helmholtz domains. For example, we prove that the class of links with weakly Helmholtz complement coincides with the well-known class of *homology boundary links*.
© 2012 Elsevier GmbH. All rights reserved.

MSC 2010: primary 57-02; secondary 57M05; 57M25; 57R19; 76-02

Keywords: Method of cutting surfaces; Hodge decomposition; Homology boundary link; Corank; Cut number

* Corresponding author.
E-mail addresses: benedett@dm.unipi.it (R. Benedetti), frigerio@dm.unipi.it (R. Frigerio), ghiloni@science.unitn.it (R. Ghiloni).

0723-0869/$ - see front matter © 2012 Elsevier GmbH. All rights reserved.
doi:10.1016/j.exmath.2012.09.001

1. Introduction

Hodge decomposition is an important analytic structure widely occurring in the theoretical and applied literature on *electromagnetism*, *fluid dynamics* and *elasticity* in domains of the Euclidean space $\mathbb{R}^3$. In [8], one can find a friendly introduction to this topic, including a historical account concerning its origins. In 1858, Helmholtz [14] first investigated the motion of an ideal fluid in a spatial domain Ω without assuming that its velocity field V admits a potential function. He defined the notion of curl of V to measure the local rotation of the fluid and proved that every vector field on Ω decomposes in its curl-free and divergence-free parts. This is the first manifestation of the Hodge decomposition theorem. In his study, Helmholtz recognized the importance of the role played by the topological properties of Ω. He defined simply-connected and multiply-connected 3-dimensional domains, extending the corresponding notions used by Riemann for surfaces. In particular, the concept of multiply-connectedness for Ω is given in terms of the maximum number of cross-sectional surfaces $(\Sigma, \partial\Sigma)$ in $(\overline{\Omega}, \partial\Omega)$ that one can "cut away" from Ω without disconnecting it. These topological notions were developed by Thomson and reconsidered by Maxwell in the study of fluid dynamics and electromagnetism. Quoting from p. 439 of [8]:

"Thomson introduced an embryonic version of the one-dimensional homology $H_1(\Omega)$ in which one counted the number of "irreconcilable" closed paths inside the domain Ω. This was subject to the standard confusion of the time between homology and homotopy of paths: homology was the appropriate notion in this setting, but the definitions were those of homotopy. He also introduced a primitive version of two-dimensional relative homology $H_2(\overline{\Omega}, \partial\Omega)$ in which one counted the maximum number of "barriers", meaning cross-sectional surfaces $(\Sigma, \partial\Sigma) \subset (\overline{\Omega}, \partial\Omega)$, that one could erect without disconnecting the domain Ω. Thomson pointed out that while these barriers might be disjoint in simple cases, in general one must expect them to intersect one another."

Thomson's insight concerning the fact that, in general, one must expect the "barriers" to intersect one another was almost completely disregarded in the literature on electromagnetism, fluid dynamics and elasticity, leading to the extensively used *method of cutting surfaces* (see "Section A" of our References for a selection of titles).

The method of cutting surfaces is aimed at providing an effective construction of a basis of the space of harmonic vector fields on the domain Ω. Such a space appears as a summand in the Hodge decomposition of the space of vector fields and contains a lot of topological information about Ω. We refer the reader to Section 4.1 for more information about the method of cutting surfaces in the standard setting of square-summable vector fields.

In this paper we discuss in detail the class of domains to which this method can be successfully applied. We call such domains *Helmholtz domains*. The characterizing property of Helmholtz domains is that they become "simple" after a finite number of cuts along *pairwise disjoint* surfaces. It turns out that there is a bit of indeterminacy in the literature about the right meaning of "simple" in this definition. Requiring the domain to be *simply connected* certainly suffices. However, the (possibly weaker) condition consisting in the *existence of potentials for curl-free smooth vector fields* is more appropriate with respect to the usual applications of the method. Apparently, the relationship between

Please cite this article in press as: R. Benedetti, et al., The topology of Helmholtz domains, Expo. Math. (2012), doi:10.1016/j.exmath.2012.09.001

these (a priori different) notions is not widely well-established. One could say that the confusion of the early times between homology and homotopy somehow propagated until now, sometimes giving rise to true misunderstandings (see e.g. Example 3.6).

The first aim of this expository paper is to provide a complete description of the topology of Helmholtz domains. We achieve this result just by applying a few classical results of 3-dimensional topology. It is worth recalling that spatial domains (whose study includes, for example, *knot theory*) represent a nontrivial instance of 3-dimensional manifolds. Since Poincaré's *Analysis Situs* (1895) (see [41] for a useful historical account), ideas and techniques of (3-dimensional) *geometric and algebraic topology* have been developed and successfully applied to the study of spatial domains.

Theorem 3.2 shows that, under mild assumptions on the boundary (e.g. when the boundary is Lipschitz regular), the homological and the homotopical notions of "simplicity" mentioned above are indeed equivalent to each other. Moreover, it turns out that simple domains admit a clear and easy description: they are just the complement of a finite number of disjoint closed balls in a larger open ball. For domains with polyhedral boundary, this result is due to Borsuk [28] (1934). The (more general) case of domains with *locally flat* topological boundary is settled here thanks to later deep results that will be recalled in Theorem 2.8. Our proof is based on elementary properties of the *Euler–Poincaré characteristic* of compact surfaces and 3-manifolds and (like in [28]) eventually reduces to the celebrated Alexander Theorem [23] (1924), that ensures that every locally flat 2-sphere in $\mathbb{R}^3$ bounds a 3-ball. In [39] (1948), Fox obtained Borsuk Theorem as a corollary of his *re-embedding theorem* (see Section 4.4). However, Fox's arguments are admittedly inspired by Alexander's results and techniques.

Once simple domains have been completely described, it is rather easy to give an exhaustive characterization of general Helmholtz domains (see Theorem 4.5). In a sense, this is a disappointing result, since it shows that the topology of Helmholtz domains is forced to be quite elementary. For example, Helmholtz domains with connected boundary are just (possibly knotted) handlebodies, and the complement of any nontrivial link is not Helmholtz. This seems to suggest that the range of application of the method of cutting surfaces is quite limited.

In Section 5, we introduce and discuss the strictly larger class of *weakly Helmholtz* domains. Roughly speaking, such a domain can be cut along a finite number of disjoint surfaces into subdomains on which every curl-free smooth vector field that is defined on the *whole* original domain admits a potential. We believe that this requirement naturally weakens the Helmholtz condition, thus allowing to apply the method of cutting surfaces to topologically richer classes of domains. Unlike in the case of Helmholtz domains, we are not able to give an exhaustive classification of weakly Helmholtz ones. However, we provide several interesting characterizations of weakly Helmholtz domains. In particular and remarkably, we prove that the class of links with weakly Helmholtz complement coincides with the class of *homology boundary links*. In particular, every knot and every boundary link has weakly Helmholtz complement. Homology boundary links are broadly studied in knot theory, and it is a nice occurrence that the method of cutting surfaces naturally leads to this distinguished class of links.

Paper [13] is a sort of complement to the present one. It deals with a rather explicit description of the Hodge decomposition of the space of L^2-vector fields on any bounded

Please cite this article in press as: R. Benedetti, et al., The topology of Helmholtz domains, Expo. Math. (2012), doi:10.1016/j.exmath.2012.09.001

domain of $\mathbb{R}^3$ with sufficiently regular boundary, and it does not make use of methods relying on cutting surfaces. In paper [1], taking inspiration from the results of [13], the authors devise an efficient algorithm computing a finite element discrete basis of the space of harmonic vector fields on a general domain Ω, without the assumption that Ω is Helmholtz.

Fox re-embedding theorem (see Theorem 4.9) implies that the study of spatial domains may be often reduced to the study of knotted handlebodies in Euclidean 3-space. Recent results about knotted handlebodies may be found in [26], where a thorough discussion of handlebodies with weakly Helmholtz complements is carried out.

We stress that, from the viewpoint of 3-dimensional topology, most results of this paper follow from direct applications of classical and well-known facts of differential/algebraic/geometric topology, that are usually covered by basic courses on these subjects. Accordingly, "Section B" of our References contains well-established books on these subjects (see e.g. [42,61]). The classical results described in such books are sufficient to our needs, and will be recalled time by time just when they are needed. We mentioned above that the discussion about Helmholtz domains only relies on simple facts about the Euler–Poincaré characteristic (see Section 3.4), together with Alexander Theorem. Very clear and accessible proofs of this last result are available (e.g. in [48]). The discussion about weakly Helmholtz domains is a bit more advanced. More information on the algebraic topology of spatial domains is developed in Section 5.2, where we make an intensive use of Poincaré–Lefschetz duality, and in Section 6.2.

We hope that this paper could be helpful to people interested in the research areas mentioned at the beginning of this introduction. The role of the (algebraic) topology of domains had already been stressed in [8,15] (for example in order to justify the dimension of the summands of the Hodge decomposition). Hopefully, the present work should supplement the papers just mentioned, by unfolding some aspects of 3-dimensional topology underlying the method of cutting surfaces.

2. Domains

Let us first introduce a bit of terminology. In what follows, *smooth* maps (whence, in particular, diffeomorphisms) and smooth manifolds are assumed to be of class C^∞. In the literature, the terms "disk" and "ball" are often used without distinction. We prefer here to use both terms, and we agree that a disk is closed and a ball is the open interior of a disk. More precisely:

Definition 2.1. Let (x_1, x_2, x_3) be the usual coordinates of $\mathbb{R}^3$ and let D^3 be the standard 3-disk $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 \leq 1\}$ of $\mathbb{R}^3$. We also denote by D^2 the standard 2-disk in the plane $\mathbb{R}^2$ defined by $D^2 := D^3 \cap \mathbb{R}^2$, where we identify $\mathbb{R}^2$ with the plane $x_3 = 0$ of $\mathbb{R}^3$. A subset X of a manifold M homeomorphic to $\mathbb{R}^3$ is a (*topological*) 3-*disk* if, up to homeomorphism, the pair (M, X) is equivalent to $(\mathbb{R}^3, D^3)$, i.e. there exists a homeomorphism $\psi : M \longrightarrow \mathbb{R}^3$ such that $\psi(X) = D^3$. A (*topological*) 3-*ball* of M is the internal part of a 3-disk. We say that a subset Y of M is a (*topological*) 2-*disk* if, up to homeomorphism, the pair (M, Y) is equivalent to $(\mathbb{R}^3, D^2)$. Smooth disks

Please cite this article in press as: R. Benedetti, et al., The topology of Helmholtz domains, Expo. Math. (2012), doi:10.1016/j.exmath.2012.09.001

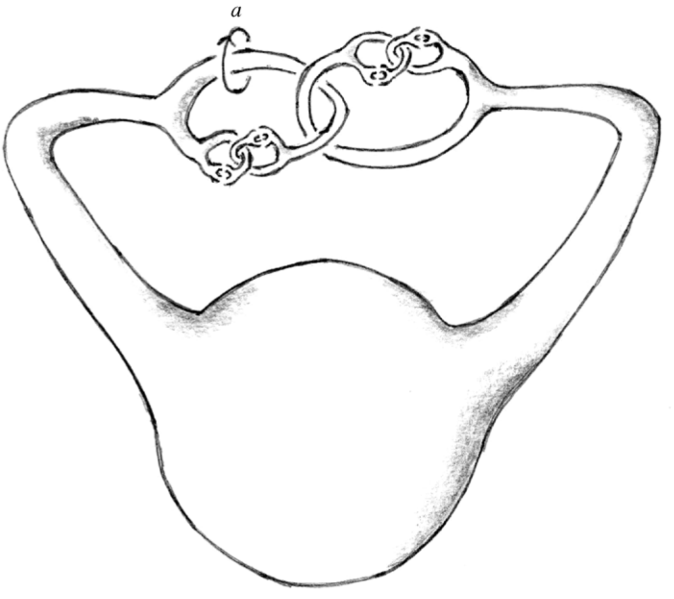


Fig. 1. The Alexander horned sphere S_{Alex}. The loop a is not homotopically trivial in $\mathbb{R}^3 \setminus S_{\mathrm{Alex}}$.

or balls in a smooth M diffeomorphic to $\mathbb{R}^3$ are defined in the same way by replacing "homeomorphism" with "diffeomorphism". Disks and balls in an arbitrary 3-manifold W are contained, by definition, in some chart M homeomorphic (or diffeomorphic, in the smooth case) to $\mathbb{R}^3$.

A *domain* in $\mathbb{R}^3$ is a nonempty connected open set $\Omega \subset \mathbb{R}^3$ such that $\mathrm{Int}\,\overline{\Omega} = \Omega$ (i.e. Ω coincides with the interior of its closure in $\mathbb{R}^3$). Moreover, throughout the whole paper, domains are assumed to be *bounded*, whence with compact closure.

Sometimes it is convenient to identify $\mathbb{R}^3$ with an open subset of the 3-sphere $S^3 = \mathbb{R}^3 \cup \{\infty\}$ via the stereographic projection from the point "at infinity". A nonempty connected open subset Ω of S^3 is a domain if $\mathrm{Int}\,\overline{\Omega} = \Omega$. Of course every domain in S^3 has compact closure, and the stereographic projection induces a bijection between domains in $\mathbb{R}^3$ and domains in S^3 whose closure does not contain the point at infinity.

We denote by $\partial\Omega$ the usual (topological) boundary of Ω, i.e. the set

$$\partial\Omega = \overline{\Omega} \setminus \Omega.$$

It turns out (see e.g. Remark 3.8) that domains with "wild" boundary can display pathological behaviors that we would like to exclude from our investigation. Therefore, we will concentrate our attention on domains with "tame" boundary, carefully specifying what "tame" means in our context. However, before going on we would like to mention what is probably the most famous example of wild surface in S^3: the Alexander horned sphere, which is described in Fig. 1.

The Alexander horned sphere is a subset $S_{\mathrm{Alex}} \subset \mathbb{R}^3$ homeomorphic to the 2-dimensional sphere. It was constructed by Alexander in 1924 (see [24]), and it is one

Please cite this article in press as: R. Benedetti, et al., The topology of Helmholtz domains, Expo. Math. (2012), doi:10.1016/j.exmath.2012.09.001

of the most famous pathological examples in mathematics. The celebrated (and quite sophisticated) Jordan–Brouwer Separation Theorem (see [30]) asserts that every topological 2-sphere S embedded in $\mathbb{R}^3$ disconnects $\mathbb{R}^3$ in two components, and each of these components is in fact a domain, according to our definition. Moreover, exactly one of these components is unbounded. Among the peculiar properties of the Alexander horned sphere, there is the remarkable fact that the unbounded component of $\mathbb{R}^3 \setminus S_{\mathrm{Alex}}$ is *not* simply connected (see Section 2.6 for a brief discussion of this notion).

2.1. Smooth surfaces, tubular neighborhoods and separation theorems

We begin by defining the tamest class of domains one could consider. A *smooth surface* S in $\mathbb{R}^3$ is a compact and connected subset of $\mathbb{R}^3$ such that the following condition holds: for every point $p \in S$, there exist a neighborhood U_p of p in $\mathbb{R}^3$ and a diffeomorphism $\varphi : U_p \longrightarrow \mathbb{R}^3$ such that $\varphi(U_p \cap S) = P$, where P is an affine plane. In other words, $S \subset \mathbb{R}^3$ is a smooth surface if the pair $(\mathbb{R}^3, S)$ is locally modeled, up to diffeomorphism, on the pair $(\mathbb{R}^3, \mathbb{R}^2)$. For $i = 1, 2, 3$, let $H_i := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_i = 0\}$, where (x_1, x_2, x_3) are linear coordinates on $\mathbb{R}^3$. By the Inverse Function Theorem, S is a smooth surface if and only if it is locally the graph of a real smooth function defined on an open subset of some H_i.

We have already mentioned that the Jordan–Brouwer Separation Theorem ensures that every *topological* 2-sphere S embedded in S^3 disconnects S^3 in two domains. Using Alexander duality (see e.g. [47, Theorem 3.44]), it is possible to generalize this result to the case of embedded *topological* compact surfaces. In Proposition 2.2 we show how, in the less general case of smooth surfaces, a much easier proof of this result may be obtained using *transversality*, which is one of the main tools of differential topology. To this aim we first introduce the notion of *tubular neighborhood*. We refer the reader, for instance, to [65,50] for a proof of the existence of tubular neighborhoods.

Let S be a compact smooth surface in S^3. For every $\epsilon > 0$, let us define the ϵ-*neighborhood* $N_\epsilon(S)$ of S in $\mathbb{R}^3$ by setting

$$N_\epsilon(S) := \{x \in \mathbb{R}^3 \mid \mathrm{dist}(x, S) \leq \epsilon\}.$$

If ϵ is small enough, then there exists a natural retraction $r : N_\epsilon(S) \longrightarrow S$ such that $r(x)$ is the nearest point to x in S, and the pair $(N_\epsilon(S), S)$ is locally modeled, up to diffeomorphism, on the pair $(\mathbb{R}^2 \times [-1, 1], \mathbb{R}^2)$ (we will see in Section 2.2 that the pair $(N_\epsilon(S), S)$ is in fact globally diffeomorphic to $(S \times [-1, 1], S)$). Moreover, for every $x \in S$, the set $r^{-1}(x)$ is a straight copy of $[-\epsilon, \epsilon]$ intersecting S in x and $\partial N_\epsilon(S)$ exactly in its endpoints. If ϵ is such that the properties just described are satisfied, then $N_\epsilon(S)$ is a tubular neighborhood of S.

Proposition 2.2. *Every smooth surface S in $\mathbb{R}^3$ disconnects S^3 in two domains $\Omega(S)$ and $\Omega^*(S)$.*

Proof. Let us fix a point $p \in S$ and a tubular neighborhood $N_\epsilon(S)$ of S, and set $\{p_1, p_2\} = r^{-1}(p) \cap \partial N_\epsilon(S)$. We first show that every point $x \in N_\epsilon(S) \setminus S$ may be joined either to p_1 or to p_2 by a path in $N_\epsilon(S) \setminus S$. Let $q = r(x)$. Recall that $r^{-1}(q)$ is a closed interval intersecting S in one point, so it is possible to join x to a point $x' \in r^{-1}(q) \cap \partial N_\epsilon(S)$ by a path α in $N_\epsilon(S) \setminus S$. Since S is connected, there exists a path γ on S joining q to p. Using that $(N_\epsilon(S), S)$ is locally modeled, up to diffeomorphism, on the pair $(\mathbb{R}^2 \times [-1, 1], \mathbb{R}^2)$, it is not difficult to show that γ may be pushed into a path γ' in $\partial N_\epsilon(S)$ starting at x' and ending either at p_1 or at p_2. By concatenating α with γ' we have thus obtained the desired path in $N_\epsilon(S) \setminus S$ joining x to p_1 or to p_2.

Let us now show that $S^3 \setminus S$ has at most two connected components. It is sufficient to prove that every point $x \in S^3 \setminus S$ can be joined either to p_1 or to p_2 by a path supported in $S^3 \setminus S$. The case when $x \in N_\epsilon(S) \setminus S$ has already been settled, so we may suppose that $x \notin N_\epsilon(S)$. Let β be a path in S^3 joining x to p_1. If β does not intersect S, we are done. Otherwise, there exists a subpath β' of β supported in $S^3 \setminus S$ joining x to a point $y \in N(S) \setminus S$. We have seen that y may be joined to p_i for some $i = 1, 2$ by a path β'' in $N_\epsilon(S) \setminus S$, so the conclusion follows by considering the concatenation of β' and β''. We have thus proved that $S^3 \setminus S$ has at most two connected components.

Suppose now, by contradiction, that $S^3 \setminus S$ is connected. Then any closed interval transverse to S in a local model can be completed in $S^3 \setminus S$ to an embedded smooth circle $f_0 : S^1 \longrightarrow C_0 \subset S^3$ that transversely intersects S in exactly one point. Since S^3 is simply connected, f_0 is smoothly homotopic to an embedded circle $f_1 : S^1 \longrightarrow C_1 \subset S^3$ that does not intersect S. Moreover, we can assume that there exists a smooth homotopy $F : S^1 \times [0, 1] \longrightarrow S^3$ between f_0 and f_1, which is transverse to S. Then the set $F^{-1}(S)$ consists of a finite disjoint union of smooth circles or closed intervals having $F^{-1}(S) \cap (S^1 \times \{0, 1\})$ as set of end-points. In particular, $F^{-1}(S) \cap (S^1 \times \{0, 1\})$ should be given by an *even* number of points, while we know that it consists of just one point. This gives the desired contradiction. $\square$

Notation. Henceforth, whenever $S \subset \mathbb{R}^3 \subset S^3$ is a smooth surface, we denote by $\Omega(S)$ and $\Omega^*(S)$ the connected components of $S^3 \setminus S$. We also assume that $\infty \in \Omega^*(S)$, so $\Omega(S)$ is the unique bounded component of $\mathbb{R}^3 \setminus S$, while $\Omega'(S) := \Omega^*(S) \setminus \{\infty\}$ is the unique unbounded component of $\mathbb{R}^3 \setminus S$. In particular, $\Omega(S)$ is a domain in $\mathbb{R}^3$ and $\partial\Omega(S) = S$. The *local model* of $(\Omega(S), S)$ at every boundary point is given by (P_+, P), where P is an affine plane as above, and $P_+ \subset \mathbb{R}^3$ is a half-space bounded by P.

Definition 2.3. A domain Ω in $\mathbb{R}^3$ has *smooth boundary* if $\partial\Omega$ consists of the disjoint union of a finite number of smooth surfaces.

It readily follows from the definitions that the closure of a domain with smooth boundary admits a natural structure of compact smooth manifold with boundary.

The following lemma is an immediate consequence of the previous discussion.

Lemma 2.4. *Let Ω be a domain with smooth boundary. Then we can order the boundary surfaces $S_0, S_1, \ldots, S_h$ in such a way that:*

(1) *The $\overline{\Omega(S_j)}$'s, $j = 1, \ldots, h$, are contained in $\Omega(S_0)$ and are pairwise disjoint.*
(2) *Ω is given by the following intersection:*

$$\Omega = \Omega(S_0) \cap \bigcap_{j=1}^{h} \Omega^*(S_j).$$

Please cite this article in press as: R. Benedetti, et al., The topology of Helmholtz domains, Expo. Math. (2012), doi:10.1016/j.exmath.2012.09.001

2.2. Orientation

Let $S \subset \mathbb{R}^3$ be a smooth surface. We claim that S is *orientable*. In fact, if $\mathbb{R}^3$ is oriented by means of the equivalence class of its standard basis (e_1, e_2, e_3), then S can be oriented as the boundary of $\Omega(S)$, via the rule "first the outgoing normal vector". More explicitly, for each $p \in S$, one can consistently declare that a basis (v_1, v_2) of the tangent space $T_p S$ of S at p is positively oriented if and only if (n, v_1, v_2) is a positively oriented basis of $\mathbb{R}^3$, where n is a vector orthogonal to $T_p S$ and pointing outward $\Omega(S)$.

Let now $N_\epsilon(S)$ be a tubular neighborhood of S. Using the fact that $N_\epsilon(S) \setminus S$ has exactly two connected components, it is not difficult to show that the pair $(N_\epsilon(S), S)$ is diffeomorphic to $(S \times [-1, 1], S)$. Moreover, under this identification the subset $N_\epsilon(S) \cap \overline{\Omega(S)}$ corresponds to $S \times [-1, 0]$, hence it is a *collar* of S in $\overline{\Omega(S)}$. In the same way, $N_\epsilon(S) \cap \overline{\Omega^*(S)}$ is a collar of S in $\overline{\Omega^*(S)}$.

Similar results hold for tubular neighborhoods of 1-submanifolds of the Euclidean 3-space: if C is a smoothly embedded circle in $\mathbb{R}^3$ and ϵ is small enough, then $N_\epsilon(C)$ is a tubular neighborhood of C, diffeomorphic to a (closed) solid torus $D^2 \times S^1$ and having C as a *core*.

2.3. Link complements

A *link* $L = C_0 \cup \cdots \cup C_h$ in S^3 is the union of a finite family of smoothly embedded disjoint circles C_j. If $h = 0$, then L is a *knot*. Suppose that $\infty \in C_0$, hence $A(L) = S^3 \setminus L$ is a connected open set in $\mathbb{R}^3$. With our definitions, since $\overline{A(L)} = \mathbb{R}^3$, the internal part of $\overline{A(L)}$ does not coincide with $A(L)$ and $A(L)$ is not a domain. However, to L there is associated the domain $\mathsf{C}(L) = S^3 \setminus U(L)$, where $U(L)$ is the union of small disjoint closed tubular neighborhoods of the C_j's. We call $\mathsf{C}(L)$ the *complement-domain of* L. The boundary component of $\mathsf{C}(L)$ corresponding to C_j is a smooth torus T_j and, with the above notations, $\Omega^*(T_0)$ and $\Omega(T_j)$, $j = 1, \ldots, h$, are open solid tori. It is clear that $\mathsf{C}(L)$ is *homotopically equivalent to* $A(L)$ (see e.g. [47] for the definition of homotopy equivalence), hence $\mathsf{C}(L)$ and $A(L)$ share all their homotopy invariants (like the fundamental group). A knot $C = C_0$ is *unknotted* if also $\Omega(T_0)$ is a solid torus or, equivalently, if C bounds a 2-disk of S^3. A link has *geometrically unlinked components* if its components are contained in pairwise disjoint 3-disks of S^3. A link is *trivial* if it has geometrically unlinked unknotted components.

Suppose now that $\infty \notin L$, i.e. consider L as a link of $\mathbb{R}^3$. We use the symbol $U(L)$ again to indicate the union of small disjoint closed tubular neighborhoods of the C_j's in $\mathbb{R}^3$. Choose a smooth 3-ball B of $\mathbb{R}^3$ containing $U(L)$ and define $\mathsf{B}(L) := B \setminus U(L)$. We call $\mathsf{B}(L)$ the *box-domain* of L. Any self-diffeomorphism of S^3 that takes L onto a link L' containing the point at infinity establishes a diffeomorphism between the box-domain $\mathsf{B}(L)$ and the complement-domain $\mathsf{C}(L')$ with a 3-disk removed.

The reader may observe that the complement- and the box-domains of a link are well-defined, up to diffeomorphism (up to ambient isotopy indeed).

2.4. Cutting along surfaces

Let Ω be a domain with smooth boundary. A *properly embedded surface* Σ *in* $(\overline{\Omega}, \partial\Omega)$ is a compact and connected subset of $\overline{\Omega}$ such that:

Please cite this article in press as: R. Benedetti, et al., The topology of Helmholtz domains, Expo. Math. (2012), doi:10.1016/j.exmath.2012.09.001

(1) At points in $\Sigma \setminus \partial\Omega$, Σ has the same local model of a smooth surface.
(2) If $\Sigma \cap \partial\Omega \neq \emptyset$, then at every point of this intersection, up to local diffeomorphism, the triple $(\overline{\Omega}, \partial\Omega, \Sigma)$ is equivalent to the local model (P_+, P, T_+), where the pair (P_+, P) is as in Section 2.1, and $T_+ = T \cap P_+$, T being a plane orthogonal to P. It follows that Σ is a *smooth surface with boundary* $\partial\Sigma = \Sigma \cap \partial\Omega$. This boundary is a (not necessarily connected) smooth curve embedded in $\partial\Omega$.
(3) $(\Sigma, \partial\Sigma)$ admits a *bicollar* in $(\overline{\Omega}, \partial\Omega)$, i.e. there exists a closed neighborhood U of Σ in $\overline{\Omega}$ such that $(U, U \cap \partial\Omega)$ is diffeomorphic to $(\Sigma \times [-1, 1], (\partial\Sigma) \times [-1, 1])$, via a diffeomorphism sending each point $x \in \Sigma$ into $(x, 0) \in \Sigma \times \{0\}$. It is not hard to see that the existence of a bicollar is equivalent to the fact that Σ is orientable. Any orientation on Σ induces an orientation on $\partial\Sigma$, via the rule "first the outgoing normal vector" mentioned above.

Let Σ be properly embedded in $(\overline{\Omega}, \partial\Omega)$. Then the result $\Omega_C(\Sigma)$ of the *cut/open* operation along Σ is the internal part in $\mathbb{R}^3$ of the complement in $\overline{\Omega}$ of a bicollar of $(\Sigma, \partial\Sigma)$. In general, $\Omega_C(\Sigma)$ is not connected. However, every connected component of $\Omega_C(\Sigma)$ is a domain. The boundary of $\Omega_C(\Sigma)$ is no longer smooth, because some corner lines arise along $\partial\Sigma$. However, by means of a standard "rounding the corners" procedure, we can assume that the class of domains with smooth boundary is closed under the cut/open operation.

Remark 2.5. We could define cuts more directly just by setting $A(\Sigma) = \Omega \setminus \Sigma$. The components of $A(\Sigma)$ are not domains in general. On the other hand, each component of $\Omega_C(\Sigma)$ is contained in and is homotopically equivalent to one component of $A(\Sigma)$. This establishes a bijection between the set of components of $A(\Sigma)$ and the set of components of $\Omega_C(\Sigma)$, such that corresponding components share all their homotopy invariants.

Example 2.6. Given a knot K in S^3, a *Seifert surface* of K is a connected orientable smoothly embedded surface S with boundary equal to K. Every knot has a Seifert surface (see [69]). Given the domain $\mathsf{C}(K)$ as in Section 2.3, we can assume that such a surface S is transverse to the boundary torus of $\mathsf{C}(K)$ along a *preferred longitude* parallel to K (it is well-known that the isotopy class of this preferred longitude does not depend on the chosen Seifert surface—see Remark 6.3). Hence, $\Sigma := S \cap \mathsf{C}(K)$ is properly embedded in $\mathsf{C}(K)$ and the space $(\mathsf{C}(K))_C(\Sigma)$ obtained by applying to $\mathsf{C}(K)$ the cut/open operation along Σ, being connected, is a domain.

2.5. Domains with locally flat boundary

In order to perform constructions and develop arguments which use tools from differential topology, it is very convenient to work with domains with smooth boundaries. Such a choice allows us, for instance, to exploit the powerful notion of transversality, that has already appeared in the proof of Proposition 2.2. Using transversality, in Section 5.2 we will be able to approach in an elementary and geometric way some fundamental results about *duality* (such results are usually established in more general settings by using more sophisticated tools from algebraic topology). On the other hand, in the literature about the method of cutting surfaces, classes of domains with weaker regularity properties are

Please cite this article in press as: R. Benedetti, et al., The topology of Helmholtz domains, Expo. Math. (2012), doi:10.1016/j.exmath.2012.09.001

often considered. For example, it is common to require the boundary of a domain to be locally the graph of a Lipschitz function. In this case, the domain is said to have *Lipschitz boundary*. There is a natural way to deal with more general classes of boundaries, keeping nevertheless the same qualitative local pictures as in the case of smooth domains. In fact, we may consider triples $(\overline{\Omega}, \partial\Omega, \Sigma)$ that admit the same local models as in the smooth case, where local models are now considered only "up to local homeomorphism" rather than "up to local diffeomorphism". Such topological triples are called *locally flat*. Note that, according to these definitions, our topological disks in 3-manifolds are locally flat. The following lemma is immediate.

Lemma 2.7. *A compact connected subset of* $\mathbb{R}^3$*, which is locally the graph of* continuous *functions, is a locally flat surface.*

Several deep fundamental results of 3-dimensional geometric topology [66,27,30] imply that, up to homeomorphism, there is not a real difference between the smooth and the locally flat topological case:

Theorem 2.8. *For every locally flat triple* $(\overline{\Omega}, \partial\Omega, \Sigma)$*, the following statements hold.*

(1) Triangulation. *There is a homeomorphism* $t : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ *that maps the given triple onto a* polyhedral *triple (i.e. the piecewise linear realization in* $\mathbb{R}^3$ *of a finite* simplicial complex *with distinguished subcomplexes).*
(2) Smoothing. *There is a homeomorphism* $s : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ *that maps the given triple onto a smooth one.*

Summarizing:

In order to study the geometric topology of arbitrary *locally flat topological triples, it is not restrictive to consider only smooth ones. Moreover, if useful, we can use also tools from* 3*-dimensional* polyhedral (PL) geometry.

It might be worth mentioning that no analogous of Theorem 2.8 holds in higher dimensions. Every smooth manifold admits a unique polyhedral structure, but there exist PL-manifolds that do not admit smoothings, and it may happen that a single PL-manifold admits nondiffeomorphic smoothings. Moreover, there exist topological manifolds that cannot be triangulated.

2.6. Isotopy, homotopy and homology

We conclude the section with a brief and intuitive description of some concepts that will be extensively used throughout the paper (see Sections 3.4 and 5.2 for more details). Let M be a smooth connected n-manifold with (possibly empty) boundary (for our purposes, it is sufficient to consider the cases in which M is a 3-dimensional domain as above or the whole spaces $\mathbb{R}^3$, S^3, or a smooth surface). Two smooth simple oriented loops $C_0, C_1 \subset M$ are *isotopic* if they are related by a smooth isotopy, i.e. by a smooth map $F : S^1 \times [0, 1] \longrightarrow M$ such that, if $F_t := F(\cdot, t) : S^1 \longrightarrow M$, then F_0, F_1 are oriented parameterizations of C_0, C_1 respectively, and F_t is a smooth embedding for every $t \in [0, 1]$. In other words, C_0 is isotopic to C_1 if it can be smoothly deformed into C_1 without crossing itself.

Please cite this article in press as: R. Benedetti, et al., The topology of Helmholtz domains, Expo. Math. (2012), doi:10.1016/j.exmath.2012.09.001

A *homotopy* between C_0 and C_1 is just the same as an isotopy, provided that we do *not* require F_t to be an embedding for every t. More precisely, if C_0, C_1 are *continuous* (possibly self-intersecting) loops in M, we say that C_0 is homotopic to C_1 if it can be taken into C_1 by a *continuous* deformation along which self-crossings are allowed. In particular, C_0 is homotopically trivial if it is homotopic to a constant loop, or, equivalently, if a parameterization of C_0 can be extended to a continuous map from the 2-disk D^2 to M (where we are identifying S^1 with ∂D^2). The manifold M is *simply connected* if (it is connected and) every loop in M is homotopically trivial. It is well-known (and very easy) that $\mathbb{R}^3$ and S^3 are simply connected, so every knot, when considered as a parameterized loop, is homotopically trivial. On the other hand, by definition nontrivial knots in S^3 provide examples of loops that are not smoothly isotopic to any unknotted knot.

More in general, let us define a 1-*cycle* (*with integer coefficients*) in M as the union L of a finite number of (not necessarily embedded nor disjoint) oriented loops in M. We say that L is a *boundary* if there exist an oriented (possibly disconnected) surface with boundary S and a continuous map $f : S \longrightarrow M$ such that the restriction of f to the boundary of S defines an orientation-preserving parameterization of L (the orientation of S canonically induces an orientation of ∂S also in the topological setting): with a slight abuse, in this case we say that L *bounds* $f(S)$. Of course, knots and links in S^3 are particular instances of 1-cycles in S^3, and every knot is a boundary, since it bounds a (possibly singular) 2-disk, or a Seifert surface. If L, L' are 1-cycles in M and $-L'$ is the 1-cycle obtained by reversing all the orientations of the loops of L', we say that L is *homologous* to L' if the 1-cycle $L \cup -L'$ is a boundary, and that L is homologically trivial if it bounds or, equivalently, if it is homologous to the empty 1-cycle. It readily follows from the definitions that homotopic loops define homologous 1-cycles. The space of equivalence classes of 1-cycles (with respect to the relation of being homologous) is the *singular* 1-*homology module of* M (*with integer coefficients*) and it is usually denoted by $H_1(M; \mathbb{Z})$. The union of 1-cycles induces a sum on $H_1(M; \mathbb{Z})$, which is therefore an Abelian group. It is not difficult to show that, since M is connected, every 1-cycle in M is homologous to a single loop, and this readily implies that, if M is simply connected, then $H_1(M; \mathbb{Z}) = 0$. The converse statement is not true in general (see Remark 3.9), but turns out to hold for tame domains in $\mathbb{R}^3$ (see Corollary 3.5). Note, however, that even if $M = \Omega$ is a domain in S^3 with locally flat boundary, then there may exist a loop of M which is homologically trivial, but not homotopically trivial: if $K \subset S^3$ is a nontrivial knot with complement-domain $\mathsf{C}(K)$, then a Seifert surface Σ for K defines a preferred longitude $\gamma = \Sigma \cap \partial \overline{\mathsf{C}(K)} \subset \partial \overline{\mathsf{C}(K)}$. Such a longitude bounds the surface with boundary $\Sigma \cap \overline{\mathsf{C}(K)}$ and is therefore homologically trivial in $\overline{\mathsf{C}(K)}$. However, as a consequence of the classical Dehn's Lemma (see [69, p. 101]), if γ were homotopically trivial in $\overline{\mathsf{C}(K)}$, it would bound a (embedded locally flat) 2-disk in $\overline{\mathsf{C}(K)}$, and this would imply in turn that K is trivial, a contradiction.

The singular 2-homology module of M can be described in a similar way as the set of equivalence classes of maps of compact smooth oriented (possibly disconnected) surfaces in M, up to 3-dimensional "bordism". A nice, nontrivial fact in the situations of our interest is that every 1- or 2-homology class can be represented by *submanifolds* (i.e. the above maps may be chosen to be embeddings). In the polyhedral setting, this is a consequence of *Kneser's method* (1924) for eliminating singularities (see [43, p. 32]). By Theorem 2.8 (or even by classical results within the smooth framework), this also holds in the smooth case.

Please cite this article in press as: R. Benedetti, et al., The topology of Helmholtz domains, Expo. Math. (2012), doi:10.1016/j.exmath.2012.09.001

3. Simple domains

Let Ω be a domain. In the theoretical and applied literature about Helmholtz domains, two main notions are employed in order to declare that Ω is "simple":

(a) *Ω is simply connected* (i.e. it has trivial fundamental group).
(b) *Every curl-free smooth vector field on Ω is the gradient of a smooth function on Ω.*

Since our discussion is mainly motivated by the applications mentioned in the introduction, we believe that condition (b) is more relevant here, so henceforth we use the term "simple" according to the following:

Definition 3.1. Let Ω be a domain. Then Ω is simple if it satisfies condition (b) above.

It is widely known (and proved in Corollary 3.5) that a simply connected domain is simple. In fact, as described in the following paragraphs, condition (b) above is equivalent to the vanishing of the first de Rham cohomology module of Ω, which is in turn implied by the request that Ω be simply connected. The converse implication seems to have risen some misunderstandings (see Example 3.6). The main result of this section ensures that conditions (a) and (b) above are indeed equivalent, if we restrict to domains with locally flat boundary (but see also Remark 3.8). In fact, in Section 3.5 we will prove the following result (where we keep notations from Lemma 2.4):

Theorem 3.2. *Let Ω be a simple domain of $\mathbb{R}^3$ with locally flat boundary $\partial\Omega = S_0 \cup \cdots \cup S_h$. Then, for every $j \in \{0, 1, \ldots, h\}$, both $\Omega(S_j)$ and $\Omega^*(S_j)$ are 3-balls of S^3 bounded by the locally flat 2-sphere S_j. Therefore, the domain*

$$\Omega = \Omega(S_0) \cap \bigcap_{j=1}^{h} \Omega^*(S_j)$$

is just an "external" 3-ball with a finite number of "internal" pairwise disjoint 3-disks removed. In particular, Ω is simply connected.

Other characterizations of simple domains are given in Corollary 3.5.

3.1. Vector fields, differential forms and de Rham cohomology

We begin by reformulating condition (b) more conveniently in terms of differential forms. Every nondegenerate scalar product $\langle\ ,\ \rangle$ on a finite dimensional real vector space V determines an isomorphism $\psi : V \longrightarrow V^*$ between V and its *dual space* $V^* := \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$, by the formula $\psi(v)(w) = \langle v, w\rangle$, for every $v, w \in V$. A Riemannian metric on a smooth manifold M is just a smooth field $\{\langle\ ,\ \rangle_p\}_{p\in M}$ of positive definite (hence nondegenerate) scalar products on the tangent spaces T_pM. The above formula applied pointwise on M determines a canonical isomorphism between the space of smooth tangent vector fields and the space of smooth 1-forms on M (from now on, even when not explicitly stated, differential forms are assumed to be smooth). Let us apply this general fact to the standard flat Riemannian metric $ds^2 = dx_1^2 + dx_2^2 + dx_3^2$ on $\mathbb{R}^3$ (and to its restriction to any domain). In this case, if $V = (V_1, V_2, V_3)$ is a smooth vector field on a

Please cite this article in press as: R. Benedetti, et al., The topology of Helmholtz domains, Expo. Math. (2012), doi:10.1016/j.exmath.2012.09.001

domain Ω, then $\omega := \sum_{j=1}^{3} V_j dx_j$ is the associated 1-form. The differential of ω is the 2-form

$$d\omega = \left(-\frac{\partial V_2}{\partial x_3} + \frac{\partial V_3}{\partial x_2}\right) dx_2 \wedge dx_3 - \left(\frac{\partial V_1}{\partial x_3} - \frac{\partial V_3}{\partial x_1}\right) dx_1 \wedge dx_3 + \left(-\frac{\partial V_1}{\partial x_2} + \frac{\partial V_2}{\partial x_1}\right) dx_1 \wedge dx_2.$$

Since

$$\operatorname{curl}(V) = \left(-\frac{\partial V_2}{\partial x_3} + \frac{\partial V_3}{\partial x_2}, \frac{\partial V_1}{\partial x_3} - \frac{\partial V_3}{\partial x_1}, -\frac{\partial V_1}{\partial x_2} + \frac{\partial V_2}{\partial x_1}\right),$$

V is curl-free if and only if $d\omega = 0$.

If $f\colon \Omega \longrightarrow \mathbb{R}$ is a smooth function, the differential of f is the 1-form

$$df = \sum_{j=1}^{3} \frac{\partial f}{\partial x_j} dx_j.$$

By the very definitions, the gradient ∇f corresponds to df, via the above canonical isomorphism determined by ds^2.

A 1-form is *closed* if its differential vanishes, and it is *exact* if it is the differential of a smooth function. Since $d(df) = 0$ for every smooth function f (or, equivalently, every gradient field is curl-free), every exact 1-form is closed. If Ω is a domain, then the first de Rham cohomology group $H^1_{DR}(\Omega)$ is defined as the quotient vector space of closed 1-forms defined on Ω modulo exact 1-forms defined on Ω. Condition (b) above is then equivalent to condition

(b′) *Every closed* 1*-form on Ω is exact, i.e.* $H^1_{\mathrm{DR}}(\Omega) = 0$.

While condition (b) involves the Riemannian metric of Ω, condition (b′) only depends on the differential structure of Ω: therefore, property (b), which is obviously an isometric invariant of domains, is in fact a diffeomorphism invariant. Moreover, as a very particular case of de Rham Theorem (see e.g. [29]), we know that

$$H^1_{\mathrm{DR}}(\Omega) \cong H^1(\Omega; \mathbb{R}),$$

where the vector space on the right-hand side is the *singular* 1*-cohomology module with real coefficients*, which is a topological (homotopic indeed) invariant (see e.g. [47]). Hence, we may reformulate condition (b) in the language of (basic) algebraic topology as follows:

(b″) $H^1(\Omega; \mathbb{R}) = 0$.

3.2. The Universal Coefficient Theorem

Singular homology and singular cohomology with real and integer coefficients are closely related to each other. In fact, the Universal Coefficient Theorem expresses singular homology and singular cohomology with arbitrary coefficients in terms of singular homology with integer coefficients (see e.g. [47, Theorems 3.2 and 3A.3]). In order to state the consequences of the Universal Coefficient Theorem that are relevant to our

Please cite this article in press as: R. Benedetti, et al., The topology of Helmholtz domains, Expo. Math. (2012), doi:10.1016/j.exmath.2012.09.001

purposes, we first extend our notations for the 1- and 2-dimensional (co)homology modules to the case of arbitrary dimension. Let X be a topological space, and let R be either the ring of integers $\mathbb{Z}$ or the field of real numbers $\mathbb{R}$. For every $i \in \mathbb{N}$ we denote by $H_i(X; R)$ (resp. by $H^i(X; R)$) the singular i-th homology module (resp. the singular i-th cohomology module) of X with coefficients in R. We also denote by $T_i(X)$ the submodule of finite-order elements of $H_i(X, \mathbb{Z})$, and we observe that $T_i(X)$ is finite if $H_i(X; \mathbb{Z})$ is finitely generated. If G, G' are Abelian groups, we denote by $\mathrm{Hom}_{\mathbb{Z}}(G, G')$ the space of group homomorphisms between G and G' (the subscript $\mathbb{Z}$ is due to the fact that every Abelian group is naturally a $\mathbb{Z}$-module, and group homomorphisms coincide with $\mathbb{Z}$-linear homomorphisms of $\mathbb{Z}$-modules). On the other hand, if V is a real vector space, we denote by $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ the *dual space* of V, i.e. the space of $\mathbb{R}$-linear maps from V to $\mathbb{R}$. Since $\mathbb{R}$ is a field, in our cases of interest the Universal Coefficient Theorem specializes to the following results:

Theorem 3.3 (*Universal Coefficient Theorem for Homology*)**.** *For every $i \in \mathbb{N}$, there exists a canonical isomorphism*

$$H_i(X; \mathbb{R}) \cong H_i(X; \mathbb{Z}) \otimes \mathbb{R}.$$

Theorem 3.4 (*Universal Coefficient Theorem for Cohomology*)**.** *For every $i \in \mathbb{N}$, there exist canonical isomorphisms*

$$H^i(X; \mathbb{R}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_i(X; \mathbb{Z}), \mathbb{R}) \cong \mathrm{Hom}_{\mathbb{R}}(H_i(X; \mathbb{R}), \mathbb{R}).$$

In addition, if $H_{i-1}(X; \mathbb{Z})$ is finitely generated, then

$$H^i(X; \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_i(X; \mathbb{Z}), \mathbb{Z}) \oplus T_{i-1}(X).$$

In fact, the last statement of Theorem 3.4 may be strengthened as follows. The tautological pairing between singular cochains and singular chains induces a well-defined $\mathbb{Z}$-bilinear pairing

$$H^i(X; \mathbb{Z}) \times H_i(X; \mathbb{Z}) \to \mathbb{Z},$$

usually called *Kronecker pairing*, which in turn induces a $\mathbb{Z}$-linear map

$$H^i(X; \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(H_i(X; \mathbb{Z}), \mathbb{Z}).$$

This map is always surjective, and its kernel is isomorphic to $T_{i-1}(X)$ whenever $H_{i-1}(X; \mathbb{Z})$ is finitely generated. Since $\mathrm{Hom}_{\mathbb{Z}}(H_i(X; \mathbb{Z}), \mathbb{Z})$ is a free $\mathbb{Z}$-module, it is not difficult to show that these facts imply the last statement of Theorem 3.4.

Using Theorem 3.2, we may now establish the following characterizations of simple domains. We refer the reader to Section 4.1 for the definitions of weak derivatives of vector fields and of the Sobolev space $W^{1,2}(\Omega)$. These notions are involved in the statement of condition (f) below, and they naturally come into play in several applications of the method of cutting surfaces (see again Section 4.1).

Corollary 3.5. *Let Ω be a domain with locally flat boundary. Then the following properties are equivalent:*

Please cite this article in press as: R. Benedetti, et al., The topology of Helmholtz domains, Expo. Math. (2012), doi:10.1016/j.exmath.2012.09.001

(a) *Ω is simply connected.*
(b) *Ω is simple, i.e. every curl-free smooth vector field on Ω is the gradient of a smooth function.*
(b″) $H^1(\Omega;\mathbb{R}) = 0$.
(c) $H_1(\Omega;\mathbb{Z}) = 0$.
(d) $H_1(\Omega;\mathbb{R}) = 0$.
(e) *For every curl-free smooth vector field V and every divergence-free smooth vector field W on Ω with compact support, the integral $\int_\Omega V \bullet W\, dx$ is null, where $V \bullet W = \sum_{j=1}^{3} V_j \cdot W_j$ if $V = (V_1, V_2, V_3)$ and $W = (W_1, W_2, W_3)$.*

Moreover, if Ω has Lipschitz boundary, then we can add the following equivalent condition to the list:

(f) *Every curl-free vector field in $L^2(\Omega)^3$ is the weak gradient of a function in $W^{1,2}(\Omega)$.*

Proof. As observed in Section 2.6, if Ω is simply connected, then every 1-cycle in Ω is a boundary, so $H_1(\Omega;\mathbb{Z}) = 0$. Therefore, the Universal Coefficient Theorem implies that $H_1(\Omega;\mathbb{R}) = 0$ and $H^1(\Omega;\mathbb{R}) = 0$. We have thus proved that

$$\text{(a)} \Longrightarrow \text{(c)} \Longrightarrow \text{(d)} \Longrightarrow \text{(b}''\text{)}\ (\Longleftrightarrow \text{(b)})\ .$$

On the other hand, Theorem 3.2 ensures that (b) implies (a). We have thus proved that the first five conditions are equivalent to each other.

If (b) holds, then (e) follows immediately from the Green formula. Suppose now that (e) holds, let V be a curl-free smooth vector field on Ω and let ω be the 1-form corresponding to V via the duality described above. Let now φ be any fixed compactly supported closed 2-form on Ω. As a direct consequence of Stokes' Theorem, the map which associates to every class $[\psi] \in H^1_{DR}(\Omega)$ the real number

$$\int_\Omega \psi \wedge \varphi$$

is well-defined and determines a linear map $f_\varphi\colon H^1_{DR}(\Omega) \longrightarrow \mathbb{R}$. Now a classical result in de Rham Cohomology Theory (see e.g. [29, p. 44]) ensures that every linear map $H^1_{DR}(\Omega) \longrightarrow \mathbb{R}$ arises in this way, i.e. it is of the form f_φ for some closed compactly supported 2-form φ. Therefore condition (e) translates into the fact that every linear map $H^1_{DR}(\Omega) \longrightarrow \mathbb{R}$ vanishes on the cohomology class $[\omega]$ of ω, and this readily implies that $[\omega] = 0$, i.e. ω is exact. This is in turn equivalent to the fact that V is the gradient of a smooth function.

Finally, (b″) $\Longleftrightarrow$ (f) is immediate from the version of de Rham Theorem given in [20, Assertion (11.7), p. 85]. □

3.3. A fallacious counterexample

Before going into the proof of Theorem 3.2, we discuss a *fake counterexample* to the implication (b) $\Longrightarrow$ (a).

Example 3.6. We briefly analyze the example given by Vourdas and Binns in their response to Kotiuga in the correspondence [19, p. 232] (see also [7], [16, Section 2.1] and [17, Section 1]). Let C be the oriented trefoil knot of $\mathbb{R}^3$ and let Σ be the Seifert surface of C drawn in Fig. 2 (on the left). Denote by $\Omega_C(\Sigma)$ the domain of $\mathbb{R}^3$ obtained by applying to the complement-domain $\mathsf{C}(C)$ of C the cut/open operation along Σ.

In [19, p. 232], Vourdas and Binns assert that $S^3 \setminus \Sigma$ (or, equivalently, $\Omega_C(\Sigma)$) is not simply connected, while $H_1(S^3 \setminus \Sigma; \mathbb{R}) = 0$ (or, equivalently, $H_1(\Omega_C(\Sigma); \mathbb{R}) = 0$). The second claim is wrong. In fact, consider the two oriented loops a and b contained in Σ and the two oriented loops R and T contained in $\Omega_C(\Sigma)$ drawn in Fig. 2 (on the right). The surface Σ is homeomorphic to a torus minus an open 2-ball, and the homology classes of a and of b in Σ form a basis of $H_1(\Sigma; \mathbb{R})$ (see also Fig. 8.12 of [6, p. 243] to visualize these facts). Alexander Duality Theorem immediately implies that the homology classes of R and of T form a basis of $H_1(\Omega_C(\Sigma); \mathbb{R})$. In particular, this last space is nontrivial. Moreover, the trefoil knot is an example of *fibered knot* having the given Seifert surface as a fiber (this is carefully described in [69, p. 327]). Hence, $\Omega_C(\Sigma)$ is homeomorphic to $(\Sigma \setminus \partial\Sigma) \times (0, 1)$ and has therefore the same homotopy type of Σ. Note that this fact confirms the above claim that $H_1(\Omega_C(\Sigma); \mathbb{R})$ and $H_1(\Sigma; \mathbb{R})$ are isomorphic.

The first argument above can be rephrased in a more physical fashion. Suppose a is an ideally thin conductor, carrying a current of unitary intensity. Let $\mathbf{H}_a$ be the corresponding magnetic field. The restriction $\mathbf{H}'_a$ of $\mathbf{H}_a$ to $S^3 \setminus \Sigma$ is a curl-free smooth vector field, which does not have any scalar potential. In fact, the circulation of $\mathbf{H}'_a$ along R is 1. In particular, by Stokes' Theorem, the homology class of R in $S^3 \setminus \Sigma$ is not null. Similar considerations can be repeated for b and T.

We believe that the following observation points out a possible source for this mistake. In Fig. 3, it is drawn a compact connected orientable surface B of $\mathbb{R}^3$ with boundary R contained in $S^3 \setminus C$ (see also Fig. 8.13 of [6, p. 244]). The existence of such a surface implies that R represents the null homology class in $H_1(S^3 \setminus C; \mathbb{R})$. Then the restriction to $S^3 \setminus \Sigma$ of any curl-free smooth vector field defined on the *whole* of $S^3 \setminus C$ has null circulation along R. On the other hand, not every curl-free smooth vector fields on $S^3 \setminus \Sigma$ can be extended to $S^3 \setminus C$. Note also that the surface B intersects in an essential way the Seifert surface Σ. These facts explain why the homology class of R in $S^3 \setminus C$ is null, while the homology class of R in $S^3 \setminus \Sigma$ is not. We refer the reader to Section 5 for further elaborations.

Example 3.7. In their discussion about the relationship between homotopy and homology [19], Vourdas and Binns also consider the case of the *Whitehead link* (see the top of Fig. 4, on the left). With notations as in Fig. 4, they claim that the loop R is homologically trivial and homotopically nontrivial in the complement of C (see [7]). On the contrary, the sequence of moves described in Fig. 4 shows that R is homotopic (in the complement of C) to a loop R' which is clearly null-homotopic. As discussed in Section 2.6, the fact that R is homotopic to R' in $\mathbb{R}^3 \setminus C$ follows from the fact that R can be continuously deformed into R' without crossing C (but crossing itself!). This implies, in particular, that R bounds a singular 2-disk in $\mathbb{R}^3 \setminus C$. In fact, since R and C are not geometrically unlinked, R *cannot* bound an *embedded* locally flat 2-disk in $\mathbb{R}^3 \setminus C$. As a consequence, it can be shown that R and R' are not isotopic in $\mathbb{R}^3 \setminus C$.

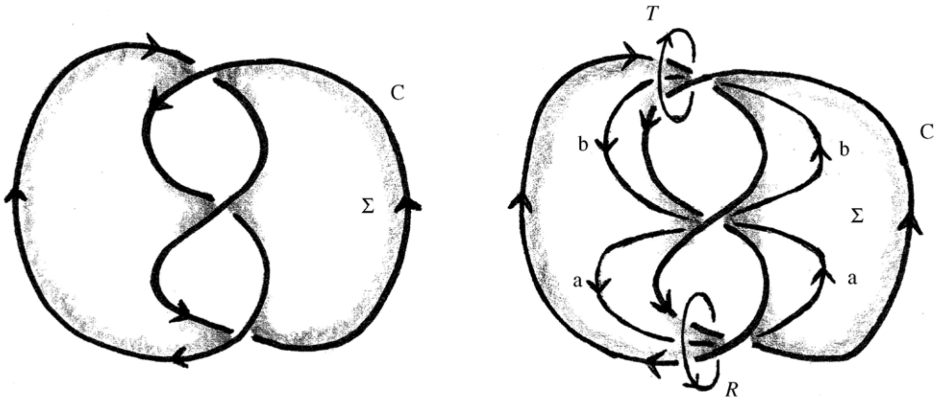


Fig. 2. The trefoil knot with one of its Seifert surfaces.

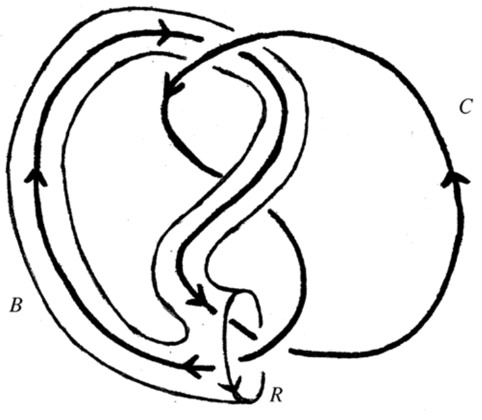


Fig. 3. A null homologous cycle.

3.4. Elementary results about the algebraic topology of domains

Let M be a compact smooth manifold. We say that M is *closed* if its boundary is empty. By the classical *Morse theory* (see [63,50]), if M is closed, then it has the homotopy type of a finite CW complex of dimension $m = \dim M$, which can be constructed starting from any Morse function on M. If M is connected with *nonempty* boundary, then it has the homotopy type of a CW complex of dimension $<m$, which can be determined by any Morse function $f: (M, \partial M) \longrightarrow ([0, 1], \{1\})$ without local maxima. The same facts hold if M is polyhedral. One can get a unified treatment of the smooth and of the polyhedral case by reformulating Morse theory in terms of *handle decomposition theory* (see [64,70]). By Theorem 2.8, in our favorite case of spatial domains, we can adopt both points of view.

Recall that the Universal Coefficient Theorem for cohomology implies that, for every $k \in \mathbb{N}$, the singular k-cohomology module $H^k(M; \mathbb{R})$ of M with real coefficients is isomorphic to the dual space $\mathrm{Hom}_{\mathbb{R}}(H_k(M; \mathbb{R}), \mathbb{R})$ of the corresponding singular

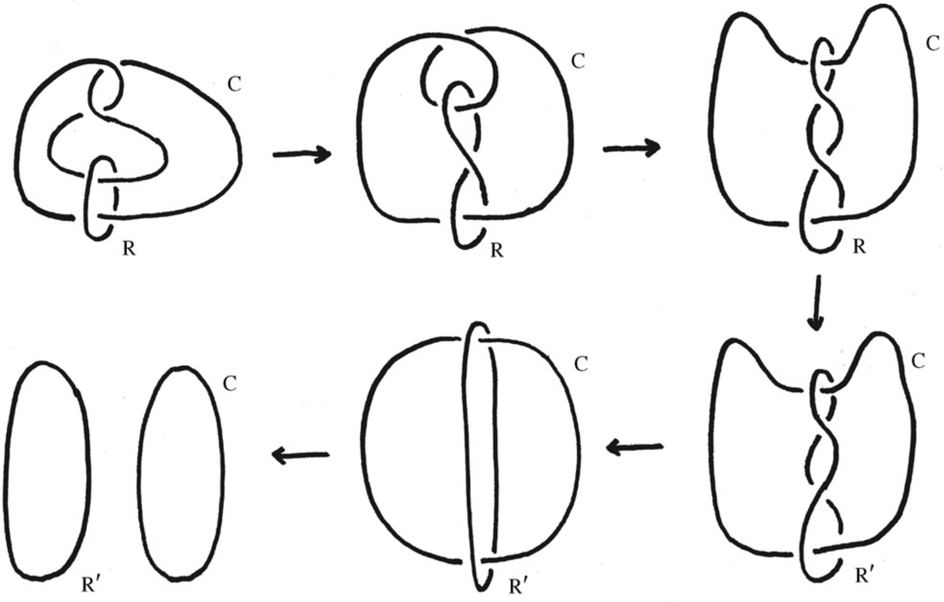


Fig. 4. A proof that R is homotopically trivial in $\mathbb{R}^3 \setminus C$. The horizontal moves are realized by smooth isotopies, while the third move describes a homotopy between R and R' in $\mathbb{R}^3 \setminus C$. During this homotopy, the loop crosses itself.

homology module $H_k(M;\mathbb{R})$. Moreover, compactness of M implies that, for every $k \in \mathbb{N}$, the *k-th Betti number* $b_k(M) := \dim H_k(M;\mathbb{R})$ of M is finite, whence equal to $\dim H^k(M;\mathbb{R})$. In fact, the fundamental isomorphism between *cellular* (or *simplicial*) and singular homologies implies that $\dim H_k(M;\mathbb{R})$ is finite for every $k \in \mathbb{N}$ and vanishes if $k > \dim M$. Similar results also hold for homology and cohomology with integer coefficients: $H_n(M;\mathbb{Z})$ and $H^n(M;\mathbb{Z})$ are finitely generated for every $n \in \mathbb{N}$ and trivial for $n > \dim M$. Hence, the submodule $T_n(M)$ of finite-order elements of $H_n(M,\mathbb{Z})$ is finite, and

$$H_n(M;\mathbb{Z}) = \big(H_n(M;\mathbb{Z})/T_n(M)\big) \oplus T_n(M).$$

Being finitely generated and torsion-free, the quotient $H_n(M;\mathbb{Z})/T_n(M)$ is isomorphic to $\mathbb{Z}^r$ for some $r \geq 0$; such an r will be called the *rank* of $H_n(M;\mathbb{Z})$ and will be denoted by $r_n(M)$.

The Universal Coefficient Theorem for homology ensures that $H_n(M;\mathbb{R}) = H_n(M;\mathbb{Z}) \otimes \mathbb{R}$, and this implies in turn that $r_n(M) = b_n(M)$. Let us now recall the definition of the *Euler–Poincaré characteristic* $\chi(M)$ of M:

$$\chi(M) := \sum_{n=0}^{\dim M} (-1)^n b_n(M).$$

It is well-known that, if c_n is the number of n-cells (n-simplices) of any finite CW complex homotopy equivalent to (any triangulation of) M, then $\chi(M)$ admits the following combinatorial description:

$$\chi(M) = \sum_{n=0}^{\dim M} (-1)^n c_n.$$

We now list some elementary results that will prove useful later.

(1) Assume that M is connected. Then $b_0(M) = 1$. If $\dim M = m$ and M has *nonempty* boundary, then $b_m(M) = 0$. The last claim follows from the fact that M has the homotopy type of a CW complex of strictly smaller dimension.

(2) If M is a closed manifold of *odd* dimension $m = 2n+1$, then $\chi(M) = 0$. In fact, the "dual" CW complexes associated to f and $-f$, where f is a suitable Morse function on M, are such that the respective numbers of cells verify the relations $c_i = c^*_{m-i}$. Then the result easily follows from the combinatorial formula for $\chi(M)$. If M is triangulated, one can use the *dual cell decomposition* of a given triangulation. These facts may be thought as primitive manifestations of the *Poincaré duality* for M.

(3) If M is a connected manifold with *nonempty* boundary ∂M, then we can construct the *double* $D(M)$ of M, by glueing two copies of M along their boundaries via the identity map. Then $D(M)$ is closed and

$$\chi(D(M)) = 2\chi(M) - \chi(\partial M).$$

In the case of triangulable manifolds (like spatial domains), the latter equality follows easily by considering a triangulation of $(M, \partial M)$, that induces a triangulation of the double, and by using the combinatorial formula for χ. Hence, if $\dim M$ is odd, then $\chi(\partial M) = 2\chi(M)$ is even. Moreover, we observe that

$$\chi(\partial M) = \sum_i \chi(S_i),$$

where the S_i's are the boundary components of M.

Let us now specialize to domains.

(4) As already mentioned, if $\Omega \subset \mathbb{R}^3$ is a domain with smooth boundary, then Ω is homotopically equivalent to $\overline{\Omega}$, so $b_n(\Omega) = b_n(\overline{\Omega})$ for every $n \in \mathbb{N}$. Since $\overline{\Omega}$ is a compact smooth 3-manifold with nonempty boundary, we deduce from point (1) above that

$$\chi(\Omega) = \chi(\overline{\Omega}) = 1 - b_1(\Omega) + b_2(\Omega).$$

(5) If $M = S$ is a smooth surface in $\mathbb{R}^3$, then $b_0(S) = 1 = b_2(S)$, and S bounds $\overline{\Omega(S)}$. In particular, by point (3) above, $b_1(S) = 2 - \chi(S)$ is even. The nonnegative integer

$$g(S) := \frac{b_1(S)}{2}$$

is called *genus* of S. A basic classification theorem of orientable surfaces (see [50]) says that two compact orientable surfaces are diffeomorphic if and only if they have the same genus. In particular, S is a smooth 2-sphere if and only if $g(S) = 0$.

Please cite this article in press as: R. Benedetti, et al., The topology of Helmholtz domains, Expo. Math. (2012), doi:10.1016/j.exmath.2012.09.001

(6) If $\Omega \subset \mathbb{R}^3$ is a domain whose boundary consists of the disjoint union of smooth surfaces $S_0, \ldots, S_h$, then points (3) and (5) above imply that:

$$\chi(\Omega) = \frac{\chi(\partial M)}{2} = \frac{1}{2}\sum_{i=0}^{h} \chi(S_i) = \frac{1}{2}\sum_{i=0}^{h}\big(2 - 2g(S_i)\big) = h + 1 - \sum_{i=0}^{h} g(S_i).$$

3.5. Proof of Theorem 3.2

Let Ω be a simple domain with locally flat boundary. We already know that the fact that Ω is simple is equivalent to the condition $H^1(\Omega; \mathbb{R}) = 0$. Moreover, it is not restrictive to assume that Ω has smooth boundary. We denote by $S_0, \ldots, S_h$ the boundary components of $\partial\Omega$, keeping notations from Lemma 2.4.

Let us set $b_1 := b_1(\Omega), b_2 := b_2(\Omega)$. As a consequence of the Universal Coefficient Theorem for cohomology, the fact that $H^1(\Omega; \mathbb{R}) = 0$ is equivalent to the condition $b_1 = 0$. By point (4) above, this is equivalent to $\chi(\Omega) = 1 + b_2$ as well. Together with the equality $\chi(\Omega) = h + 1 - \sum_{i=0}^{h} g(S_i)$ proved above, this implies that

$$h - \sum_{i=0}^{h} g(S_i) = b_2 \geq 0. \tag{1}$$

The proof proceeds now by induction on $h \geq 0$. If $h = 0$, then we have $-g(S_0) \geq 0$, so $g(S_0) = 0$ and S_0 is a smooth 2-sphere embedded in S^3. Hence, in this case, our theorem reduces to the celebrated Alexander Theorem (1924) [23] (see also [48] for a very accessible proof in the case of smooth spheres, rather than polyhedral ones as in the original paper by Alexander). If $h \geq 1$, then Eq. (1) implies that $g(S_{j_0}) = 0$ for at least one $j_0 \in \{0, \ldots, h\}$. Suppose first that $j_0 \geq 1$. Let us denote by Ω^0 the domain $\Omega^0 = \Omega \cup \overline{\Omega(S_{j_0})}$ obtained by capping-off the boundary sphere S_{j_0} of Ω with the 3-disk $\overline{\Omega(S_{j_0})}$. An elementary application of the Mayer–Vietoris Theorem (see e.g. [47]) shows that Ω^0 is a domain with $(h-1)$ boundary components such that $H^1(\Omega^0; \mathbb{R}) = 0$, and this allows us to conclude by induction. If $j_0 = 0$, then the same proof applies, after defining Ω^0 as the domain obtained by filling Ω (in S^3) with the 3-disk $\overline{\Omega^*(S_0)}$. $\square$

Remark 3.8. Theorem 3.2 does not hold in general if we do not assume Ω to have locally flat boundary. In fact, on one hand, the Jordan–Brouwer Separation Theorem establishes that every *topological* 2-sphere S embedded in S^3 disconnects S^3 in two domains each of which has trivial singular 1-homology module. On the other hand, as we already mentioned above, Alexander ([24,25], see also [69, p. 76 and p. 81]) produced celebrated examples of nonlocally flat topological 2-spheres whose complement in S^3 consists of domains one of which (or even both of which) is not simply connected.

Remark 3.9. A smooth compact connected 3-manifold M with nonempty boundary is a $\mathbb{Z}$-*homology disk* (resp. $\mathbb{R}$-*homology disk*) if its homology modules with coefficients in $\mathbb{Z}$ (resp. in $\mathbb{R}$) are trivial, except that in dimension 0 (so a $\mathbb{Z}$-homology disk is necessarily an $\mathbb{R}$-homology disk). Non-simply connected $\mathbb{R}$-homology disks are easily constructed by removing a small genuine 3-ball from closed 3-manifolds with finite (but nontrivial) fundamental group such as the projective space $\mathbb{P}^3(\mathbb{R})$ or any lens space $L(p, q)$ (see

[69, p. 233]). In the same spirit, a $\mathbb{Z}$-homology disk which is not simply connected can be obtained by removing a genuine 3-ball from a closed connected 3-manifold that has trivial 1-dimensional $\mathbb{Z}$-homology but is not simply connected. The first example of such a manifold is due to Poincaré. Theorem 3.2 implies that connected $\mathbb{R}$-homology disks that are not simply connected cannot be smoothly embedded in S^3.

Remark 3.10. Even in the locally flat case, the conclusions of Theorem 3.2 are no longer true when dealing with domains in higher dimensional Euclidean space. For example, the projective plane $\mathbb{P}^2(\mathbb{R})$ can be embedded in $\mathbb{R}^4$, and a tubular neighborhood of the image of such an embedding is a 4-dimensional $\mathbb{R}$-homology disk with fundamental group isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

We end this section with an open question (as far as we know):

Question 3.11. Let Ω be a *not necessarily bounded* domain with smooth boundary. Assume that $H^1(\Omega;\mathbb{R}) = 0$. Is it true that Ω is simply connected?

4. Helmholtz domains

Before giving the precise definition of Helmholtz domain, we briefly describe the method of cutting surfaces in the standard L^2-setting (see [11,10,3]), by considering the basic problem of "superconductive walls" in magnetostatics.

4.1. The standard setting of the method of cutting surfaces

Let Ω be a bounded open domain of $\mathbb{R}^3$ with Lipschitz boundary, let $L^2(\Omega)$ be the usual Hilbert space of square-summable functions on Ω and let $L^2(\Omega)^3$ be the corresponding Hilbert space of L^2-vector fields on Ω whose scalar product is given by

$$\langle V, W\rangle := \int_\Omega V \bullet W \, dx,$$

where $V \bullet W := \sum_{i=1}^3 V_i W_i$ if $V = (V_1, V_2, V_3)$ and $W = (W_1, W_2, W_3)$. Denote by $W^{1,2}(\Omega)$ the Sobolev space consisting of all functions in $L^2(\Omega)$ whose weak gradient is a well-defined element of $L^2(\Omega)^3$. Let $\mathscr{C}_0^\infty(\Omega)$ be the set of all real-valued smooth functions on Ω with compact support and let $V \in L^2(\Omega)^3$. We recall that V has weak curl in $L^2(\Omega)^3$, denoted by $\mathrm{curl}(V)$, if $\mathrm{curl}(V)$ is a vector field in $L^2(\Omega)^3$ such that

$$\langle \mathrm{curl}(V), \Phi\rangle = \langle V, \mathrm{curl}(\Phi)\rangle \quad \text{for every } \Phi \in \mathscr{C}_0^\infty(\Omega)^3.$$

Moreover, V has weak divergence $\mathrm{div}(V)$ in $L^2(\Omega)$ if $\mathrm{div}(V)$ is a function in $L^2(\Omega)$ such that

$$\int_\Omega \mathrm{div}(V)\cdot\varphi \, dx = -\langle V, \nabla\varphi\rangle \quad \text{for every } \varphi \in \mathscr{C}_0^\infty(\Omega).$$

The vector field V is called *curl-free* if $\mathrm{curl}(V) = 0$ and *divergence-free* if $\mathrm{div}(V) = 0$. If V is both curl- and divergence-free, then it is called *harmonic*. Let $n_{\partial\Omega}$ be the outward unit normal L^∞-vector field of the boundary $\partial\Omega$ of Ω (see Lemma 4.2 of [21, p. 88]) and let $\mathbb{H}(\Omega)$ be the space of harmonic L^2-vector fields of Ω tangent to $\partial\Omega$, i.e.

Please cite this article in press as: R. Benedetti, et al., The topology of Helmholtz domains, Expo. Math. (2012), doi:10.1016/j.exmath.2012.09.001

$$\mathbb{H}(\Omega) := \left\{V \in L^2(\Omega)^3 \mid \mathrm{curl}(V) = 0,\ \mathrm{div}(V) = 0,\ V \bullet n_{\partial\Omega} = 0\right\},$$

where $V \bullet n_{\partial\Omega}$ is the normal component of V on $\partial\Omega$ in the sense of traces (see Part A, Section 1 in Chapter IX of [9]).

The mentioned problem of "superconductive walls" in magnetostatics can be stated as follows (see Section 4.2 of [5]): *given a divergence-free vector field J in $L^2(\Omega)^3$ having null flux across every connected component of $\partial\Omega$, find $H \in L^2(\Omega)^3$ in such a way that*

$$\begin{cases} \mathrm{curl}(H) = J & \text{on } \Omega, \\ \mathrm{div}(H) = 0 & \text{on } \Omega, \\ H \bullet n_{\partial\Omega} = 0 & \text{on } \partial\Omega, \\ \langle H, V\rangle = 0 & \text{for every } V \in \mathbb{H}(\Omega). \end{cases}$$

The Hodge decomposition theorem for $L^2(\Omega)^3$ ensures that such a system has a unique solution, and that $\mathbb{H}(\Omega)$ is isomorphic to the first de Rham cohomology group of Ω. The latter fact implies that $\mathbb{H}(\Omega)$ is finite-dimensional, so that the system can be reformulated in a form suitable for numerical computations. Indeed, if $\{V_1, \ldots, V_g\}$ is a basis of $\mathbb{H}(\Omega)$, then one can rewrite the system replacing the last equation (which encodes an infinite number of conditions) with the following finite number of conditions:

$$\langle H, V_i\rangle = 0 \quad \text{for every } i \in \{1, \ldots, g\}.$$

The method of cutting surfaces allows to construct a basis of $\mathbb{H}(\Omega)$ as follows. Suppose that there exist *pairwise disjoint* oriented Lipschitz connected surfaces $\Sigma_1, \ldots, \Sigma_g$ in $\overline{\Omega}$, called *cutting surfaces of* Ω, such that each surface Σ_i intersects transversally $\partial\Omega$ in its boundary $\partial\Sigma_i$ and the open set $\dot{\Omega} := \Omega \setminus \bigcup_{i=1}^{g} \Sigma_i$ is "simple" in the following sense:

$$\text{every curl-free vector field in } L^2(\dot{\Omega})^3 \text{ is the weak gradient of a function in } W^{1,2}(\dot{\Omega}). \tag{2}$$

Let $i \in \{1, \ldots, g\}$ and let n_{Σ_i} be the unit normal L^∞-vector field of Σ_i. Given $\phi \in W^{1,2}(\dot{\Omega})$, we denote by $[\phi]_{\Sigma_i}$ the jump function $\phi|_{\Sigma_i^+} - \phi|_{\Sigma_i^-}$ of ϕ across Σ_i. Similarly, we denote by $[\partial\phi/\partial n_{\Sigma_i}]_{\Sigma_i}$ the jump function $(\partial\phi/\partial n_{\Sigma_i})|_{\Sigma_i^+} - (\partial\phi/\partial n_{\Sigma_i})|_{\Sigma_i^-}$ of the normal derivative $\partial\phi/\partial n_{\Sigma_i}$ of ϕ (see Remark 2.1 of [10] for further details).

Consider the following problem: *find $\phi_i \in W^{1,2}(\dot{\Omega})$ such that*

$$\begin{cases} \Delta\phi_i = 0 & \text{on } \dot{\Omega}, \\ \partial\phi_i/\partial n_{\partial\Omega} = 0 & \text{on } \partial\Omega \setminus \displaystyle\bigcup_{i=1}^{g} \partial\Sigma_i, \\ [\partial\phi_i/\partial n_{\Sigma_j}]_{\Sigma_j} = 0 & \text{for every } j \in \{1, \ldots, g\}, \\ [\phi_i]_{\Sigma_j} = \delta_{ij} & \text{for every } j \in \{1, \ldots, g\}. \end{cases}$$

This system has a simple variational formulation, which implies the existence of a solution, that is unique up to an additive constant (see Lemma 1.2 of [11]). It follows that the weak gradient V_i of ϕ_i on $\dot{\Omega}$ uniquely defines an element of $L^2(\Omega)^3$. The set $\{V_1, \ldots, V_g\}$ is a basis of $\mathbb{H}(\Omega)$ (see Lemma 1.3 of [11]). The fact that the cutting surfaces Σ_i of Ω are pairwise disjoint is of crucial importance in the above procedure. We refer the reader to

[2–5,10–12,18] for some applications and further results concerning the method of cutting surfaces.

4.2. Helmholtz domains

The previous discussion motivates the following:

Definition 4.1. A domain $\Omega \subset \mathbb{R}^3$ with locally flat boundary is *Helmholtz* if there exists a finite family $\mathcal{F} = \{\Sigma_i\}$ (called *cut system for* Ω) of disjoint properly embedded (connected) surfaces in $(\overline{\Omega}, \partial\Omega)$, such that every connected component of $\Omega_C(\mathcal{F})$ is a simple domain. The symbol $\Omega_C(\mathcal{F})$ denotes the open subset of $\mathbb{R}^3$ obtained from Ω by the cut/open operation along the Σ_i's.

We are going to provide an exhaustive and simple characterization of Helmholtz domains (and of their cut systems). We say that a cut system for Ω is *minimal* if it does not properly contain any cut system for Ω. Of course, every cut system contains a minimal cut system.

Lemma 4.2. *Suppose $\mathcal{F}$ is a minimal cut system for Ω. Then $\Omega_C(\mathcal{F})$ is connected. Moreover, every surface of $\mathcal{F}$ has nonempty boundary.*

Proof. Let $\Omega_1, \ldots, \Omega_k$ be the connected components of $\Omega_C(\mathcal{F})$ and suppose by contradiction $k \geq 2$. Then we can find a connected surface $\Sigma_0 \in \mathcal{F}$ which lies "between" two distinct Ω_i's. We now show that the family $\mathcal{F}' = \mathcal{F} \setminus \{\Sigma_0\}$ is a cut system for Ω, thus obtaining the desired contradiction.

Up to reordering the Ω_i's, we may suppose that (parallel copies of) Σ_0 lie in the boundary of both Ω_{k-1} and Ω_k, so that $\Omega_C(\mathcal{F}') = \Omega'_1 \cup \cdots \cup \Omega'_{k-1}$, where $\Omega'_i = \Omega_i$ for every $i \in \{1, \ldots, k-2\}$, Σ_0 is properly embedded in Ω'_{k-1} and $\Omega_{k-1} \cup \Omega_k$ is obtained by cutting Ω'_{k-1} along Σ_0. Since $\mathcal{F}$ is a cut system for Ω, the modules $H^1(\Omega_{k-1}; \mathbb{R})$ and $H^1(\Omega_k; \mathbb{R})$ are null. By Theorem 3.2, it follows that Ω_{k-1} and Ω_k are simply connected. But Σ_0 is connected, so an easy application of van Kampen's Theorem (see e.g. [47]) ensures that Ω'_{k-1} is also simply connected, whence simple. Therefore $\mathcal{F}'$ is a cut system for Ω.

We have thus proved the first statement of the lemma. Suppose by contradiction that an element of $\mathcal{F}$, say Σ_0, has nonempty boundary. Then Proposition 2.2 implies that Σ_0 disconnects S^3, and this implies in turn that Σ_0 disconnects Ω, a contradiction. □

Definition 4.3. A (3-dimensional) 1-*handle* is a topological pair (M, A) homeomorphic to the pair $(D^2 \times [0, 1], D^2 \times \{0, 1\})$. Equivalently, the pair (M, A) is a 1-handle if M is homeomorphic to the 3-disk D^3 (which, of course, is in turn homeomorphic to $D^2 \times [0, 1]$) and A is the union of two disjoint 2-disks in ∂M. The connected components of A are the *attaching* 2-*disks* of M, while if $B \subset M$ corresponds to $D^2 \times \{1/2\}$ under a homeomorphism $(M, A) \cong (D^2 \times [0, 1], D^2 \times \{0, 1\})$, then B is a *co-core* of M.

A *handlebody* $\overline{H}$ in S^3 is a compact submanifold with boundary of S^3 which is constructed by attaching a finite number of 1-handles to a finite number of 3-disks of S^3. More precisely, a handlebody is a compact connected submanifold $\overline{H} \subseteq S^3$ with locally flat boundary that can be decomposed as the union of a finite number of 3-disks (that are usually called the 0-*handles* of $\overline{H}$) and a finite number of 1-handles in such a way that the following conditions hold: the 0-handles are pairwise disjoint; the 1-handles are pairwise disjoint; if Z is the union of the 0-handles and U is the union of the 1-handles of the decomposition, then $Z \cap U$ coincides with the union of the attaching 2-disks of the 1-handles; the previous conditions imply that these attaching 2-disks are pairwise disjoint subsets that lie on the union of the boundaries of the 0-handles. We notice that the decomposition of a handlebody $\overline{H}$ into handles is not unique, even up to isotopy. An *open handlebody* is just a domain H whose closure $\overline{H}$ in S^3 is a handlebody.

The very same definition also works in the smooth (rather than locally flat) case, provided that a "rounding the corners" procedure is carried out along the boundaries of the attaching 2-disks.

Remark 4.4. It is readily seen that a subset $\overline{H}$ of S^3 is a handlebody if and only if it is isotopic to a regular neighborhood of a finite connected spatial graph Γ (i.e. a 1-dimensional compact connected polyhedron) in S^3. Such a graph is a *spine* of $\overline{H}$.

Every open handlebody H is Helmholtz: a cut system $\mathcal{M}$ for H is easily constructed by taking one co-core for every 1-handle of $\overline{H}$, since in this case the result $H_C(\mathcal{M})$ of cutting H along $\mathcal{M}$ is just the union of the internal parts of the 0-handles of $\overline{H}$, that are 3-balls. It is not hard to see that $\mathcal{M}$ contains a subfamily $\mathcal{M}'$ of co-cores such that $H_C(\mathcal{M}')$ is just one 3-ball. We call such an $\mathcal{M}'$ a *minimal system of meridian 2-disks for* H. An easy argument using the Euler–Poincaré characteristic shows that the number $g(\overline{H})$ of 2-disks in a minimal system of meridian 2-disks for H is equal to the genus $g(\partial H)$ of ∂H. In particular, this number does not depend on the handle-decomposition of $\overline{H}$, it is denoted by $g(\overline{H})$ and called the *genus* of $\overline{H}$. Via "handle sliding", it can be easily shown that two handlebodies are (abstractly) homeomorphic if and only if they have the same genus. It follows from the definitions that 3-disks are the handlebodies of genus 0, while solid tori are the handlebodies of genus 1.

4.3. A characterization of Helmholtz domains

We are now ready to state the main result of this section.

Theorem 4.5. *Let Ω be a domain with locally flat boundary and let $S_0, \ldots, S_h$ be the connected components of $\partial\Omega$, ordered as in Lemma 2.4. Then Ω is a Helmholtz domain if and only if the following two conditions hold:*

(1) *The domains $\Omega(S_0)$ and $\Omega^*(S_j)$, $j = 1, \ldots, h$, are open handlebodies in S^3.*
(2) *Every $\Omega(S_j)$, $j = 1, \ldots, h$, is contained in a 3-disk of S^3 embedded in $\Omega(S_0)$, and these 3-disks are pairwise disjoint.*

Moreover, if Ω is Helmholtz, then there exists a cut system $\mathcal{F}$ for Ω such that each element of $\mathcal{F}$ is a properly embedded 2-disk in $(\overline{\Omega}, \partial\Omega)$, and $\Omega_C(\mathcal{F})$ consists of one "external" 3-ball with a finite number of "internal" pairwise disjoint 3-disks removed. In particular, $\Omega_C(\mathcal{F})$ is connected and simply connected.

Proof. We can suppose as usual that Ω has smooth boundary. Assume that Ω verifies (1) and (2). Thanks to these conditions, it is possible to choose a minimal system $\mathcal{M}_0$ of meridian 2-disks for $\Omega(S_0)$ and, for every $i \in \{1, \ldots, h\}$, a minimal system $\mathcal{M}_i$

of meridian 2-disks for $\Omega^*(S_i)$ in such a way that 2-disks belonging to distinct $\mathcal{M}_i$'s, $i = 0, 1, \ldots, h$, are pairwise disjoint. It is now readily seen that $\bigcup_{i=0}^{h} \mathcal{M}_i$ provides the cut system required in the last statement of the theorem. In particular, Ω is Helmholtz.

Let us concentrate on the converse implication. Denote by $\mathcal{F}$ an arbitrary cut system for the Helmholtz domain Ω. According to the definition of the cut/open operation along $\mathcal{F}$, we have $\Omega_C(\mathcal{F}) = \Omega \setminus \bigcup_{\Sigma \in \mathcal{F}} U_\Sigma$, where each U_Σ is a bicollar of $(\Sigma, \partial\Sigma)$ in $(\overline{\Omega}, \partial\Omega)$, and these bicollars are pairwise disjoint. Hence $\overline{\Omega}$ can be reconstructed starting from $\overline{\Omega_C(\mathcal{F})}$ by attaching to its boundary the U_Σ's along the surfaces Σ^+ and Σ^- corresponding to $\Sigma \times \{\pm 1\}$ in $\Sigma \times [-1, 1] \cong U_\Sigma$. By Theorem 3.2, every component of $\Omega_C(\mathcal{F})$ consists of an "external" 3-ball with a finite number of "internal" pairwise disjoint 3-disks removed, so the boundary components of $\Omega_C(\mathcal{F})$ are spheres. It follows that every surface Σ is planar, whence homeomorphic either to the 2-sphere or to D_k^2 for some nonnegative integer k, where D_k^2 is the closure in $\mathbb{R}^2$ of a 2-disk D^2 with k disjoint 2-disks removed from its interior.

We conclude the proof of the theorem in two steps. We first assume that all the surfaces of a given cut system $\mathcal{F}$ of the Helmholtz domain Ω are 2-disks. Next we show how every arbitrarily given cut system $\mathcal{F}$ can be replaced by one consisting of 2-disks only.

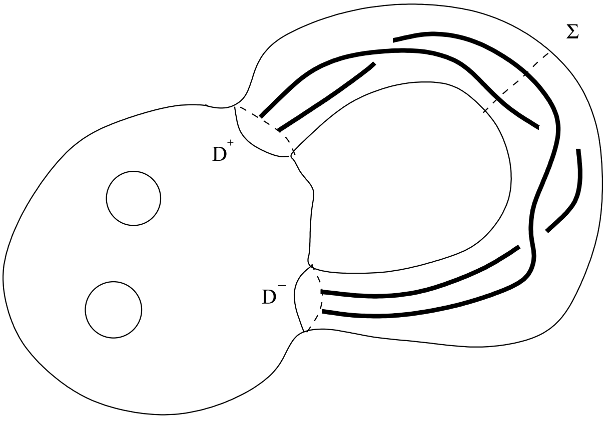
Step 1. Suppose that every surface in $\mathcal{F}$ is a 2-disk. By Lemma 4.2, up to replacing $\mathcal{F}$ with a minimal cut system contained in $\mathcal{F}$, we may suppose that $\Omega_C(\mathcal{F})$ is connected, so that it consists of just one "external" 3-ball B_0 with a finite number of "internal" pairwise disjoint 3-disks removed. Observe that we can reconstruct $\overline{\Omega}$ starting from $\overline{\Omega_C(\mathcal{F})}$ just by attaching to $\overline{\Omega_C(\mathcal{F})}$ one 1-handle for each 2-disk in $\mathcal{F}$: if D is such a disk, the corresponding 1-handle coincides with the removed tubular neighborhood $D \times [-1, 1]$ of D in $\overline{\Omega}$, in such a way that the attaching 2-disks are identified with $D \times \{-1, 1\}$. Let us first consider the 1-handles attached to $\overline{B}_0$. By the very definitions, the internal part $\Omega(S_0)$ of the union of $\overline{B}_0$ with these 1-handles is an open handlebody. Let $T_1, \ldots, T_h$ be the internal boundary spheres of $\Omega_C(\mathcal{F})$ and, for each $j \in \{1, \ldots, h\}$, let B_j be the internal part of the 3-disk D_j bounded by T_j. Now $\overline{\Omega}$ is obtained by attaching to each T_j a finite number of 1-handles contained in the corresponding D_j. This description provides a realization of each $\Omega^*(S_j)$, $j = 1, \ldots, h$, as an open handlebody. Note that every $\Omega(S_j)$, $j = 1, \ldots, h$, is contained in the corresponding B_j. Moreover, $\mathcal{F}$ coincides with the family obtained by taking one co-core 2-disk for each added 1-handle. This completes the proof under the assumption that every surface in $\mathcal{F}$ is a 2-disk.

Step 2. Suppose now that $\mathcal{F}$ is arbitrary, and let us show that it is possible to replace $\mathcal{F}$ with a cut system containing only 2-disks.

Up to replacing $\mathcal{F}$ with a minimal cut system, we may assume that every element of $\mathcal{F}$ is homeomorphic to D_k^2 for some nonnegative k, and that $\Omega_C(\mathcal{F})$ is connected, so that it consists of just one external 3-ball B_0 with a finite number of internal pairwise disjoint 3-disks $D_1, \ldots, D_l$ removed. We denote by T_0 the 2-sphere bounding B_0 and by T_i the 2-sphere bounding D_i, $i = 1, \ldots, l$, and we observe that, under the above assumptions, for every surface $\Sigma \in \mathcal{F}$, there exists $i \in \{0, \ldots, l\}$ such that both Σ^+ and Σ^- are contained in T_i.

We now show that, if $\Sigma \in \mathcal{F}$ is homeomorphic to D_k^2 for some $k \geq 1$, then we can obtain a new cut system $\mathcal{F}'$ from $\mathcal{F}$ by replacing Σ with two properly embedded 2-disks. Such a cut system contains a minimal cut system $\mathcal{F}''$ with a smaller number (with respect to $\mathcal{F}$) of

Please cite this article in press as: R. Benedetti, et al., The topology of Helmholtz domains, Expo. Math. (2012), doi:10.1016/j.exmath.2012.09.001



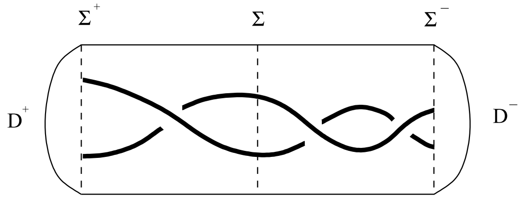


Fig. 5. Diskal vs. planar co-cores: the dashed lines represent Σ, Σ^+ and Σ^-, while the thickened strings represent the "holes" of $D^2_k \times [-1, 1]$ (here $k = 2$).

surfaces that are not disks. Together with an obvious inductive argument, this easily implies that, if Ω is Helmholtz, then it admits a cut system consisting of 2-disks only, whence the conclusion. So let T_i be the component of $\partial \Omega_C(\mathcal{F})$ containing Σ^+ and Σ^-, choose a boundary component γ of D^2_k and denote by γ^+, γ^- the curves on T_i corresponding to $\gamma \times \{-1\}$, $\gamma \times \{1\}$ under the identification of $\Sigma \times \{\pm 1\}$ with $\Sigma^+ \subset T_i$ and $\Sigma^- \subset T_i$. If D_{γ^+} is the 2-disk on T_i bounded by γ^+ and containing Σ^+, we slightly push the internal part of D_{γ^+} into $\Omega_C(\mathcal{F})$ thus obtaining a 2-disk D^+ properly embedded in $\Omega_C(\mathcal{F})$ such that $\partial D^+ = \gamma^+$ (see Fig. 5). The same procedure applies to γ^- providing a 2-disk D^- properly embedded in $\Omega_C(\mathcal{F})$, and of course we may also assume that D^+ and D^- are disjoint. Also observe that by construction both D^+ and D^- are disjoint from every surface in $\mathcal{F}$.

We now set $\mathcal{F}' = (\mathcal{F} \setminus \{\Sigma\}) \cup \{D^+, D^-\}$. It is easy to see that $\Omega_C(\mathcal{F}')$ is given by the disjoint union of a domain homeomorphic to $\Omega_C(\mathcal{F})$ and a domain Ω' homeomorphic to the internal part of

$$\Big(D^2 \times [-1, -1+\varepsilon]\Big) \cup \Big(D^2_k \times [\varepsilon, 1-\varepsilon]\Big) \cup \Big(D^2 \times [1-\varepsilon, 1]\Big).$$

Now Ω' is homeomorphic to a 3-ball with k pairwise disjoint 3-disks removed, so it is simple. Together with the fact that $\Omega_C(\mathcal{F})$ is simple, this implies that $\mathcal{F}'$ is a cut system for Ω. $\square$

Remark 4.6. Building on Theorem 3.2, and bearing in mind the proof of Theorem 4.5, we can now list some equivalent reformulations of the Helmholtz condition for spatial domains.

(1) A domain Ω of $\mathbb{R}^3$ with locally flat boundary is Helmholtz if and only if there exists a finite family $\{\Omega_i\}_{i\in I}$ of simple domains of $\mathbb{R}^3$ with locally flat boundary such that the following conditions hold: the closures of the Ω_i's are pairwise disjoint, and $\overline{\Omega}$ can be constructed by attaching a finite number of pairwise disjoint 1-handles to $\bigcup_{i\in I}\overline{\Omega}_i$ along the boundary spheres of the Ω_i's. In addition, one may suppose that $\{\Omega_i\}_{i\in I}$ consists of a single simple domain.
(2) A domain Ω of $\mathbb{R}^3$ with locally flat boundary is Helmholtz if and only if there exists a finite family $\{D_i\}_{i\in I}$ of properly embedded 2-disks in $(\overline{\Omega}, \partial\Omega)$ such that $\Omega \setminus \bigcup_{i\in I} D_i$ is simply connected. In particular, as already mentioned in Lemma 4.2, we get an equivalent definition of Helmholtz domains if we admit only cutting surfaces with *nonempty* boundary.
(3) Suppose that Ω is Helmholtz. Then Ω is *weakly Helmholtz*, and every cut system for Ω is a *weak cut system* for Ω (see Section 5 for the definitions of weakly Helmholtz domain and weak cut system). In particular, Proposition 5.19 implies that every cut system for Ω contains at least $b_1(\Omega)$ surfaces. On the other hand, if $\mathcal{F} = \{D_1, \ldots, D_\ell\}$ is a cut system for Ω consisting of properly embedded 2-disks in $(\overline{\Omega}, \partial\Omega)$ such that $\Omega \setminus \bigcup_{i=1}^{\ell} D_i$ is simply connected, then an easy application of the Mayer–Vietoris Theorem implies that ℓ is equal to $b_1(\Omega)$. Therefore, $b_1(\Omega)$ provides the optimal lower bound on the number of surfaces contained in a cut system for Ω.

In Fig. 6, it is drawn a "typical example" of Helmholtz domain: each big circle containing smaller circles represents an "external" 3-ball with a finite number of "internal" pairwise disjoint 3-disks removed, and the remaining bands represent the attached 1-handles.

In some sense, Theorem 4.5 should be considered a *negative* result, as it shows that the topology of Helmholtz domains is necessarily very simple. The following corollary provides an evidence for this claim. Its proof follows immediately from Theorem 4.5 and the discussion in Section 2.3. For simplicity, we say that a link L of S^3 is *Helmholtz* if its complement-domain $\mathsf{C}(L)$ is.

Corollary 4.7. *Given a link L in S^3, the following assertions are equivalent:*

(1) *L is Helmholtz.*
(2) *L is trivial.*
(3) *$\mathsf{B}(L)$ is Helmholtz.*

In Fig. 7, it is drawn a trivial knot (on the left) and its box-domain (on the right). By the preceding corollary, such a box-domain is Helmholtz. The trefoil knot is not trivial, so its box-domain, drawn in Fig. 8, is not Helmholtz. Other examples of Helmholtz and non-Helmholtz domains are described in Fig. 9.

4.4. On re-embeddings of Helmholtz domains

In general, the handlebodies occurring in Theorem 4.5 are knotted. Let us give a precise definition of knotting for handlebodies. A handlebody $\overline{H}$ is *unknotted* if, up to ambient isotopy, it admits a *planar* spine (in the sense of Remark 4.4) contained in $\mathbb{R}^2 \subset \mathbb{R}^3 \subset S^3$.

Please cite this article in press as: R. Benedetti, et al., The topology of Helmholtz domains, Expo. Math. (2012), doi:10.1016/j.exmath.2012.09.001

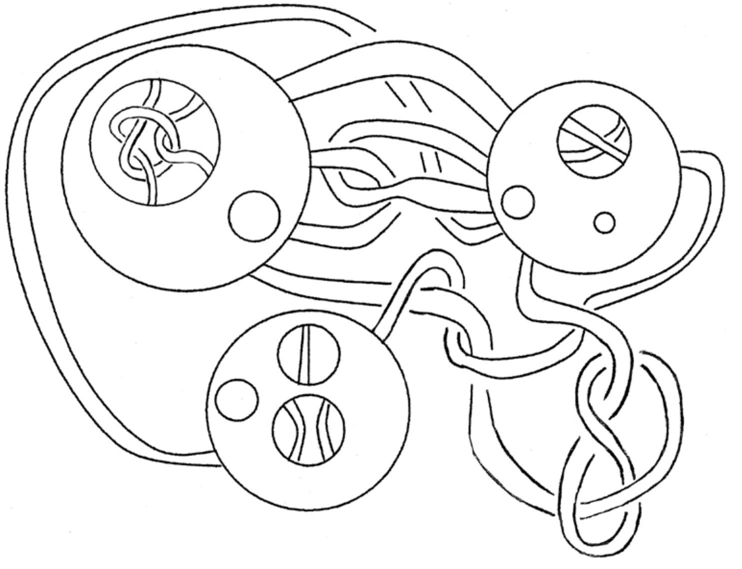


Fig. 6. A Helmholtz domain.

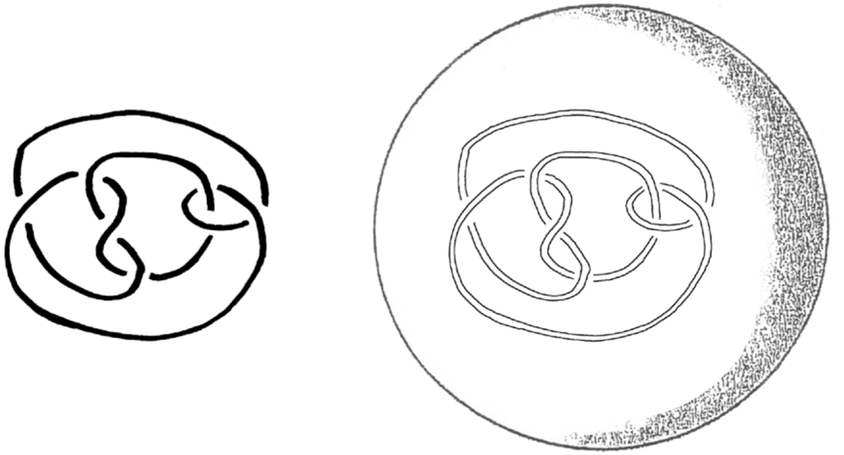


Fig. 7. The box-domain of the trivial knot is Helmholtz.

Thanks to a celebrated theorem of Waldhausen [80,72], this is equivalent to the fact that also $S^3 \setminus \overline{H}$ is a(n open) handlebody: in fact, a realization of S^3 as the union of two handlebodies along their boundaries is a so-called *Heegaard splitting* of S^3, and Waldhausen proved that S^3 admits a unique (up to isotopy) Heegaard splitting of any fixed genus. Just as in the case of knots, we define a *link of handlebodies* in S^3 to be the union of a finite family of disjoint handlebodies. A link of handlebodies is *geometrically*

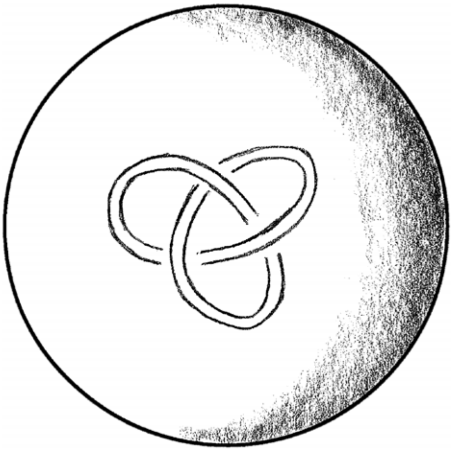


Fig. 8. A box-domain of a trefoil knot is not Helmholtz.

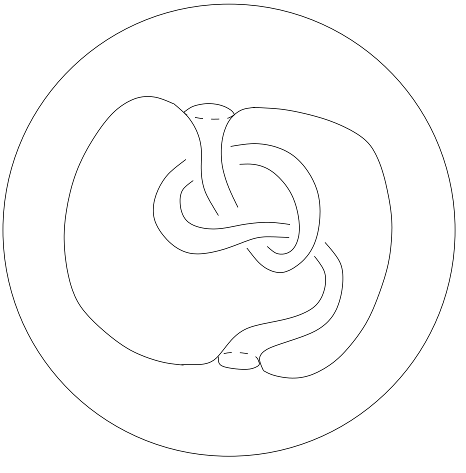


Fig. 9. A knotted torus inside a sphere. The union of these two surfaces disconnects $\mathbb{R}^3$ into three connected components, two of which, being bounded, are domains. The "most internal" domain is homeomorphic to the complement domain of the trefoil knot, so it is not Helmholtz. The domain with two boundary components is homeomorphic to a solid torus with a disk removed, so it is Helmholtz.

unlinked if its components are contained in pairwise disjoint 3-disks of S^3, and it is *trivial* if it is geometrically unlinked and has unknotted components. Equivalently, a link of handlebodies is trivial if it admits a planar spine (of course, such a spine is disconnected if the link has at least two components).

Every (possibly knotted) handlebody can be re-embedded in S^3 onto an unknotted one. We can perform such re-embeddings separately for the handlebodies $\Omega(S_0)$ and $\Omega^*(S_j),\ j = 1, \ldots, h$, of Theorem 4.5, thus getting the following:

Corollary 4.8. *A domain Ω of $\mathbb{R}^3$ with locally flat boundary is Helmholtz if and only if it can be re-embedded in S^3 onto a domain Ω', which is the complement of a trivial link of handlebodies.*

It is easily seen that, in the case of the domain shown in Fig. 6, such a re-embedding can be performed just by switching some crossings of the bands representing the 1-handles.

By comparing the previous corollary with the following general (and nontrivial) re-embedding theorem due to Fox [39], we have a further evidence of the fact that the topology of Helmholtz domains is quite elementary.

Theorem 4.9 (*Fox Re-embedding Theorem*)**.** *Every domain Ω of S^3 with locally flat boundary can be re-embedded in S^3 onto a domain Ω', which is the complement of a link of handlebodies.*

5. Weakly Helmholtz domain

In this section, we propose and discuss a strictly weaker notion of "domains that simplify after suitable cuts". The notion we are introducing defines a class of domains that are very well-behaved with respect to the method of cutting surfaces. Moreover, this class has the advantage of covering a much wider range of topological models.

In order to save words, from now on, if M is a compact oriented 3-manifold with (locally flat) boundary, we call *system of surfaces* in M any finite family $\mathcal{F} = \{\Sigma_i\}$ of disjoint oriented connected surfaces properly embedded in M. We stress that every element of a system of surfaces is connected and oriented, and that the elements of such a system are pairwise disjoint. We begin with a definition in the spirit of condition (b) of Section 3.

Definition 5.1. A domain Ω with locally flat boundary is *weakly Helmholtz* if it admits a system of surfaces $\mathcal{F}$ (called a *weak cut system for Ω*) such that, for every connected component Ω^0 of $\Omega_C(\mathcal{F})$, the following condition holds: the restriction to Ω^0 of every curl-free smooth vector field defined *on the whole of Ω* is the gradient of a smooth function on Ω^0.

It readily follows from the preceding definition and from Theorem 4.5 that every Helmholtz domain is weakly Helmholtz. As already mentioned in the introduction, we are not able to provide a description of weakly Helmholtz domains as effective as the classification of Helmholtz domains given by Theorem 4.5. Nevertheless, Theorem 5.4 provides an interesting characterization of weakly Helmholtz domains in terms of classical properties of manifolds and of their fundamental group.

We devote the whole section to the proof of Theorem 5.4. In fact, by taking advantage of the techniques developed below, at the end of the section we will be able to establish a slightly stronger result (see Theorem 5.20). Further obstructions to be weakly Helmholtz and several examples will be discussed in Section 6.

In order to properly state our characterization of Helmholtz domains we first need the following definitions, which in the case of closed manifolds date back to [76] (see also [46,74]).

Please cite this article in press as: R. Benedetti, et al., The topology of Helmholtz domains, Expo. Math. (2012), doi:10.1016/j.exmath.2012.09.001

Definition 5.2. Let M be a (possibly nonorientable) connected compact 3-manifold with (possibly empty) boundary. The *cut number* $c(M)$ of M is the maximal number of disjoint properly embedded bicollared (connected) surfaces $\Sigma_1, \ldots, \Sigma_k$ in $(M, \partial M)$ such that $M \setminus \bigcup_{i=1}^{k} \Sigma_i$ is connected.

A properly embedded surface in an orientable 3-manifold is bicollared if and only if it is orientable. Therefore, if Ω is a domain with locally flat boundary, then the cut number of $\overline{\Omega}$ is just the maximal cardinality of a system of surfaces that does not disconnect $\overline{\Omega}$.

Definition 5.3. For each nonnegative integer r, we denote by $\mathbb{Z}^{*r}$ the free group of rank r. Given a group Γ, the *corank* of Γ is the maximal nonnegative integer r such that $\mathbb{Z}^{*r}$ is isomorphic to a quotient of Γ.

Recall that $b_1(\overline{\Omega}) = \dim H_1(\overline{\Omega}; \mathbb{R})$ is the first Betti number of $\overline{\Omega}$. We are now ready to state the main result of the section:

Theorem 5.4. *Let $\Omega \subset \mathbb{R}^3$ be a domain with locally flat boundary and let $r := b_1(\overline{\Omega})$. Then the following conditions are equivalent:*

(1) *Ω is weakly Helmholtz.*
(2) *There exists a system of surfaces $\mathcal{F} = \{\Sigma_1, \ldots, \Sigma_r\}$ in $\overline{\Omega}$ such that $\overline{\Omega} \setminus \bigcup_{i=1}^{r} \Sigma_i$ is connected (i.e. $c(\overline{\Omega}) \geq r$).*
(3) *There exists a surjective homomorphism from $\pi_1(\Omega)$ onto $\mathbb{Z}^r$ (i.e. $\operatorname{corank} \pi_1(\overline{\Omega}) \geq r$).*
(4) *$c(\overline{\Omega}) = \operatorname{corank} \pi_1(\overline{\Omega}) = r$.*

Moreover, if Ω is weakly Helmholtz, then every weak cut system for Ω contains a weak cut system $\mathcal{F} = \{\Sigma_1, \ldots, \Sigma_r\}$ such that $\overline{\Omega} \setminus \bigcup_{i=1}^{r} \Sigma_i$ is connected, and every $\Sigma_i \in \mathcal{F}$ has nonempty boundary.

It was first proved by Stallings [76] that the equality $c(M) = \operatorname{corank} \pi_1(M)$ holds in general for every M as in Definition 5.2 (see also [74] for a detailed proof). For the sake of completeness, in Proposition 5.13 we give a proof of such an equality in the case we are interested in, i.e. when $M = \overline{\Omega}$ for some domain Ω with smooth boundary. Our proof of Proposition 5.13 closely follows Stallings' original argument.

5.1. Topological reformulations of the weakly Helmholtz condition

Just as we did in Section 3, we begin by giving some topological reformulations of the definition of weakly Helmholtz domain. As usual, it is not restrictive to work in the framework of domains with smooth boundary.

If $f: M \to N$ is a smooth map between smooth manifolds, we denote by f_* and f^* the induced maps in homology and cohomology. With a common abuse, we denote by the same symbol f^* both the map induced on singular cohomology (with real or integer coefficients) and the map induced on de Rham cohomology. In the same way, f_* denotes both the map on homology with integer coefficients, and the map on homology with real coefficients. It is worth mentioning that these choices do not lead to any ambiguity, due to the easy (but very important) fact that both the de Rham isomorphism and the isomorphisms provided by the Universal Coefficient Theorem are *natural*, in the following sense:

Please cite this article in press as: R. Benedetti, et al., The topology of Helmholtz domains, Expo. Math. (2012), doi:10.1016/j.exmath.2012.09.001

Proposition 5.5. *Let $f: M \to N$ be a smooth map, and fix $i \in \mathbb{N}$. Then, under the identifications*

$$H^i(M;\mathbb{R}) \cong H^i_{DR}(M), \qquad H^i(N;\mathbb{R}) \cong H^i_{DR}(N)$$

provided by de Rham Theorem, the maps

$$f^*: H^i(N;\mathbb{R}) \to H^i(M;\mathbb{R}), \qquad f^*: H^i_{DR}(N) \to H^i_{DR}(M)$$

coincide. Moreover, under the identifications

$$H^i(M;\mathbb{R}) \cong \mathrm{Hom}_{\mathbb{R}}(H_i(M;\mathbb{R}),\mathbb{R}), \qquad H^i(N;\mathbb{R}) \cong \mathrm{Hom}_{\mathbb{R}}(H_i(N;\mathbb{R}),\mathbb{R})$$

provided by the Universal Coefficient Theorem, the map $f^*: H^i(N;\mathbb{R}) \to H^i(M;\mathbb{R})$ coincides with the dual map of $f_*: H_i(M;\mathbb{R}) \to H_i(N;\mathbb{R})$.

In particular, we have $f^* = 0$ on de Rham cohomology if and only if $f^* = 0$ on singular cohomology with real coefficients, and this last condition is in turn equivalent to the fact that $f_* = 0$ on singular homology with real coefficients.

Easy proofs of Proposition 5.5 may be found e.g. in [29,47]. We are now ready to express in (co)homological terms the condition defining weakly Helmholtz domains:

Proposition 5.6. *Let Ω be a domain with smooth boundary, let $\mathcal{F}$ be a system of surfaces in $\overline{\Omega}$ and let $\Omega_1, \ldots, \Omega_k$ be the connected components of $\Omega_C(\mathcal{F})$. For $j \in \{1,\ldots,k\}$, let also $i_j: \Omega_j \longrightarrow \Omega$ be the inclusion. Then $\mathcal{F}$ is a weak cut system for Ω if and only if one of the following equivalent conditions hold:*

(β_1) *For every $j \in \{1,\ldots,k\}$, the image of $i_j^*: H^1_{DR}(\Omega) \longrightarrow H^1_{DR}(\Omega_j)$ vanishes.*

(β_2) *For every $j \in \{1,\ldots,k\}$, the image of $i_j^*: H^1(\Omega;\mathbb{R}) \longrightarrow H^1(\Omega_j;\mathbb{R})$ vanishes.*

(β_3) *For every $j \in \{1,\ldots,k\}$, the image of $(i_j)_*: H_1(\Omega_j;\mathbb{R}) \longrightarrow H_1(\Omega;\mathbb{R})$ vanishes.*

Proof. The fact that $\mathcal{F}$ is a weak cut system for Ω if and only if (β_1) holds is a consequence of the canonical isomorphism between vector fields and 1-forms, and conditions (β_1), (β_2) and (β_3) are equivalent by Proposition 5.5. □

5.2. Poincaré–Lefschetz duality

In order to study weakly Helmholtz domains, it is now convenient to establish some more results about the algebraic topology of an arbitrary domain. In what follows we will try to keep our arguments as self-contained and elementary as possible. Nevertheless, the description of weakly Helmholtz domains naturally leads to subtleties that can be faced only with the help of more advanced techniques. In particular, we will use less elementary (but still "classical") tools such as *relative* homology and *Poincaré–Lefschetz duality*. The reader may find in [47] a complete proof of Poincaré–Lefschetz Duality Theorem, and a thorough discussion of all the notions and the results needed below. For the convenience of the reader we recall a version of Poincaré–Lefschetz duality that will be extensively used in the sequel:

Theorem 5.7 (*Poincaré–Lefschetz Duality*)**.** *Let M be a compact orientable manifold with (possibly empty) boundary ∂M, and let $n = \dim M$. Then for every $i \in \mathbb{N}$ there exists a*

Please cite this article in press as: R. Benedetti, et al., The topology of Helmholtz domains, Expo. Math. (2012), doi:10.1016/j.exmath.2012.09.001

canonical isomorphism

$$H^i(M;\mathbb{Z}) \to H_{n-i}(M,\partial M;\mathbb{Z}).$$

Since the contents of the following sections are less elementary than the ones introduced so far, in the sequel we are forced to switch to a bit more formal approach, because otherwise our discussion would probably become quite lengthy. However, in order to preserve as much as possible the geometric (rather than algebraic) flavor of our arguments, we will often describe algebraic notions in terms of geometric ones via an extensive use of transversality. More precisely, we will often exploit the fact that, if M is a smooth oriented n-dimensional manifold with (possibly empty) boundary ∂M, where $n = 2, 3$, then every k-dimensional (relative) homology class in $(M, \partial M)$ with integer coefficients can be geometrically represented by a smooth oriented closed k-manifold (properly) embedded in M. Moreover, the *algebraic intersection* between a k-dimensional and an $(n-k)$-dimensional class (which plays a fundamental role in Poincaré–Lefschetz duality) can be realized geometrically by taking transverse geometric representatives of the classes involved and counting the intersection points with suitable signs depending on orientations.

Before going on, we need to recall some definitions and elementary results about finitely generated $\mathbb{Z}$-modules (i.e. finitely generated Abelian groups). Let A be any such module. The elements $a_1,\dots,a_r$ of A are *linearly independent* if, whenever $c_1,\dots,c_r \in \mathbb{Z}$ are such that $\sum_{i=1}^r c_i a_i = 0$, then $c_i = 0$ for every i (in particular, a set of linearly independent elements do not contain torsion elements). A finite set $a_1,\dots,a_r$ is a *basis* of A if, for every $a \in A$, there exists a unique r-tuple of coefficients $(c_1,\dots,c_r) \in \mathbb{Z}^r$ such that $a = \sum_{i=1}^r c_i a_i$ or, equivalently, if the a_i's are linearly independent and generate A. Of course, A is free if and only if it admits a basis, and in this case the rank of A is just the cardinality of any of its bases. In general (i.e. when the finitely generated $\mathbb{Z}$-module A is not assumed to be free), A decomposes as a direct sum $F(A) \oplus T(A)$, where $F(A)$ is free and $T(A)$ is the finite submodule of torsion elements of A. Then, the rank of A is defined as the rank of $F(A)$. If Λ is a submodule of A, then $\operatorname{rank}\Lambda \le \operatorname{rank} A$, and $\operatorname{rank}\Lambda = \operatorname{rank} A$ if and only if Λ has finite-index in A.

We now fix a domain $\Omega \subset \mathbb{R}^3$ with smooth boundary. The following lemma is an immediate consequence of the Universal Coefficient Theorem and Poincaré–Lefschetz duality.

Lemma 5.8. *We have*

$$\operatorname{rank} H_2(\overline{\Omega},\partial\Omega;\mathbb{Z}) = b_1(\overline{\Omega}).$$

Proof. We have

$$\operatorname{rank} H_2(\overline{\Omega},\partial\Omega;\mathbb{Z}) = \operatorname{rank} H^1(\overline{\Omega};\mathbb{Z}) = \operatorname{rank} H_1(\overline{\Omega};\mathbb{Z}) = b_1(\overline{\Omega}),$$

where the first equality is due to Poincaré–Lefschetz duality, the second one to the Universal Coefficient Theorem for cohomology, and the third one to the Universal Coefficient Theorem for homology. □

Recall from Section 3.2 that, if $T_n(\overline{\Omega})$ is the submodule of finite-order elements of $H_n(\overline{\Omega},\mathbb{Z}) \cong H_n(\Omega;\mathbb{Z})$, then $T_n(\overline{\Omega})$ is finite for every $n \in \mathbb{N}$ and trivial for every $n > 2$. In fact, one can prove even more:

Lemma 5.9 (*See also [15]*). *We have $T_n(\overline{\Omega}) = 0$ for every $n \in \mathbb{N}$.*

Proof. Of course, it is sufficient to consider the cases $n = 0, 1, 2$. Since the 0-dimensional homology module of any topological space is free, we have $T_0(\overline{\Omega}) = 0$.

In order to prove that $T_1(\overline{\Omega}) = 0$, we define $B = S^3 \setminus \overline{\Omega}$ and observe that $\partial\Omega = \overline{\Omega} \cap \overline{B}$ is the common smooth boundary of Ω and B. Since $\partial\Omega$ admits a bicollar, we may apply the Mayer–Vietoris machinery to the splitting $S^3 = \overline{\Omega} \cup \overline{B}$, thus obtaining the short exact sequence

$$\begin{aligned} 0 = \quad & H_2(S^3; \mathbb{Z}) \longrightarrow H_1(\partial\Omega; \mathbb{Z}) \longrightarrow H_1(\overline{\Omega}; \mathbb{Z}) \oplus H_1(\overline{B}; \mathbb{Z}) \\ \longrightarrow \; & H_1(S^3; \mathbb{Z}) = 0. \end{aligned} \tag{3}$$

Therefore, $H_1(\overline{\Omega}; \mathbb{Z})$ is isomorphic to a submodule of the free $\mathbb{Z}$-module $H_1(\partial\Omega; \mathbb{Z})$, so it is free, and $T_1(\overline{\Omega}) = 0$.

Finally, since $\overline{\Omega}$ is homotopically equivalent to a 2-dimensional CW complex, we have $H^3(\overline{\Omega}; \mathbb{Z}) = 0$. Now the Universal Coefficient Theorem for cohomology gives

$$H^3(\overline{\Omega}; \mathbb{Z}) \cong \big(H_3(\overline{\Omega}; \mathbb{Z})/T_3(\overline{\Omega})\big) \oplus T_2(\overline{\Omega}),$$

so $T_2(\overline{\Omega}) = 0$. $\square$

Lemma 5.9 implies that the natural morphism $H_1(\overline{\Omega}; \mathbb{Z}) \longrightarrow H_1(\overline{\Omega}; \mathbb{Z}) \otimes \mathbb{R} \cong H_1(\overline{\Omega}; \mathbb{R})$ is injective. Therefore, keeping notations from the beginning of Section 5, we obtain that (β_3) is equivalent to condition

(β_4) *For every $j \in \{1, \dots, k\}$, the image of $(i_j)_* : H_1(\Omega_j; \mathbb{Z}) \longrightarrow H_1(\Omega; \mathbb{Z})$ vanishes.*

Assumption. Unless otherwise specified, from now on we only consider homology and cohomology with *integer* coefficients.

We are now able to describe the Poincaré–Lefschetz Duality Theorem completely in terms of intersection of cycles:

Proposition 5.10. *Let us consider the map*

$$\psi \colon H_2(\overline{\Omega}, \partial\Omega) \to \mathrm{Hom}_{\mathbb{Z}}(H_1(\overline{\Omega}), \mathbb{Z})$$

taking a class $[\alpha] \in H_2(\overline{\Omega}, \partial\Omega)$ to the homomorphism which sends every $[\gamma] \in H_1(\overline{\Omega})$ to the algebraic intersection number between $[\alpha]$ and $[\gamma]$. Then ψ establishes an isomorphism

$$H_2(\overline{\Omega}, \partial\Omega) \cong \mathrm{Hom}_{\mathbb{Z}}(H_1(\overline{\Omega}), \mathbb{Z}).$$

Proof. Under the identification $H_2(\overline{\Omega}, \partial\Omega) \cong H^1(\overline{\Omega})$ provided by Poincaré–Lefschetz duality, the Kronecker pairing between $H^1(\overline{\Omega})$ and $H_1(\overline{\Omega})$ induces a pairing

$$\langle \cdot, \cdot \rangle \colon H_2(\overline{\Omega}, \partial\Omega) \times H_1(\overline{\Omega}) \to \mathbb{Z}$$

(we refer the reader to the brief discussion after Theorem 3.4 for the definition of Kronecker pairing). As explained in [47], for every $[\alpha] \in H_2(\overline{\Omega}, \partial\Omega)$, $[\gamma] \in H_1(\overline{\Omega})$, the number $\langle [\alpha], [\gamma] \rangle$ is just the algebraic intersection number between $[\alpha]$ and $[\gamma]$. On the other

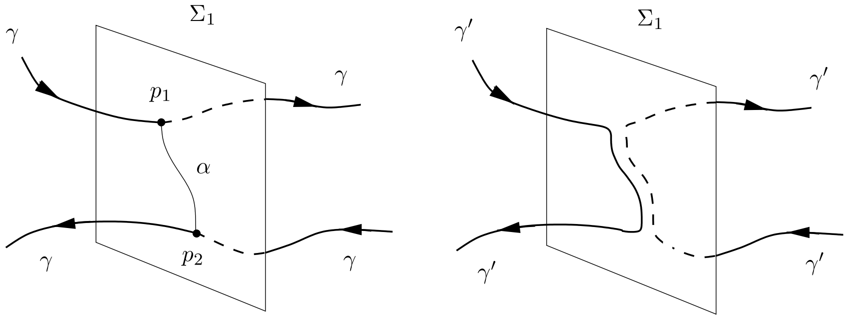


Fig. 10. Modifying γ to obtain γ'.

hand, since $T_0(\overline{\Omega}) = 0$, the Universal Coefficient Theorem for cohomology ensures that the Kronecker pairing induces an isomorphism $H^1(\overline{\Omega}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_1(\overline{\Omega}), \mathbb{Z})$, and this concludes the proof. □

Lemma 5.11. *Let $\mathcal{F} = \{\Sigma_1, \dots, \Sigma_r\}$ be a system of surfaces in $\overline{\Omega}$ and let γ be a 1-cycle (with integer coefficients) in $\overline{\Omega}$ whose algebraic intersection with every Σ_i is null. Then γ is homologous to a 1-cycle γ' supported in $\overline{\Omega} \setminus \bigcup_{i=1}^r \Sigma_i$.*

Proof. Up to homotopy, we may assume that γ is the disjoint union of a finite number of embedded disjoint loops which transversely intersect $\Sigma_1 \cup \cdots \cup \Sigma_r$ in k points $p_1, \dots, p_k \in \Omega$. By an obvious induction argument, it is sufficient to prove that, if $k > 0$, then γ is homologous to a 1-cycle γ' intersecting $\Sigma_1 \cup \cdots \cup \Sigma_r$ in $(k-2)$ points.

Up to reordering the Σ_i's, we may assume that $\gamma \cap \Sigma_1 \neq \emptyset$. Moreover, since the algebraic intersection between γ and Σ_1 is null, up to reordering the p_i's, we may suppose that $\gamma \cap \Sigma_1 = \{p_j,\ 1 \le j \le h\}$ for some $2 \le h \le k$, and that γ intersects Σ_1 in p_1 and p_2 with *opposite* orientations.

Let us choose $\epsilon > 0$ in such a way that γ intersects the tubular neighborhood $N_\epsilon(\Sigma_1)$ of Σ_1 (in $\overline{\Omega}$) in h small segments $\gamma_1, \dots, \gamma_h$ with $p_i \in \gamma_i$ for every i. Since Σ_1 is connected, if α is a path on Σ_1 connecting p_1 and p_2, then we can define the 1-cycle γ' as follows (see Fig. 10): we remove γ_1 and γ_2 from γ and we concatenate the resulting paths with the paths obtained by pushing α on the boundary components of $N_\epsilon(\Sigma_1)$ in Ω. Since γ intersects Σ_1 in p_1 and p_2 with opposite orientations, the 1-cycle γ' is the disjoint union of a finite number of embedded loops which can be oriented in such a way that $[\gamma'] = [\gamma]$ in $H_1(\overline{\Omega})$. This concludes the proof. □

If Λ is a submodule of the finitely generated $\mathbb{Z}$-module A, then we say that Λ is *full* if it is not a proper finite-index submodule of any other submodule of A. Therefore, if Λ is full and rank Λ = rank A, then $\Lambda = A$.

Lemma 5.12. *Let $\mathcal{F} = \{\Sigma_1, \dots, \Sigma_r\}$ be a system of surfaces in $\overline{\Omega}$ and let $[\Sigma_i] \in H_2(\overline{\Omega}, \partial\Omega)$ be the class represented by Σ_i, $i = 1, \dots, r$. Then the following conditions are equivalent:*

(1) *The* $[\Sigma_i]$*'s are linearly independent in* $H_2(\overline{\Omega}, \partial\Omega)$.
(2) *The* $[\Sigma_i]$*'s are linearly independent and generate a full submodule of* $H_2(\overline{\Omega}, \partial\Omega)$.
(3) *The set* $\Omega_C(\mathcal{F})$ *is connected.*

Proof. (1) $\Longrightarrow$ (3) Let $\Omega' := \overline{\Omega} \setminus \bigcup_{i=1}^r \Sigma_i$. Since $\overline{\Omega_C(\mathcal{F})}$ is a strong deformation retract of Ω', it is sufficient to show that Ω' is connected. Suppose by contradiction that Ω' is disconnected and let Ω^0 be a connected component of Ω' with $\partial\overline{\Omega^0} \setminus \partial\Omega = (\Sigma_{j_1} \cup \cdots \cup \Sigma_{j_l}) \setminus \partial\Omega$ (where $j_h \neq j_k$ if $h \neq k$). Then $[\Sigma_{j_1}] + \cdots + [\Sigma_{j_l}] = 0$ in $H_2(\overline{\Omega}, \partial\Omega)$, a contradiction.

(3) $\Longrightarrow$ (2) Recall that, under the isomorphism

$$H_2(\overline{\Omega}, \partial\Omega) \cong \mathrm{Hom}_{\mathbb{Z}}(H_1(\overline{\Omega}), \mathbb{Z})$$

described in Proposition 5.10, the class $[\Sigma_j] \in H_2(\overline{\Omega}, \partial\Omega)$ is identified with the linear map $f_j\colon H_1(\overline{\Omega}) \longrightarrow \mathbb{Z}$ which sends every $[\gamma] \in H_1(\overline{\Omega})$ to the algebraic intersection between Σ_j and γ. Now, since $\Omega_C(\mathcal{F})$ is connected, for every $i \in \{1, \ldots, r\}$ we can construct a loop $\gamma_i \subset \Omega$ which intersects Σ_i transversely in one point and is disjoint from Σ_j for every $j \neq i$. It readily follows that, if $\sum_{j=1}^r c_j f_j = 0$, then, for every $i \in \{1, \ldots, r\}$, we have that $c_i = (\sum_{j=1}^r c_j f_j)(\gamma_i) = 0$, so the $[\Sigma_i]$'s are linearly independent. Let now Λ be the submodule of $\mathrm{Hom}_{\mathbb{Z}}(H_1(\overline{\Omega}), \mathbb{Z})$ generated by the f_j's and suppose that Λ' is a submodule of $\mathrm{Hom}_{\mathbb{Z}}(H_1(\overline{\Omega}), \mathbb{Z})$ with $\Lambda \subset \Lambda'$. Also suppose that Λ has finite-index in Λ', and take an element $f \in \Lambda'$. Our assumptions imply that there exists $n \in \mathbb{Z} \setminus \{0\}$ such that $n \cdot f$ lies in Λ, so $n \cdot f$ is a linear combination $\sum_{i=1}^r c_i f_i$ of the f_i's. It follows that $c_i = nf(\gamma_i)$ for every $i = 1, \ldots, r$, so $c_i = nc_i'$ for some $c_i' \in \mathbb{Z}$ and $f = \sum_{i=1}^r c_i' f_i \in \Lambda$. We have thus proved that Λ is full.

(2) $\Longrightarrow$ (1) is obvious. □

5.3. Cut number and corank

Let Ω be a domain with smooth boundary. We define $d(\overline{\Omega})$ as the maximal cardinality of a system of surfaces $\mathcal{F} = \{\Sigma_1, \ldots, \Sigma_r\}$ whose elements define linearly independent elements $[\Sigma_1], \ldots, [\Sigma_r]$ of $H_2(\overline{\Omega}, \partial\Omega)$. Of course we have

$$d(\overline{\Omega}) \leq \mathrm{rank}\, H_2(\overline{\Omega}, \partial\Omega) = b_1(\overline{\Omega}).$$

The following result describes how the invariants $c(\overline{\Omega})$, $\mathrm{corank}\,\pi_1(\overline{\Omega})$ and $d(\overline{\Omega})$ are related to each other.

Proposition 5.13. *The following equalities hold:*

$$d(\overline{\Omega}) = c(\overline{\Omega}) = \mathrm{corank}\,\pi_1(\overline{\Omega}).$$

Proof. The equality $d(\overline{\Omega}) = c(\overline{\Omega})$ is an immediate consequence of Lemma 5.12. Therefore, in order to conclude, it is sufficient to prove the inequalities $c(\overline{\Omega}) \leq \mathrm{corank}\pi_1(\overline{\Omega}) \leq d(\overline{\Omega})$.

Let $\mathcal{F} = \{\Sigma_1, \ldots, \Sigma_r\}$ be a system of surfaces in $\overline{\Omega}$ such that $\overline{\Omega} \setminus \bigcup_{i=1}^r \Sigma_i$ is connected and let B_r be the wedge of r copies $S_1^1, \ldots, S_r^1$ of the circle, with base point x_0. Also recall that the fundamental group $\pi_1(B_r, x_0)$ is freely generated by the (classes of the) loops $\gamma_1, \ldots, \gamma_r$, where $\gamma_j\colon [0,1] \longrightarrow S^1_j$ is a generator of $\pi_1(S^1_j, x_0)$ (in particular, $\gamma(0) = \gamma(1) = x_0$). By a classical Pontryagin–Thom construction (see [65]), we can construct a continuous map

$$f = f_{\mathcal{F}}\colon \overline{\Omega} \longrightarrow B_r$$

as follows. Consider a system of disjoint closed bicollars U_j of the Σ_j's in $\overline{\Omega}$ and fix diffeomorphic identifications $U_j \cong \Sigma_j \times [0,1]$, $j = 1, \ldots, r$. Then we set $f(x,t) = \gamma_j(t)$ if $(x,t) \in U_j$, and $f(q) = x_0$ if $q \in M \setminus \bigcup_{j=1}^r U_j$. Since $\overline{\Omega} \setminus \bigcup_{j=1}^r U_j$ is connected, it is easily seen that, if p is any basepoint in $\overline{\Omega} \setminus \bigcup_{j=1}^r U_j$, then the map $f_*\colon \pi_1(\overline{\Omega}, p) \longrightarrow \pi_1(B_r, x_0)$ is surjective. We have thus shown that $c(\overline{\Omega}) \leq \operatorname{corank} \pi_1(\overline{\Omega})$.

In order to prove that $\operatorname{corank} \pi_1(\overline{\Omega}) \leq d(\overline{\Omega})$, we can argue as follows. Let $r = \operatorname{corank} \pi_1(\overline{\Omega})$ and take a surjective homomorphism $\phi : \pi_1(\overline{\Omega}) \longrightarrow \mathbb{Z}^{*r}$. As B_r is a $K(\mathbb{Z}^{*r}, 1)$ space with contractible universal covering (see [47]), there exists a continuous surjective map $f : \overline{\Omega} \longrightarrow B_r$ such that $\phi = f_*$. Up to homotopy, we can assume that the restriction of f to $f^{-1}(B_r \setminus \{x_0\})$ is smooth. By Morse–Sard Theorem (see [65,50]), we can choose a regular value $x_j \in S^1_j \setminus \{x_0\}$ and define $N_j := f^{-1}(x_j)$ for every $j \in \{1, \ldots, r\}$. Then N_j is a finite union of disjoint properly embedded surfaces in $\overline{\Omega}$. Moreover, if we fix an orientation on every S^1_j, then we can define an orientation on N_j by the usual "first the outgoing normal vector" rule, where a vector v is outgoing in $q \in N_j$ if $df(v)$ is positively oriented as a vector of the tangent space to S^1_j in $f(q)$. Let now p be a basepoint in $f^{-1}(x_0) \subset \overline{\Omega}$ and let α_j be a loop in Ω based at p whose homotopy class $[\alpha_j] \in \pi_1(\overline{\Omega}, p)$ is sent by $\phi = f_*$ onto a generator of $\pi_1(S^1_j, x_0) < \pi_1(B_r, x_0)$. Up to homotopy, we may suppose that the intersection between α_j and N_j is transverse. Moreover, by the very construction of α_j, the algebraic intersection between α_j and N_k is equal to 1 if $j = k$ and to 0 otherwise. In particular, there exists a connected component Σ_j of N_j such that the algebraic intersection of α_j with Σ_k is not null if and only if $k \neq j$. By Poincaré–Lefschetz duality, this readily implies that $\Sigma_1, \ldots, \Sigma_r$ represent linearly independent elements of $H_2(\overline{\Omega}, \partial\Omega)$. This gives in turn the inequality $\operatorname{corank} \pi_1(\overline{\Omega}) \leq d(\overline{\Omega})$. □

Since $d(\overline{\Omega}) \leq b_1(\overline{\Omega})$, Proposition 5.13 immediately implies the following result.

Corollary 5.14. *We have* $c(\overline{\Omega}) = \operatorname{corank} \pi_1(\overline{\Omega}) \leq b_1(\overline{\Omega})$.

Remark 5.15. As mentioned above, the relations $c(M) = \operatorname{corank} \pi_1(M) \leq b_1(M)$ hold in general, i.e. even when M is any (possibly nonorientable) compact manifold. In fact, the proof of Proposition 5.13 can be easily adapted to show that $c(M) = \operatorname{corank} \pi_1(M)$. Moreover, if $\operatorname{corank} \pi_1(M) = r$, then there exists a surjective homomorphism from $\pi_1(M)$ to the Abelian group $\mathbb{Z}^r$. As a consequence of the classical Hurewicz Theorem (see e.g. [47]), such a homomorphism factors through $H_1(M)$, whose rank is therefore at least r. This readily implies the inequality $\operatorname{corank} \pi_1(M) \leq b_1(M)$.

5.4. Topological characterizations of weakly Helmholtz domains

We say that a weak cut system $\mathcal{F}$ for Ω is *minimal* if no proper subset of $\mathcal{F}$ is a weak cut system for Ω. It follows from the definitions that every system of surfaces containing a

Please cite this article in press as: R. Benedetti, et al., The topology of Helmholtz domains, Expo. Math. (2012), doi:10.1016/j.exmath.2012.09.001

weak cut system is itself a weak cut system, so a system of surfaces is a weak cut system if and only if it contains a minimal weak cut system.

The following result shows that, just as in the case of Helmholtz domains, every weakly Helmholtz domain admits a nondisconnecting cut system. Moreover, each surface of such a cut system has nonempty boundary. As usual, let $\Omega \subset \mathbb{R}^3$ be a domain with smooth boundary.

Lemma 5.16. *Let $\mathcal{F}$ be a minimal weak cut system for Ω. Then $\Omega_C(\mathcal{F})$ is connected, and every surface of $\mathcal{F}$ has nonempty boundary.*

Proof. Let $\Omega_1, \ldots, \Omega_k$ be the connected components of $\Omega_C(\mathcal{F})$, and suppose by contradiction that $k \geq 2$. Then we can find a connected surface $\Sigma_0 \in \mathcal{F}$ which lies "between" two distinct Ω_i's. We set $\mathcal{F}' = \mathcal{F} \setminus \{\Sigma_0\}$ and show that $\mathcal{F}'$ is a weak cut system for Ω, thus contradicting the minimality of $\mathcal{F}$.

Up to reordering the Ω_i's, we may suppose that (two parallel copies of) Σ_0 lie in the boundary of both Ω_{k-1} and Ω_k, so that $\Omega_C(\mathcal{F}') = \Omega'_1 \cup \cdots \cup \Omega'_{k-1}$, where $\Omega'_i = \Omega_i$ for every $i \in \{1, \ldots, k-2\}$, Σ_0 is properly embedded in Ω'_{k-1} and $\Omega_{k-1} \cup \Omega_k$ is obtained by cutting Ω'_{k-1} along Σ_0. We now claim that every 1-cycle in Ω'_{k-1} decomposes, up to boundaries, as the sum of a cycle supported in Ω_{k-1} and a cycle supported in Ω_k. In fact, since Σ_0 disconnects Ω'_{k-1}, the homology class represented by Σ_0 in $H_2(\Omega'_{k-1}, \partial\Omega'_{k-1})$ is null. This implies that the algebraic intersection between Σ_0 and any 1-cycle in Ω'_{k-1} is null, and the claim now follows from Lemma 5.11.

The claim just proved implies that the image of $(i'_{k-1})_* : H_1(\Omega'_{k-1}) \longrightarrow H_1(\Omega)$ equals the sum of the images of $(i_{k-1})_*\colon H_1(\Omega_{k-1}) \longrightarrow H_1(\Omega)$ and of $(i_k)_*\colon H_1(\Omega_k) \longrightarrow H_1(\Omega)$, which are both trivial, because $\mathcal{F}$ satisfies condition (β_4). Therefore, the image of $(i'_j)_*$ vanishes for every $j \in \{1, \ldots, k-1\}$, so $\mathcal{F}'$ is a weak cut system for Ω. We have thus obtained the desired contradiction and proved that $\Omega_C(\mathcal{F})$ is connected.

Suppose now that an element of $\mathcal{F}$, say Σ_0, has nonempty boundary. Then Proposition 2.2 implies that Σ_0 disconnects S^3, and this implies in turn that S disconnects Ω, a contradiction. □

Lemma 5.17. *Let $\mathcal{F} = \{\Sigma_1, \ldots, \Sigma_r\}$ be a system of surfaces in $\overline{\Omega}$ and let $\Lambda \subset H_2(\overline{\Omega}, \partial\Omega)$ be the submodule generated by the classes $[\Sigma_1], \ldots, [\Sigma_r]$ represented by the Σ_i's. The system $\mathcal{F}$ is a weak cut system if and only if* rank $\Lambda = b_1(\overline{\Omega})$.

Proof. We claim that $\mathcal{F}$ is a weak cut system for Ω if and only if the following condition holds:

- if $[\gamma] \in H_1(\overline{\Omega})$ has null algebraic intersection with every $[\Sigma_i]$, $i = 1, \ldots, r$, then $[\gamma] = 0$ in $H_1(\overline{\Omega})$.

In fact, suppose that $\mathcal{F}$ is a weak cut-system and let $[\gamma] \in H_1(\overline{\Omega})$ have null algebraic intersection with every $[\Sigma_i]$, $i = 1, \ldots, r$. Then, by Lemma 5.11, we can suppose that $[\gamma]$ is represented by a 1-cycle supported in $\Omega_C(\mathcal{F})$. This implies that, if $\Omega_1, \ldots, \Omega_k$ are the connected components of $\Omega_C(\mathcal{F})$, then $[\gamma] = \sum_{i=1}^{k} [\gamma_i]$ in $H_1(\overline{\Omega})$, where the 1-cycle γ_i is supported in Ω_i for every i. But condition (β_4) implies that, if $\mathcal{F}$ is a weak cut system, then $[\gamma_i] = 0$ in $H_1(\overline{\Omega})$ for every i, so $[\gamma]$ is homologically trivial in Ω.

On the other hand, if the inclusion $i_j\colon \Omega_j \longrightarrow \overline{\Omega}$ induces a nontrivial homomorphism $(i_j)_*\colon H_1(\Omega_j) \longrightarrow H_1(\overline{\Omega})$, then every nonnull class $[\gamma]$ in $\mathrm{Im}\,(i_j)_*$ has null algebraic intersection with every $[\Sigma_i]$, $i = 1, \ldots, r$. This concludes the proof of the claim.

For every $j \in \{1, \ldots, r\}$, let now $f_j\colon H_1(\overline{\Omega}) \longrightarrow \mathbb{Z}$ be the linear map corresponding to $[\Sigma_j]$ under the identification

$$H_2(\overline{\Omega}, \partial\Omega) \cong \mathrm{Hom}_{\mathbb{Z}}\,(H_1(\overline{\Omega}), \mathbb{Z})$$

provided by Proposition 5.10. The claim above shows that $\mathcal{F}$ is a weak cut system for Ω if and only if

$$\bigcap_{i=1}^{r} \mathrm{Ker}(f_i) = \{0\}.$$

It is now easy to check that this last condition is satisfied if and only if the f_i's generate a finite-index submodule of $\mathrm{Hom}_{\mathbb{Z}}\,(H_1(\overline{\Omega}), \mathbb{Z})$, i.e. if and only if $\mathrm{rank}\,\Lambda = \mathrm{rank}\,H_2(\overline{\Omega}, \partial\Omega) = b_1(\overline{\Omega})$. □

Corollary 5.18. *Every weak cut system for Ω contains at least $b_1(\overline{\Omega})$ surfaces.*

We summarize the results obtained so far in Proposition 5.19 and Theorem 5.20, which provide a characterization of weakly Helmholtz domains and of their weak cut systems.

Proposition 5.19. *Let $\mathcal{F} = \{\Sigma_1, \ldots, \Sigma_r\}$ be a system of surfaces in $\overline{\Omega}$, and let $[\Sigma_i] \in H_2(\overline{\Omega}, \partial\Omega)$ be the class represented by Σ_i, $i = 1, \ldots, r$. Then the following conditions are equivalent:*

(1) *$\mathcal{F}$ is a minimal weak cut system for Ω.*
(2) *$r = b_1(\overline{\Omega})$ and $\Omega_C(\mathcal{F})$ is connected.*
(3) *The $[\Sigma_i]$'s provide a basis of $H_2(\overline{\Omega}, \partial\Omega)$.*
(4) *$r = b_1(\overline{\Omega})$ and the $[\Sigma_i]$'s are linearly independent elements in $H_2(\overline{\Omega}, \partial\Omega)$.*

Proof. Let us denote by Λ the submodule of $H_2(\overline{\Omega}, \partial\Omega)$ generated by the $[\Sigma_i]$'s.

(1) $\Longrightarrow$ (2) By Lemma 5.16, the minimality of $\mathcal{F}$ implies that $\Omega_C(\mathcal{F})$ is connected. Moreover, Lemma 5.12 implies that Λ is free of rank r, while Lemma 5.17 gives $\mathrm{rank}\,\Lambda = b_1(\overline{\Omega})$.

(2) $\Longrightarrow$ (3) Since $\Omega_C(\mathcal{F})$ is connected, Lemma 5.12 implies that Λ is freely generated by the $[\Sigma_i]$'s, so condition (2) ensures that $\mathrm{rank}\,\Lambda = r = b_1(\overline{\Omega}) = \mathrm{rank}\,H_2(\overline{\Omega}, \partial\Omega)$. Moreover, again by Lemma 5.12, the submodule Λ is full, so it is equal to the whole of $H_2(\overline{\Omega}, \partial\Omega)$, and the $[\Sigma_i]$'s provide a basis of $H_2(\overline{\Omega}, \partial\Omega)$.

(3) $\Longrightarrow$ (4) is obvious.

(4) $\Longrightarrow$ (1) Condition (4) implies that $\mathrm{rank}\,\Lambda = \mathrm{rank}\,H_2(\overline{\Omega}, \partial\Omega)$, so Lemma 5.17 implies that $\mathcal{F}$ is a weak cut system for Ω. Moreover, $\mathcal{F}$ is minimal by Corollary 5.18. □

As a consequence of Propositions 5.13 and 5.19, we obtain the following characterization of weakly Helmholtz domains, which implies in particular Theorem 5.4 stated at the beginning of the section.

Theorem 5.20. *Let $\Omega \subset \mathbb{R}^3$ be a domain with locally flat boundary and set $r := b_1(\overline{\Omega})$. Then the following conditions are equivalent:*

ARTICLE IN PRESS

(1) *Ω is weakly Helmholtz.*
(2) *There exists a system of surfaces $\mathcal{F} = \{\Sigma_1, \ldots, \Sigma_r\}$ in $\overline{\Omega}$ such that $\overline{\Omega} \setminus \bigcup_{i=1}^{r} \Sigma_i$ is connected.*
(3) *There exists a basis of $H_2(\overline{\Omega}, \partial\Omega)$ represented by a system of surfaces in $\overline{\Omega}$.*
(4) *There exists a surjective homomorphism from $\pi_1(\Omega)$ onto $\mathbb{Z}^{*r}$.*
(5) $c(\overline{\Omega}) = d(\overline{\Omega}) = \operatorname{corank} \pi_1(\overline{\Omega}) = r$.

Moreover, if Ω is weakly Helmholtz, then every weak cut system for Ω contains a weak cut system $\mathcal{F} = \{\Sigma_1, \ldots, \Sigma_r\}$ such that $\overline{\Omega} \setminus \bigcup_{i=1}^{r} \Sigma_i$ is connected, and every $\Sigma_i \in \mathcal{F}$ has nonempty boundary.

Proof. The equivalence of conditions (1)–(3) is proved in Proposition 5.19. The equivalence between (2) and (4) follows from the equality $c(\overline{\Omega}) = \operatorname{corank} \pi_1(\overline{\Omega})$, and (5) is equivalent to (2) (or to (4)) by Proposition 5.13 and Corollary 5.14. Finally, the last statement of the theorem follows from Lemma 5.16. □

6. Weakly Helmholtz domains: obstructions and examples

In this section we describe some effective obstructions that prevent a domain from being weakly Helmholtz. We then apply these obstructions to the study of families of examples, paying a particular attention to the case of link complements. We begin by introducing a tool that will prove useful to our purposes.

6.1. The intersection form on surfaces

Let S be a connected compact oriented surface. If α, β are 1-cycles on S, up to homotopy, we can suppose that α and β transversely intersect in a finite number of points. We can then define the algebraic intersection between α and β as the difference between the number of points in which they intersect "positively" and the number of points in which they intersect "negatively", with respect to the fixed orientation on S. It is not difficult to show that the algebraic intersection defines a bilinear *skew-symmetric* product on the space of 1-cycles, and that, if a 1-cycle is a boundary, then it has null algebraic intersection with any other 1-cycle. It follows that this bilinear product descends to homology, thus defining a bilinear skew-symmetric intersection form

$$\langle \cdot , \cdot \rangle : H_1(S) \times H_1(S) \longrightarrow \mathbb{Z}.$$

Proposition 5.10 may be adapted to the context of surfaces to show that the identification between $H_1(S)$ and $\operatorname{Hom}_{\mathbb{Z}}(H_1(S), \mathbb{Z})$ induced by $\langle \cdot , \cdot \rangle$ coincides with the identification $H_1(S) \cong H^1(S) \cong \operatorname{Hom}_{\mathbb{Z}}(H_1(S), \mathbb{Z})$ induced by Poincaré–Lefschetz duality and the Universal Coefficient Theorem for cohomology. Using this, it is easy to show that $\langle \cdot , \cdot \rangle$ is nondegenerate. Then, general results about nondegenerate skew-symmetric bilinear forms imply that $H_1(S)$ admits a *symplectic basis*, i.e. a free basis $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ such that $\langle \alpha_i, \alpha_j \rangle = \langle \beta_i, \beta_j \rangle = 0$ and $\langle \alpha_i, \beta_j \rangle = \delta_{ij}$ for every $i, j \in \{1, \ldots, g\}$, where $g = g(S)$ is the genus of S.

A submodule A of $H_1(S)$ is said to be *Lagrangian* if the intersection form of S identically vanishes on $A \times A$.

Please cite this article in press as: R. Benedetti, et al., The topology of Helmholtz domains, Expo. Math. (2012), doi:10.1016/j.exmath.2012.09.001

6.2. More algebraic topology of spatial domains

We are interested in studying the intersection form on the boundary components of domains with smooth boundary. So, let Ω be such a domain, and let $S_0, \dots, S_h$ be the components of $\partial\Omega$. We are now going to describe some important relationships between the homology of Ω and the homology of $\partial\Omega$. We begin with the following:

Lemma 6.1. *The maps* $i_*\colon H_1(\partial\Omega;\mathbb{Z}) \longrightarrow H_1(\overline{\Omega};\mathbb{Z})$ *and* $i_*\colon H_2(\partial\Omega;\mathbb{Z}) \longrightarrow H_2(\overline{\Omega};\mathbb{Z})$, *induced by the inclusion* $i\colon \partial\Omega \hookrightarrow \overline{\Omega}$, *are surjective.*

Proof. Every class in $H_1(\overline{\Omega};\mathbb{Z})$ can be represented by a knot C embedded in Ω. Let $S \subset S^3$ be a Seifert surface for C, which we can assume to be transverse to $\partial\Omega$. Then $S \cap \overline{\Omega}$ realizes a cobordism between C and a smooth curve contained in $\partial\Omega$, thus proving that C is homologous to a 1-cycle in $\partial\Omega$.

Every class in $H_2(\overline{\Omega};\mathbb{Z})$ can be represented by the disjoint union of a finite number of compact smooth orientable surfaces embedded in Ω. Every such surface necessarily separates S^3 (see Proposition 2.2), whence Ω, and is therefore homologically equivalent to a linear combination of boundary components. □

We now consider the following portion of the homology exact sequence of the pair $(\overline{\Omega}, \partial\Omega)$:

$$H_2(\partial\Omega) \longrightarrow H_2(\overline{\Omega}) \xrightarrow{\pi_*} H_2(\overline{\Omega},\partial\Omega) \xrightarrow{\partial} H_1(\partial\Omega) \xrightarrow{i_*} H_1(\overline{\Omega}) \longrightarrow H_1(\overline{\Omega},\partial\Omega)\,. \quad (4)$$

Lemma 6.2. *We have the short exact sequence of free modules:*

$$0 \longrightarrow H_2(\overline{\Omega},\partial\Omega) \xrightarrow{\partial} H_1(\partial\Omega) \xrightarrow{i_*} H_1(\overline{\Omega}) \longrightarrow 0.$$

Moreover, $\operatorname{rank} H_2(\overline{\Omega},\partial\Omega) = \operatorname{rank}\operatorname{Ker}(i_*) = b_1(\overline{\Omega}) = \sum_{j=0}^{h} g(S_j)$.

Proof. By Lemma 6.1, the map π_* in sequence (4) is trivial, so ∂ is injective. Surjectivity of i_* and the fact that $i_*\partial = 0$ follow respectively from Lemma 6.1 and from the exactness of sequence (4). Moreover, we already know that $H_1(\partial\Omega)$ and $H_1(\overline{\Omega})$ are free, so the sequence splits and $H_2(\overline{\Omega},\partial\Omega)$ is also free.

As a consequence of the exactness of the sequence in the statement, we have

$$\operatorname{rank} H_2(\overline{\Omega},\partial\Omega) = \operatorname{rank}\operatorname{Ker}(i_*),$$
$$\operatorname{rank} H_1(\partial\Omega) = \operatorname{rank} H_2(\overline{\Omega},\partial\Omega) + \operatorname{rank} H_1(\overline{\Omega}).$$

Moreover, we have proved in Lemma 5.8 that $\operatorname{rank} H_2(\overline{\Omega},\partial\Omega) = b_1(\overline{\Omega})$, so $\operatorname{rank} H_1(\partial\Omega) = 2\operatorname{rank} H_1(\overline{\Omega})$, i.e. $b_1(\partial\Omega) = 2b_1(\overline{\Omega})$. But homology is additive with respect to the disjoint union of topological spaces, so $b_1(\partial\Omega) = 2\sum_{j=0}^{h} g(S_j)$, whence the conclusion. □

Remark 6.3. Let $K \subset S^3$ be a knot with complement-domain $\Omega = \mathsf{C}(K)$. Lemma 6.2 implies that the kernel of the map $i_*\colon H_1(\partial\Omega) \to H_1(\overline{\Omega})$ is freely generated by the class $[\gamma]$ of a nontrivial loop on $\partial\Omega$. Let S be a Seifert surface for K intersecting $\partial\mathsf{C}(K)$ in a simple loop α parallel to K. Since α bounds the surface $S \cap \overline{\Omega}$ properly embedded in Ω, the class $[\alpha]$ is a multiple of $[\gamma]$, and using that α is simple and not homologically trivial it is not difficult to show that in fact $[\alpha] = \pm[\gamma]$. Finally, two simple closed loops on a torus define the same homology class if and only if they are isotopic, so we can conclude that the isotopy class of the loop obtained as the transverse intersection of $\partial\Omega$ with a Seifert surface for K does not depend on the chosen surface, as claimed in Example 2.6.

6.3. An obstruction to be weakly Helmholtz

As usual, let Ω be a domain with smooth boundary and let $S_0, \ldots, S_h$ be the connected components of $\partial\Omega$. Since homology is additive with respect to the disjoint union of topological spaces, we have a canonical isomorphism $H_1(\partial\Omega) \cong \bigoplus_j H_1(S_j)$, which allows us to define canonical projections $p_j : H_1(\partial\Omega) \longrightarrow H_1(S_j)$, $j = 0, \ldots, h$. If $i_* : H_1(\partial\Omega) \longrightarrow H_1(\overline{\Omega})$ is the homomorphism induced by the inclusion, then we set

$$P_j := p_j(\mathrm{Ker}(i_*)) \subset H_1(S_j), \quad j = 0, \ldots, h.$$

Proposition 6.4. *If Ω is weakly Helmholtz, then P_j is a Lagrangian submodule of $H_1(S_j)$ for every $j \in \{0, \ldots, h\}$.*

Proof. By Theorem 5.20, we can choose a basis of $H_2(\overline{\Omega}, \partial\Omega)$ represented by a system of surfaces $\mathcal{F} = \{\Sigma_1, \ldots, \Sigma_r\}$. By Lemma 6.2, we have that $\mathrm{Ker}(i_*) = \mathrm{Im}\,\partial$, where $\partial : H_2(\overline{\Omega}, \partial\Omega) \longrightarrow H_1(\partial\Omega)$ is the usual "boundary map" of the sequence of the pair $(\overline{\Omega}, \partial\Omega)$. This readily implies that, for every $j \in \{0, \ldots, h\}$, the module P_j is generated by a set of classes which are represented by *pairwise disjoint* 1-cycles, whence the conclusion. $\square$

Example 6.5. As an application of the previous proposition, we can show that the open tubular neighborhood (homeomorphic to $S \times (0, 1)$) of a smooth surface S of genus $g > 0$ is *not* weakly Helmholtz. In fact, if γ is any simple loop on $S \times \{1\}$, then the 1-cycle $(\gamma \times \{1\}) \sqcup (-\gamma \times \{0\})$ bounds the annulus $\gamma \times [0, 1]$, so the class $[\gamma \times \{1\}] - [\gamma \times \{0\}]$ lies in $\mathrm{Im}\,\partial = \mathrm{Ker}(i_*)$. After setting $S_i = S \times \{i\}$, $i = 0, 1$, we have then $P_i = H_1(S_i)$, and P_i is *not* Lagrangian. In Fig. 11, we have drawn an open tubular neighborhood of a torus in $\mathbb{R}^3$ corresponding to the case $g = 1$: such a domain is not weakly Helmholtz.

The following lemma shows that, if $\partial\Omega$ is connected, then Proposition 6.4 does not provide any effective obstruction to be weakly-Helmholtz.

Lemma 6.6. *If the boundary $\partial\Omega = S_0$ is connected, then* $\mathrm{Ker}(i_*) \subset H_1(S_0)$ *is a Lagrangian submodule of $H_1(S_0)$.*

Proof. Let α be a 1-cycle in $\mathrm{Ker}(i_*)$ represented by a smooth loop $C_1 \subset S_0$. If $[\beta]$ is any class in $\mathrm{Ker}(i_*) = \mathrm{Im}\,\partial$, then $[\beta] = \partial[\Sigma]$, where Σ is a properly embedded surface in $(\overline{\Omega}, \partial\Omega)$. Since S_0 admits a collar in $\overline{\Omega}$, we can push α a bit inside Ω and obtain a 1-cycle α' transverse to Σ. Since $[\alpha'] = i_*([\alpha]) = 0$, the algebraic intersection between α' and Σ is null, and this easily implies in turn that $\langle[\alpha], [\beta]\rangle = 0$, whence the conclusion. $\square$

Please cite this article in press as: R. Benedetti, et al., The topology of Helmholtz domains, Expo. Math. (2012), doi:10.1016/j.exmath.2012.09.001

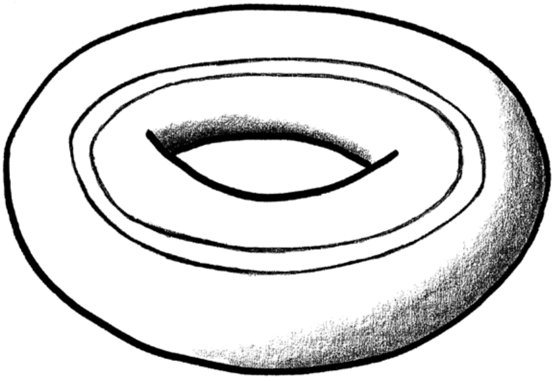


Fig. 11. An open solid torus with a coaxial smaller closed solid torus removed is not weakly Helmholtz.

6.4. Weakly-Helmholtz links

We now concentrate our attention on the study of link complements in S^3. Let L be a link in S^3. We say that L is *weakly Helmholtz* if the complement-domain $\mathsf{C}(L)$ of L is (see Section 2.3). We have the following easy:

Lemma 6.7. *The link L is weakly Helmholtz if and only if its box-domain* $\mathsf{B}(L)$ *is.*

Proof. Recall that $\mathsf{B}(L)$ is obtained by removing a small 3-disk D from $\mathsf{C}(L)$. An easy application of the Mayer–Vietoris machinery now implies that the modules $H_1(\mathsf{C}(L))$ and $H_1(\mathsf{B}(L))$ are isomorphic, so $b_1(\mathsf{C}(L)) = b_1(\mathsf{B}(L))$. On the other hand, an easy application of Van Kampen's Theorem (see e.g. [47]) ensures that the fundamental groups $\pi_1(\mathsf{C}(L))$ and $\pi_1(\mathsf{B}(L))$ are also isomorphic, so $b_1(\mathsf{C}(L)) = \operatorname{corank} \pi_1(\mathsf{C}(L))$ if and only if $b_1(\mathsf{B}(L)) = \operatorname{corank} \pi_1(B(L))$. Now the conclusion follows from Theorem 5.4. □

As a consequence of Corollary 4.7, we know that a knot in S^3 is Helmholtz if and only if it is trivial. On the contrary, the following result shows that every knot is weakly Helmholtz:

Corollary 6.8. *The following statements hold.*

(1) *Every knot in S^3 is weakly Helmholtz.*
(2) *The box-domain of any knot in S^3 is weakly Helmholtz.*

Proof. Let K be a knot, and observe that Lemma 6.2 implies that $b_1(\mathsf{C}(K)) = 1$. Let S be a Seifert surface of a knot K in S^3. Since S does not disconnect the complement-domain $\mathsf{C}(K)$ of K, the equivalence (1) $\Longleftrightarrow$ (2) in Theorem 5.4 immediately implies that K is weakly Helmholtz. This proves (1), and (2) now follows from Lemma 6.7. □

Remark 6.9. The box-domain of a trefoil knot, drawn in above Fig. 8, is an example of weakly Helmholtz domain, which is not Helmholtz.

We have seen in Corollary 6.8 that all knots and all the box-domains of knots are weakly Helmholtz. On the other hand, if L is the Hopf link (see Fig. 12, on the left), then $\mathsf{C}(L)$

Please cite this article in press as: R. Benedetti, et al., The topology of Helmholtz domains, Expo. Math. (2012), doi:10.1016/j.exmath.2012.09.001

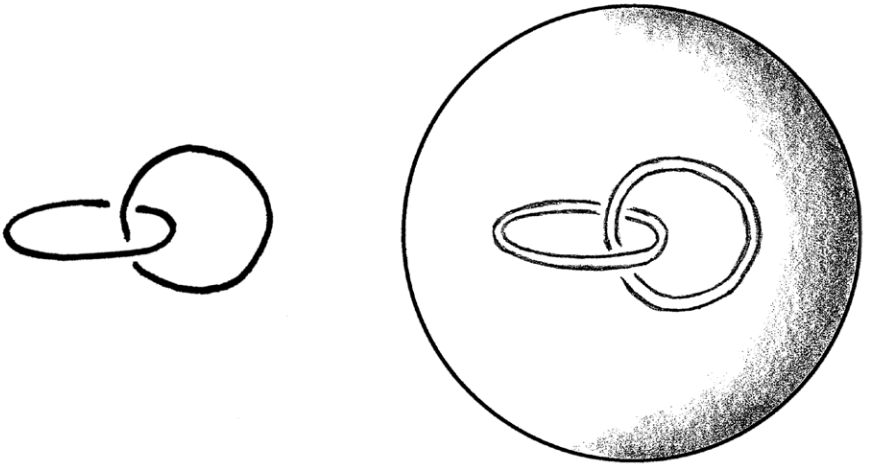


Fig. 12. The box-domain of the Hopf link is not weakly Helmholtz.

is diffeomorphic to an open tubular neighborhood of the standard torus in $\mathbb{R}^3$, so $\mathsf{C}(L)$ is not weakly Helmholtz (see Example 6.5). By Lemma 6.7, the same is true for $\mathsf{B}(L)$. Lemma 6.10 generalizes this result to a large class of links.

We say that two components K_1 and K_2 of L are *algebraically unlinked* if K_1 is homologically trivial in $\mathsf{C}(K_2)$. It turns out that $[K_1] = 0$ in $H_1(\mathsf{C}(K_2))$ if and only if $[K_2] = 0$ in $H_1(\mathsf{C}(K_1))$, so the definition just given is indeed symmetric in K_1 and K_2. Equivalently, K_1 and K_2 are algebraically unlinked if and only if their *linking number* vanishes; moreover the linking number can be easily computed by using any planar link diagram as half the sum of the signs at the crossing points between the two components (see e.g. [69, Section D of Chapter 5]). Clearly, if two components of L are geometrically unlinked (see Section 2.3), then a fortiori they are also algebraically unlinked. The Whitehead link (see Fig. 4 above on the left) is a celebrated example with two components that are algebraically, but not geometrically, unlinked. The components K_1 and K_2 are said to be *algebraically linked* it they are not algebraically unlinked. For example, the Hopf link has algebraically linked components.

Lemma 6.10. *If L has algebraically linked components, then it is not weakly Helmholtz.*

Proof. Take two algebraically linked components C_0 and C_1 of L and let F_0 be an oriented Seifert surface for C_0. As usual, we can assume that F_0 is transverse to C_1 and to the corresponding toric boundary component S_1 of $\partial\mathsf{C}(L) = \partial U(L)$, where $\mathsf{C}(L) = S^3 \setminus U(L)$. Then the class $[\alpha] = p_1(\partial[F_0 \setminus \mathrm{Int}(U(L))]) \in p_1(\mathrm{Ker}(i_*)) \subset H_1(S_1)$ is represented by the oriented intersection between F_0 and S_1, which is given by a finite number of copies of the meridian of S_1 (with possibly different orientations). Since C_0 and C_1 are linked, the class $[\alpha]$ is not null in $H_1(S_1)$, so it is equal to a nontrivial multiple of the class represented by the meridian of S_1. On the other hand, also the class $[\beta]$ of the

Please cite this article in press as: R. Benedetti, et al., The topology of Helmholtz domains, Expo. Math. (2012), doi:10.1016/j.exmath.2012.09.001

preferred longitude on S_1, determined by any Seifert surface of C_1, belongs to $p_1(\mathrm{Ker}(i_*))$, and $\langle[\alpha],[\beta]\rangle \neq 0$, so Proposition 6.4 implies that L is not weakly Helmholtz. □

The following lemma investigates the case of links with algebraically unlinked components.

Lemma 6.11. *Suppose that the components $C_0,\ldots,C_k$ of a link L are pairwise algebraically unlinked. Then there exists a family of smooth surfaces $F_0,\ldots,F_k$ with boundary such that each F_j is a Seifert surface for C_j and, if $i \neq j$, then F_i and F_j (transversely) intersect only in $\mathsf{C}(L)$. Moreover, if $i_j\colon S_j \longrightarrow \overline{\mathsf{C}(L)}$ is the inclusion of the boundary component corresponding to C_j and Q_j is the kernel of $(i_j)_*\colon H_1(S_j) \longrightarrow H_1(\overline{\mathsf{C}(L)})$, then Q_j is generated by (the class of) the preferred longitude of C_j, and* $\mathrm{Ker}(i_*) = \bigoplus_j Q_j$.

Proof. Fix $j \in \{0,\ldots,k\}$ and take an arbitrary Seifert surface F'_j of C_j transverse to every $C_h, h \neq j$. Up to re-defining $\mathsf{C}(L)$ as the complement in S^3 of smaller tubular neighborhoods of the C_h's, we may also assume that, for each fixed $h \neq j$, F'_j intersects transversely each S_h in a finite number $m_1,\ldots,m_l$ of copies of the meridian of S_h (with possibly different orientations), in such a way that each m_i bounds a 2-disk D_i in the interior of F'_j. Since the algebraic intersection of C_j and C_h is null, we also have $[m_1]+\cdots+[m_l] = 0$ in $H_1(S_h)$, so the number of positively oriented meridians occurring in the oriented intersection $F'_j \cap S_h$ is equal to the number of negatively oriented meridians in the same intersection.

Let us now remove the D_i's, $i = 1,\ldots,l$, from the interior of F'_j. In this way, we obtain a properly embedded surface with more boundary components. We can now glue in pairs the added boundary components by attaching $l/2$ disjoint annuli parallel to S_h to $l/2$ pairs of meridians in $F'_j \cap S_h$ having opposite orientations. After applying this procedure to every $h \neq j$, we obtain the desired Seifert surface F_j that misses all the $S_h,\ h \neq j$.

Now, if $[l_j] \in H_1(S_j)$ is the class of the preferred longitude of C_j, then $[l_j] = \partial[F_j]$, so $[l_j]$ lies in Q_j and hence $\mathrm{rank} \bigoplus_j Q_j = k+1 = \mathrm{rank}\,\mathrm{Ker}(i_*)$. Then the conclusion follows from the fact that $\bigoplus_j Q_j$ is a full submodule of $H_1(\partial\Omega)$. □

One may wonder if the Seifert surfaces of the previous lemma can be chosen to be pairwise disjoint. A classical definition is in order (see [69, p. 137]).

Definition 6.12. A link L is a *boundary link* if it admits a system of *disjoint* Seifert surfaces for its components.

Of course, every knot is a boundary link. Every link L with geometrically unlinked components is a boundary link as well, since every component C of L admits a Seifert surface contained in the 3-disk that separates C from the other components (see [69]). However, there are boundary links that have geometrically linked components. For example, every 2-component links given by the union of a nontrivial knot and its preferred longitude is a boundary link (see Fig. 14).

On the other hand, the Whitehead link (see the top of Fig. 4, on the left) provides an example of a link with algebraically unlinked components which is *not* a boundary link (see again [69, p. 137], and Example 6.16 for an even stronger result). This shows that,

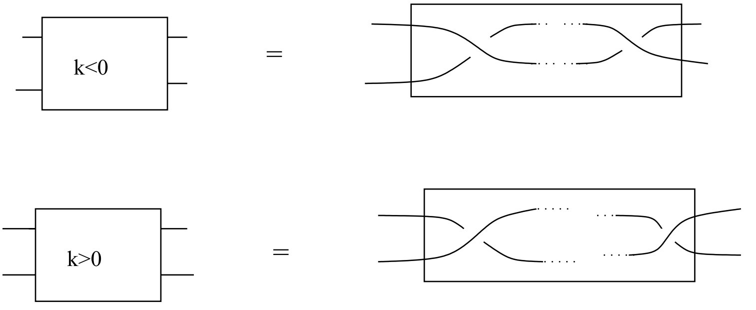


Fig. 13. The boxes used in Figs. 14 and 18: the integer k denotes the number of positive or negative half-twists.

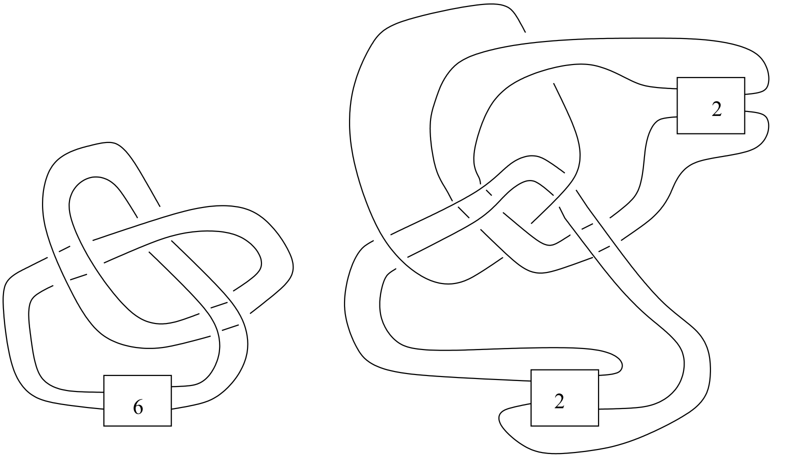


Fig. 14. On the left, the boundary link given by the union of the trefoil knot and its preferred longitude. On the right, a more complicated boundary link. The meaning of the labeled boxes is explained in Fig. 13.

in general, it is not possible to remove the internal intersections of the Seifert surfaces provided by Lemma 6.11 by local "cut and paste" operations along the intersection lines.

Let L be an r-component link. Lemma 6.2 implies that $b_1(\mathsf{C}(L)) = r$. Therefore, in the case of links we may rephrase Theorem 5.4 as follows:

Corollary 6.13. *A link L with r components is weakly Helmholtz if and only if there is a surjective homomorphism from $\pi_1(\mathsf{C}(L))$ to $\mathbb{Z}^{*r}$.*

We recognize that the condition described in the last corollary is just one possible *definition* of *homology boundary links*, so a link is weakly Helmholtz if and only if it is a homology boundary link. More precisely, putting together Corollary 6.13 and Lemma 6.7, we obtain the following:

Please cite this article in press as: R. Benedetti, et al., The topology of Helmholtz domains, Expo. Math. (2012), doi:10.1016/j.exmath.2012.09.001

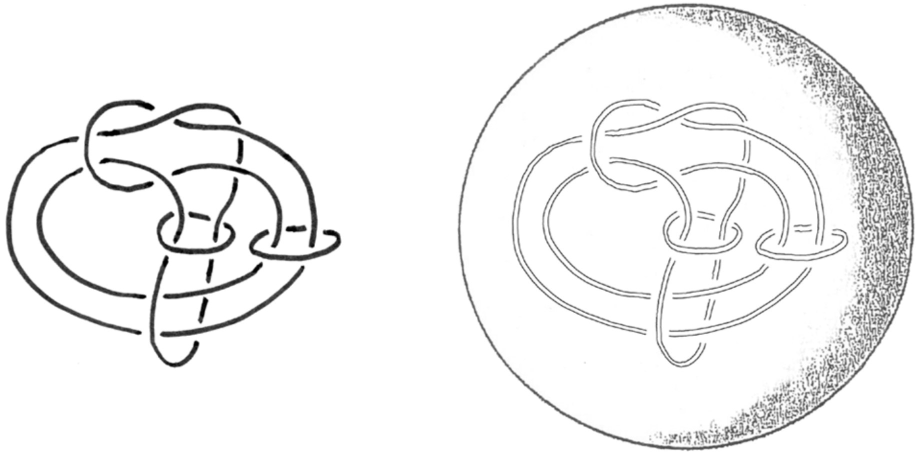


Fig. 15. A box-domain of a homology boundary link is weakly Helmholtz.

Corollary 6.14. *Given a link L in S^3, the following assertions are equivalent:*

(1) *L is weakly Helmholtz.*
(2) *L is a homology boundary link.*
(3) $\mathsf{B}(L)$ *is weakly Helmholtz.*

Every classical boundary link is a homology boundary link. In fact, L is a boundary link if and only if there exists a surjective homomorphism $\phi\colon \pi_1(\mathsf{C}(L)) \longrightarrow \mathbb{Z}^{*r}$ that sends the meridians of the link onto a set of generators of $\mathbb{Z}^{*r}$. This characterization of boundary links was originally given in [75] (see also [45]), where the definition of homology boundary links was first introduced. The link drawn in Fig. 15 (on the left) is a homology boundary link, which is not a boundary link (see [49, p. 170]). By Corollary 6.14, the box-domain of such a link (see Fig. 15, on the right) is weakly Helmholtz.

Homology boundary links are an intriguing, very important class of links widely studied in knot theory. It is a nice occurrence that the method of cutting surfaces naturally leads to this distinguished class of links.

6.5. Other examples of weakly Helmholtz domains

Getting an exhaustive description of weakly Helmholtz domains, similar to the characterization of Helmholtz ones given in Theorem 4.5, looks somehow hopeless. This already holds true in the case of links. Note that, once a concrete link L is given (for instance by means of a planar diagram), it is very easy to decide whether the components of L are pairwise algebraically linked or not. On the contrary, answering the question whether L is homology boundary or not is in general quite hard (for example, some nontrivial argument is needed even for showing that the Whitehead link is not homology boundary—see Example 6.16). The general case of arbitrary domains is even more complicated. Up to "Fox re-embedding" (see Theorem 4.9), it is not restrictive to deal with domains Ω that are the complements of links of handlebodies. As every handlebody is the regular

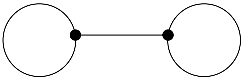


Fig. 16. Planar handcuff graph.

neighborhood of a spine, which is a compact graph embedded in S^3 (i.e. a *spatial graph*), if Γ is a *link of spines*, then we can naturally extend our previous notation and denote by $\mathsf{C}(\Gamma)$ the complement-domain of Γ. In the case of a classical link L, i.e. in the case of a link of genus 1 handlebodies, we have in some sense a "canonical" spine for $\mathsf{C}(L)$: the link L itself. This is no longer true in the general case, in the sense that a link of handlebodies, considered up to isotopy, can admit essentially different (links of) spines. This implies further complications in the study of general weakly Helmholtz domains. The analysis of these complications basically represents the starting point of the investigations developed in [26].

For example, let us consider the simple case of just one genus 2 handlebody $\overline{H}$. Every such handlebody admits a spine Γ, which is a spatial embedding of the so-called "handcuff graph" (a planar realization of which is shown in Fig. 16).

If we remove from Γ the interior of the edge that connects the two cycles (i.e. the "isthmus" of Γ), then we get a classical link L_Γ with two components. We set $\Omega = \mathsf{C}(\Gamma)$ and $\Omega' = \mathsf{C}(L_\Gamma)$. Clearly $\overline{\Omega} \subset \overline{\Omega'}$, as $\overline{\Omega}$ is obtained from $\overline{\Omega'}$ by removing a 1-handle.

The following proposition allows us to construct many examples both of links with two algebraically unlinked components that are not homology boundary links, and of knotted genus 2 handlebodies having weakly Helmholtz complements.

Proposition 6.15. *With the notations just introduced, the following results hold:*

(1) *If L_Γ is a homology boundary link, then Ω is weakly Helmholtz.*
(2) *Suppose that $\overline{H}$ is unknotted. Then L_Γ is a homology boundary link if and only if Γ is planar. In particular, if L_Γ is nontrivial, then it is not a homology boundary link.*

Proof. By a general position argument, it is easy to see that every loop in $S^3 \setminus L_\Gamma$ is homotopic to a loop that does not intersect the isthmus of Γ. This implies that $i_* : \pi_1(\overline{\Omega}) \to \pi_1(\overline{\Omega'})$ is surjective. Then (1) follows immediately from Theorem 5.4 and Corollary 6.13.

Let us now suppose that $\overline{H}$ is unknotted. We have to show that, if L_Γ is homology boundary, then Γ is planar, the converse implication being trivial. Waldhausen's result about the uniqueness of Heegaard splittings of S^3 implies that $\overline{\Omega}$ is an unknotted genus 2 handlebody, so $\pi_1(\overline{\Omega}) \cong \mathbb{Z}^{*2}$. Therefore, since L_Γ is homology boundary, we have a sequence of surjective homomorphisms

$$\mathbb{Z}^{*2} \cong \pi_1(\overline{\Omega}) \to \pi_1(\overline{\Omega'}) \to \mathbb{Z}^{*2} .$$

But free groups are Hopfian (see [59]), which means that every surjective homomorphism of $\mathbb{Z}^{*2}$ onto itself is in fact an isomorphism, so all the above maps are isomorphisms, and $\pi_1(\Omega')$ is isomorphic to $\mathbb{Z}^{*2}$. Under this hypothesis, a generalization to links (see for instance Theorem 1.1 in [49]) of Papakyriakopoulos unknotting theorem for knots [67] ensures that L_Γ is trivial. We can now apply the main theorem of [71] and conclude that Γ is planar. □

Please cite this article in press as: R. Benedetti, et al., The topology of Helmholtz domains, Expo. Math. (2012), doi:10.1016/j.exmath.2012.09.001

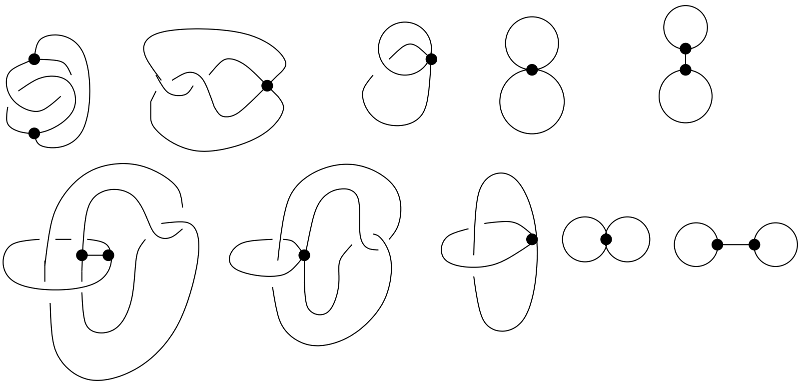


Fig. 17. All the graphs described here are spines of the same unknotted handlebody. In particular, the unknotted handlebody admits a spine whose constituent link is the Hopf link, and a spine whose constituent link is the Whitehead link.

We stress that $\overline{H}$ may admit *infinitely many* handcuff spines with pairwise *non*isotopic associated links (see the examples below). Hence, if Ω is not weakly Helmholtz, point (1) of the above proposition implies that no such link is homology boundary. However, checking whether this last condition is satisfied seems to be very demanding.

Example 6.16. (1) In Figs. 17 and 18 we show some spatial handcuff graphs Γ that become planar via a finite sequence of spine modifications that do not affect the isotopy type of the regular neighborhood $\overline{H}$.

The fact that the spines described here can be modified into planar graphs shows that, in every case, $\overline{H}$ is unknotted, so, by point (2) of Proposition 6.15, we see that all the corresponding nontrivial links L_Γ are not homology boundary. The top part of Fig. 17 provides an example of the somewhat counterintuitive fact that $\overline{H}$ may be unknotted even if L_Γ is algebraically linked. The bottom part establishes that the Whitehead link is not homology boundary. The examples described in Fig. 18 provide an infinite family of links having (for $k \neq 0, 1$) algebraically unlinked components that are not homology boundary. Note that every link in the family has one unknotted component. For $k = -1$ (resp. $k = -2$), the knotted component of the link is the trefoil knot (resp. the figure-eight knot). For $k = 2$ we get again the Whitehead link.

(2) If L_Γ has geometrically unlinked components (i.e. if it is a split-link), then $\Omega = \mathsf{C}(\Gamma)$ is weakly Helmholtz by point (1) of Proposition 6.15. If, in addition, we assume that L_Γ is nontrivial, then $\overline{H}$ is knotted by point (2). Remarkably, there exist also examples where $\overline{H}$ is knotted whereas L_Γ is trivial. In fact, it is proved in [58] that the handlebody $\overline{H}$ determined by the spine Γ of Fig. 19 is knotted.

6.6. (Non)weakly Helmholtz domains with connected boundary

As a consequence of Lemma 6.6, the obstructions to be weakly Helmholtz that we have described so far do not apply to domains with connected boundary. In fact, providing examples of domains with smooth connected boundary which are not weakly Helmholtz is

Please cite this article in press as: R. Benedetti, et al., The topology of Helmholtz domains, Expo. Math. (2012), doi:10.1016/j.exmath.2012.09.001

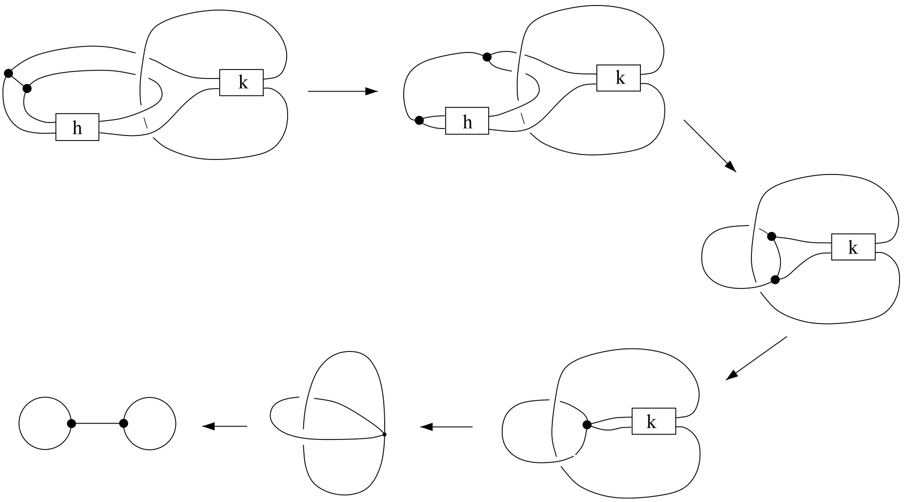


Fig. 18. Spines of the unknotted handlebody. We assume that $h = (-1)^k 2$. The meaning of the labeled boxes is explained in Fig. 13.

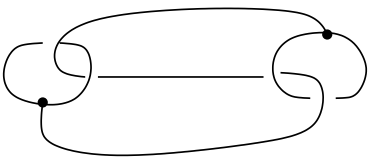


Fig. 19. A knotted handlebody whose complement is a weakly Helmholtz domain.

a quite challenging task. Fox Re-embedding Theorem implies that, in order to exhibit such examples, we may restrict our attention to handlebody complements. So, let $\Gamma \subseteq S^3$ be a connected graph and let us set $\Omega = \mathsf{C}(\Gamma)$. We also denote by g the genus of $\partial\Omega$. Alexander Theorem implies that, if $g = 0$, then Ω is a ball, so in particular it is Helmholtz (whence weakly Helmholtz). Moreover, if $g = 1$, then Ω is a knot complement, so Corollary 6.8 ensures that Ω is again weakly Helmholtz. However, it turns out that for every $g \geq 2$ there exists a domain Ω with connected boundary of genus g, which is not weakly Helmholtz.

The first example of such a domain is due to Jaco and McMillan [54], who constructed a spatial handlebody of genus 3 whose complement is not weakly Helmholtz. Via a quite sophisticated group theoretic argument, they proved in fact that the fundamental group of the complement-domain of the graph described in Fig. 20 has corank strictly smaller than 3. Subsequently, Jaco proved in [53] that the complement-domain of the graph described in Fig. 21 is not weakly Helmholtz, thus providing the first example of a domain with connected boundary of genus 2 which is not weakly Helmholtz. The proof of this fact given in [53] relies on the following (nontrivial) topological obstruction: if $c(M) \geq 2$, then every map $f : M \to S^1 \times S^1$ is homotopic to a non-surjective map. This obstruction turns

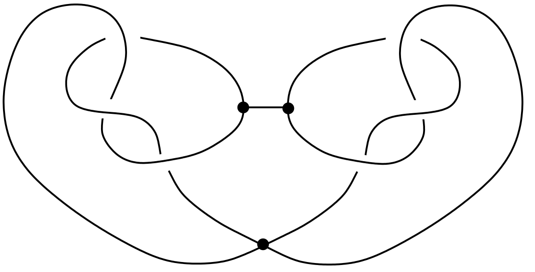


Fig. 20. A graph whose complement-domain is not weakly Helmholtz.

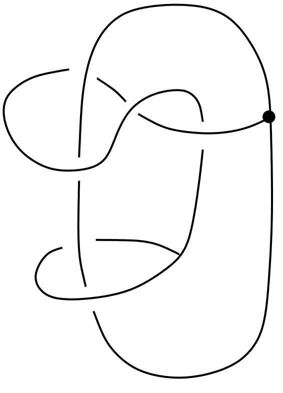


Fig. 21. The complement-domain of this graph is the first known example of a domain with connected boundary of genus 2, which is not weakly Helmholtz.

out to be very useful in the case described in Fig. 21, but it looks not so handy in discussing other examples.

Another interesting example is provided by *Kinoshita θ-graph* Γ_K, which was introduced by Kinoshita in [57], and is described in Fig. 22. It is the spine of the spatial handlebody H_K, whose complement will be denoted by M_K.

Kinoshita analyzed several interesting properties of Γ_K and of M_K. To this aim, he adapted to the case of graphs the classical theory of Alexander modules for links. More precisely, he introduced some elementary ideals $E_d(\Gamma, z)$ associated to any presentation of the Alexander module of the fundamental group of $S^3 \setminus \Gamma$. These ideals turn out to be isotopy invariants of the couple (Γ, z), where Γ is a spatial graph (not necessarily of genus 2) endowed with a $\mathbb{Z}$-cycle z. By means of these invariants he proved for example that the graph Γ_K is knotted. In [77] the author remarked that Kinoshita's invariants (and some variations of them) can be used to face questions concerning the cut number of complements of graphs. More precisely, Kinoshita's invariants can be used to study the corank of the fundamental group of a given graph complement. As an application, he proved that $c(M_K) = 1$.

Please cite this article in press as: R. Benedetti, et al., The topology of Helmholtz domains, Expo. Math. (2012), doi:10.1016/j.exmath.2012.09.001

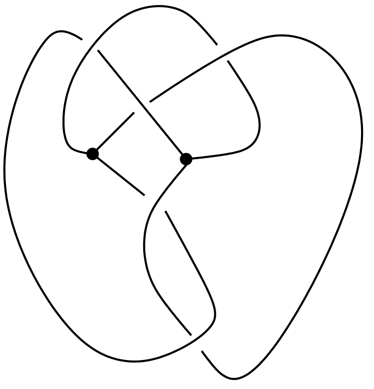


Fig. 22. Kinoshita's θ-graph.

A detailed discussion about the use of Alexander ideals in the study of handlebody complements is carried out in [26], where it is shown that there exist infinitely many spatial handlebodies of genus 2 whose complements are not weakly Helmholtz and are pairwise nonhomeomorphic. A comprehensive account on Alexander modules of groups and spaces is given in [49] (which is mainly concerned with links).

We would like to stress the remarkable fact that *easily* computable obstructions are able to recognize, at least in some cases, that a domain is not weakly Helmholtz. It is known that the corank of a group G can be computed *in principle* from a finite presentation of G by means of an algorithm (see Makanin's paper [60]). Moreover, a finite presentation of the fundamental group of the complement of a link (or of a graph) may be easily deduced from any planar diagram of the link (or of the graph) via the *Wirtinger method* (see e.g. [69]). This is an important conceptual fact, however the time of execution of Makanin's algorithm grows too fast with the input complexity, so this is not really helpful in practice, even when one deals with rather simple examples. As suggested by Stallings in [76], in some cases one can associate to the finite presentation of G some pertinent 3-dimensional (triangulated) manifold, and try to exploit geometric/topological tools in order to reduce the determination of the corank of G to the computation of the cut number of the related manifold. Note that in our situation the problem is 3-dimensional from the very beginning, and one can try to use for example the theory of *normal surfaces* in order to detect the potential weak cut systems (if any). To this respect Kinoshita's domain M_K should appear rather promising, as it admits a very simple triangulation as well as simple presentation of the fundamental group. However, even in this case it turns out that the needed computations cannot be carried out without the help of a computer.

Acknowledgments

The first two authors like to thank R. Ghiloni for having "discovered" the literature about Helmholtz domains, and involved them in the task of clarifying their topology. The third author thanks Alberto Valli, for having introduced him to these themes, and convinced him of the utility of such a task. He also thanks Ana Alonso Rodríguez, Annaliese Defranceschi, Domenico Luminati, and Valter Moretti for helpful conversations.

Please cite this article in press as: R. Benedetti, et al., The topology of Helmholtz domains, Expo. Math. (2012), doi:10.1016/j.exmath.2012.09.001

Appendix

Without aiming at being exhaustive, in this appendix we indicate to the interested reader some more advanced topics related to the previous discussion.

Let us first deal with the case of links. As mentioned above, once a diagram of a link L is given, one may easily produce a finite presentations of the fundamental group of $S^3 \setminus L$. Then, one may apply Fox's *free differential calculus* [40] to compute, for example, the Alexander ideals of $S^3 \setminus L$. As in the case of graph complements, these ideals may provide interesting information about the cut number of $S^3 \setminus L$.

As an alternative, one may recur to certain, in principle computable, increasingly discriminating sequence of invariants whose vanishing is a necessary condition in order to be homology boundary. The original definition of these invariants is given in [62], so that they are known as *Milnor $\bar{\mu}$ invariants*. Let us recall some of their properties. For every integer $q > 1$, for every link L with N *ordered and oriented* components $K_1, \dots, K_N$, for every $(l_1, \dots, l_p) \in \mathbb{N}^p$, with $1 \le l_i \le N$, $p < q$, it is defined an invariant of the form

$$\bar{\mu}(l_1, \dots, l_p)(L) = [\mu(l_1, \dots, l_p)(L)] \in \mathbb{Z}/\Delta(l_1, \dots, l_p)\mathbb{Z},$$

where:

- the integer l_j labels the component K_{l_j} (note that indices may be repeated);
- the integer $\mu(l_1, \dots, l_p)(L)$ may be determined starting from a finite presentation of $\pi_1(\mathsf{C}(L))$, and it is well-defined only up to multiples of $\Delta(l_1, \dots, l_p)$;
- the integer $\Delta(l_1, \dots, l_p)$ is inductively defined as the g.c.d. of the numbers $\mu(j_1, \dots, j_s)(L)$ where $s \ge 2$ and $(j_1, \dots, j_s)$ ranges over all the cyclic permutations of the proper subsequences of $(l_1, \dots, l_p)$.
- if $j_1 \neq j_2$, the value $\mu(j_1, j_2)(L)$ is the linking number of the corresponding components.

The last property shows that the obstructions to be homology boundary provided by Milnor invariants generalize the obstruction discussed in Lemma 6.10.

Strictly speaking, Milnor invariants are isotopy invariants for ordered and oriented links. However, their vanishing does not depend on the chosen order or orientation. The original definition of Milnor invariants, being based on the study of the group $G_1 = \pi_1(\mathsf{C}(L))$, has a strong algebraic flavor. Roughly speaking, Milnor invariants detect whether or not the (preferred) longitudes of the link components can be expressed as longer and longer commutators, i.e. they detect how deep the longitudes live in the *lower central series* of the link group, which is inductively defined as follows: $G_1 = \pi_1(\mathsf{C}(L))$, and $G_n = [G_{n-1}, G_1]$ is the subgroup of G generated by the set $\{aba^{-1}b^{-1} \mid a \in G_{n-1},\ b \in G_1\}$.

In [68,79], an equivalent definition of Milnor invariants is given in terms of the *Massey products* in the systems $\{S^3 \setminus K_{l_j}\}_{j=1}^p$. This approach makes extensive use of the cup product on singular 1-cocycles (with coefficients in suitable quotients of the ring of integers).

In [33], one can find a more geometric approach to these invariants, based on the construction of "derived links". This method is particularly suited in order to deal with the first nonvanishing invariant (if any). In some sense, the approach via derived links provides a geometric realization of Massey products. It makes use of relative 2-cycles and

Please cite this article in press as: R. Benedetti, et al., The topology of Helmholtz domains, Expo. Math. (2012), doi:10.1016/j.exmath.2012.09.001

transverse intersection rather than of 1-cocycles and cup products. The naive idea of a derived link is as follows. If L is a link as in Lemma 6.11, then we can construct a system of Seifert surfaces transversely intersecting only in $\mathsf{C}(L)$. In fact, we can arrange things in such a way that the intersection of each pair of surfaces is a single connected knot in $\mathsf{C}(L)$. Each such knot splits in two parallel copies by slightly isotoping it inside both surfaces. By taking all the knots obtained in this way, we get a derived link L' of the given link L. One can define "higher order" invariants of L by using the linking numbers of the pairs of components of L'. If all these linking numbers vanish, then the procedure may be iterated.

In [33,68], one finds some examples of computations of nontrivial Milnor invariants. In particular, when L is the Whitehead link, it turns out that $\bar{\mu}(1,1,2,2)(L) = 1$, according to the fact that L is not homology boundary.

Milnor invariants with pairwise distinct indices l_j have a particular meaning. In fact, they are invariant up to *link homotopy equivalence* (see [62]). Two links L, L' are link homotopy equivalent if they may be obtained one from the other via a homotopy where self-crossings of each link component are allowed, while crossings of different components are not. If a link L is link homotopy equivalent to the trivial link, then Milnor invariants with pairwise distinct indices vanish. Note, for example, that the Whitehead link becomes trivial just by performing one crossing change on one of its components (see Fig. 4). It is known that every boundary link is link homotopic to a trivial link (see [32] or [37]), so Milnor invariants may be used to detect links that are not boundary links. Milnor invariants have been widely exploited also in the study of homology boundary links (up to link homotopy) (see e.g. [33–36,31]).

The theory of spatial graphs as well as of links of handlebodies is considerably less developed than the classical link theory. Several equivalence relations coming from link theory ("homotopy", "cobordism", "homology", ...) have been extended to the context of spatial graphs [38,78,73], and particular efforts have been devoted to detect whether or not a spatial graph is planar (up to isotopy) [71,81]. A largely diffused approach to this sort of problems consists in associating to every graph some invariant families of classical links [55,56,44], in order to exploit results from the theory of links.

The theory of links of handlebodies is even less developed. A natural approach consists in considering links of spines, up to suitable *moves* on spines that do not alter the carried handlebodies. This strategy is developed e.g. in [72,51,52]. Then, every invariant of spatial graphs that takes the same value on spines carrying the same handlebody defines an invariant of the handlebody itself. This is the case, for example, for the invariants defined by Kinoshita and mentioned above, or for the *quandle coloring invariants* recently introduced by Ishii in [51].

Recall that a spatial handlebody is *knotted* if it does not admit any planar spine. However, one could wonder whether it makes sense to state that a fixed knotted handlebody is "more knotted" than another knotted handlebody. For example, a knotted handlebody does not admit an unknotted (i.e. planar) spine, but may still admit a handcuff spine with associated trivial link, so it make sense to assign a higher level of knotting to spatial handlebodies that do not admit such a peculiar spine. Following this approach, in [26] several *levels of knotting* are defined in terms of the *non*-existence of any spine enjoying less and less restrictive properties. Kinoshita's and Ishii's invariants are extensively used in [26] in order to investigate the relationships between these different levels of knotting, and to

Please cite this article in press as: R. Benedetti, et al., The topology of Helmholtz domains, Expo. Math. (2012), doi:10.1016/j.exmath.2012.09.001

relate the level of knotting of a spatial handlebody to classical topological properties of its complement.

Note on the bibliography. Refs. [1] to [22] form the *Section A* relative to electromagnetism, hydrodynamics and elasticity on domains in $\mathbb{R}^3$. Refs. [24] to [81] form the *Section B* on (3-dimensional) differential/algebraic/geometric topology.

References

[1] A. Alonso Rodríguez, E. Bertolazzi, R. Ghiloni, A. Valli, Construction of a finite element basis of the first de Rham cohomology group and numerical solution of 3D magnetostatic problems, Preprint UTM 753, Department of Mathematics, University of Trento, 2012.

[2] A. Alonso Rodríguez, P. Fernandes, A. Valli, The time-harmonic eddy-current problem in general domains: solvability via scalar potentials, in: Computational Electromagnetics (Kiel, 2001), in: Lect. Notes Comput. Sci. Eng., vol. 28, Springer, Berlin, 2003, pp. 143–163.

[3] C. Amrouche, C. Bernardi, M. Dauge, V. Girault, Vector potentials in three-dimensional non-smooth domains, Math. Methods Appl. Sci. 21 (1998) 823–864.

[4] A. Bermúdez, R. Rodríguez, P. Salgado, A finite element method with Lagrange multipliers for low-frequency harmonic Maxwell equations, SIAM J. Numer. Anal. 40 (2002) 1823–1849.

[5] A. Bossavit, Magnetostatic problems in multiply connected regions: some properties of the curl operator, IEE Proc. A 135 (1988) 179–187.

[6] A. Bossavit, Computational Electromagnetism Variational Formulations, Complementarity, Edge Elements, in: Electromagnetism, Academic Press, Inc., San Diego, CA, 1998.

[7] A. Bossavit, A. Vourdas, K.J. Binns, Magnetostatics with scalar potentials in multiply connected regions, IEE Proc. A 136 (1989) 260–261.

[8] J. Cantarella, D. De Turck, H. Gluck, Vector calculus and the topology of domains in 3-spaces, Amer. Math. Monthly 109 (2002) 409–442.

[9] R. Dautray, J.-L. Lions, Mathematical Analysis and Numerical Methods for Science and Technology, in: Spectral Theory and Applications, vol. 3, Springer-Verlag, Berlin, 1990, With the Collaboration of Michel Artola and Michel Cessenat. Translated from the French by John C. Amson.

[10] P. Fernandes, G. Gilardi, Magnetostatic and electrostatic problems in inhomogeneous anisotropic media with irregular boundary and mixed boundary conditions, Math. Models Methods Appl. Sci. 7 (1997) 957–991.

[11] C. Foias, R. Temam, Remarques sur les équations de Navier–Stokes stationnaires et les phénomènes successifs de bifurcation, Ann. Sc. Norm. Super Pisa Cl. Sci. 5 (1978) 28–63.

[12] G. Geymonat, F. Krasucki, Hodge decomposition for symmetric matrix fields and the elasticity complex in Lipschitz domains, Commun. Pure Appl. Anal. 8 (2009) 295–309.

[13] R. Ghiloni, The Hodge decomposition theorem for general three-dimensional vector fields, without cuts, Preprint UTM 731, Department of Mathematics, University of Trento, 2010.

[14] H. Helmholtz, Über integrale der hydrodynamische gleichungen, welche der wirbelbewegung entsprechen, J. Reine Angew. Math. 55 (1858) 25–55.

[15] P.W. Gross, P.R. Kotiuga, Electromagnetic Theory and Computation: A Topological Approach, in: Mathematical Sciences Research Institute Publications, vol. 48, Cambridge University Press, Cambridge, 2004.

[16] L. Kettunen, K. Forsman, A. Bossavit, Formulation of the eddy current problem in multiply connected regions in terms of h, Internat. J. Numer. Methods Engrg. 41 (1998) 935–954.

[17] L. Kettunen, K. Forsman, A. Bossavit, Discrete spaces for Div and Curl-free fields, IEEE Trans. Magn. 34 (1998) 2551–2554.

[18] P.R. Kotiuga, On making cuts for magnetic scalar potentials in multiply connected regions, J. Appl. Phys. 61 (1987) 3916–3918.

[19] P.R. Kotiuga, A. Vourdas, K.J. Binns, Correspondence on magnetostatics with scalar potentials in multiply connected regions, IEE Proc. A 137 (1990) 231–232.

[20] D. Mitrea, M. Mitrea, M. Taylor, Layer potentials, the Hodge Laplacian, and global boundary problems in nonsmooth Riemannian manifolds, Mem. Amer. Math. Soc. 150 (2001).

[21] J. Necǎs, Les méthodes directes en théorie des équations elliptiques, 1967 (in French), Masson et Cie, Éditeurs, Paris; Academia, Éditeurs, Prague.
[22] R. Temam, Navier–Stokes Equations. Theory and Numerical Analysis, third ed., in: Studies in Mathematics and its Applications, vol. 2, North-Holland Publishing Co., Amsterdam, 1984, With an Appendix by F. Thomasset.

****

[23] J.W. Alexander, On the subdivision of 3-space by a polyhedron, Proc. Natl. Acad. Sci. 10 (1924) 6–8.
[24] J.W. Alexander, An example of a simply-connected surface bounding a region which is not simply-connected, Proc. Natl. Acad. Sci. 10 (1924) 8–10.
[25] J.W. Alexander, Remarks on a point set constructed by Antoine, Proc. Natl. Acad. Sci. 10 (1924) 10–12.
[26] R. Benedetti, R. Frigerio, Levels of knotting of spatial handlebodies, Trans. Amer. Math. Soc. (2012) http://dx.doi.org/10.1090/S0002-9947-2012-05707-2.
[27] R.H. Bing, Locally tame sets are tame, Ann. of Math. 59 (1954) 145–158.
[28] K. Borsuk, Über die fundamentalgruppe der polyeder im euklidischen dreidimensionalen Raume, Monatsh. Math. Phys. 41 (1934) 64–77.
[29] R. Bott, L. Tu, Differential Forms in Algebraic Topology, Springer, 1982.
[30] M. Brown, Locally flat imbeddings of topological manifolds, Ann. of Math. 75 (1962) 331–341.
[31] A. Casson, Link cobordism and Milnor's invariants, Bull. London Math. Soc. 7 (1975) 39–40.
[32] L. Cervantes, R.A. Fenn, Boundary links are homotopy trivial, Quart. J. Math. Oxford Ser. (2) 39 (1988) 151–158.
[33] T. Cochran, Derivatives of links: Milnor's concordance invariants and Massey products, Mem. Amer. Math. Soc. 84 (1990).
[34] T. Cochran, J. Levine, Homolopy boundary links and the Andrews Curtis conjecture, Topology 30 (1991) 231–240.
[35] T. Cochran, K.E. Orr, Not all links are concordant to boundary links, Ann. of Math. 138 (1993) 519–554.
[36] T. Cochran, K.E. Orr, Homology boundary links and Blanchfield forms: concordance classification and new tangle-theoretic constructions, Topology 33 (1994) 397–427.
[37] D. Dimovski, A geometric proof that boundary links are homotopically trivial, Topology Appl. 29 (1988) 237–244.
[38] T. Fleming, R. Nikkuni, Homotopy on spatial graphs and the Sato–Levine invariant, Trans. Amer. Math. Soc. 361 (2009) 1885–1902.
[39] R.H. Fox, On the imbedding of polyhedra in 3-space, Ann. of Math. 49 (1948) 462–470.
[40] R.H. Fox, Free differential calculus, Ann. of Math. 57 (1953) 547–560.
[41] C.McA. Gordon, 3-dimensional topology up to 1960, in: I.M. James (Ed.), History of Topology, North-Holland, Elsevier, 1999.
[42] M.J. Greenberg, Lectures on Algebraic Topology, W. A. Benjamin, Inc., New York, Amsterdam, 1967.
[43] L. Guillou, A. Marin, A la Recherche de la Topologie Perdue, in: Progr. Math., vol. 62, Birkhäuser, 1986.
[44] S.M. Ghuman, Seifert surfaces for graphs, J. Knot Theory Ramifications 8 (1999) 985–999.
[45] M.A. Gutierrez, Boundary links and an unlinking theorem, Trans. Amer. Math. Soc. 17 (1972) 491–499.
[46] S. Harvey, On the cut number of a 3-manifold, Geom. Topol. 6 (2002) 409–424.
[47] A. Hatcher, Algebraic Topology, Cambridge University Press, 2002, Single electronic copy freely available from: http://www.math.cornell.edu/~hatcher/.
[48] A. Hatcher, Basic 3-manifold topology. http://www.math.cornell.edu/~hatcher/3M/3Mdownloads.html.
[49] J.A. Hillman, Algebraic Invariants of Links, in: Series on Knots and Everything, vol. 32, World Scientific, Singapore, 2002.
[50] M.W. Hirsch, Differential Topology, in: Graduate Texts in Mathematics, vol. 33, Springer-Verlag, New York, 1994.
[51] A. Ishii, Moves and invariants for knotted handlebodies, Algebr. Geom. Topol. 8 (2008) 1403–1418.
[52] A. Ishii, M. Iwakiri, Quandle cocycle invariants for spatial graphs and knotted handlebodies, Canad. J. Math. 64 (2012) 102–122.
[53] W. Jaco, Nonretractable cubes-with-holes, Michigan Math. J. 18 (1971) 193–201.
[54] W. Jaco, D.R. McMillan, Retracting three-manifolds onto finite graphs, Illinois J. Math. 14 (1970) 150–158.

Please cite this article in press as: R. Benedetti, et al., The topology of Helmholtz domains, Expo. Math. (2012), doi:10.1016/j.exmath.2012.09.001

[55] L.H. Kauffman, Invariants of graphs in 3-space, Trans. Amer. Math. Soc. 311 (1989) 697–710.
[56] L.H. Kauffman, J. Simon, K. Wolcott, P. Zhao, Invariants of theta-curves and other graphs in 3-space, Topology Appl. 49 (1993) 193–216.
[57] S. Kinoshita, On elementary ideals of polyhedra in the 3-sphere, Pacific J. Math. 42 (1972) 89–98.
[58] H.W. Lambert, Mapping cubes with holes onto cubes with handles, Illinois J. Math. 13 (1969) 606–615.
[59] W. Magnus, A. Karras, B. Solitar, Combinatorial Group Theory: Presentations in Terms of Generators and Relators, Wiley (Interscience), 1966.
[60] G.S. Makanin, Equations in a free group, Izv. Akad. Nauk SSSR Ser. Mat. 46 (1982) 1199–1273.
[61] W.S. Massey, Algebraic Topology: An Introduction, Harcourt, Brace & World, Inc., New York, 1967.
[62] J.W. Milnor, Link groups, Ann. of Math. 59 (1954) 177–195.
[63] J.W. Milnor, Morse Theory, in: Annals Studies, vol. 51, Princeton Univ. Press, 1963.
[64] J.W. Milnor, Lectures on the H-Cobordism Theorem, Princeton University Press, Princeton, NJ, 1965, Notes by L. Siebenmann and J. Sondow.
[65] J.W. Milnor, Topology from the Differentiable Viewpoint, The University Press of Virginia, 1969.
[66] E.E. Moise, Geometric Topology in Dimensions 2 and 3, Springer-Verlag, 1977.
[67] C.D. Papakyriakopoulous, On Dehn's lemma and the asphericity of knots, Ann. of Math. 66 (1957) 1–26.
[68] R. Porter, Milnor's $\bar{\mu}$ invariants and Massey products, Trans. Amer. Math. Soc. 257 (1980) 30–71.
[69] D. Rolfsen, Knots and Links, in: Mathematics Lecture Series, vol. 7, Publish or Perish, Inc., Berkeley, Calif., 1976.
[70] C.P. Rourke, B.J. Sanderson, Introduction to Piecewise-Linear Topology, Springer, 1972.
[71] M. Scharlemann, A. Thompson, Detecting unknotted graphs in 3-spaces, J. Differential Geom. 34 (1991) 539–560.
[72] M. Scharlemann, A. Thompson, Thin position and Heegaard splittings of the 3-sphere, J. Differential Geom. 39 (1994) 343–357.
[73] R. Shinjo, K. Taniyama, Homology classification of spatial graphs by linking numbers and Simon invariants, Topology Appl. 134 (2003) 53–67.
[74] A.S. Sikora, Cut numbers of 3-manifolds, Trans. Amer. Math. Soc. 357 (2004) 2007–2020.
[75] N. Smythe, Boundary links, in: R.H. Bing (Ed.), Topology Seminars, in: Ann. Math. Studies, vol. 60, Princeton Univ. Press, 1965.
[76] J.R. Stallings, Problems about free quotients of groups, in: Geometric Group Theory, Columbus, OH, 1992, in: Ohio State Univ. Math. Res. Inst. Publ., vol. 3, de Gruyter, Berlin, 1995, pp. 165–182.
[77] S. Suzuki, Alexander ideals of graphs in the 3-sphere, Tokyo J. Math. 7 (1984) 233–247.
[78] K. Taniyama, Cobordism, homotopy and homology of graphs in $\mathbb{R}^3$, Topology 33 (1994) 509–523.
[79] V.G. Turaev, The Milnor invariants and Massey products, in: Studies in Topology II, in: Acad. Nauk USSR, vol. 66, 1976, In Translation J. Sov. Math..
[80] F. Waldhausen, Heegaard-Zerlegungen der 3-Sphäre, Topology 7 (1968) 195–203.
[81] P. Zhao, Is a knotted graph determined by its associated links? Topology Appl. 57 (1994) 23–30.

Please cite this article in press as: R. Benedetti, et al., The topology of Helmholtz domains, Expo. Math. (2012), doi:10.1016/j.exmath.2012.09.001