Δ_0 complexity of the relation $y = \Pi_{i \leq n} F(i)$

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Abstract

We prove that if G is a Δ_0-definable function on the natural numbers and $F(n) = \Pi_{i=0}^{n} G(i)$, then F is also Δ_0-definable. Moreover the inductive properties of F can be proved inside the theory IΔ_0.

1 Introduction

We will use the standard notation $I\Delta_0$ (see [10]) to denote the fragment of Peano Arithmetic where induction is restricted to bounded formulas (Δ_0-formulas). It is known (see [11]) that the Δ_0-definable sets are exactly the sets belonging to the linear time hierarchy (i.e. the sets recognizable in linear time by an alternating Turing machine).

Bennett showed in [1] that the graph of the function $x \mapsto 2^x$ (on $\mathbf{N}$) is Δ_0-definable. Later Paris ([4]) and Pudlák ([9]) found a Δ_0-definition of $y = 2^x$ for which $2^{x+1} = 2^x 2$ is provable in $I\Delta_0$. In [2] it is shown that the factorial function is Δ_0-definable and its properties are provable in $I\Delta_0$. We generalize these results by showing that if $F: \mathbf{N} \to \mathbf{N}$ is a Δ_0-definable function (i.e. a function with a Δ_0-definable graph), then the function $G(n) = \Pi_{i \leq n} F(i)$ is also Δ_0-definable (and $G(n+1) = G(n)F(n+1)$ is provable in $I\Delta_0$).

So, while Δ_0 is not known to be closed under $\Sigma_{i \leq n}$ (see for instance [6]), our result shows that it is closed under $\Pi_{i \leq n}$. An explanation of this phenomenon is that it is often easier to give a Δ_0-definition of the graph of an exponentially

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growing function than of a function of slower growth rate since one can use the output of the function itself to bound the quantifiers in the definition. However, one should be aware of the fact that if $F(i)$ takes the value 1 for too many i's, then the function $G(n) = \Pi_{i \leq n} F(i)$ may not grow exponentially in n. To overcome this difficulty we will use a result of Paris and Wilkie [6] about counting Δ_0-sets. We recall that the following problem, considered in [7, 6], is still open.

Let $A \subseteq \mathbf{N}$ be Δ_0-definable. Is the function H defined by $H(n) = |A \cap n|$ also Δ_0-definable?

Paris and Wilkie [6] gave a partial positive answer by showing that:

$$H(n) = |A \cap n| \text{ is } \Delta_0\text{-definable provided } \forall n \ |A \cap n| \leq \log(n) \tag{1}$$

Actually, they proved that $(\log n)^k$ would also work for any fixed $k \in \mathbf{N}$.

Roughly speaking, the idea for defining $y = \Pi_{i \leq n} F(i)$ is to use (1) to count the number of i's less than n for which $F(i) > 1$ in order to reduce the computation of $G(n) = \Pi_{i \leq n} F(i)$ to the case in which F is always > 1 and therefore G grows exponentially.

Viceversa, our result implies (1). To see this, let A be a Δ_0-definable set satisfying $|A \cap n| \leq \log(n)$ for every n. Define a function F by

$$F(i) = \begin{cases} 1 & \text{if } i \notin A \\ 2 & \text{if } i \in A \end{cases}$$

By our result $z = \Pi_{i<n} F(i)$ is Δ_0-definable. We can now give the following Δ_0-definition of $y = |A \cap n|$:

$$\exists z \leq n \ (z = \Pi_{i<n} F(i) \ \& \ 2^y = z).$$

So (1) is equivalent to the closure of Δ_0-definable functions under $\Pi_{i \leq n}$.

We will also use previous work of Paris, Wilkie and Woods [8] which shows that if F is Δ_0-definable, then $H(n) = \Sigma_{i \leq \log(n)} F(i)$ is also Δ_0-definable.

Notations. If $\mathcal{M}$ is a model of $I\Delta_0$ the symbol $\Delta_0^{\mathcal{M}}$ will denote those subsets of $\mathcal{M}^k$, $k \in \mathbf{N}$, definable in $\mathcal{M}$ by a Δ_0-formula with parameters from $\mathcal{M}$. For any $a \in \mathcal{M}$ we will identify the interval $[0, a) = \{x | 0 \leq x < a\}$ with a. $\log(x)$ will denote the (total) Δ_0-definable functionsuch that $\log(0) = 0$ and for $y > 0$, $\log(y) = x$ iff $2^x \leq y < 2^{x+1}$. Note that $x \leq \log(y)$ iff $2^x \leq y$.

2 Counting Δ_0-sets

In this section we will formulate and prove the results of Paris and Wilkie about counting sparse Δ_0-sets in a form suitable for our purposes.

Let $A \in {\Delta_0}^{\mathbf{N}}$, $A \subseteq \mathbf{N}^{s+2}$. Following [6] we define

$$A_n(\vec{x}) = \{m < n | < \vec{x}, m, n > \in A\}.$$

Here $\vec{x}$ play the role of parameters and their mention will be often omitted in the sequel (thus writing A_n instead of $A_n(\vec{x})$). Note that the above definition gives the most general form of a Δ_0-family $\{A_n\}$ of Δ_0-sets $A_n \subseteq n$. It is not known whether the cardinality $y = |A_n(\vec{x})|$ can be defined by a Δ_0-formula in the variables $y, n, \vec{x}$. However, Paris and Wilkie proved that this can be done if $|A_n(\vec{x})| \leq \log(n)$ for every n.

Theorem 2.1 ([**6**]) *Let $A \in \Delta_0^{\mathbf{N}}$ and suppose that for some $k \in \mathbf{N}$, $|A_n| \leq \log(n)^k$ for all n. Then the function defined by $H(n) = |A_n|$ is Δ_0.*

Given a binary function H we denote by H_n the unary function $H_n(y) = H(n, y)$. Theorem 2.1 can be formalized in $I\Delta_0$ as follows.

Theorem 2.2 ([**6**]) *Let $\mathcal{M}$ be a model of $I\Delta_0$, $k \in \mathbf{N}$, and let $A \subseteq \mathcal{M}$ be Δ_0-definable. Then there is a Δ_0-definable function H such that for all $n \in \mathcal{M}$, either H_n injects $(\log n)^k$ into A_n or there is $\beta < (\log n)^k$ such that H_n is a bijection from β onto A_n.*

Theorem 2.2 holds uniformly in the parameters occurring in the definition of A.

Since it is not known if the pigeon-hole principle (Δ_0-PHP) is provable in $I\Delta_0$ it could happen that in a model $\mathcal{M}$ of $I\Delta_0$, A_n has different "sizes". In the hypothesis of Theorem 2.2 this cannot happen because of the following result (see [7]):

($\log - \Delta_0$PHP). Let $F \in \Delta_0^{\mathcal{M}}$, $a \in \mathcal{M}$, $k \in \mathbf{N}$. Then F does not map $(\log a)^k + 1$ one-to-one into $(\log a)^k$.

By introducing a new variable y we can prove a more uniform version of Theorem 2.2 which will be needed in the sequel.

Lemma 2.3 *Let $k \in \mathbf{N}$ and $A \subseteq \mathcal{M}$ be Δ_0-definable. Then there exists a Δ_0-definable function H such that for all $n, y \in \mathcal{M}$ either $H_{n,y} : (\log y)^k \to A_n$ injectively, or there exists $\beta < (\log y)^k$ such that $H_{n,y} : \beta \to A_n$ bijectively.*

Proof. Let p_1 be the projection on the first coordinate of a pair, that is $p_1(< u, v >) = u$. Let $B = \{< x, m > \ | \ < x, p_1(m) > \in A \ \& \ x < p_1(m)\}$. Note that if $m =< n, y >$, then $B_m = \{x < m | < x, m > \in B\} = \{x < m | < x, n > \in A \ \& \ x < n\} = A_n$. By Theorem 2.2 there exists a $\Delta_0^{\mathcal{M}}$ function F such that either F_m injects $(\log m)^k$ into B_m or there exists $\beta < (\log m)^k$ such that F_m is a bijection from β onto B_m. Now it suffices to define $H_{n,y}$ as the restriction of F_m to $(\log y)^k$. QED

Let $Card(A, y) = min(\log(y), |A|)$. The next lemma shows that if $A \in \Delta_0$ then the function $(n, y) \mapsto Card(A_n, y)$ is Δ_0-definable, and moreover its natural properties are provable inside $I\Delta_0$ (and can be stated without explicit mention of bijections onto initial segments).

Lemma 2.4 *Let $\mathcal{M} \models I\Delta_0$ and let $\{A_n\}$ be a $\Delta_0^{\mathcal{M}}$-family of $\Delta_0^{\mathcal{M}}$-sets $A_n \subseteq n$. Then there exists a $\Delta_0^{\mathcal{M}}$-definable total function $(n, y) \mapsto Card(A_n, y)$ such that:*

1. *$Card(A_n, y) \leq \log(y)$.*
2. *If $Card(A_n, y) < \log y$ then $Card(A_n, y) = Card(A_n, z)$ for all $z \geq y$.*
3. *If A_n is empty, then $Card(A_n, y) = 0$.*
4. *If A_n has only one element, then $Card(A_n, y) = 1$ for $y \geq 2$ and $Card(A_n, y) = 0$ for $y < 2$.*
5. *Let $\{B_n\}$ be another Δ_0-family of $\Delta_0^{\mathcal{M}}$-sets $B_n \subseteq n$. If $A_n \cap B_n = \emptyset$ then $Card(A_n \cup B_n, y) = Card(A_n, y) + Card(B_n, y)$ provided one of the two sides of the equality (hence both) is $< \log(y)$.*
6. *If $\sigma_n : A_n \to B_n$ is a Δ_0-family of Δ_0-bijections, then $Card(A_n, y) = Card(B_n, y)$.*

Proof. Apply Lemma 2.3 with $k = 1$. Define $Card(A_n, y) = \log(y)$ if $H_{n,y}$ injects $\log(y)$ into A_n, and $Card(A_n, y) = \beta$ if $H_{n,y}$ is a bijection from β onto A_n. Using the $\log - \Delta_0$PHP it follows that the function $(n, y) \mapsto Card(A_n, y)$ satisfies the desired properties. QED

So in the standard model $Card(A_n, y) = min(\log(y), |A_n|)$. $Card(\cdot, y)$ behaves well as long as it is $< \log(y)$. It is easy to see that if in point 5) B_n has only one element, then $Card(A_n \cup B_n, y) = Card(A_n, y) + 1$ provided $Card(A_n, y) < \log(y)$.

3 Δ_0-definition of $y = \Pi_{i \leq n} F(i)$.

The following result shows that Δ_0 is closed under logarithmic summations, i.e. if F is Δ_0-definable then the relation $z = \Sigma_{i \leq y} F(i)$ is Δ_0-definable.

Theorem 3.1 ([8]) *Let $k \in \mathbf{N}$, $a, b \in \mathcal{M} \models I\Delta_0$, and $F : (\log a)^k \to b$ be $\Delta_0^{\mathcal{M}}$-definable. Then there is a $\Delta_0^{\mathcal{M}}$-definable function $G : (\log a)^k \to \mathcal{M}$ (uniformly) such that $G(0) = F(0)$ and for all $i < (\log a)^k$, $G(i+1) = G(i) + F(i+1)$.*

The uniformity in Theorem 3.1 is connected with the possible presence of parameters in the formula defining F. In this case the parameters are carried into the Δ_0-definition of G.

In the hypothesis of the theorem all the usual properties of the sum (such as the associative and the commutative law, distributivity with respect to the product) can be easily proved in $I\Delta_0$. We will use them without explicit mention.

We recall that if $\mathcal{M} \models I\Delta_0$ and $a, b \in \mathcal{M}$ we can define in a Δ_0-way the relation "b is the p-adic valuation of a" as follows:

$$v_p(a) = b \quad \text{iff} \quad p^b | a \ \& \ p^{b+1} \not| a.$$

This is Δ_0-definable since exponentiation is such (see [5]). In the standard model $\mathbf{N}$ we have $y = \Pi_{i \leq n} F(i)$ if and only if $\forall p \leq y$ if p is prime, then $v_p(y) = \sum_{i \leq n} v_p(F(i))$.

The problem is now shifted to defining the relation $z = \sum_{i \leq n} v_p(F(i))$, and if there are too many i's for which $F(i) = 1$ then the length of the sum,namely n, may not be $\leq \log(y)$, and we cannot apply Theorem 3.1.

To avoid this problem we partition the segment $[0, n]$ into disjoint sets putting together those i's for which the p-adic valuations $v_p(F(i))$ (for $i \leq n$) are equal, i.e. in $\mathbf{N}$ we have $y = \Pi_{i \leq n} F(i)$ if and only if

$$\forall p \leq y \text{ if } p \text{ is prime, then } v_p(y) = \textstyle\sum_{h \leq \log(y)} h|\{i \leq n \mid v_p(F(i)) = h\}|.$$

Everything now is Δ_0-definable except the cardinality $|\{i \leq n \mid v_p(F(i)) = h\}|$. To give a Δ_0-definition of $y = \Pi_{i \leq n} F(i)$ we will replace $|\{i \leq n \mid v_p(F(i)) = h\}|$ by $Card(\{i \leq n \mid v_p(F(i)) = h\}, 2y)$. This will give a Δ_0-definition of $y = \Pi_{i \leq n} F(i)$ in the standard model because if $v_p(y) = \sum_{h \leq \log(y)} hCard(\{i \leq n \mid v_p(F(i)) = h\}, 2y)$, then each term of the summation is $\leq v_p(y) < \log(2y)$ and hence we can conclude that $|\{i \leq n \mid v_p(F(i)) = h\}| = Card(\{i \leq n \mid v_p(F(i)) = h\}, 2y)$. We will show that this idea gives a reasonable Δ_0-definition of $y = \Pi_{i \leq n} F(i)$ also in any non-standard model $\mathcal{M}$ of $I\Delta_0$.

Definition 3.2 Let $F : \mathcal{M} \to \mathcal{M}$ be a $\Delta_0^{\mathcal{M}}$ definable function. We define the relation $G(n, y)$ (whose intended meaning is $y = \Pi_{i \leq n} F(i)$) as follows:

$$y = 0, \quad \text{if } F(i) = 0 \text{ for some } i \leq n, \qquad \text{otherwise}$$

$$\forall p \leq y \text{ if } p \text{ is prime, then } v_p(y) = \textstyle\sum_{h \leq \log(y)} hCard(\{i \leq n \mid v_p(F(i)) = h\}, 2y).$$

Remark 3.3

1. From the uniqueness of the decomposition in product of powers of primes, which is provable in $I\Delta_0$ it follows that $G(n, y)$ & $G(n, z) \to y = z$. In what follows we will write $G(n) = y$ instead of $G(n, y)$.

2. Notice that $\max_{i \leq n} v_p(F(i)) \leq \log y$ and if h satisfies $\max_{i \leq n} v_p(F(i)) < h \leq \log y$, then $\{i \leq n \mid v_p(F(i)) = h\} = \emptyset$, so we can restrict h to vary up to $\max_{i \leq n} v_p(F(i))$.

Lemma 3.4 *In $\mathcal{M}$, if $v_p(y) = \sum_{1 \leq h \leq \log(y)} hCard(\{i \leq n \mid v_p(F(i)) = h\}, 2y)$, then for $1 \leq h \leq \log(y)$, $Card(\{i \leq n \mid v_p(F(i)) = h\}, 2y) < \log(2y)$.*

Proof. We have $Card(\{i \leq n \mid v_p(F(i)) = h\}, 2y) \leq v_p(y) < \log(2y)$. QED

The following theorem shows that the definition of $y = \Pi_{i \leq n} F(i)$ behaves well in $I\Delta_0$.

Theorem 3.5 *Given a Δ_0-formula $F(x, y)$ (possibly containing other variables) there is a Δ_0-formula $G(x, z)$ (with the same number of variables as F), such that $I\Delta_0$ proves the universal closure of:*
0) if $\forall x \exists! y F(x, y)$, *then* $\forall x \exists! z G(x, z)$,
1) $G(0) = F(0)$,
2) $G(n) = y \rightarrow G(n+1) = yF(n+1)$,
3) $G(n+1) = z \rightarrow \exists y \leq z (G(n) = y \;\&\; z = yF(n+1))$,
where $F(x) = y$ stands for $F(x, y)$ and $G(x, z)$ stands for $G(x) = z$.

Proof. Let $\mathcal{M} \models I\Delta_0$ and $G(x, z)$ be the formula of Definition 3.2. We can assume $F(i) > 0$ for all i, since the result follows trivially if there is an i such that $F(i) = 0$.

1) We have to show that $v_p(F(0)) = v_p(G(0))$, i.e.

$$v_p(F(0)) = \sum_{h \leq \log(F(0))} hCard(\{i \leq 0 \mid v_p(F(i)) = h\}, 2F(0))$$

for every prime $p \leq F(0)$. It is clear that for $p \leq F(0)$ and $h \leq \log(F(0))$:

$$\{i \leq 0 \mid v_p(F(i)) = h\} = \begin{cases} \{0\} & \text{if } v_p(F(0)) = h \\ \emptyset & \text{if } v_p(F(0)) \neq h \end{cases}$$

hence

$$Card(\{i \leq 0 \mid v_p(F(i)) = h\}, 2F(0)) = \begin{cases} 1 & \text{if } v_p(F(0)) = h \\ 0 & \text{if } v_p(F(0)) \neq h \end{cases}$$

This implies $v_p(F(0)) = v_p(G(0))$.

2) Assume $y = G(n)$. Then $v_p(y) = \sum_{h \leq \log(y)} hCard(\{i \leq n \mid v_p(F(i)) = h\}, 2y)$. It is enough to show that

$$v_p(yF(n+1)) = \sum_{h \leq \log(yF(n+1))} hCard(\{i \leq n+1 \mid v_p(F(i)) = h\}, 2yF(n+1))$$

for every prime p. We distiguish two cases:

i) Suppose $p \not| F(n+1)$. So $v_p(F(n+1)) = 0$. It follows that $v_p(yF(n+1)) = v_p(y)$. By Lemma 3.4 , $Card(\{i \leq n \mid v_p(F(i)) = h\}, 2y) \leq \log(2y)$. The hypothesis $p \not| F(n+1)$ implies that $\{i \leq n \mid v_p(F(i)) = h\} = \{i \leq n+1 \mid v_p(F(i)) = h\}$ for all $h \leq \log y$. Moreover, $yF(n+1) \geq y$, so applying Theorem 2.4 we can deduce that $Card(\{i \leq n \mid v_p(F(i)) = h\}, 2y) = Card(\{i \leq n+1 \mid v_p(F(i)) = h\}, 2yF(n+1))$. In this case $\max_{i \leq n} v_p(F(i)) = \max_{i \leq n+1} v_p(F(i))$, and we will denote it by λ. So

$$\sum_{h\le\lambda} hCard(\{i \le n \mid v_p(F(i)) = h\}, 2y) =$$

$$\sum_{h\le\lambda} hCard(\{i \le n+1 \mid v_p(F(i)) = h\}, 2yF(n+1)),$$

which is equivalent to say that $v_p(yF(n+1)) = v_p(G(n+1))$.

ii) Suppose $p|F(n+1)$, and let $k = v_p(F(n+1))$. Let $\lambda = \max_{i\le n} v_p(F(i))$. If $k > \lambda$ then $\{i \le n+1 \mid v_p(F(i)) = k\} = \{n+1\}$, and Theorem 2.4 implies $Card(\{i \le n+1 \mid v_p(F(i)) = k\}, z) = 1$ for all $z \ge 2$. It is easy to show now that the following equalities are true:

$$v_p(yF(n+1)) = v_p(F(n+1)) + v_p(y)$$

$$= k + \sum_{h\le\lambda} hCard(\{i \le n \mid v_p(F(i)) = h\}, 2y) \quad (1)$$

$$= kCard(\{i \le n+1 \mid v_p(F(i)) = k\}, 2yF(n+1)) +$$

$$\sum_{h\le\lambda} hCard(\{i \le n+1 \mid v_p(F(i)) = h\}, 2yF(n+1))$$

$$= \sum_{h\le k} hCard(\{i \le n+1 \mid v_p(F(i)) = h\}, 2yF(n+1))$$

$$= \sum_{h\le\log(yF(n+1))} hCard(\{i \le n+1 \mid v_p(F(i)) = h\}, 2yF(n+1)).$$

Assume now that $k \le \lambda$. Using the associativity of the sum and again Theorem 2.4 we have that (1) is equal to

$$k + \sum_{\substack{h\le\lambda\\ h\ne k}} hCard(\{i \le n \mid v_p(F(i)) = h\}, 2y) + kCard(\{i \le n \mid v_p(F(i)) = k\}, 2y)$$

$$= \sum_{\substack{h\le\lambda\\ h\ne k}} hCard(\{i \le n \mid v_p(F(i)) = h\}, 2y)$$

$$+ kCard(\{i \le n+1 \mid v_p(F(i)) = k\}, 2y) \quad (2)$$

where we have used the fact that $\{i \le n+1 \mid v_p(F(i)) = k\} = \{i \le n \mid v_p(F(i)) = k\} \cup \{n+1\}$. Now we can increase $2y$ to $2yF(n+1)$, so (2) is equal to

$$= \sum_{\substack{h \le \lambda \\ h \ne k}} hCard(\{i \le n \mid v_p(F(i)) = h\}, 2yF(n+1))$$

$$+kCard(\{i \le n+1 \mid v_p(F(i)) = k\}, 2yF(n+1)) \quad (3)$$

Finally we merge the two terms into a single sum getting

$$= \sum_{h \le \lambda} hCard(\{i \le n+1 \mid v_p(F(i)) = h\}, 2yF(n+1))$$

$$= \sum_{h \le \log(yF(n+1))} hCard(\{i \le n+1 \mid v_p(F(i)) = h\}, 2yF(n+1))$$

and we are done.

3) In an analogous way we can show that if $z = G(n+1)$ then $G(n) = \frac{z}{F(n+1)}$. QED

For the Δ_0-notion of product we gave we can prove some basic properties of the product function such as the associative and commutative law. We give below a version of the commutative law.

Lemma 3.6 *Let $F: \mathcal{M} \to \mathcal{M}$ be Δ_0-definable. Then for every Δ_0-family of Δ_0-permutations $\sigma_n : [0,n] \to [0,n]$ $(n \in \mathcal{M})$ if $\prod_{i \le n} F(i)$ is defined then*

$$\prod_{i \le n} F(i) = \prod_{i \le n} F(\sigma_n(i)).$$

Proof. Let $\prod_{i \le n} F(i) = y$. It is enough to use Theorem 2.4 (point 6) together with the observation that σ_n is a Δ_0-bijection between $\{i \le n : v_p(F(\sigma_n(i))) = h\}$ and $\{i \le n \mid v_p(F(i)) = h\}$ for all primes p and $h \le \log(y)$. QED

We do not know whether our main result holds for the ring of integers.

Question 3.7 *Let $F: \mathbf{N} \to \mathbf{Z}$ be Δ_0-definable. Is $G(n) = \prod_{i \le n} F(i)$ also Δ_0-definable?*

A positive answer would imply (it is actually equivalent) that Δ_0-definable sets are closed under counting $mod(2)$, which is a well known open question [7]. To see this let $A \in \Delta_0^{\mathbf{N}}$ and define $F(i) = -1$ if $i \in A$ and $F(i) = 1$ if $i \notin A$. Then $\Pi_{i \le n} F(i) \equiv |\{i \le n \mid i \in A\}| \pmod 2$.

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