Some new results on easy lambda-terms *

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May 16, 1993

A Corrado Böhm,
il maestro e l'amico

Abstract

We prove that there is a closed lambda term X such that for every closed normal form N, $X = N$ is consistent (with the $\lambda\beta$-calculus), and yet there is some closed lambda term Y such that $X = Y$ is not consistent. We also prove that there is a closed lambda term X which can be consistently equated to every closed lambda term with the exclusion of $\lambda x.x$. Both results are consequences of a general method to obtain Church Rosser extensions of lambda calculus by which we obtain various other consistency results, for instance we prove that for every $m, n, r \geq 2$, $Con(\mathbf{Y}\Omega_m = \Omega_n = \omega_r)$ and $\mathbf{Y}\Omega_m$ can be consistently equated to every closed normal form.

Abstract

Given two closed λ-terms A and B we consider the question whether $A = B$ is consistent with the $\lambda\beta$-calculus. In general the problem is undecidable, however if A is a 0-term we can give good sufficient conditions for the consistency of $\lambda\beta + \{A = B\}$. This allows us to prove some counterintuitive results such as: 1) there is a closed λ-term X which can be consistently equated to every closed λ-term with the exception of the identity $\lambda x.x$; 2) there is a closed λ-term which can be consistently equated to every closed normal form, but not to the Curry fixed point operator $\mathbf{Y}$.

*Work partially supported by the research projects 60% and 40% of the Italian Ministero dell' Università e della Ricerca Scientifica e Tecnologica.

1 Introduction

An *easy* term, is a lambda term X such that for every closed lambda term Y, the $\lambda\beta$-calculus plus the equation $X = Y$ is a consistent lambda theory, i.e. $\lambda\beta + \{X = Y\} \not\vdash \mathbf{K} = \mathbf{S}$, written $Con(X = Y)$. Easy terms were introduced and studied by G.Jacopini and M.Venturini Zilli, two former students of Corrado Böhm (see [6], [7], [8]). A classical theorem, due to Jacopini, states that $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$ is easy. Easy terms can be considered computational processes of a completely non-informative kind. So they are suitable candidates for representing inside lambda calculus the undefined value of a partial recursive function (see [12, 11, 3]). However this class deserves further investigations. In fact, in some contrast with the intuition "easy = completely undefined", in [5] it is proved that if we identify all the easy terms, then we obtain new equalities which equate an easy term with a non-easy term. A useful sufficient condition for non-easiness is given by Theorem 9.2: if $N_1X = N_2X$ where N_1 and N_2 are distinct normal forms, then X is non-easy, actually there is a closed normal form N such that $\neg Con(X = N)$. Motivated by this result we introduce the following notion: a term is *n.f.- easy* if for every closed normal form N, $Con(X = N)$. It turns out that in the $\lambda\mathbf{I}$-calculus, $n.f. - easy$ implies easy. However, as a further confirmation of the deceptive nature of easy terms, we prove the quite counterintuitive result that (in the full λ-calculus) n.f.-easy does not imply easy, namely there is a non-easy term X which can be consistently equated to every closed normal form (Theorem 9.4). This is to be contrasted with the continuity theorem in lambda calculus which asserts that, with respect to the topology induced by the Böhm trees (cf. [2]) every lambda definable function is continuous and the normal forms are dense. We also prove (Theorem 9.6) that there is a closed lambda term X which can be consistently equated to every closed lambda term with the exclusion of the identity $\lambda x.x$. (More generally one could take any projector $\lambda x_1 \dots x_n.x_i$ instead of the identity.) In particular $\lambda\beta + \{X = \lambda x.x\}$ is consistent while $\lambda\beta + \{X = \lambda xy.xy\}$ is not consistent, showing that the extensionality rule may turn a consistent $\lambda\beta$-theory into an inconsistent one. (This fact had already been proved by Jacopini using easy terms.) Both results are immediate consequences of a "separation theorem" (Theorem **??**) which states that given two closed terms A and B with incompatible Böhm trees, there is a closed term X depending only on A, which can be consistently equated to B but not to A. (More precisely: for every $A \in \Lambda_0$, there is $X \in \Lambda_0$ such that for every $B \in \Lambda_0$ whose Böhm tree is incompatible with that of A, $Con(X = B)$ and $\neg Con(X = A)$.) Although n.f.-easy does not imply easy, we prove that if a closed term X can be consistently equated with every closed term with a finite Böhm tree, then X is easy. Actually for X to be easy it suffices that X can be consistently equated with every closed lambda term whose Böhm tree has height at most one (Theorem 9.5). The separation theorem is proved with the help of a general syntactical method to obtain Church-Rosser extensions of lambda calculus (Theorem 3.4), which seems to have a wide range of applicability especially when used in connection with the continuity properties of Böhm trees (see section 5). As further examples we prove by this method many consistency results concerning lambda terms which can be written with only one variable. Examples of such terms are $\omega_n = \lambda x.xx\dots x$ (n occurrences of x after λx) and $\Omega_n = \omega_n\omega_n$. Note that the Curry fixed point combinator $\mathbf{Y}$ does not lead outside of this class, in the sense that if A can be written with one variable, so does $\mathbf{Y}A$. We prove in particular that for every $m, n, r \geq 2$, $Con(\mathbf{Y}\Omega_m = \Omega_n = \omega_r)$ (Theorem 5.6) and $\mathbf{Y}\Omega_n$ is n.f.-easy (Theorem 7.1). It remains open whether $\mathbf{Y}\Omega_n$ is easy. We do not know if one can obtain such results by semantical methods of the kind considered in [1, 13]. We did not consider the problem of whether our results can be extended to the $\lambda\beta\eta$-calculus.

2 Notations and preliminaries

With minor differences, we refer to [2] for notation and prerequisites. We denote by $\to$ one-step β-reduction (also written $\to_\beta$), and by $\to^*$ multistep β-reduction, namely the transitive closure of $\to$. The sign $=$ between lambda terms means β-convertibility, and $\equiv$ means α-convertibility, namely identity up to renaming of bound variables. Λ is the set of all lambda terms and Λ_0 is the set of all closed lambda terms. An *abstraction term* is a term of the form $\lambda x.U$, and an *application term* is a term of the form UV. Clearly every lambda-term is either an abstraction or an application term. We recall that a 0-term is a lambda term A which cannot be β-reduced to an abstraction term. Clearly not all the unsolvable terms are 0-terms (e.g. if A is a 0-term, $\lambda x.A$ is an unsolvable term which is not a 0-term). It was proved by G. Jacopini and M. Venturini Zilli that there are easy terms which are not 0-terms. However all the natural examples seem to be 0-terms and we concentrate mainly on 0-terms. Roman letters such as $A, B, C, \dots, a, b, c, \dots$ will denote lambda terms. We use instead $\mathcal{A}, \mathcal{B}, \mathcal{C}$ etc. to denote sets of lambda terms. We introduce some notations for sets of lambda terms and relations between sets of lambda terms:

Definition 2.1

- $\mathcal{Z}_0$ = the set of all closed 0-terms.
- $[\mathcal{A}]^\beta$ = the smallest set containing $\mathcal{A}$ and closed under beta reduction.
- $\mathcal{A}^*$ = the smallest set containing $\mathcal{A}$ and closed under application.
- $\mathcal{A}\mathcal{B} = \{ab \mid a \in \mathcal{A}, b \in \mathcal{B}\}$.
- $\lambda x.\mathcal{A} = \{\lambda x.a \mid a \in \mathcal{A}\}$.
- For $n \geq 1$ let $\mathcal{A}^n\mathcal{B} = \{a_1(a_2(\dots(a_n b)\dots)) \mid$ each a_i is in $\mathcal{A}$ and $b \in \mathcal{B}\}$.
- $\mathcal{A}^\infty\mathcal{B}$ is the union for $n \geq 1$ of the sets $\mathcal{A}^n\mathcal{B}$.

- $S_0(\mathcal{A})$ is the smallest set containing $\mathcal{A}$ and all the closed substerms of terms in $\mathcal{A}$.
- Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \subseteq \Lambda_0$. $\mathcal{A}[\mathcal{B} := \mathcal{C}]$ is defined as the set of all terms t such that for some $n \geq 0$ there exist $b_1, \ldots, b_n \in \mathcal{B}$, $c_1, \ldots, c_n \in \mathcal{C}$ and a context $C[\ , \ldots, \ ]$ with n holes, such that $C[b_1, \ldots, b_n] \in \mathcal{A}$ and $t \equiv C[c_1, \ldots, c_n]$.

  Clearly $\mathcal{A} \subseteq \mathcal{A}[\mathcal{B} := \mathcal{C}]$ (take a context with zero holes). Note that in general $\mathcal{C}[\mathcal{C} := \mathcal{C}] \not\subseteq \mathcal{C}$, for instance if $\mathcal{C} = \{ab, a, b\}$, then $aa \in \mathcal{C}[\mathcal{C} := \mathcal{C}]$.
- $\mathcal{A}[\mathcal{B} := \mathcal{C}]^{\text{strict}}$ is defined similarly but assuming that the context $C[\ , \ldots, \ ]$ is not the trivial context $\Box$.

  So $\mathcal{A}[\mathcal{B} := \mathcal{C}]^{\text{strict}}$ is the set of all terms which can be obtained from some term of $\mathcal{A}$ by replacing some *proper* subterms belonging to $\mathcal{B}$ with some terms belonging to $\mathcal{C}$. Note that, in order to avoid difficulties with renaming of bound variables (and also to avoid confusions with the next definition), we use the notations $\mathcal{A}[\mathcal{B} := \mathcal{C}]$ and $\mathcal{A}[\mathcal{B} := \mathcal{C}]^{\text{strict}}$ only when $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are sets of *closed* terms.
- If x is a variable and $A, B \in \Lambda$, we employ the usual notation $A[x := B]$ to denote the term obtained from A replacing all the free occurrences of x with B and renaming bound variables to avoid conflicts.

If q is a lambda term we will often identify, by abuse of notation, q with the singleton set $\{q\}$. Thus $\mathcal{A}q$ is $\mathcal{A}\{q\}$, $\mathcal{A}^n q$ is $\mathcal{A}^n\{q\}$, etc. Similarly $S_0(q)$ is the set of all closed subterms of q (including q itself, if it is closed). Note that if $\mathcal{A}$ and $\mathcal{B}$ are sets of closed lambda-terms with $S_0(\mathcal{A}) \cap \mathcal{B} = \emptyset$, then $\mathcal{A}[\mathcal{B} := \mathcal{C}] \subseteq \mathcal{A}$ holds vacuosly since there are no substitutions to be performed. Similarly for $\mathcal{A}[\mathcal{B} := \mathcal{C}]^{\text{strict}}$.

3 Church-Rosser extensions of lambda calculus

In this section we study the problem of determining, given two closed lambda-terms X and M, whether $Con(X = M)$ holds. By Böhm's theorem [4], if X and M are distinct $\beta\eta$-normal forms, then $Con(X = M)$ never holds. On the other hand if X and M are both unsolvable, then $Con(X = M)$ does hold [2]. So the only interesting case is when X is unsolvable and M is solvable. We concentrate on the subcase in which X is a 0-term, and we show that in this situation we can give good sufficient conditions for $Con(X = M)$ to hold. We start with the following observations:

Remark 3.1 1. If $X = M$ holds (in a given model), then for every β-reduct X' of X, $X' = M$ holds.

2. If $X = M$ holds and $C[\ ]$ is a non-trivial context such that $C[X] = M$ holds, then $C[M] = M$ also holds.

A repeated application of the above remark generates, starting from X, a recursively enumerable set of lambda-terms, call it $\text{Closure}_M(X)$, which we are forced to equate to M if we wish to equate X to M. More precisely define:

Definition 3.2 Given $X, M \in \Lambda_0$ let $\mathit{Closure}_M(X)$ be the smallest set $\mathcal{C}$ containing X, closed under β reduction, and such that $\mathcal{C}[\mathcal{C} := M]^{\text{strict}} \subseteq \mathcal{C}$.

Definition 3.3 If $\mathcal{C} \subseteq \Lambda_0$, define $\text{Closure}_M(\mathcal{C}) = \bigcup_{X \in \mathcal{C}} \text{Closure}_M(X)$.

Note that Closure_M is a closure operation, i.e. it is monotone and satisfies $\text{Closure}_M(\text{Closure}_M(\mathcal{C})) = \text{Closure}_M(\mathcal{C})$.

Theorem 3.4 *If $\mathit{Closure}_M(X)$ consists entirely of application terms (i.e. terms of the form UV), then $Con(X = M)$ holds. More generally let $\mathcal{C}$ and $\mathcal{M}$ be sets of closed lambda terms with the properties:*

1. *$\mathcal{C}$ contains only application terms;*
2. *$\mathcal{C}$ is closed under β-reduction;*
3. *$\mathcal{C}[\mathcal{C} := \mathcal{M}]^{strict} \subseteq \mathcal{C}$.*

Then for every $M \in \mathcal{M}$ it is consistent with the $\lambda\beta$-calculus to identify all the terms in $\mathcal{C}$ with M, i.e. $Con(\{X = M \mid X \in \mathcal{C}\})$.

Before proving the theorem note that in the presence of clause 2, clause 1 is equivalent to $\mathcal{C} \subseteq \mathcal{Z}_0$. Also note that the theorem follows trivially from the particular case in which $\mathcal{M}$ consists of a single lambda-term M. In this case the hypothesis of the theorem simply mean that $\text{Closure}_M(\mathcal{C}) = \mathcal{C} \subseteq \mathcal{Z}_0$. Finally note that the following conditions are equivalent:

1. $\text{Closure}_M(X) \subseteq \mathcal{Z}_0$;
2. There exists $\mathcal{C} \subseteq *$, with $X \in \mathcal{C}$ and $\text{Closure}_M(\mathcal{C}) = \mathcal{C} \subseteq \mathcal{Z}_0$.

In fact if 1 holds, one can take $\mathcal{C} = \text{Closure}_M(X)$, while if 2 holds, then $\text{Closure}_M(X) \subseteq \mathcal{C} \subseteq \mathcal{Z}_0$. The theorem says that if these two equivalent conditions are satisfied, then for every $X \in \mathcal{C}$ $Con(X = M)$ holds. In the applications it is often convenient to work with condition 2, since it gives us more freedom in the choice of $\mathcal{C}$. In fact the main difficulty in applying Theorem 3.4 is to make the rigth choice of $\mathcal{C}$, i.e. to choose $\mathcal{C}$ in such a way that its closure properties are easy to verify.

Proof of Theorem 3.4. Define a notion of reduction, called μ-reduction, by letting $X \to_\mu M$ for every $X \in \mathcal{C}$. More precisely $\to_\mu$ is the smallest binary relation closed under substitutions and contexts and containing all the pairs (X, M) for $X \in \mathcal{C}$. $\overset{=}{\to}_\mu$ is the reflexive closure of $\to_\mu$, and $\to^*{}_\mu$ is its reflexive and transitive closure. We say that Δ is a *μ-redex* if $\Delta \in \mathcal{C}$. So every μ-redex is a 0-term.

Claim: *$\Rrightarrow_\mu$ is CR (i.e. it has the Church Rosser property).*

To prove the claim consider two nested occurrences of μ-redexes $T \in \mathcal{C}$ and $C[T] \in \mathcal{C}$, and the corresponding μ-reductions $C[T] \to_\mu M$ and $C[T] \to_\mu C[M]$. Since $\mathcal{C}[\mathcal{C} := M]^{\text{strict}} \subseteq \mathcal{C}$, we have $C[M] \in \mathcal{C}$. Hence we can find a common μ-reduct of $C[M]$ and M by μ-reducing $C[M]$ to M. The claim easily follows (the case of disjoint μ-redexes being trivial).

Since the Church Rosser property is preserved under taking the transitive closure, we have:

Claim: *$\to^*{}_\mu$ is CR.*

Now we must consider the interaction between β-reduction and μ-reduction.

Claim: *Given two reductions $A \to_\beta B$ and $A \to_\mu C$ there exists D such that $B {\to^*}_\mu D$ and $C \Rrightarrow_\beta D$.*

Let $\Delta \equiv (\lambda x.T)Q$ be the β-redex which has been reduced in $A \to_\beta B$, and let Δ' be the μ-redex which has been reduced in $A \to_\mu C$. Since the case of disjoint redexes is trivial, we can assume that Δ and Δ' are nested or coincident, and that the external one is A itself (otherwise we can remove the outer context).

Consider the case in which $A \equiv \Delta \equiv (\lambda x.T)Q$ and Δ' is a proper subterm of A. Since $\mathcal{C}$ consists entirely of application terms, Δ' cannot be $(\lambda x.T)$, so it must be contained in either T or in Q. Suppose for instance that Δ' is contained in Q. Then our two reductions have the form $(\lambda x.T)Q \to_\beta T[x := Q]$ and $(\lambda x.T)Q \to_\mu (\lambda x.T)Q^*$. We can now find a common reduct by: $T[x := Q] \to^*_\mu T[x := Q^*]$ and $(\lambda x.T)Q^* \to_\beta T[x := Q^*]$. So we can take $D \equiv T[x := Q^*]$. The case in which Δ' is contained in T is similar.

Suppose now that $A \equiv \Delta' \in \mathcal{C}$. Then $A \to_\mu M \equiv C$. Since $\mathcal{C}$ is closed under β-reduction and $A \to_\beta B$, $B \in \mathcal{C}$. Hence $B \to_\mu M$ and we can take $D \equiv M \equiv C$. The claim is thus proved.

By a diagram chase (Lemma 3.3.6 of [2]) it follows that $\to^*{}_\mu$ commutes with $\to^*{}_\beta$. (The fact that in the above claim $C \Rrightarrow_\beta D$ is a single-step or empty reduction, ensures that the diagram chase does not degenerate into an Escher-like picture). Since $\to^*_\mu$ and $\to^*_\beta$ are separately CR and commute, we can conclude by the "Hindley-Rosen lemma" (Lemma 3.3.5 in [2]) that $\to^*{}_{\beta\mu}$ is CR. If $\lambda\beta + \{c = M \mid c \in \mathcal{C}\} \vdash \mathbf{K} = \mathbf{S}$, then $\mathbf{K} =_{\beta\mu} \mathbf{S}$. Absurd since $\mathbf{K}$ and $\mathbf{S}$ are $\beta\mu$-normal forms. QED

The above theorem is to be compared with Theorems 15.3.3 and 15.3.5 in [2]. Theorem 15.3.3 [2] is due to Mitschke [9] and was used by him to give an alternative proof that Ω is easy. Note thatif Closure$(X) \subseteq \mathcal{Z}_0$, our proof shows not only that $Con(X = M)$ holds, but that $X = M$ holds in a Church Rosser extension of the $\lambda\beta$-calculus, in fact we have defined a Church Rosser notion of reduction $\to^*_{\beta\mu}$ extending $\to^*_\beta$, such that for every $T \in \text{Closure}_M(X)$ $T \to^*_{\beta\mu} M$.

Example 3.5 Let $\mathcal{C} = \{\Omega\}$ and let $M \in \Lambda_0$. Since the only reduct of Ω is Ω itself, Ω is a 0-term, and $\mathcal{C}$ is closed under β-reduction. Since Ω does not contain any closed 0-term as a proper subterm, $\mathcal{C}[\mathcal{C} := M]^{\text{strict}} = \mathcal{C}$ (empty substitution). So the hypothesis of the theorem are satisfied and we can conclude $Con(\Omega = M)$.

In the above example $\{\Omega\} = \text{Closure}_M(\Omega)$. In many cases however $\text{Closure}_M(X)$ is a very difficult set to describe and it turns out to be easier to look for a larger set $\mathcal{C} \supseteq \text{Closure}_M(X)$ which contains X and satisfies $\text{Closure}_M(\mathcal{C}) = \mathcal{C} \subseteq \mathcal{Z}_0$ (then apply Theorem 3.4). So the difficulty of the method is to choose the right $\mathcal{C}$. We will see an illustration of this fact in section 5.

4 Candidates for easy terms

Ω is certainly the "easiest" easy term. G. jacopini and M. Venturini Zilli [7] showed that any "recurrent" lambda-term is easy. Other examples are given by the observation that if U is easy, so is UV for any V (for $U = \mathbf{K}M$ implies $UV = M$). Besides these, there are not many natural examples of easy lambda-terms. The term Ω_3 (defined in the introduction) is a 0-term which is not easy. In fact Ω_3, being equal to $\Omega_3\omega_3$, cannot be equated to the identity $\mathbf{I}$. A natural candidate for an easy term is the Curry fixed point $\mathbf{Y}A$ of a closed 0-term A (recall that $\mathbf{Y} \equiv \lambda f.(\lambda y.f(yy))(\lambda y.f(yy))$).

Question 4.1 *1. Is it the case that for every closed 0-term A, $\mathbf{Y}A$ is easy?*

2. Is $\mathbf{Y}\Omega_3$ easy?

3. More generally suppose $X = \Omega_3 X$. Is X easy?

The difference between questions 2 and 3 is that in 2 we ask whether the Curry fixed point of Ω_3 is easy, while in 3 we ask whether *any* fixed point of Ω_3 is easy. We conjecture that questions 2 and 3 have a positive answer. For 1 the answer is negative since we have a counterexample: $\mathbf{YY}$ is a 0-term, but its fixed point $\mathbf{Y}(\mathbf{YY})$ (which is β-convertible to $\mathbf{YY}$) is not easy since it cannot be consistently equated to $\mathbf{KK}$. For 2 we have a partial answer which motivates our conjecture: $\mathbf{Y}\Omega_3$ can be consistently equated to every closed normal form and to every closed term A which is not of the form $\lambda x.T_1T_2T_3$ where each T_i is either the variable x or it is an unsolvable closed term. In particular $\mathbf{Y}\Omega_3$ is n.f.-easy.

Although such a result is not a complete answer to our question, we include a detailed proof for two reasons: first, in the course of our efforts we will prove some consistency results of independent interest about lambda-terms which can be written with only one variable. Secondly, the argument illustrates many of the techniques which will be employed in more complicated situations, for instance in the proof that n.f.-easy does not imply easy.

To prove $Con(\mathbf{Y}\Omega_3 = A)$ it turns out to be convenient to distinguish the two cases $A = \omega_3$ and $A \neq \omega_3$. The proof of $Con(\mathbf{Y}\Omega_3 = \omega_3)$ (Theorem 5.6) is a good example of the difficulties involved in making the right choice of $\mathcal{C}$ in Theorem 3.4. We will take $\mathcal{C}$ to be a suitable subset of the 0-terms which can be written with only one variable.

The proof of $Con(\mathbf{Y}\Omega_3 = A)$ in the remaining cases requires a new idea, namely to exploit the continuity properties of Böhm trees in order to find the right $\mathcal{C}$ (see sections 6 and 7).

5 Terms which can be written using only one variable

Note that $\mathbf{Y}\Omega_3 \to X$ where $X \equiv (\lambda y.\Omega_3(yy))(\lambda y.\Omega_3(yy))$. So to prove $Con(\mathbf{Y}\Omega_3 = \omega_3)$ it suffices to prove $Con(X = \omega_3)$. To apply Theorem 3.4 we must find a set $\mathcal{C}$ such that $X \in \mathcal{C}$ and $\text{Closure}_{\omega_3}(\mathcal{C}) = \mathcal{C} \subseteq \mathcal{Z}_0$. A noticeable property of X is that it can be written using only one variable (since $\Omega_3 \equiv (\lambda y.yyy)(\lambda y.yyy)$ can be written using only the variable y and so does X). So one could try to take $\mathcal{C}$ to be the set of all the 0-terms which can be written using only the variable y. But this does not work: $\Omega_3\mathbf{I}$ is such a term, but $\omega_3\mathbf{I}$ is not a 0-term although it belongs to $\text{Closure}_{\omega_3}(\Omega_3\mathbf{I})$. It turns out that what causes troubles is the presence of the identity $\mathbf{I}$. The correct choice of $\mathcal{C}$ is the following:

Definition 5.1 Define $\mathcal{C}$ as $\mathcal{Q} \cap \mathcal{Z}_0$ where $\mathcal{Q}$ is the set of all closed terms which 1) do not contain free or bound occurrences of variables different from y (or are α-convertible to such a term); 2) do not contain $\mathbf{I}$ as a subterm; 3) do not contain vacuous abstractions (i.e. they belong to the $\lambda\mathbf{I}$-calculus).

It remains to show that with this definition we do have $\text{Closure}_{\omega_3}(\mathcal{C}) = \mathcal{C} \subseteq \mathcal{Z}_0$. Note that $\mathcal{Q}$ is closed under application, i.e. $\mathcal{Q}\mathcal{Q} \subseteq \mathcal{Q}$. Conversely if $UV \in \mathcal{Q}$ where U and V are closed terms, then U and V belong to $\mathcal{Q}$.

Lemma 5.2 *$\mathcal{Q}$ is closed under β-reduction.*

Proof. An easy induction on the length of the terms. The crucial case is to show that if $\lambda y.R$ and V belong to $\mathcal{Q}$, then $R[y := V]$ belongs to $\mathcal{Q}$. This is clear from the definition of $\mathcal{Q}$. QED

The crucial property of $\mathcal{Q}$ is that every applicative term in $\mathcal{Q}$ is a 0-term, so that $\mathcal{C}$ coincides with the set of all the applicative terms in $\mathcal{Q}$, namely:

Lemma 5.3 *$\mathcal{Q}\mathcal{Q} = \mathcal{C}$.*

Proof. It suffices to show that $\mathcal{Q}\mathcal{Q}$ is closed under β-reduction. So let $U, V \in \mathcal{Q}$. We must show that if $UV \to T$ is a (one-step) β-reduction, then

$T \in \mathcal{Q}\mathcal{Q}$. If the redex being contracted is contained in either U or V, then we use the fact that $\mathcal{Q}$ is closed under β-reduction. So we can assume that U has the form $\lambda y.R$ and the redex being contracted is UV itself. Then $T \equiv R[y := V]$. Since $\lambda y.R \in \mathcal{Q}$, by the definition of $\mathcal{Q}$ the term R must be of the form $T_1 \dots T_n$ where $n \geq 2$, each T_i belongs to $\mathcal{Q} \cup \{y\}$ and at least one T_i is the variable y. But then $R[y := V]$ is an application of at least two terms in $\mathcal{Q}$, hence it belongs to $\mathcal{Q}\mathcal{Q}$ as desired. QED

Lemma 5.4 *Let $\mathcal{M}$ be the set of all the abstraction terms in $\mathcal{Q}$, i.e. $\mathcal{M} = \mathcal{Q} - \mathcal{C}$. We have:*

1. $\mathcal{C}[\mathcal{C} := \mathcal{M}]^{strict} \subseteq \mathcal{C}$ *(hence $\mathcal{C}[\mathcal{C} := \mathcal{M}] \subseteq \mathcal{C} \cup \mathcal{M}$);*
2. $\mathcal{M}[\mathcal{C} := \mathcal{M}] \subseteq \mathcal{M}$.

Proof. By a simultaneus induction on the length of the terms. More precisely suppose that $c \in \mathcal{C}$, $m \in \mathcal{M}$ and $C[\,]$ is a non trivial context with $C[c] \in \mathcal{M} \cup \mathcal{C}$. We show by induction on the length of $C[c]$ that if $C[c] \in \mathcal{C}$, then $C[m] \in \mathcal{C}$, and if $C[c] \in \mathcal{M}$, then $C[m] \in \mathcal{M}$. Suppose for instance that $C[c] \in \mathcal{C}$. Notice that

$$\mathcal{C} = \mathcal{M}\mathcal{M} \ \cup \ \mathcal{M}\mathcal{C} \ \cup \ \mathcal{C}\mathcal{M} \ \cup \ \mathcal{C}\mathcal{C}$$

If, say, $C[c] \equiv UV \in \mathcal{M}\mathcal{C}$, then c must be contained in either U or V (since c is properly contained in $C[c]$), and we can apply the induction hypothesis to U or to V to show that $C[m] \in \mathcal{M}\mathcal{C} \subseteq \mathcal{C}$. The other three cases are similar.

One the other hand if $C[c] \in \mathcal{M}$, then $C[c]$ must be of the form $\lambda y.T_1 \dots T_n$, where $n \geq 2$, each T_i belongs to $\mathcal{M} \cup \mathcal{C} \cup \{y\}$ and at least one T_i is the variable y. The displayed occurrence of c in $C[c]$ must be contained in one of the T_i's and applying the induction hypothesis to T_i we see that $C[m] \in \mathcal{M}$. QED

We have thus proved that $\mathcal{C}$ and $\mathcal{M}$ satisfy the hypothesis of Theorem 3.4, hence we have:

Theorem 5.5 *It is consistent to equate all the lambda terms in $\mathcal{C}$ with any given lambda term M in $\mathcal{M}$ (or even with a given M in $\mathcal{C}$, but this comes for free since all the unsolvables can be consistently equated).*

Corollary 5.6 *For every $m, n, r \geq 2$, $Con(\mathbf{Y}\Omega_m = \Omega_n = \omega_r)$.*

Proof. It is enough to observe that $\omega_r \in \mathcal{M}$, $\Omega_n \in \mathcal{C}$ and $\mathbf{Y}\Omega_m$ is reducible to a term in $\mathcal{C}$, namely $\mathbf{Y}\Omega_m \to (\lambda y.\Omega_m(yy))(\lambda y.\Omega_m(yy)) \in \mathcal{C}$. QED

6 Böhm trees

$BT(A)$ denotes the Böhm tree of the lambda term A. Böhm trees can be contrued as partial functions and they can be partially ordered by inclusion as in [2], namely $BT(A) \subseteq BT(B)$ iff the Böhm tree of B can be obtained by the Böhm tree of A by replacing some $\perp$'s by some Böhm trees. Note that the Böhm tree of every normal form is maximal. More generally every $\perp$-free Böhm tree is maximal. The crucial property of Böhm trees that we need is the following:

Proposition 6.1 *If $BT(A) \subseteq BT(B)$, then $BT(C[A]) \subseteq BT(C[B])$.*

Proof. See [2] Corollary 14.3.20 (iii). QED

Corollary 6.2 *Let $A \in \Lambda_0$ and let $\mathcal{A} = \{t \in \Lambda_0 \mid BT(A) \subseteq BT(t)\}$. Then $\mathcal{A}$ is closed under β-reduction and $\mathcal{A}[\mathcal{Z}_0 := \Lambda_0] \subseteq \mathcal{A}$.*

Proof. Closure under β-reduction is obvious since β-convertible terms have the same Böhm tree. Closure under $[\mathcal{Z}_0 := \Lambda_0]$ follows from the previous proposition together with the fact that every $t \in \mathcal{Z}_0$ is unsolvable, so it has a trivial Böhm tree, i.e. $BT(t) = \perp$. QED

The above corollary says that sets of lambda-terms defined using Böhm trees have closure properties very similar to those needed to apply Theorem 3.4 (except that they do not consist of 0-terms). This fact will be used in the following sections to define the appropriate sets $\mathcal{C} \subseteq \mathcal{Z}_\prime$ needed to prove our results concerning n.f.-easy terms. In particular in section 8 $\mathcal{C}$ will be defined by a grammar to generate sets starting from other sets, taking as a starting point sets of the form $\mathcal{A} = \{t \in \Lambda_0 \mid BT(A) \subseteq BT(t)\}$.

7 $\mathbf{Y}\Omega_n$ is n.f.-easy

Theorem 7.1 *$\mathbf{Y}\Omega_n$ is n.f.-easy ($n \geq 2$). Moreover $\mathbf{Y}\Omega_n$ can be consistently equated to every closed term A such that $BT(A) \not\subseteq BT(\omega_n)$.*

Proof. We already proved that $Con(\mathbf{Y}\Omega_n = \omega_n)$. Since the Böhm tree of a normal form is maximal, it follows that if A is a closed normal form different from ω_n, then $BT(A) \not\subseteq BT(\omega_n)$. So to prove the theorem it suffices to show that if A is a closed lambda term with $BT(A) \not\subseteq BT(\omega_n)$ (not necessarily a normal form), then $Con(\mathbf{Y}\Omega_n = A)$. Let $\mathcal{A} = \{t \in \Lambda_0 \mid BT(A) \subseteq BT(t)\}$. By the properties of Böhm trees we have:

Claim: *$\mathcal{A}$ is closed under β-reduction, $\mathcal{A}[\mathcal{Z}_0 := \Lambda_0] \subseteq \mathcal{A}$, and $\omega_n \notin \mathcal{A}$.*

Define:

1. $X \equiv (\lambda y.\Omega_n(yy))(\lambda y.\Omega_n(yy))$;

2. $\mathcal{O}$ is the set of all terms of the form $\Omega_n \omega_n \ldots \omega_n$ (zero or more ω_n's);

3. $\mathcal{X} = (\lambda y.\mathcal{O}(yy))(\lambda y.\mathcal{O}(yy))$;

Note that $\mathbf{Y}\Omega_n \to X$, so it suffices to show $Con(X = A)$. To apply Theorem 3.4 we must find $\mathcal{C}$ with $X \in \mathcal{C}$ and $\text{Closure}_A(\mathcal{C}) = \mathcal{C} \subseteq \mathcal{Z}_0$. From the definition of Ω_n it is clear that $\mathcal{O}$ is closed under β-reduction. $\mathcal{X}$ is not closed under β-reduction but every one-step β-reduct of a term in $\mathcal{X}$ belongs to $\mathcal{X} \cup \mathcal{O}\mathcal{X}$. It follows that:

Claim: *$\mathcal{O}^\infty \mathcal{X}$ and $\mathcal{X} \cup \mathcal{O}^\infty \mathcal{X}$ are closed under β-reduction.*

Now define:

$$\mathcal{C} = \mathcal{X} \cup \mathcal{O}^\infty \mathcal{X} \cup \mathcal{O}^\infty \mathcal{A}.$$

From the above claims it follows that $\mathcal{C}$ is closed under β-reduction and therefore $\mathcal{C} \subseteq \mathcal{Z}_0$ (since any term in $\mathcal{C}$ is of the form UV).

Claim: $\mathcal{C} \cap S_0(\mathcal{C} \cup \mathcal{O}) = \emptyset$.

To prove the claim note that every element of $\mathcal{C}$ is a closed term of the form UV. Now if U and V are closed terms with $UV \in S_0(\mathcal{O})$, then by definition of $\mathcal{O}$, $V \equiv \omega_n$. On the other hand no element of $\mathcal{C}$ has the form $U\omega_n$ (since $\omega_n \notin \mathcal{A}$), hence $\mathcal{C} \cap S_0(\mathcal{O}) = \emptyset$. Since $\mathcal{X} = (\lambda y.\mathcal{O}(yy))(\lambda y.\mathcal{O}(yy))$ it also follows $\mathcal{C} \cap S_0(\mathcal{X}) = \emptyset$ and the claim is proved.

To finish the proof of $\text{Closure}_A(\mathcal{C}) = \mathcal{C} \subseteq \mathcal{Z}_0$, it suffices to show:

Claim: $\mathcal{C}[\mathcal{C} := A]^{strict} \subseteq \mathcal{C}$.

Note that $\mathcal{C}$ can be generated by the grammar:

$$\mathcal{C} ::= \mathcal{X} \mid \mathcal{O}\mathcal{A} \mid \mathcal{O}\mathcal{C}$$

So to prove the claim it suffices to show:

1. $\mathcal{A}[\mathcal{C} := A] \subseteq \mathcal{A}$;

2. $\mathcal{X}[\mathcal{C} := A]^{\text{strict}} \subseteq \mathcal{X}$ (hence $\mathcal{X}[\mathcal{C} := A] \subseteq \mathcal{X} \cup \mathcal{A}$);

3. $\mathcal{O}[\mathcal{C} := A] \subseteq \mathcal{O}$.

Part 1 follows from $\mathcal{A}[\mathcal{Z}_0 := \Lambda_0] \subseteq \mathcal{A}$. Parts 2 and 3 follows from the fact that $\mathcal{C} \cap S_0(\mathcal{X} \cup \mathcal{O}) = \emptyset$. QED

8 A separation theorem in lambda-calculus

Definition 8.1 Two closed terms A and B have *incompatible Böhm trees* if there is no term t such that $BT(A) \subseteq BT(t)$ and $BT(B) \subseteq BT(t)$.

Theorem 8.2 *Given a solvable term $A \in \Lambda_0$ there is a term $X \equiv X_A \in \Lambda_0$ such that $\neg Con(X = A)$ and for every $B \in \Lambda_0$ whose Böhm tree is incompatible with that of A, $Con(X = B)$.*

The definition of $X \equiv X_A$ turns out to be simpler in the case $\neg Con(A = AA)$ holds, for instance when $A \equiv \mathbf{K}$.

Definition 8.3 Let $A \in \Lambda_0$ be such that $\neg Con(A = AA)$. Define:

1. $P \equiv (\lambda xy.xxA(xxy))(\lambda xy.xxA(xxy))$;
2. $X \equiv (\lambda x.P(xx))(\lambda x.P(xx))$.

The idea is that P is defined by fixed point in such a way that $Py \to^* PA(Py)$ and X is defined by fixed point so that $X \to PX$.

Proposition 8.4 *If A and X are as in Definition 8.3, then $\neg Con(X = A)$.*

Proof. By definition of P, $PA = PA(PA)$. So if $A = X$ (in some model), then $PX = PX(PX)$. But $PX = X = A$, hence $A = AA$. Contradiction. QED

Now we want to define $X \equiv X_A$ for an arbitrary solvable term A without the assumption $\neg Con(A = AA)$. Since A is solvable, A can be reduced to a term of the form $\lambda x_1 \dots x_n.x_i T_1 \dots T_s$. If k is chosen sufficiently large, for instance $k = s + n + 2$, and we define $N_1 \equiv \dots N_n \equiv \lambda x_1 \dots x_k.x_k$, then $\neg Con(A = AN_1, \dots, N_n A)$ holds. So we have "almost" reduced the general case to the case $\neg Con(A = AA)$.

Definition 8.5 Let $A \equiv \lambda x_1 \dots x_n.x_i T_1 \dots T_s$ be a solvable closed term. If $\neg Con(A = AA)$ define $X \equiv X_A$ as in Definition 8.3. Otherwise let $k = s+n+2$ and define:

1. $N_1 \equiv \dots \equiv N_n \equiv \lambda x_1 \dots x_k.x_k$.
2. $P \equiv (\lambda xy.xxAN_1 \dots N_n(xxy))(\lambda xy.xxAN_1 \dots N_n(xxy))$;
3. $X \equiv (\lambda x.P(xx))(\lambda x.P(xx))$.

Note that $Py \to^* PAN_1 \dots N_n(Py)$ and $X \to PX$.

Proposition 8.6 *If A and X are as in Definition 8.5, then $\neg Con(X = A)$.*

Proof. If $X = A$ (in some model), then $PX = PAN_1 \dots N_n(PX) = PXN_1 \dots N_n(PX)$. But $PX = X = A$, hence $A = AN_1 \dots N_n A$. Contradiction. QED

Proof of Theorem ??. Given a solvable closed term A define $X = X_A$ as in Definition 8.5. We have already proved $\neg Con(X = A)$. It remains to show that if $BT(A)$ and $BT(B)$ are incompatible, then $Con(X = B)$. To apply Theorem 3.4 we must find $\mathcal{C}$ with $X \in \mathcal{C}$ and $\text{Closure}_B(\mathcal{C}) = \mathcal{C} \subseteq \mathcal{Z}_0$.

We need several auxiliary sets. Let $\mathcal{B} = \{t \in \Lambda_0 \mid BT(B) \subseteq BT(t)\}$ and $\mathcal{A} = \{t \in \Lambda_0 \mid BT(A) \subseteq BT(t)\}$. Since A and B have incompatible Böhm trees, $\mathcal{B} \cap \mathcal{A} = \emptyset$. Moreover by Corollary 6.2, $\mathcal{A}$ and $\mathcal{B}$ are closed under β-reduction. Note that if A and B are normal forms, then $\mathcal{A} = \{A\}$ and $\mathcal{B} = \{B\}$.

Define:

$$\mathcal{P} = (\lambda xy.xx\mathcal{A}N_1 \dots N_n(xxy))(\lambda xy.xx\mathcal{A}N_1 \dots N_n(xxy)).$$

Note that $\mathcal{P}$ contains P but it is not closed under β-reduction. The set of all β-reducts of P is quite difficult to describe, but it is certainly contained in the set $\mathcal{D}$ defined by the following grammar.

$$\mathcal{D} ::= \mathcal{P} \mid \mathcal{D}\mathcal{A} \mid \mathcal{D}\mathcal{D} \mid \lambda y.\mathcal{D}^\infty y \mid \mathcal{D}\mathcal{A}N_1 \dots N_n \mid \mathcal{D}\mathcal{D}N_1 \dots N_n$$

Claim: *$\mathcal{D}$ is closed under β-reductions.*

To prove the claim let $d \in \mathcal{D}$ and let $d \to q$ be a one-step β-reduction obtained by contracting the redex $\Delta \leq d$. To show $q \in \mathcal{D}$ we reason by induction on the length of the derivation $\mathcal{D} ::= d$. If $d \in \mathcal{P} = (\lambda xy.xx\mathcal{A}N_1 \dots N_n(xxy))(\lambda xy.xx\mathcal{A}N_1 \dots N_n(xxy))$ and $d \equiv \Delta$, then $q \in \lambda y.\mathcal{P}\mathcal{A}N_1 \dots N_n(\mathcal{P}y) \subseteq \lambda y.\mathcal{D}^\infty y \subseteq \mathcal{D}$. If $d \in \mathcal{P}$ and $d \not\equiv \Delta$, then $\Delta \leq \mathcal{A}$ and we can use the fact that $\mathcal{A}$ is closed under β-reduction ($[\mathcal{A}]^\beta \subseteq \mathcal{A}$). If $d \in \mathcal{D}\mathcal{A}$ and $d \not\equiv \Delta$, then d must be of the form UV with $U \in \mathcal{D}$ and $V \in \mathcal{A}$ and Δ must be contained either in U or in V. In the former case use the induction hypothesis, in the latter use $[\mathcal{A}]^\beta \subseteq \mathcal{A}$. If $d \in \mathcal{D}\mathcal{A}$ and $d \equiv \Delta$, then $d \in (\lambda y.\mathcal{D}^\infty y)\mathcal{A}$ and $q \in \mathcal{D}^\infty \mathcal{A} \subseteq \mathcal{D}$. If $d \in \mathcal{D}\mathcal{D}$ and $d \not\equiv \Delta$ we apply the induction hypothesis. If $d \in \mathcal{D}\mathcal{D}$ and $d \equiv \Delta$, then $d \in (\lambda y.\mathcal{D}^\infty y)\mathcal{D}$ and $q \in \mathcal{D}^\infty \mathcal{D} \subseteq \mathcal{D}$. If $d \in \lambda y.\mathcal{D}^\infty y$ and Δ is a closed term, then Δ must be contained in one of the $\mathcal{D}$'s of $\lambda y.\mathcal{D}^\infty y$ and we apply the induction hypothesis. If instead Δ is an open term, then Δ must belong to $\mathcal{D}y$, and its contractum is in $\mathcal{D}^\infty y$. Since $\mathcal{D}^\infty(\mathcal{D}^\infty y) \subseteq \mathcal{D}^\infty y$, $d \to q \in \lambda y.\mathcal{D}^\infty y$. Finally if $d \in \mathcal{D}\mathcal{D}N_1 \dots N_n \cup \mathcal{D}\mathcal{A}N_1 \dots N_n$, then since each N_i is a normal form, Δ must be contained in $\mathcal{D}\mathcal{D} \cup \mathcal{D}\mathcal{A}$ and we have reduced to a previous case. The claim is thus proved.

Claim: *All the terms in $\mathcal{D}$ are unsolvable.*

To see this it is enough to notice that no term in $\mathcal{D}$ is in solved form and $\mathcal{D}$ is closed under β-reduction.

Note that since A and B have incompatible Böhm tree, A and B are solvable. Hence $\mathcal{A}$ and $\mathcal{B}$ consist entirely of solvable terms. It follows that:

Claim: $\mathcal{D} \cap \mathcal{A} = \mathcal{D} \cap \mathcal{B} = \emptyset$.

Note that the fact that $\mathcal{D}$ is closed under β-reduction does not imply that $\mathcal{D}x$ is closed under β-reduction (where x is a variable). The exact situation is expressed by the following:

Claim: *If $\mathcal{Q} \subseteq \Lambda$ is closed under β- reduction, every one-step β-reduct of $\mathcal{D}\mathcal{Q}$ belongs to $\mathcal{D}^\infty\mathcal{Q}$.*

To prove the claim let $t \in \mathcal{D}\mathcal{Q}$ and let $t \to q$ be a one-step reduction obtained by contracting the redex $\Delta \leq t$. If $t \not\equiv \Delta$, then $q \in \mathcal{D}\mathcal{Q} \subseteq \mathcal{D}^\infty\mathcal{Q}$ using the fact that both $\mathcal{D}$ and $\mathcal{Q}$ are closed under β- reduction. If $t \equiv \Delta$, then $t \in (\lambda x.\mathcal{D}^\infty x)\mathcal{Q}$ and therefore $q \in \mathcal{D}^\infty\mathcal{Q}$ as desired.

An immediate consequence is:

Claim: *If $\mathcal{Q}$ is closed under β-reduction, so is $\mathcal{D}^\infty\mathcal{Q}$. In particular $\mathcal{D}^\infty\mathcal{B}$ is closed under β-reduction (hence it consists entirely of 0-terms).*

Now let $\mathcal{M} = \lambda x.\mathcal{D}^\infty(xx)$. Note that $\lambda x.P(xx) \in \mathcal{M}$ and by the previous claims $[\mathcal{M}]^\beta \subseteq \mathcal{M}$. Define:

$$\mathcal{X} = \mathcal{M}\mathcal{M} = (\lambda x.\mathcal{D}^\infty(xx))(\lambda x.\mathcal{D}^\infty(xx)).$$

Note that $X \in \mathcal{X}$. $\mathcal{X}$ is not closed under β-reduction, but every one-step reduct of a term in $\mathcal{X}$ belongs to $\mathcal{X} \cup \mathcal{D}^\infty\mathcal{X}$. It follows that:

Claim: *$\mathcal{D}^\infty\mathcal{X}$ and $\mathcal{X} \cup \mathcal{D}^\infty\mathcal{X}$ are closed under β-reduction.*

Finally define:

$$\mathcal{C} = \mathcal{X} \cup \mathcal{D}^\infty\mathcal{X} \cup \mathcal{D}^\infty\mathcal{B}.$$

Then $[\mathcal{C}]^\beta \subseteq \mathcal{C}$ and therefore $\mathcal{C} \subseteq \mathcal{Z}_0$ (since no term in $\mathcal{C}$ has the form $\lambda x.U$). Note that $X \in \mathcal{C}$ and that $\mathcal{C}$ can be generated by the grammar:

$$\mathcal{C} ::= \mathcal{X} \mid \mathcal{D}\mathcal{B} \mid \mathcal{D}\mathcal{C}$$

.

To prove $\text{Closure}_B(\mathcal{C}) = \mathcal{C} \subseteq \mathcal{Z}_0$ it only remains to show $\mathcal{C}[\mathcal{C} := B]^{\text{strict}} \subseteq \mathcal{C}$. By definition of $\mathcal{C}$ it suffices to prove:

1. $\mathcal{X}[\mathcal{C} := B]^{\text{strict}} \subseteq \mathcal{X}$ (hence $\mathcal{X}[\mathcal{C} := B] \subseteq \mathcal{X} \cup \mathcal{B}$);
2. $\mathcal{D}[\mathcal{C} := B] \subseteq \mathcal{D}$;
3. $\mathcal{B}[\mathcal{C} := B] \subseteq \mathcal{B}$.

Part 3 follows from $\mathcal{B}[\mathcal{Z}_0 := \Lambda_0] \subseteq \mathcal{B}$ which holds by the definition of $\mathcal{B}$ and the properties of Böhm trees (i.e. Corollary 6.2).

Claim: *$\mathcal{X} \cap \mathcal{D} = \emptyset$.*

We must show that $\mathcal{X} = (\lambda x.\mathcal{D}^\infty(xx))(\lambda x.\mathcal{D}^\infty(xx))$ does not intersect $\mathcal{D} = \mathcal{P} \cup \mathcal{D}\mathcal{A} \cup \mathcal{D}\mathcal{D} \cup \lambda x.\mathcal{D}^\infty x \cup \mathcal{D}\mathcal{A}N_1 \dots N_n \cup \mathcal{D}\mathcal{D}N_1 \dots N_n$. Since $\mathcal{P} = (\lambda xy.xx\mathcal{A}(xxy))(\lambda xy.xx\mathcal{A}(xxy))$ clearly $\mathcal{X} \cap \mathcal{P} = \emptyset$. Now it is enough to observe that $(\lambda x.\mathcal{D}^\infty(xx)) \cap \mathcal{D} = \emptyset$ (since the only elements of $\mathcal{D}$ of the form $\lambda x.U$ are those of $\lambda x.\mathcal{D}^\infty x$). The claim is thus proved.

Claim: $\mathcal{P} \cap \mathcal{C} = \emptyset$.

We have $\mathcal{C} = \mathcal{X} \cup \mathcal{D}\mathcal{B} \cup \mathcal{D}\mathcal{C}$ and $\mathcal{P} = (\lambda xy.xx\mathcal{A}(xxy))(\lambda xy.xx\mathcal{A}(xxy))$. Clearly $\mathcal{P} \cap \mathcal{X} = \emptyset$. To prove the claim it is enough to observe that $\mathcal{D}$ contains no elements of the form $\lambda xy.U$.

Claim: $\mathcal{C} \cap \mathcal{D} = \emptyset$.

Suppose $\mathcal{C} \cup \mathcal{D} \neq \emptyset$ and choose $t \in \mathcal{C} \cap \mathcal{D}$ which minimazes the sum of the lengths of the derivations $\mathcal{C} ::= t$ and $\mathcal{D} ::= t$. Recall that $\mathcal{C} = \mathcal{X} \cup \mathcal{D}\mathcal{B} \cup \mathcal{D}\mathcal{C}$ and $\mathcal{D} = \mathcal{P} \cup \mathcal{D}\mathcal{A} \cup \mathcal{D}\mathcal{D} \cup \lambda y.\mathcal{D}^\infty y \cup \mathcal{D}\mathcal{A}N_1 \ldots N_n \cup \mathcal{D}\mathcal{D}N_1 \ldots N_n$. If the intersection is non-empty, then since $\mathcal{X}$ does not intersect $\mathcal{D}$ and $\mathcal{P}$ does not intersect $\mathcal{C}$, the only possibility is: $t \in (\mathcal{D}\mathcal{B} \cup \mathcal{D}\mathcal{C}) \cap (\mathcal{D}\mathcal{A} \cup \mathcal{D}\mathcal{D} \cup \mathcal{D}\mathcal{A}N_1 \ldots N_n \cup \mathcal{D}\mathcal{D}N_1 \ldots N_n)$ (we can exclude the case that $\lambda y.\mathcal{D}^\infty y$ intersects $\mathcal{C}$ since $\mathcal{C}$ consists entirely of 0-terms). Now $t \in \mathcal{D}\mathcal{C} \cap \mathcal{D}\mathcal{D}$ is excluded by the minimality of t. $t \in \mathcal{D}\mathcal{B} \cap \mathcal{D}\mathcal{A}$ is also excluded since $\mathcal{B} \cap \mathcal{A} = \emptyset$. $t \in \mathcal{D}\mathcal{C} \cap \mathcal{D}\mathcal{A}$ is excluded since $\mathcal{C} \cap \mathcal{A} = \emptyset$ (as $\mathcal{A}$ consists entirely of solvable terms, and $\mathcal{C} \subseteq \mathcal{Z}_0$). Finally suppose that $t \in (\mathcal{D}\mathcal{B} \cup \mathcal{D}\mathcal{C}) \cap (\mathcal{D}\mathcal{A}N_1 \ldots N_n \cup \mathcal{D}\mathcal{D}N_1 \ldots N_n)$. Then $\mathcal{D} \cap (\mathcal{D}\mathcal{A}N_1 \ldots N_{n-1} \cup \mathcal{D}\mathcal{D}N_1 \ldots N_{n-1}) \neq \emptyset$. But this is visibly impossible since by the definition of N_i we have $N_1 \equiv \ldots \equiv N_n \notin \mathcal{A}$ and no term in $\mathcal{D}$ has the form $UVN_1 \ldots N_{n-1}$ with $V \not\equiv N_1$.

Claim: $\mathcal{D}[\mathcal{C} := B] \subseteq \mathcal{D}$ *(if B is a normal form we have the stronger claim $\mathcal{C} \cap S_0(\mathcal{D}) = \emptyset$).*

To prove the claim let $c \in \mathcal{C}$ and let $d \equiv C[c] \in \mathcal{D}$. We must prove that $C[B] \in \mathcal{D}$. The proof is by induction on the length of d. Note that $d \not\equiv c$ (since $\mathcal{C} \cap \mathcal{D} = \emptyset$). We distinguish three cases:

1) If $d \in \mathcal{P} = (\lambda xy.xx\mathcal{A}(xxy))(\lambda xy.xx\mathcal{A}(xxy))$, then c must appear inside one of the two $\mathcal{A}$'s. Now use the fact that $\mathcal{A}[\mathcal{C} := \Lambda_0] \subseteq \mathcal{A}$ (by Corollary 6.2 and the fact that $\mathcal{C} \subseteq \mathcal{Z}_0$).

2) Suppose $d \in \mathcal{D}\mathcal{A}$. If c occurs inside the $\mathcal{D}$-part we apply the induction hypothesis. If c occurs inside the $\mathcal{A}$-part we use the fact that $\mathcal{A}[\mathcal{C} := \Lambda_0] \subseteq \mathcal{A}$.

3) If $d \in \mathcal{D}\mathcal{D} \cup \lambda x.\mathcal{D}^\infty x \cup \mathcal{D}N_i$, then c must be contained in one of the $\mathcal{D}$'s and we apply the induction hypothesis.

The claim is thus proved.

Claim: $\mathcal{X}[\mathcal{C} := B]^{strict} \subseteq \mathcal{X}$.

To see this recall the definition $\mathcal{X} = (\lambda x.\mathcal{D}^\infty(xx))(\lambda x.\mathcal{D}^\infty(xx))$. Since $\mathcal{C} \subseteq \mathcal{Z}_0$, $\mathcal{C} \cap \lambda x.\mathcal{D}^\infty(xx) = \emptyset$. So if $c \in \mathcal{C}$ and $c < \mathcal{X}$, c must occur inside one of the $\mathcal{D}$'s in the definition of $\mathcal{X}$. Now apply the previous claim.

The proof of $Con(X = B)$ is now complete. QED

Remark 8.7 From the proof of Theorem **??** we see that the term X can be taken to be a 0-term.

In the next section we will use Theorem **??** to show that n.f.-easy does not imply easy. Actually this was the main motivation behind Theorem **??**.

9 n.f.-Easy terms

Lemma 9.1 *Given two distinct closed normal forms N_1 and N_2, there exists a closed normal form U such that $N_1U \neq N_2U$ and both N_1U and N_2U have a normal form.*

Proof. By a renaming of bound variables we can assume that $N_1 \equiv \lambda x.M_1$, $N_2 \equiv \lambda x.M_2$. Let $k \geq 0$ be so big that for every subterm of M_1 or M_2 of the form $xT_1 \ldots T_n$ we have $n \leq k$. Then $U \equiv \lambda y_1 \ldots y_k z.zy_1 \ldots y_k$ is as desired. QED

One of the motivation for the notion of n.f.-easy term is the following test for non-easiness.

Theorem 9.2 *If $X \in \Lambda_0$ is such that $N_1X = N_2X$ where N_1 and N_2 are distinct closed normal forms, then X is not n.f.-easy (hence it is not easy).*

Proof. Given N_1 and N_2 distinct closed normal forms, by the previous lemma we can find U be such that $N_1U \neq N_2U$ and both N_1U and N_2U have a normal form. It follows $\neg Con(N_1U = N_2U)$ and therefore $\neg Con(X = U)$. QED

As an example if $X = XX$, then X is not easy: take $N_1 \equiv \lambda x.x$ and $N_2 \equiv \lambda x.xx$. An immediate consequences of Theorem **??** is:

Theorem 9.3 *Let $Y \in \Lambda_0$ be a term without normal form such that $BT(Y)$ is $\perp$-free. Then there is a term $X \in \Lambda_0$ such that $\neg Con(X = Y)$ and X can be consistently equated to every closed normal form N (or even to every closed term N such that $BT(N) \not\subseteq BT(Y)$).*

Proof. Just note that if N is a normal form and Y is a term without normal form such that $BT(Y)$ is $\perp$-free, then $BT(N)$ and $BT(Y)$ are (both maximal and) incompatible. QED

Corollary 9.4 *n.f.-easy does not imply easy.*

Corollary 9.4 is to be contrasted with the following theorem which implies in particular that if a term X can be consistently equated to every term with a finite Böhm tree, then X is easy.

Theorem 9.5 *Suppose that $X \in \Lambda$ can be consistently equated to every closed lambda term with Böhm tree of height at most 1 (namely to every closed term of the form $\lambda x_1 \ldots x_m.x_i\ Q_1 \ldots Q_n$ where all the Q_i's are unsolvable). Then X is easy.*

Proof. Suppose X is as in the hypothesis of the theorem. Then X must be unsolvable (because if X is solvable there is $k > 0$ such that it is inconsistent to equate X and $\lambda x_1 \ldots x_k.x_1$). Now let $Y \in \Lambda_0$. We must show $Con(X = Y)$. We can assume that Y is solvable, otherwise $X = Y$ holds in any model which equates all the unsolvables. So Y has the form $\lambda x_1 \ldots x_m.x_i M_1 \ldots M_n$. Let $N_1 \equiv \ldots \equiv N_m \equiv \lambda y_1 \ldots y_n.\mathbf{I}$. Then for every term t of the form $\lambda x_1 \ldots x_m.x_i P_1 \ldots P_n$ (where m and n are fixed and the P_i's are arbitrary), we have $tN_1 \ldots N_m = \mathbf{I}$. Define $Y^* = \lambda x_1 \ldots x_m.x_i(XN_1 \ldots N_m M_1) \ldots (XN_1 \ldots N_m M_n)$. Since X is unsolvable, each $XN_1 \ldots N_m M_i$ is unsolvable, and so by our assumptions $Con(X = Y^*)$. Now it is enough to observe that in any model of $X = Y^*$ we also have $X = Y$. To see this assume $X = Y^*$. Then we have the following equalities $X = \lambda x_1 \ldots x_m.x_i(XN_1 \ldots N_m M_1) \ldots (XN_1 \ldots N_m M_n) = \lambda x_1 \ldots x_m.x_i(Y^* N_1 \ldots N_m M_1) \ldots (Y^* N_1 \ldots N_m M_n) = \lambda x_1 \ldots x_m.x_i(\mathbf{I} M_1) \ldots (\mathbf{I} M_n) = Y$. QED

Another consequence of Theorem **??** is the fact that there is a non-easy term which can be consistently equated to every closed term except $\lambda x.x$. More generally:

Theorem 9.6 *Let $A \equiv \lambda x_1 \ldots x_n.x_i$ (a projector). Then there is a closed term $X \in \Lambda_0$ such that $\neg Con(X = A)$ and such that X can be consistently equated to every closed lambda term $B \neq A$.*

Proof. If A and B have incompatible Böhm trees we can apply Theorem **??** and Remark 8.7 to find a closed 0-term X with $\neg Con(X = A)$ and $Con(X = B)$. If A and B have compatible Böhm trees, then since A is a normal form and $BT(A)$ has height 1, we must have $BT(B) = \bot$, i.e. B is unsolvable. But then we still have $\neg Con(X = A)$ (for the same X as before) and we also have $Con(X = B)$ because both X and B are unsolvable. QED

It can be shown that if A is not a projector the above result might fail. In fact it does fail even for $A \equiv \lambda xy.xy$.

Proposition 9.7 *Let $\mathbf{1} \equiv \lambda xy.xy$. Suppose that $X \in \Lambda_0$ can be consistently equated to every closed term $B \neq \mathbf{1}$. Then X is easy.*

Proof. Let X be as in our assumptions. Then clearly X is unsolvable. Let $\mathbf{0} \equiv \lambda xy.y$ and let $\mathbf{1}^* \equiv \lambda xy.x(X\mathbf{00}y)$. Note that $X\mathbf{00}y$ is unsolvable, hence $\mathbf{1}^* \neq \mathbf{1}$. Thus $X = \mathbf{1}^*$ is consistent. Now it suffices to show that from $X = \mathbf{1}^*$ we can derive $X = \mathbf{1}$. Assume $X = \mathbf{1}^*$. Then $\mathbf{1}^* \equiv \lambda xy.x(\mathbf{1}^*\mathbf{00}y)$. But $\mathbf{1}^*\mathbf{00}y = y$, hence $\mathbf{1}^* = \lambda xy.xy = \mathbf{1}$ and the desired result follows. QED

We finish with some results showing the close connections between easy and n.f.-easy.

Proposition 9.8 *In the $\lambda\mathbf{I}$-calculus, if X is n.f.-easy, then it is easy.*

Proof. First note that if X is n.f.-easy, then X has no normal form (this holds also in the $\lambda\mathbf{I}$-calculus). Now the desired result follows from the fact that the $\lambda\mathbf{I}$-calculus has a model which equates all the closed terms without normal form (Theorem 16.1.13 of [2]). QED

Theorem 9.9 *Let $X \in \Lambda_0$ be n.f.-easy, then for every closed term Y with head normal form, the term XY is easy.*

Proof. Let Z be such that not $Con(XY = Z)$. Let Z^* be a such that Z^* is a closed normal form and $Z^*Y = Z$. To obtain such a Z^* observe that there exists a sequence $T_1, \ldots, T_n$, such that $YT_1 \ldots T_n = \mathbf{I}$, where every T_i is a closed normal form (see [2], 8.3.14) Let us call $T_1 \ldots T_n$ *a solving sequence for* Y. Now let x be a fresh variable. From inside out replace in Z every subterm of the form (UV) with the term $xT_1 \ldots T_n UV$ and let $Z^{(x)}$ be the resulting term. Such a construction can be found in [10] and for further reference we call $Z^{(x)}$ the *Statman transform* of Z (with respect to the given solving sequence). Now set $Z^* \equiv \lambda x.Z^{(x)}$ then in the theory obtained by adding the axiom $X = Z^*$ we can derive $XY = Z$. Hence not $Con(X = Z^*)$ contradicting the hypothesis that X is n.f.-easy.QED

So for example the term $\mathbf{Y}\Omega_3\mathbf{I}$ is easy. The next theorem says that it is consistent for a n.f.-easy term to define an arbitrary function, provided the arguments of the function have head-normal form.

Theorem 9.10 *Let $X \in \Lambda_0$ be n.f. easy. For every closed term Y, the theory $\{XZ = YZ | Z \in \Lambda_0$ and Z has head normal form$\}$ is consistent.*

Proof. Suppose that the theory $\{XZ = YZ | Z$ has head normal form$\}$ is not consistent. Then $\mathbf{K} = \mathbf{S}$ can be derived by a finite number of equations, say $XZ_1 = YZ_1, \ldots, XZ_n = YZ_n$. It is then possible to find a solving sequence $T_1, \ldots, T_k$ that acts simultaneously for all Z_i's. Let $Y^{(x)}$ be the Statman transform of Y with respect to this solving sequence. Let $Y^* \equiv \lambda x.xT_1 \ldots T_k Y^{(x)} x$. Then Y^* is in normal form and for every i, $Y^* Z_i = YZ_i$. Now we obtain a contradiction as in Theorem 9.9: from $X = Y^*$ we derive $XZ_i = YZ_i$ for all $i = 1, \ldots, n$, hence not $Con(X = Y^*)$ contradicting the fact that X is n.f.-easy. QED

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