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ON 3-MANIFOLDS UP TO BLOWING-UP

Riccardo Benedetti
Dipartimento di Matematica
Università di Pisa - Italia

INTRODUCTION

This paper has been expressly written for these Proceedings. It contains essentially the text of my talk, together with the report of some facts occurred after a first version entitled "Classification of 3-manifolds up to blowing-up and existence of rational real algebraic structures" containing complete proofs of the main results has circulated.

I was originally motivated, actually many years ago, to introduce and study smooth manifolds up to blowing-up along smooth centres, by an old question posed by J.Nash. In his pioneeristic paper [Na] he asked, among other, if every compact closed connected smooth manifold of dimension n admitted any structure of real algebraic variety birationally equivalent to S^n (see also [Iv]). As blowing-up along non singular centres is the basic construction in algebraic geometry to produce birationally equivalent varieties, and as this construction naturally extends to the smooth case as blowing-up along smooth centres, it suggested to me, by analogy, to consider the following equivalence relation on the set V_n of compact closed connected smooth manifolds of dimension n.

1.1 DEFINITION $X,Y \in V_n$ are saied to be m-equivalent if there exists a finite sequence :

$$X=M_0 \leftrightarrow M_1 \leftrightarrow M_2 \leftrightarrow \dots \leftrightarrow M_k=Y$$

where each $\leftrightarrow$ is either a diffeomorphism or a blowing-up (or down) along a smooth centre (i.e. along a manifold $N \in V_m$, $m<n$, eventually embedded in M_i or in M_{i+1}).

In other words we mean each blowing-up (down) as a simple modification and we consider the relation generated by diffeomorphisms and such simple modifications.

Thus a smooth analogous of Nash's question arises :
is every $X \in V_n$ m-equivalent to S^n ?
Probably more reasonably , one could ask :
how does the quotient set $V_n/\sim_m$ look ?

1.2 REMARK The relationship between Nash's question and its smooth analogous should be seen just at the analogy level . The implications between them is a subtler matter. Let $X \in V_n$ be a non singular projective real algebraic variety birationally equivalent to S^n . Are X and S^n m-equivalent as smooth manifolds ?
An answer should presumably make use of Hironaka's theorem on the resolution of singularities. However , even among people having familiarity with Hironaka's work , I have encountered two attitudes : someone seems to consider such a positive answer as a "moral" corollary of Hironaka proof ; someother seems to be much more prudent ; shortly I did not get a definitive answer .
On the other hand , we shall see in the present paper that even in low dimension , the fact that $X \in V_n$ is m-equivalent to S^n allows only a weak answer to Nash's question (in fact the best possible) : actually Nash asked for non singular real algebraic models , and we get such models having in general non empty singular set .

I can state now the main results of the paper .

THEOREM A (main theorem-weak formulation) Every $M \in V_3$ is m-equivalent to S^3 .

For the strong formulation see the next paragraph 2 .

As an application I get

THEOREM B (An answer in low dimension to Nash's question) Every $M \in V_3$ is homeomorphic to some affine real algebraic variety M' , having , in general , non empty singular set Σ , and containing a proper algebraic subset Y ($\Sigma \subset Y$) such that M'-Y is algebraically isomorphic to a Zariski open set of S^3 .

The analogous results for V_2 hold in a easier way .

Actually we can obtain some more precision : we may get Σ of dimension 1 (0 for V_2) and the homeomorphism as a "stratified smooth isomorphism" with respect to the stratification $(M'-\Sigma , \Sigma)$ of M' ....
Moreover , the singularities seem to be not eliminable , in the general case ; for example , in dimension 2 :

a non singular rational real algebraic surface in V_2 is necessarily homeomorphic to S^2 or to the 2-torus , providing it is orientable (see [Kh] and also [S]) .

The case of V_3 was the first interesting open one ; for V_2 the result is an easy consequence of the classification of surfa ces up to diffeomorphism . Moreover , it was not hopeless : a general well-known method to construct <u>all</u> the manifolds of V_3 (that is via <u>surgery along framed links</u> in S^3 or in $S^2 \tilde{\times} S^1$, the twisted S^2-bundle over S^1 , according to the orientability or non-orientability) looked very suited to approach the problem : in fact one had , in particular , to blow-up along embedded circles. It was a nice fact (but actually not so hard , all one needed is contained in the simple lemma 3.1. below) to remark that the well-known Kirby's calculus on framed links , which preserves the diffeomorphism type , could be quite naturally extended to a very "flexible" <u>m-calculus</u> preserving the m-equivalence type . I remar ked it with Alexis Marin ; using it we constructed several non-trivial examples of 3-manifolds m-equivalent to S^3 (enough to conjecture the main result) ; but after some fruitless attem- pts we were contented to collect our remarks (not so strong in our opinion) in the note published at the Dipartimento di Mat. of Pisa ([BM]) .
It has turned out that essentially the same path had been follo- wed , independently and with the same motivations , by S.Akbulut who also remarked the m-calculus and used it to produce examples ; this is what one can deduce from the Abstract of A.M.S.(1/30/1987 832-57-118) .

Since 1987 no progress had been made .
On last September I was aware of Nakanishi's paper ([N]) concer- ning (unframed) links in S^3 up to <u>Fox congruences</u> : I realized

soon that his result for the congruence mod(2,1) , together with the m-calculus actually gave a proof of the main theorem in the orientable case. By a non immediate extra-work not depending on Nakanishi's theorem in [N] , I reduced the non-orientable case to the orientable case , finally completing the proof .
This is the proof I had presented in my talk .
While it has to my eyes the nice feature of the first complete solution of a longstanding problem I had been interested in from the very beginning of its own formulation , however it could be considered not completely satisfactory : one could try to find a more self-contained proof , relaxing the dependence on [N] ; moreover Nakanishi's theorem , while being very nice , is largely proved by "pictures" in 2-dimensional space of links projections . One could look for a more conceptual and purely 3-dimensional proof .
After our symposium I had realized that the argument of the reduction from the non-orientable to the orientable case , could be used to get a proof of the main theorem , at least in its weak formulation , without using the m-calculus and Nakanishi's result ; but it was not yet completely satisfactory .

Finally I received a letter (dated 3 Nov.1989) from Alexis Marin , containing several friendly comments on my paper and a sketch of a proof , extracting the basic 3-dimensional meaning of the business , not using [N] , which I consider the definitive proof as it is short and clean . I will reproduce this sketch .

On the other hand , I believe that the formulation of the results in terms of m-calculus (see the parag. 4 and 5), allowed by the use of [N] , maintains its own interest and deserves to be pointed out .

I take the occasion to thank the JSPS for giving me the occasion to spend some time in Japan , the Nagoya University and mostly Masahiro Shiota for the friendly hospitality .

2. Strong formulation of the main theorem .

Decompose $V_3 = V_3^+ \cup V_3^-$, "+" means orientable and "-" non-orientable . For every $M \in V_3$ let $b=b_M=\dim H_2(M,Z_2)$.

In fact we shall obtain :

THEOREM A^+ (main theorem-strong formulation for V_3^+)
For every $M \in V_3^+$, there exist a link L in M and a link L' in $\#_{1 \leq i \leq b} \mathbb{P}^3$ (i.e. the connected sum of b copies of the projective 3-space) such that by blowing-up along the components of L and L' respectively one obtains diffeomorphic 3-manifolds .

Note that $\#_i \mathbb{P}^3$ is nothing else than the result of a blowing-up of S^3 at b points .

Before stating the analogous result for V_3^- we need a definition.

2.1. DEFINITION A simple tower of blowing-up of lenght s over S^3 is obtained as follows : consider an ordered family $C_1, \ldots, C_s$ of unknots in S^3 , looking like in this picture :

We mean that at the common point C_i and C_{i+1} have independent tangents . Then make a sequence of blowing-up :

$$S^3 = B_0 \leftarrow B_1 \leftarrow B_2 \leftarrow \ldots \leftarrow B_s$$

such that C_1 is the centre of the first one π_1 , the "strict transform" of C_{i+1} in B_i (i.e. the closure in B_i of $(\pi_i \cdots \pi_1)^{-1}(C_{i+1} - C_i)$) is the centre of π_{i+1} .
For example $S^2 \tilde{\times} S^1$ is diffeomorphic to B_1 , $\mathbb{P}^2 \times S^1$ to B_2 .

THEOREM A^- (main theorem-strong formulation for V_3^-)
For every $M \in V_3^-$, there exist a link L in M and a link L' in $B_s \# (\#_{1 \leq i \leq h} \mathbb{P}^3)$, s+h=b , every component of L and L' being a non-reversing orientation circle , such that by blowing-up along the components of L and L' respectively , one obtains diffeomorphic 3-manifolds .

The proof shall give further nice corollaries . We limit to remark the following one , not at all evident a priori .
Let us denote by m^+-equivalence the relation generated on V_3 by diffeomorphisms and blowing-up along non-reversing orientation embedded circles . In paricular no blowing-up at points is allowed . Then we have :

2.2 COROLLARY m-equivalence and m^+-equivalence coincide on V_3. That is every $M \in V_3$ is m^+-equivalent to S^3.

3. The m-calculus.

Dehn surgery data on a manifold $M \in V_3$ consist of $\Delta = (C, U, m, \ell, q/p)$ where : C is an embedded circle in M with U as orientable tubular neighbourhood ; m and ℓ are fixed meridian and longitude in ∂U ; q/p is a rational number with $(q,p)=1$ and also $\infty = q/0$ is allowed . The manifold M_Δ obtained from M by surgery with data Δ, is well defined (up to diffeo.) as :

$$M_\Delta = \overline{M-U} \cup_h U$$

$h: \partial U \to \partial U$ being any diffeomorphism such that $h_*([m]) = q[m] + p[\ell]$ in $\pi_1(\partial U)$. The key remark is contained in the following simple lemma .

3.1 LEMMA Let A,B be obtained from $M \in V_3$ by surgeries with data differring only by q/p and b/a . Let A',B' be the blowing-up of A and B respectively along C . Assume that either :

1) $p \equiv a \equiv 0 \mod 2$; or

2) $p \equiv a \equiv 1 \mod 2$ and $q \equiv b \mod 2$.

Then there exists a diffeomorphism between A' and B' which is the identity on M-U .

The proof is an easy consequence of the fact that the inverse image of U , via the blowing-up , U' say , is diffeomorphic to $\mathcal{M} \times S^1$, $\mathcal{M}$ being the Moebius strip , and that a diffeomorphism $g: \partial U' \to \partial U'$ extends to a diffeomorphism $G: U' \to U'$ if $g_*([m]) \equiv [m] \mod 2$.

By definition , a framed link in $M \in V_3$, is a link L embedded in M with a system of surgery data for each its component , with the restriction that q/p is actually equal to q/1 .If a component C of a link L is contained in an embedded ball , then its framing is completely determined by an integer number , that is the linking number lk(C,C') between C and its suitable "parallel" C'.

It is well-known ([L_1],[L_2]) , that every orientable $M\in V_3^+$ is diffeomorphic to some manifold $\chi(L)$ obtained from S^3 by surgery along the components of some framed link L , and that every $M\in V_3^-$ can be ontained as such a $\chi(L)$ for some framed link in $S^2\tilde{\times}S^1$. Such representation being not unique , *Kirby's calculus* on framed links , preserving the diffeomorphism type of $\chi(L)$ is well-known (we refer to [K] ,[R] , and [FR]). We consider the following extension of this calculus , called in the sequel *χ-calculus* , to an *m-calculus* preserving the m-equivalence type of $\chi(L)$. The"moves" generating the m-calculus we need are :

m_1 : the "moves" of χ-calculus ;

m_2 : introducing (or deleting) $\bigcirc_2$ contained in a separating 3-ball .

m_3 : changing the framing according to lemma 3.1. . In particular if the framing is given by an integer number (like in the case of links in S^3) k , then we may replace k by any $k'\equiv k$ mod 2 .

Note that m_2 corresponds to the blowing-up at a point , because this is equivalent to make the connected sum with $\mathbb{P}^3$ and $\bigcirc_2$ is the simplest representation of $\mathbb{P}^3$ via framed links in S^3 .

We shall denote by *m_R-calculus* the reduced m-calculus generated only by the moves m_1 and m_3 .

3.2. LEMMA *$\dim H_2(\chi(L),Z_2)$ is invariant by m_R-calculus* .
The proof is immediate .

3.3. LEMMA *If two framed links L,L' (for fixing the ideas in S^3 or in $S^2\tilde{\times}S^1$) are related by the m-calculus , then $\chi(L)$ and $\chi(L')$ are actually m^+-equivalent* .

Proof. If L and L' are related by an m_3 move , the they are evidently m^+-equivalent. Then $\bigcirc_2$ is m^+-equivalent to $\bigcirc_0$ which is diffeomorphic to $S^2\times S^1$. Then it is easy to see that $S^2\times S^1$ is m^+-equivalent to $(S^2\times S^1)\#(S^2\tilde{\times}S^1)$ which is diffeomorphic to $(S^2\tilde{\times}S^1)\#(S^2\tilde{\times}S^1)$, that is to $B_1\#B_1$ (with the notation of def. 2.1.) , thus it is m^+-equivalent to S^3 .

*

4. Formulation of THEOREM A^+ in terms of m-calculus .

For every framed link L in S^3 , set $b=b_L=\dim H_2(\chi(L),Z_2)$.

THEOREM C^+ Framed links in S^3 up to m_R-calculus are classified by b_L . Every link L is m_R-related to the framed link :

b-times

Consequently every framed link L in S^3 is related to the empty link by the m-calculus .

It is not hard that Theorem C^+ implies Theorem A^+ .
Moreover , by lemma 3.3. we get immediately the corollary 2.2. in the orientable case .

This is a good point to recall Nakanishi's result ([N]) .
We shall formulate it in a convenient way to our purpose .

4.1. DEFINITION Let L be a link in S^3 . Let $H:D^2\times[0,1]\to S^3$ be a smooth embedding such that $H|(S^1\times[0,1])$ (here $S^1=\partial D^2$) is an embedding in S^3-L . Then for every finite set $\{x_1,..,x_k\}\subset[0,1]$, $\cup_i H(S^1\times\{x_i\})$ is called a set of parallel components of the link $L\cup(\cup_i H(S^1\times\{x_i\}))=L\cup P$.

Let Λ_n be the set of (unframed) links in S^3 with n components.

4.2. DEFINITION Fox congruence mod (2,1) on Λ_n is the relation generated by the following elementary "move" :

(a) First introduce an even number k of parallel components getting $L\cup P\in\Lambda_{n+k}$. Give all components of P the framing 1 (or -1) .

(b) Make k Kirby's moves along the so framed components of P (forgetting the framing for L) , getting $L'\in\Lambda_n$.

If L and L' are related by a finite sequence of such moves , we say shortly that they are congruent .

4.3. DEFINITION $L=C_1\cup\ldots\cup C_n$, $L'=C_1'\cup\ldots\cup C_n'$ in Λ_n are Z_2-link-homologous if the linking numbers $lk(C_i,C_j)=lk(C_i',C_j')$ for $i\neq j$.

4.4. THEOREM ([N]) $L, L' \in \Lambda_n$ are congruent if and only if they are Z_2-link-homologous . In particular every knot is congruent to the unknot .

The key remark to us is :

4.5. LEMMA The elementary move generating the congruence is composition of moves of the m_R-calculus .

Proof. By χ-calculus we can introduce any number of pairs of parallel components one with framing 1 , the other with -1 . By m_3 we can change 1 in -1 .

*

Sketch of proof of theorem C^+.

It is a rather easy induction on the number of components of framed links in S^3 , denote it by n . If n=1 then (by 4.4.) the framed knot is m_R-equivalent either to $\bigcirc_1$ or $\bigcirc_0$.
For general n , (again by 4.4.) , we are free to work in the Z_2-link-homology class of any given link L with n components (forgetting the framing) ; considering a simple representative and reintroducing the possible framings reduced mod 2 , it is not hard , using the χ-calculus , to pass to a framed link with a strictly smaller number of components , unless we are in the final "normal form" .

*

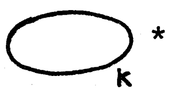
This concludes the discussion in the orientable case .

5. The non-orientable case .

We whant to prove :

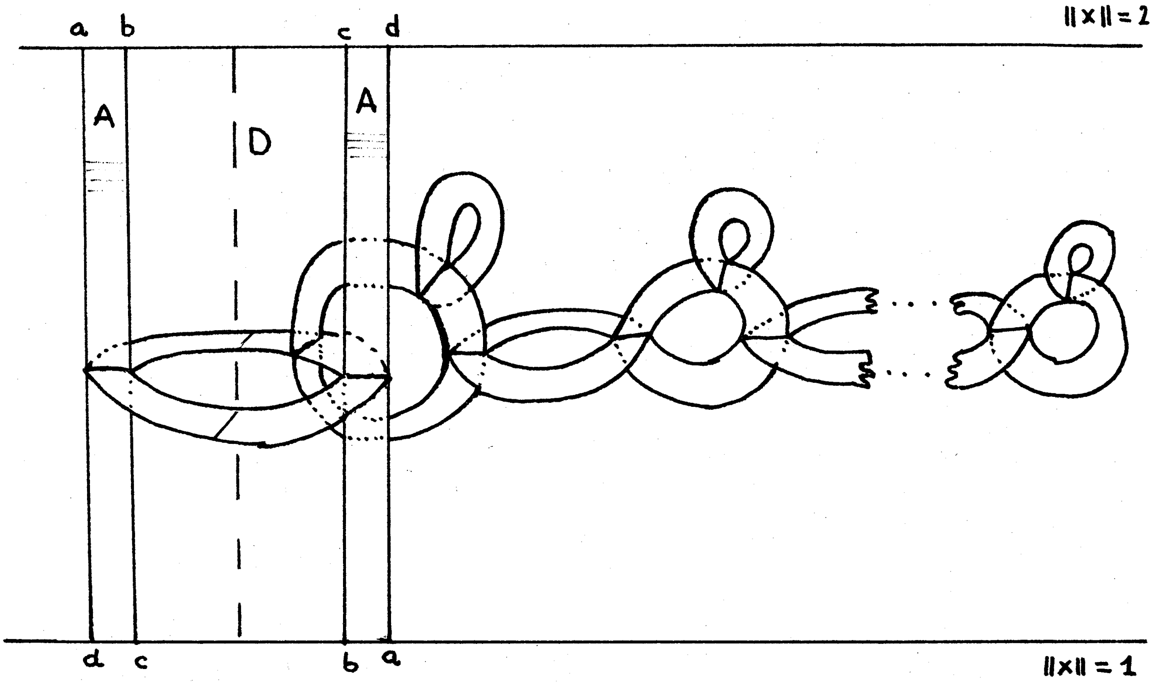
5.1. PROPOSITION Every $M \in V_3^-$ is m-equivalent to some $M' \in V_3^+$.

I will sketch the proof.
First it is convenient to consider two models for $S^2 \tilde{\times} S^1$.
One consists in considering it as B_1 (with the notation of def. 2.1.) , that is as the result of the blowing-up of S^3 along an unknot K .

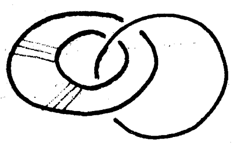


"*" means to be centre of the blowing-up .

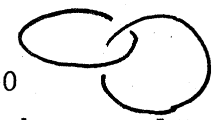
The other one is obtained by identifying the boundary components of $S = \{x=(x_1,x_2,x_3)\in \mathbb{R}^3 \mid 1\leq\|x\|\leq 2\}$, via a dilatation followed by the reflection along the plane $\{x_2=0\}$. Thus we mean $S^2\tilde{\times}S^1$ as $S/\sim$. Call D the inverse image of the unknot K , by the blowing-up in the former model . In $S/\sim$, D can be identified with $(\{x_2=0\}\cap S)/\sim$.
Then , using the finite set of generators for the mapping class group of any given non-orientable surface of fixed genus , constructed in [C] , and arguing similarly to the orientable case as made , for example, in [R] , we deduce that every $M\in V_3^-$ can be obtained by surgery along some framed link in $S^2\tilde{\times}S^1$, looking in a certain <u>special configuration</u> . More precisely consider the family of annuli embedded in $S/\sim$ as shown in this picture :



For simplicity S is shown "stright" ; k-1 annuli looks "round and planar" in the picture and are fibred by concentric circles which are homotopically trivial in $S^2\tilde{\times}S^1$. Exactly one annulus , called A , is fibred by non-trivial circles and looks "round and planar" in the other model :

A  *

Note that A cuts only one of the other annuli (the one intersecting D) along two segments .
Every component of our special link is contained in and concentric to one of these annuli . The components on A are called of A-type , the other of N-type .
Evidently , if no A-type components do occur , then we could isotopy the link far from D and , after a blowing-down , to see it as a link in S^3 . In some sense the problem is to eliminate the A-type components . We may assume that at least one of these A-type components has the property that the Moebius strip "naturally bounded" by it does not intersect the other components of the link . For the framing of such a component we have (mod 2) two possibilities : if it is 1 we can eliminate the component by χ-calculus . If the framing is 0 , we use the following argument .

0  * and B_2 (see def. 2.1.)

(in fact B_2 is the result of a blowing-up of $S^2\tilde{\times}S^1$ along a section over S^1) are diffeomorphic and in fact diffeomorphic to $\mathbb{P}^2\times S^1$. Using a suitable diffeomorphism , we see that the image of the remaining components of the link are contained in a copy of $\mathcal{M}\times S^1$, embedded in B_2 and look inside it like a link in special configuration , preserving the type . So we have one A-type component less and we can conclude by induction .
This concludes the sketch of the proof of 5.1. .

It is not hard that this proof with the one of Theorem C^+ , actually implies the theorem A^- . Using Theorem A^- , and "reversing" the proof of 5.1. , one can obtain the proof of corollary 2.2. also in the non-orientable case .

Finally we can deduce the following version in terms of m-calculus on framed links in $S^2\tilde{\times}S^1$.
Consider the <u>m_E-calculus</u> , that is the extended calculus generated by the usual m-calculus of paragraph 3 together with the further move :

" introduce or delete any A-type component with framing 0 , such that the Moebius strip naturally bounded by it does not intersect the other components " .

THEOREM C^- Every framed link L in $S^2 \tilde{\times} S^1$ is related to the empty link by the m_E-calculus .

6. A sketch of proof of THEOREM B .

It uses arguments currently emploied in Nash-Tognoli type theorem. I refer to [AK],[BCR],[Iv],[To] for an ample account about this circle of ideas .

Let $M \in V_3$. Use , to simplify , the strong version of the main theorem. First we may assume that the blowing-up over S^3 is made along non singular algebraic centres . Thus we get a non singular projective real algebraic variety M" birationally equivalent to S^3 . Moreover it has the property that $H_2(M",Z_2)$ is generated by the algebraic hypersurfaces in M" . Thus we can approximate in M" any compact smooth closed hypersurface by some non singular algebraic ones ; moreover also a relative version of this fact holds. Assume , for simplicity , that we have only one blowing-down to do in order to pass from M" to M . Let C be its smooth centre in M and D be its inverse image in M" . D is a smooth 2-torus and , by the above discussion , we may assume that D is a non singular algebraic subvariety of M" having the property that $H_1(D,Z_2)$ is generated by the algebraic curves contained in D . Give C the algebraic structure of the standard unit circle S^1. By the so recalled assumptions on D , we know that the map $\pi|D$ (π is the blowing-down) can be approximated by rational regular maps $g: D \to S^1$. Note that , topologically, $M = (M" \cup C)/\pi|D$ and that if g is close enough to $\pi|D$, then M and $(M" \cup C)/g$ are homeomorphic . Finally one recalls the fact that , since g is a regular rational map , the last quotient space can be realized by a (singular) real algebraic,affine variety M'and that there exists a regular rational map from M" onto M' which is an algebraic isomorphism between M"-D and M'-C .

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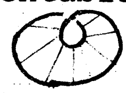
7. A sketch of A.Marin's proof of the main theorem .

Here is the key remark :

7.1. REMARK Let C_1 and C_2 be knots in $M \in V_3$, making the boundary of an embedded surface F having an orientable neighbourhood in M . Then C_1 and C_2 become isotopic in a manifold M' obtained from M by some Dehn surgeries having in the data q/p of the form $2k+1/2$, along circles embedded in a neighbourhood of $F-(C_1 \cup C_2)$ in $M-(C_1 \cup C_2)$.

In fact , we may assume that F is non-orientable (like the unknot in this picture bounds a Moebius strip ) . Then F is diffeomorphic to the connected sum of some copies of $\mathbb{P}^2$, with two disks removed . Thus we can pass from F to the 2-disk with two holes by replacing some Moebius strip embedded in F by 2-disks . Actually we can realize this operation in a manifold M' obtained by Dehn surgeries of the type saied above along the base line of each Moebius strip .

Actually it is enough to prove the main theorem in the orientable case . We may assume that $M \in V_3^+$ is obtained by surgery along a framed link in S^3 having at least one component with <u>odd</u> framing . Using the classification of odd bilinear forms over Z_2 and by "handle slides" (if one prefers, by χ-calculus) , we may assume that the linking numbers of any pair of different components of the link , is <u>even</u> . Thus we can see M as the last of a sequence $S^3=M_1, M_2, \ldots, M_k=M$, of 3-manifolds such that M_{i+1} is obtained from M_i , by a surgery along a knot C_i which bounds a surface F_i (not necessarily orientable) embedded in M_i. Now the proof of the main theorem is an easy consequence of the above remark and of lemma 3.1. : we see , by induction , that M is m-equivalent (in fact m^+-equivalent) to $\#_{1 \leq i \leq d} \mathbb{P}^3$, where $d = \dim H_1(M, Z_2)$, that is the nullity of the bilinear form over Z_2 of the given link . Note that $d = \dim H_2(M, Z_2)$, so we have obtained nothing else than Theorem A^+ .

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