Intersection theory for o-minimal manifolds

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April 3, 2000

Abstract

We develop an intersection theory for definable C^p-manifolds in an o-minimal expansion of a real closed field and we prove the invariance of the intersection numbers under definable C^p-homotopies ($p > 2$). In particular we define the intersection number of two definable submanifolds of complementary dimensions, the Brouwer degree and the winding numbers. We illustrate the theory by deriving in the o-minimal context the Brouwer fixed point theorem, the Jordan-Brouwer separation theorem and the invariance of the Lefschetz numbers under definable C^p-homotopies.

A. Pillay has shown that any definable group admits an abstract manifold structure. We apply the intersection theory to definable groups after proving an embedding theorem for abstract definably compact C^p-manifolds. In particular using the Lefschetz fixed point theorem we show that the Lefschetz number of the identity map on a definably compact group, which in the classical case coincides with the Euler characteristic, is zero.

1 Introduction

Definable sets in an o-minimal structure provide a generalization of the semialgebraic sets. Many classical results on the semialgebraic sets, including

*Partially supported by DGES PB980756/C02/01.

curve selection, cell decomposition, a good dimension theory, triangulation theorems, trivialization of functions, uniformity properties of the fibers, can be extended to definable sets over an o-minimal expansion of an ordered field (necessarily real closed). The most important examples, besides the semi-algebraic case, are based on the ordered field of the real numbers expanded with extra structure, most notably the exponential function [W]. Another example is provided by the analytic functions restricted to a compact ball [DD], in which case the definable sets coincide with the traces on $\mathbf{R}^n$ of the subanalytic sets in the projective space. Despite the fact that most examples are based on the reals, it turns out to be convenient to develop the theory over an arbitrary o-minimal expansion of an ordered field, not necessarily archimedean, because some of the results in the real case depend upon results established in the more general setting. A nice example of this phenomenon, pointed out by M. Coste, is a derivation of the trivialization theorem for definable functions, using a notion of "o-minimal spectrum". Various basic results of differential geometry, like the inverse function theorem, can be extended to the o-minimal setting; however, some other tools, such as integration, solutions of differential equations and covering spaces, are not available.

In the sequel $R = (R, <, +, \cdot, 0, 1, \dots)$ will denote an **o-minimal expansion of an ordered field**, i.e. an ordered real closed field possibly endowed with more structure such that every definable subset of R is a finite union of points and intervals (a, b) with $a, b \in R \cup \{+\infty, -\infty\}$. Thanks to the field structure of R, one can differentiate maps and introduce the notion of **definable C^p-submanifold of R^k**, for $p > 0$ (see §2). A definable manifold in general is not locally compact (if R is non-archimedean) or locally connected.

A notion of **abstract definable C^p-manifold** (see §10), not necessarily embedded in some R^k, can be introduced [PeSt] in a way completely analogous to the classical case, with the difference that one insists that an abstract manifold has a **finite atlas** and the transition maps are definable.

Pillay has shown that for every p, every definable group G admits a unique natural topology and differential structure which makes it an abstract definable C^p-manifold with group operations of class C^p (see [Pi] and [OPP]). Hence definable groups over the reals are Lie groups. An example of a definable group is $G = [0, 1) \subset R$, where the group operation sends (x, y) into $x + y$ if $x + y < 1$, and to $x + y - 1$ otherwise. The natural topology on $G = [0, 1)$ is not the one induced by its inclusion in R, but the one which makes G into an abstract manifold definably diffeomorphic to the unit circle in R^2.

Definable groups have also been studied by [NPR, PeSt, PPS, R, S1, S2]. In particular Y. Peterzil and C. Steinhorn in [PeSt] showed that every non-definably-compact definable group contains a definable torsion free subgroup of dimension one. We were motivated by a question in [PeSt] asking whether every definably compact definable group G (with the natural manifold topology), necessarily has torsion elements. Using results of Strzebonski (see [S1]) the problem reduces to showing that $E(G)$ (the Euler characteristic of G in the model theoretic sense, see [D]) is zero. In fact Strzebonski in [S1] proves that if a prime number p divides $E(G)$ then G has an element of order p. In the classical case the Euler characteristic is equal to the Lefschetz number of the identity map, which is zero for definably compact groups (Theorem 11.4). If this characterization of the Euler characteristic held for definably compact manifolds (as we conjecture), then by Strzebonski's results a definably compact definable group G would have torsion elements[1].

We develop an oriented intersection theory for definable C^p-manifolds over an o-minimal expansion R of an ordered field (see §7), following essentially [GP] without any claim of originality except for some adaptations that we will explain below. Our contribution is meant to be a preparatory work towards the problem of studying definably compact groups in an o-minimal expansion of a real closed field. For the applications to groups (see §11) one needs to develop the theory for abstract manifolds. However for technical reasons it is convenient to consider first manifolds embedded in R^k because, for instance, it is then easy to prove the basic result that counterimage of a regular value under a definable C^p-map is a definable C^p-manifold (Theorem 3.2). In the abstract case the difficulty is to prove the finiteness of the atlas for the counterimage. For our purposes we loose little generality in working with embedded manifolds because every definably compact abstract C^p-manifold is definably C^p-diffeomorphic to a C^p-submanifold of some R^k (Theorem 10.7). So a posteriori we do have a preimage theorem for definably compact abstract manifolds. In [D] L. van den Dries adapts the semialgebraic proof of [Ro] to prove an embedding theorem for definable regular C^0-manifolds.

We outline the geometric ideas of the intersection theory and the adaptations we have to do in our context. If $X \subset Y$ and $Z \subset Y$ are two definably

[1] After this paper was submitted M. Edmundo announced a proof of the fact that any definably compact definable group G has Euler characteristic zero, and hence torsion elements. His proof uses co-homologica methods.

compact boundaryless C^p-submanifolds of a definable C^p-manifold Y which meet transversally, then their intersection is a definable C^p-manifold of the expected dimension (as if they were linear spaces). This is a special case of Proposition 5.2 and the proof goes as in the classical case. In particular if X and Z have complementary dimensions in Y, their intersection has dimension zero, so it is a finite set. If one deforms X by a definable C^p-homotopy of the inclusion map $f: X \to Y$, the cardinality of the intersection will not change modulo 2, provided the two manifolds are still transversal after the deformation. This invariant is called the intersection number modulo 2 of X and Z (or of f and Z). If the manifolds are oriented, one can count each point of intersection with an algebraic sign, depending on whether the orientations of the two tangent spaces add up to the orientation of the ambient space. The algebraic sum will then be a relative integer, again invariant under definable C^p-homotopy (Corollary 8.3). When the intersection is not transversal then, using Sard's theorem and a tubular neighbourhood theorem, one can still define an intersection number, by reducing to the transversal situation after a small deformation at the expense of losing one degree of differentiability. For Sard's theorem (Theorem 3.5) one can adapt the semialgebraic proof [BCR], but we give a more direct proof in following a suggestion of P. Speissegger. In §4 we prove the version we need of the tubular neighbourhood theorem (Theorem 4.4), namely that each definably compact C^p-manifold $Y \subset R^k$ has a neighbourhood which admits a definable C^{p-1}-submersion onto Y (M. Coste has given a proof of a tubular neighbourhood theorem in the non-definably compact case during a course he recently gave in Pisa [C].) To show the invariance of the intersection numbers, the idea is to trace the points of intersection during the homotopy. Using again Sard's theorem and the tubular neighbourhood theorem, one can assume that the homotopy itself is transversal to Z. The points of intersection will then move along a definably compact one dimensional C^p-submanifold of the "homotopy cylinder" $I \times X$. We classify the one-dimensional definably compact C^p-manifolds with non-empty boundary in section §6 (Theorem 6.5). The main difficulty here is that we cannot use the arclength function. By the classification, the one-dimensional submanifold of the homotopy cylinder mentioned above has an even number of boundary points which, by transversality of the homotopy, will lie on the two sides $0 \times X$ and $1 \times X$ of the cylinder. So the parity at both sides will be equal, proving the invariance modulo 2 of the intersection numbers. The oriented case, which is the one we develop, is similar, using the fact that the sum of the orientation numbers at the two end-points of a definably compact definably connected one-dimensional C^p-manifold with non-empty boundary is zero.

After setting the foundations one can derive applications like the **Brouwer fixed point theorem for definable C^p-maps with** $p > 0$ (see Theorem 9.1), or the **Jordan-Brouwer separation theorem** (see Theorem 9.6) as in the classical case.

Part of the work on this paper was done while the authors were visiting the Mathematical Sciences Research Institute during Spring 1998.

We thank the referee for his very careful reading of the paper.

2 Smooth manifolds with boundary

2.1 Definable C^p-manifolds with boundary

Let R be an o-minimal expansion of a real closed field. Consider $p \in \mathbf{N}$, $p > 0$, definable subsets $X \subset R^k$ and $Y \subset R^l$, and a definable map $f: X \to Y$. We say that f **is a definable C^p-map** if for each $x \in X$ there is a definable open $U \subset R^k$ with $x \in U$ and a definable map $F: U \to R^l$ extending $f_{|U \cap X}$ such that all the partial derivatives $\partial^s F / \partial x_{i_1} \cdots \partial x_{i_s}$ exist and are continuous for each $s \leq p$. We say that f is a **definable C^p-diffeomorphism** if f is a homeomorphism onto Y and both f and f^{-1} are definable C^p-maps.

The closed half-space H^k is the definable set of all $(x_1, \ldots, x_k) \in R^k$ such that $x_k \geq 0$. A definable subset X of R^k is a **definable C^p-manifold of dimension** m if for each $x \in X$ there is a definable open neighbourhood W of x in X and a definable C^p-diffeomorphism $\phi: U \to W$ where U is an open definable subset of H^m. Such a ϕ is called a **definable parametrization** of the neighbourhood W of x, its inverse g a **definable local coordinate system** and the couple (W, g) a **definable chart** around x. A collection of definable charts (W, g) such that the union of the W's covers X is called an **atlas** for X. The **boundary** ∂X of X consists of the points x in X for which $g(x) \in R^{m-1} \times \{0\}$, for some definable local coordinate system g. In Corollary 2.14 we shall show that ∂X is definable. We say that X is **boundaryless** if $\partial X = \emptyset$. The **interior** $Int\, X$ of X is the set $X \backslash \partial X$.

In what follows $X \subset R^k$ **will denote a definable C^p-manifold of dimension** m.

2.2 Tangent spaces and derivatives

For any open definable subset U of R^k, for any definable C^p-map $f: U \to R^l$, and for any $x \in U$ the **differential** of f at x is the linear map $d_x f: R^k \to R^l$ given by $d_x f(h) := \lim_{t \to 0} \frac{f(x+th)-f(x)}{t}$ (the directional derivative of f at x in the direction h). The matrix associated to this linear map is the Jacobian matrix of f at x. The map $(x, h) \to d_x f(h)$ is definable and continuous.

We extend the definition of $d_x f$ to definable C^p-maps $f: U \to R^l$ defined on an open definable subset U of H^k. Let $x \in U$. By definition of definable C^p-map, f can be extended to a definable C^p-map F on a definable open neighbourhood of x in R^k. We define $d_x f: R^k \to R^l$ to be $d_x F$. This definition is independent of the extension (because given $h \in R^k$, for some $t \neq 0$, U contains the segment joining x and $x + th$, and $d_x F(h)$ depends only on the restriction of F to this segment).

Let $X \subset R^k$ be a definable C^p-manifold of dimension m and let $x \in X$. The **tangent space** $T_x X$ of X at x is defined as the image of the linear map $d_u \phi: R^m \to R^k$ where ϕ is a definable parametrization of a neighbourhood of x sending $u \in R^m$ to x. By the chain rule $T_x X$ does not depend on the choice of the definable parametrization. Since $d_u \phi$ is invertible, $T_x X$ is a linear subspace of R^k of dimension m. Note that $T_x H^m = R^m$ for all $x \in H^m$.

Using the tangent spaces one extends the notion of differential to definable C^p-maps between definable C^p-manifolds. Let $f: X \to Y$ be a definable C^p-map between definable C^p-manifolds $X \subset R^k$ and $Y \subset R^l$. Given $x \in X$, let $F: U \to R^l$ be a definable C^p-map on an open definable neighbourhood U of x in R^k which coincides with f on $X \cap U$. The **differential** of f at x is the linear map $d_x f: T_x X \to T_{fx} Y$ defined by $d_x f(v) = d_x F(v)$ for all $v \in T_x X$. This is well defined, and the chain rule continues to hold.

In some cases it is useful to consider vectors of the tangent space as velocity vectors of curves. A **definable C^p-curve on** X is a definable C^p-map $\gamma: I \to X$ where I is an interval on R. The **velocity** (**vector**) **of** γ **at time** $t \in I$ is the vector $\gamma'(t) := d_t \gamma(1) \in T_{\gamma(t)} X$. Any vector $v \in T_x X$ is the velocity of some definable C^p-curve on X passing through x (we allow I to be a half-open interval of the form $[0, \varepsilon)$ or $(-\varepsilon, 0]$ to take into account the case when $x \in \partial X$). The differential of $f: X \to Y$ can be described in terms of velocity vectors as follows: if $v \in T_x X$ is the velocity of γ at time t, then $d_x f(v) \in T_{fx} Y$ is the velocity of $f \circ \gamma$ at time t.

We shall see that if X is a definable C^p-manifold of dimension m, then ∂X is a definable C^p-manifold of dimension $m-1$ (Corollary 2.14). It follows that if x is in ∂X, then $T_x(\partial X)$ is a linear subspace of T_xX of codimension 1 which splits T_xX in two half-spaces as follows.

Definition 2.1 A vector of T_xX which is not in $T_x(\partial X)$ is called an **inward vector** if it is the velocity vector at time 0 of a definable C^p-curve $\gamma: [0, \varepsilon) \to X$ (so the curve *starts* at x), and it is called an **outward vector** if it is the velocity vector at time 0 of a definable C^p-curve $\gamma: (-\varepsilon, 0] \to X$ (so the curve *ends* at x).

An equivalent definition in terms of definable local coordinate systems is the following. Given $x \in \partial X$ and $v \in T_xX$, consider a definable local coordinate system $h: W \to h(W) \subset H^m$ around x. The last coordinate of $d_xh(v) \in R^m$ is positive if v is an inward vector, negative if v is an outward vector, and zero if v is in $T_x(\partial X)$.

2.3 The inverse function theorem

One can prove (see [D]) the **Inverse Function Theorem** for definable C^p-maps: let $f: U \to R^k$ be a definable C^p-map, with U definable and open in R^k and $x \in U$. If $d_xf: R^k \to R^k$ is invertible, then there is an open definable neighbourhood V of x, such that f maps V, C^p-diffeomorphically onto a definable open subset $f(V)$ of R^k. The inverse function theorem implies that a definable subset of R^m which is definably C^p-diffeomorphic to an open subset of R^m, is open in R^m. It then follows that ∂X is the set of those points of X which do not have a definable open neighbourhood in X that is definably C^p-diffeomorphic to an open set of R^m. In particular $\partial H^m = R^{m-1} \times \{0\}$.

We fix for the rest of this section a definable C^p-manifold $X \subset R^k$ of dimension m, a definable boundaryless C^p-manifold $Y \subset R^l$ of dimension n, and a definable C^p-map $f: X \to Y$.

Since the inverse function theorem is a local result it extends in the obvious way to definable maps between definable manifolds:

Theorem 2.2 (The Inverse Function Theorem) *Let $x \in IntX$. If $d_xf: T_xX \to T_{fx}Y$ is an isomorphism, then f maps a definable open neighbourhood of x in X, C^p diffeomorphically onto a definable open neighbourhood of $f(x)$ in Y.*

Viceversa, if the conclusion of the above theorem holds, then using the chain rule it follows that d_xf is an isomorphism. If d_xf is an isomorphism, we must have $\dim X = \dim Y$.

Now, consider the case in which the dimensions of X and Y do not need to coincide. We say f is an **immersion at** x if $d_x f$ is injective, and we say f is an **immersion** if it is so at every point of X.

We have the following corollary to the inverse function theorem (see e.g. [M]):

Corollary 2.3 (Local immersion theorem) *If $f: X \to Y$ is an immersion at $x \in X$, then there are definable parametrizations ϕ around x and ψ around $f(x)$ such that $\psi^{-1} \circ f \circ \phi$ is defined on an open definable neighbourhood U of 0 in H^m (or in R^m if $x \notin \partial X$) and $(\psi^{-1} \circ f \circ \phi)(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0) \in H^n$ (resp. in R^m), for all $(x_1, \ldots, x_m)$ in U.*

An immersion $f: X \to Y$ which is a homeomorphism onto its image is called **an embedding**. In the case $X \subset Y$ we say that X is a **submanifold of** Y if the inclusion map is an embedding.

Remark 2.4 Let $f: X \to Y$ be a definable C^p-embedding. Then f is a definable C^p-diffeomorphism onto its image.

Finally we consider the case $\dim X \geq \dim Y$. We say that a definable C^p-map $f: X \to Y$ is **a submersion at** x if $d_x f$ is onto. And we say f **is a submersion** if it is so at every point of X. In this case we also have the following corollary to the inverse function theorem (see e.g. [M]):

Corollary 2.5 (Local submersion theorem) *If $f: X \to Y$ is a definable submersion at $x \in IntX$, then there are definable parametrizations ϕ around x and ψ around $f(x)$ such that $\psi^{-1} \circ f \circ \phi$ is defined on an open definable neighbourhood U of 0 in R^m and $(\psi^{-1} \circ f \circ \phi)(x_1, \ldots, x_m) = (x_1, \ldots, x_n) \in R^n$, for all $(x_1, \ldots, x_m)$ in U.*

Note that in the inverse function theorem and in the local submersion theorem we need x to be in the interior of X while in the local immersion theorem we can allow x to be in the boundary of X

2.4 C^p-cells and existence of finite atlases

We assume familiarity with the notion of cell decomposition (see [PiSt] or [D]). A cell decomposition where all the functions involved are definable C^p-maps is called a C^p-cell decomposition. A map $(x_1, \ldots, x_k) \mapsto (x_{i_1}, \ldots, x_{i_m}) : R^k \to R^m$ with $1 \leq i_1 \leq \cdots \leq i_m \leq k$ is called a projection. The following result is stated in [DM].

Lemma 2.6 *Let C be a cell of dimension m of a definable C^p-cell decomposition of R^k. Then there is a projection $\theta: R^k \to R^m$, such that θ maps C C^p-diffeomorphically onto an open cell of R^m. In particular C is a boundaryless definable C^p-manifold of dimension m.*

Proof. The result is clear for $k = 1$. If $k > 1$ let $\pi: R^k \to R^{k-1}$ be the projection on the first $k-1$ coordinates. Then $\pi(C) \subset R^{k-1}$ is a C^p-cell of dimension m or $m-1$. By induction there is a projection $\sigma: R^{k-1} \to R^n$, where $n = \dim \pi(C)$, which maps $\pi(C)$ C^p-diffeomorphically onto an open cell of R^n. If $n = m$ we can take $\theta = \sigma \circ \pi$. If $n = m-1$ we can take $\theta(\vec{x}, y) = (\sigma(\vec{x}), y)$. QED

Lemma 2.7 *Let C be a cell of a definable C^p-cell decomposition of R^k, $k > 0$, and let $\pi: R^k \to R^{k-1}$ be the projection on the first $k-1$ coordinates. Then $\pi_{|C}$ has constant rank equal to the dimension of $\pi(C)$.*

Proof. Let $m = \dim C$. If $\dim \pi(C) = m$, then $\pi_{|C}$ is a definable C^p-diffeomorphism onto $\pi(C)$ and therefore it has constant rank. If $\dim \pi(C) = m-1$ then by the proof of Lemma 2.6 there are definable open sets $U \subset R^m$ and $V \subset R^{m-1}$ definably C^p-diffeomorphic to C and $\pi(C)$ respectively, so that the definable $\tilde{\pi}: U \to V$ induced by $\pi_{|C}: C \to \pi(C)$ after composing with these diffeomorphisms, sends $(x_1, \ldots, x_m, y)$ to $(x_1, \ldots, x_m)$, and therefore has constant rank m. QED

Lemma 2.8 *Let $X \subset R^k$ be a definable C^p-manifold of dimension m, and let C be a cell of dimension m included in X belonging to a C^p-cell decomposition of R^k. Then C is open in X.*

Proof. Let $p \in C$ and consider a definable local chart (W, h) around p, with $h(W)$ open in H^m. It is enough to prove that $C \cap W$ is open in X. There is a definable C^p-diffeomorphism $f: U \to C$ where U is a definable open subset of R^m. Now $f^{-1}(C \cap W)$ is open in U, hence in R^m. Each of the maps $f^{-1}(C \cap W) \xrightarrow{f} C \cap W \xrightarrow{h} H^m \xrightarrow{\iota} R^m$ is a definable C^p-diffeomorphism onto its image (ι is the inclusion), so their composition maps the open set $f^{-1}(C \cap W) \subset R^m$ onto a subset of R^m which is open by the inverse function theorem. Since $C \cap W$ is the counterimage of this open set under $\iota \circ h: W \to R^m$, $C \cap W$ is open in W, hence in X. QED

Proposition 2.9 *Every definable C^p-manifold $X \subset R^k$ of dimension m has an open and dense definable boundaryless submanifold $Y \subset X$ which has a finite atlas.*

Proof. Consider a C^p-cell decomposition of R^k compatible with X, and let Y be the union of all the cells in X of dimension m. Then Y is dense in X and by Lemma 2.8 it is open in X. Since each cell of dimension m is open and definably C^p-diffeomorphic to a definable open subset of R^m, we can take them as definable charts. QED

We shall use Proposition 2.9 to show that the tangent bundle and the boundary are definable.

In [BCR] Section 9.3 it is proved that every semialgebraic manifold without boundary has a finite atlas (with local coordinate systems given by the restrictions to X of suitable linear maps from R^k to R^m). The proof extends to definable boundaryless C^p-manifolds, but it does not extend to definable C^p-manifolds with non-empty boundary although it is reasonable to expect that the result is still true (one cannot use linear maps from R^k to R^m as in the boundaryless case, since the boundary of X would not necessarily be mapped into $R^{m-1} \times \{0\}$).

2.5 Definability of the tangent bundle and the differential map

Given a definable C^p-manifold $X \subset R^k$ of dimension m, the **tangent bundle** TX of X is the union $\bigcup_{x\in X}\{x\}\times T_xX \subset R^{2k}$. Let $Y \subset R^l$ be a definable C^p-manifold of dimension n. Any definable C^p-map $f\colon X \to Y$ induces a map (**the differential of** f) $df\colon TX \to TY$, $df(x,v) = (f(x), d_xf(v))$. The chain rule can then be expressed as $d(f \circ g) = df \circ dg$.

An R-**metric** is defined as a metric except that it takes values in R rather than $\mathbf{R}$. Any R-metric induces a topology in the obvious way. The product topology on R^k is induced by the R-metric $|a-b| = \max_i |a_i - b_i|$ where $a = (a_1, \dots, a_k) \in R^k$ and $b = (b_1, \dots, b_k) \in R^k$.

The **Grassmannian** $G_m(R^k)$ is the set of all m-dimensional linear subspaces of R^k. We shall put an R-metric on the Grassmannian.

We recall that a definable subset of R^k is **definably compact** if and only if it is closed and bounded. This notion was introduced in [PeSt] for abstract-definable manifolds (see below, Definiton 10.2), where it is proved to coincide with the notion of closed and bounded for definable submanifolds of R^k. Any definably continuous function $f\colon X \to R$ on a nonempty definably compact set assumes a maximum and a minimum value.

Given two nonempty definably compact subsets A, B of R^k, we define the **Hausdorff distance** $\mathsf{Hd}(A,B))$ as $\max(\max_{a\in A} \mathsf{d}(a,B),\ \max_{b\in B} \mathsf{d}(b,A))$ where $\mathsf{d}(a,B) = \min_{b\in B} |a-b|$. So $\mathsf{Hd}(A,B)) < \varepsilon$ if and only if for every x in one of the two sets A or B, there is some y in the other set with $|x-y| < \varepsilon$.

The distance between M and N in $G_m(R^k)$ with $k \geq m \geq 1$ is defined as $\mathsf{Hd}(M \cap S^{k-1}, N \cap S^{k-1})$, where S^{k-1} is the unit sphere in R^k. For $m = 0$ we just take the unique metric on $G_0(R^k)$. This gives an R-metric.

Lemma 2.10 *Let $X \subset R^k$ be a definable C^p-manifold of dimension m. Then the map $x \mapsto T_xX$ $(x \in X)$ is continuous with respect to the R-metric on X induced by that of R^k and the R-metric on the Grassmannian $G_m(R^k)$.*

Proof. (For the case $m \geq 1$, the case $m = 0$ being trivial.) For each $z \in X$ let $L_z = T_zX \cap S^{k-1}$. Fix $x \in X$. Consider a definable parametrization (U, ϕ) with $\phi(0) = x$. Then for every $u \in U$, $T_{\phi(u)}X = d_u\phi(R^m)$. Every $v \in L_{\phi(u)}$ has the form $\frac{d_u\phi(e)}{|d_u\phi(e)|}$ for some $e \in S^{m-1}$ and every $w \in L_x$ has the form $\frac{d_0\phi(e)}{|d_0\phi(e)|}$ for some $e \in S^{m-1}$. Hence $\mathsf{Hd}(L_x, L_{\phi(u)}) \leq \sup_{e\in S^{m-1}} g(u,e)$ where $g(u,e) = \left|\frac{d_u\phi(e)}{|d_u\phi(e)|} - \frac{d_0\phi(e)}{|d_0\phi(e)|}\right|$. Letting u range over a definably compact neighbourhood of 0, we can assume g uniformly continuous. It then follows that $\sup_{e\in S^{m-1}} g(u,e)$ is a continuous function of u at zero, with value zero at $u = 0$. Therefore when y tends to x, $\mathsf{Hd}(L_x, L_y)$ tends to zero. QED

The following theorem is proved in [DM] in the boundaryless case. We need it in the general case.

Theorem 2.11 *Let $X \subset R^k$ be a definable C^p-manifold. Then $TX \subset R^{2k}$ is definable.*

Proof. Use Proposition 2.9 to get a dense open definable submanifold Y of X with a finite atlas. Then TY is definable since $(x,v) \in TY$ if and only if there is a definable parametrization (U,ϕ) of the finite atlas with $\phi(0) = x$ and a vector $e \in R^m$, with $d_0\phi(e) = v$. By Lemma 2.10 we have $(x,v) \in TX$ if and only if $v \in \lim_{y\to x, y\in Y} T_yY$ where the limit is taken with respect to the R-metric on the Grassmannian. This shows that TX is definable. QED

Corollary 2.12 *If $f: X \to Y$ is a definable C^p-map between definable C^p-manifolds, then $df: TX \to TY$ is definable.*

Proof. The result is clear if X is an open subset of R^m for then $df(x,v) = \lim_{t\to 0}(f(x+tv)-f(x))/t$. If X is boundaryless and has a finite atlas we can reduce to the above case by composing with the definable parametrizations. In the general case, using Proposition 2.9 we can get a definable open dense boundaryless definable submanifold $Z \subset X$ with a finite atlas. Let $g: Z \to Y$ be the restriction of f. Then $dg: TZ \to TY$ is definable. Moreover we have $d_x f(v) = \lim_{z\to x, z\in Z} d_z g(v)$, so df is definable. QED

Corollary 2.13 *Let $X \subset R^k$ be a definable C^p-manifold of dimension m. Then $TX \subset R^{2k}$ is a definable C^{p-1}-manifold of dimension $2m$. If $f: X \to Y$ is a definable C^p-map between definable C^p-manifolds, then $df: TX \to TY$ is a definable C^{p-1}-map.*

Proof. Let $(x,v) \in TX$ and let $\phi: U \to W$ be a definable parametrization of a neighbourhood W of x. Then $TU = U \times R^m$ is open in $H^m \times R^m$, and $d\phi: TU \to TW$ is a definable C^{p-1}-diffeomorphism. Therefore $d\phi$ is a definable parametrization on $TW = TX \cap (W \times R^k)$ that is open on TX. This proves the first part of the corollary. The rest is clear. QED

Corollary 2.14 *Let $X \subset R^k$ be a definable C^p-manifold of dimension m. Then ∂X is either empty or a definable C^p-manifold of dimension $m-1$.*

Proof. We prove that ∂X is a definable set. Let $x \in X$. Then there is a definable projection $\pi: R^k \to R^m$ which maps $T_x X$ onto R^m, hence it is an immersion at x. So there is $\varepsilon > 0$ such that π restricted to $W^\varepsilon = \{y \in X \mid |y-x| < \varepsilon\}$ is a definable C^p-diffeomorphism onto its image. By the inverse function theorem, for any such π, $\pi(x)$ is in the interior of $\pi(W^\varepsilon)$ if and only if $x \notin \partial X$. This characterization of ∂X, together with the fact that TX is definable, gives the definability of ∂X.

To finish the proof it suffices to observe that if $g: W \cap X \to U \cap H^m$ is a definable local coordinate system around x, with W open in R^k and U open in R^m, then g restricts to a definable C^p-diffeomorphism from $W \cap \partial X$ to $U \cap (R^{m-1} \times \{0\})$. QED

The interior of $X \subset R^k$, $Int\, X := X \backslash \partial X$, is now clearly a definable C^p-manifold of dimension m. Note that $Int\, X$ and ∂X might coincide with the topological interior and boundary of X only in the case $\dim X = k$.

3 Regular values and Sard's theorem

Let $X \subset R^k$ and $Y \subset R^l$ be definable C^p-manifolds of dimensions m and n respectively. Let $f: X \to Y$ be a definable C^p-map. A point $x \in X$ is said to be **a critical point** of f if f is not a submersion at x. If C is the set of critical points of f, $f(C)$ is called the set of **critical values** of f and its complement $Y \setminus f(C)$ the set of **regular values** of f. Since the map $df: TX \to TY$, and hence the set $\{x \in X \mid rk\, d_x f < n\}$, is definable, the three sets just defined are definable. Every point of Y which is not in the image of f is trivially a regular value.

3.1 Preimage Theorem

Lemma 3.1 *Suppose S is a boundaryless definable C^p-manifold. Let $\pi: S \to R$ be a definable C^p-map having 0 as a regular value. Then the set $\{x \in S \mid \pi(x) \geq 0\}$ is a definable C^p-manifold with boundary equal to $\pi^{-1}(0)$.*

Proof. The set $\{x \in S \mid \pi(x) > 0\}$ is definable and open in S, therefore a definable submanifold of S (of the same dimension). For those $x \in S$ with $\pi(x) = 0$, π is a submersion at x, because 0 is a regular value. By the local submersion theorem 2.5, there are local coordinates around x such that in these coordinates π is the standard projection $(x_1, \ldots, x_n) \mapsto x_n$ ($n = \dim S$). Since the result to be proved holds for the standard projection, and in local cooordinates we can reduce to this case, the lemma follows. QED

Theorem 3.2 (Preimage Theorem) *Let $X \subset R^k$ and $Y \subset R^l$ be definable C^p-manifolds of dimensions m and n respectively. Let $f: X \to Y$ be a definable C^p-map. Suppose $y \in Im\, f$ is a regular value of f and of $f_{|\partial X}$ (the latter condition holds vacuously if ∂X is empty). Then $f^{-1}(y) \subset X$ is a definable C^p-manifold of dimension $m-n$ whose boundary is $\partial X \cap f^{-1}(y)$ and whose tangent space at $x \in f^{-1}(y)$ is the kernel of $d_x f: T_x X \to T_y Y$.*

Proof. Let $x \in f^{-1}(y)$. It suffices to prove the result for the restriction of f to suitable open definable neighbourhoods of x in X and y in Y (provided f maps the first into the second). Consider definable parametrizations ϕ and ψ around x and y. We have an induced definable map $\tilde{f} := \psi^{-1} \circ f \circ \phi$ into R^n and defined on an open subset of either R^m or H^m depending on whether

$x \notin X$ or $x \in \partial X$. Without loss of generality $x = \phi(0)$ and $y = \psi(0)$. It suffices to prove the result for $\tilde{f}$ instead of f, and 0 instead of y.

Case 1. Assume $x \notin \partial X$. By the local submersion theorem, changing ψ if necessary, we can arrange so that $\tilde{f}$ is the restriction to an open definable set of R^m of the standard projection $(x_1, \ldots, x_m)$ into $(x_1, \ldots, x_n)$. Since the result to be proved is clear for the standard projection, and the relevant notions are invariant under definable C^p-diffeomorphisms, we are done.

Case 2. If $x \in \partial X$, then $\tilde{f}$ is defined on an open definable subset U of H^m and we cannot apply the local submersion theorem. We can however extend $\tilde{f}$ to a definable C^p-map $h: U_1 \to R^n$, with $U_1 \supset U$ open in R^m, so that by the case already proved $h^{-1}(0) \subset U_1$ is a definable boundaryless C^p-manifold S of dimension $m-n$ with tangent spaces $T_s S = Ker\ d_s h$. Now $\tilde{f}^{-1}(0) = \{x \in S \mid \pi(x) \geq 0\}$ where $\pi: R^m \to R$ is the projection on the last coordinate. By Lemma 3.1 to conclude the proof it suffices to show that 0 is a regular value of $\pi_{|S}$. If this is not so, then there is $s \in \pi^{-1}(0) \cap S$ such that $T_s S$ is contained in $R^{m-1} \times \{0\} = T_s(\partial U)$. But then the kernel of $d_s h: R^m \to R^n$ is included in $T_s(\partial U)$, so it coincides with the kernel of $d_s h_{|\partial U} = d_s \tilde{f}_{|\partial U}$. This is absurd since $d_s h$ and $d_s \tilde{f}_{|\partial U}$ are both surjective (as 0 is a regular value of the latter), and they have domains of different dimension. QED

By the preimage theorem the counterimage of a regular value under a definable C^p-map is a definable manifold. The converse holds locally:

Proposition 3.3 *If $Y \subset R^l$ is a boundaryless definable C^p-manifold of dimension n, then for every $y \in Y$ there is an open definable set W in R^l containing y and a definable C^p-submersion $h: W \to R^{l-n}$ such that $Y \cap W = h^{-1}(0)$.*

Proof. Choose a definable parametrization $\varphi: U \to Y \cap W$ where W is an open definable neighbourhood of y in R^l and U is open in R^n. Then φ as a map from U to W is a definable immersion so by the local immersion theorem 2.3, there are definable parametrizations ϕ around $0 \in U$ and ψ around $y \in W$ such that $(\psi^{-1} \circ \varphi \circ \phi)(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0) \imath n R^l$. Hence $Y \cap W = Im(\varphi) = (\pi \circ \psi^{-1})^{-1}(0)$ where $\pi(x_1, \ldots, x_n, y_1, \ldots, y_{l-n}) = (y_1, \ldots, y_{l-n})$, so we can set $h = (\pi \circ \psi^{-1})^{-1}$. QED

Corollary 3.4 *Let $X \subset R^k$ be a definable boundaryless C^p-manifold of codimension one. Then each point $x \in X$ has definable open neighbourhood*

U in R^k such that $U \setminus X$ has exactly two definably connected components with $U \cap X$ as their common boundary.

Proof. Given $x \in X$, by Proposition 3.3 there is a definable open neighbourhood U of x in R^k and a definable C^p-submersion $h: U \to R$ with $X \cap U = h^{-1}(0)$. The result to be proved is clear if h is the restriction to U of the projection $(x_1, \ldots, x_k) \mapsto x_k$, and by the local submersion theorem 2.5 we can reduce to this case. QED

3.2 Sard's theorem

Theorem 3.5 (Sard's Theorem) *Let $X \subset R^k$ and $Y \subset R^l$ be definable C^p-manifolds of dimensions m and n respectively. Let $f: X \to Y$ be a definable C^p-map. Then the set of critical values of f has dimension less than n.*

Proof. Let $\Gamma = \{(fx, x) \mid x \in X\} \subset R^{l+k}$ and let $\pi: R^{l+k} \to R^l$ be the projection $(y, x) \mapsto y$ on the first l coordinates. Since $x \mapsto (fx, x)$ is a definable C^p-diffeomorphism from X to Γ, the set of critical values of $f: X \to Y$ coincides with the set of critical values of $\pi_{|\Gamma}: \Gamma \to Y$. Consider a definable C^p-cell-decomposition of R^{l+k} compatible with the set $S \subset \Gamma$ of critical points of $\pi_{|\Gamma}$. The set of critical values of $\pi_{|\Gamma}$ is the union of the sets $\pi(C)$ where C is a cell contained in S. So it suffices to show that if $C \subset S$, then $\pi(C)$ has dimension less than n. By Lemma 2.6 C is a definable C^p-manifold, and by Lemma 2.7 (applied k times) $\pi_{|C}$ has constant rank equal to $\dim \pi(C)$. To show that this constant rank is $< n$, it is enough to note that for $x \in C$ the vector space $d_x\pi(T_xC)$ is contained in $d_x\pi(T_x\Gamma)$, and the latter has dimension $< n$ since x is a critical point of $\pi_{|\Gamma}$ (as $C \subset S$). QED

Corollary 3.6 *Let X and Y be definable C^p-manifolds. Let $f: X \to Y$ be a definable C^p-map. The set of regular values of both f and $f_{|\partial X}$ is dense in Y.*

Proof. Apply Theorem 3.5 to f and $f_{|\partial X}$. Noting that $d_x(f_{|\partial X}) = (d_xf)_{|T_x(\partial X)}$. QED

4 Normal bundle and tubular neighbourhoods

Let $Y \subset R^l$ be a definable C^p-manifold of dimension n without boundary. Let $N_y(Y)$ be the orthogonal complement of $T_y(Y)$ in R^l with respect to the usual inner product on R^l. We define the **normal bundle** $N(Y) \subset R^{2l}$ as the union $\bigcup_{y \in Y} \{y\} \times N_y(Y)$.

Proposition 4.1 *Let $p > 1$. Let $Y \subset R^l$ be a definable boundaryless C^p-manifold of dimension n. Then $N(Y)$ is a definable boundaryless C^{p-1}-manifold of dimension l. Moreover the projection $\pi: N(Y) \to Y$, sending $(y, v) \in \{y\} \times N_y(Y)$ to y, is a definable C^{p-1}-submersion.*

Proof. By Proposition 3.3 for each $z \in Y$ there is a definable open set U in R^l containing z, and a definable C^p-submersion $h: U \to R^{l-n}$ such that $Y \cap U = h^{-1}(0)$. Then $N(Y \cap U) = N(Y) \cap (U \times R^l)$ is open in $N(Y)$. To prove that $N(Y)$ is a C^{p-1}-manifold, it is enough to show that $N(Y \cap U)$ is definably C^{p-1}-diffeomorphic to $(Y \cap U) \times R^{l-n}$. Given $y \in U$, $d_y h: R^l \to R^{l-n}$ is onto and its kernel is $T_y Y$. So the transpose $(d_y h)^t$ maps R^{l-n} isomorphically onto $N_y(Y)$. The required diffeomorphism sends $(y, v) \in (U \cap Y) \times R^{l-n}$ into $(y, (d_y h)^t v)$. The rest is easy. (See [GP] §2.3 for the details.) QED

Given a definably compact subset Z of $X \subset R^k$ and $\varepsilon \in R$ with $\varepsilon > 0$ we define the **ε-neighbourhood** of Z in X as the definable set of all points of X whose distance (in the sense of the R-metric of R^k) from some point of Z is less than ε.

Proposition 4.2 *Let $X \subset R^k$ be definable. If $Z \subset X$ is definably compact, any definable open subset U of X containing Z, also contains for some $\varepsilon > 0$ the ε-neighbourhood of Z in X.*

Proof. For a contradiction suppose that for each $\varepsilon > 0$ there is a point $x \in X$ in the complement of U at distance $< \varepsilon$ from Z. By o-minimality there is a definable continuous function $\varepsilon \mapsto x(\varepsilon)$ which selects such an x for every sufficiently small $\varepsilon > 0$. Still by o-minimality the limit $\lim_{\varepsilon \to 0} x(\varepsilon)$ exists in R^k, and it must belong to Z since Z is closed. But then for some $\varepsilon > 0$, $x(\varepsilon) \in U$, a contradiction. QED

Lemma 4.3 *Let $f: X \to Y$ be a definable C^p-map between definable boundaryless C^p-manifolds. Suppose $d_x f: T_x X \to T_{fx} Y$ is an isomorphism for each x in the definably compact submanifold $Z \subset X$, and assume that f maps Z C^p-diffeomorphically onto $f(Z)$. Then f maps a definable neighbourhood of Z in X C^p-diffeomorphically onto a neighbourhood of $f(Z)$ in Y.*

Proof. Given $\varepsilon > 0$ let Z^ε denote the ε-neighbourhood of Z in X. We first prove that for a sufficiently small ε, $f_{|Z^\varepsilon}$ is injective.

If this is not so, then for each ε there are distinct points $a(\varepsilon), b(\varepsilon)$ in Z^ε with the same image under f. By definable choice we can assume that the two maps sending ε to $a(\varepsilon)$ and $b(\varepsilon)$ respectively are definable and continuous, for ε varying in a small enough interval $(0, \varepsilon_0)$. Since $a(\varepsilon)$ and $b(\varepsilon)$ belong to the bounded set Z^ε, by o-minimality the limits $a = \lim_{\varepsilon \to 0} a(\varepsilon)$ and $b = \lim_{\varepsilon \to 0} b(\varepsilon)$ exist and they lie in Z because Z is closed. By continuity of f, $f(a) = \lim_{\varepsilon \to 0} f(a(\varepsilon)) = \lim_{\varepsilon \to 0} f(b(\varepsilon)) = f(b)$. Since f is injective on Z, $a = b$. Since $d_a f: T_a X \to T_{fa} Y$ is an isomorphism, by Theorem 2.2 f is a homeomorphism in a neighbourhood of a, contradicting the fact that any such neighbourhood contains distinct points $a(\varepsilon), b(\varepsilon)$ with the same image. Thus f is injective on some Z^ε. The desired neighbourhood is then $Z^\varepsilon \cap \{x \in X \mid d_x f: T_x X \to T_{fx} Y \text{ is an isomorphism }\}$. QED

Theorem 4.4 *Let $p > 1$. Let $Y \subset R^l$ be a definable C^p-manifold of dimension n, definably compact and boundaryless. Then there exists an open definable set U of R^l containing Y, and a definable C^{p-1}-submersion $\sigma: U \to Y$ that is the identity on Y.*

Proof. Define a C^{p-1}-map $h: N(Y) \to R^l$ by $h(y, v) = y + v$. Then the restriction of h to $Y \times \{0\}$ is a definable C^{p-1}-diffeomorphism onto Y, and $d_{(y,0)} h: T_{(y,0)} N(Y) \to R^l$ is an isomorphism. By Lemma 4.3 there is a definable open neighbourhood V of $Y \times \{0\}$ in $N(Y)$, which is mapped C^{p-1}-diffeomorphically onto an open definable neighbourhood $h(V)$ of Y in R^l. Composing h^{-1} with the canonical projection $\pi: N(Y) \to Y$ we obtain the required submersion σ. QED

5 Transversality

Let X, Y, Z be definable C^p-manifolds. Let $f: X \to Y$ and $g: Z \to Y$ be definable C^p-maps. We say that f **is transversal to** g, and write $f \pitchfork g$, if for each $x \in X$ and $z \in Z$ with $f(x) = y = g(z)$, we have

$$d_x f(T_x X) + d_z g(T_z Z) = T_y Y \tag{1}$$

In the special case when g is the inclusion of Z in Y, we say that f is transversal to Z, written $f \pitchfork Z$. In this case (1) becomes

$$d_x f(T_x X) + T_{fx} Z = T_{fx} Y \tag{2}$$

If f is also an inclusion, we say that X is transversal to Z, and we write $X \pitchfork Z$. This means that at any point of the intersection of X and Z, the tangent spaces of X and Z add up to the tangent space of Y. Note that for $Z = \{y\}$, $f \pitchfork Z$ if and only if y is a regular value of f. By Proposition 3.3 every definable boundaryless submanifold is locally the counterimage of a regular value under some map. The definition of transversality is motivated by the following remark.

Remark 5.1 Let X, Y and W be definable C^p-manifolds. Let $f: X \to Y$ and $h: Y \to W$ be definable C^p-maps. f is transversal to the counterimage $h^{-1}(w)$ of a regular value w under h, if and only if w is a regular value of $h \circ f: X \to W$.

Proposition 5.2 *Let $f: X \to Y$ be a definable C^p-map between definable C^p-manifolds, and let Z be a definable boundaryless submanifold of Y. If f and $f_{|\partial X}$ are transversal to Z, then $f^{-1}(Z)$ is a definable C^p-manifold with boundary $\partial(f^{-1}(Z)) = f^{-1}(Z) \cap \partial X$ and tangent space $T_x f^{-1}(Z) = (d_x f)^{-1}(T_{fx} Z)$ at $x \in f^{-1}(Z)$. Moreover the codimension of $f^{-1}(Z)$ in X is equal to the codimension of Z in Y.*

Proof. By Proposition 3.3 for each $z \in Z$ there is a neighbourhood W of z in Y and a definable C^p-map $h: W \to R^{\dim Y - \dim Z}$ such that $Z \cap W_z = h^{-1}(0)$ and 0 is a regular value of h. So $f^{-1}(Z \cap W)$ is the preimage of a regular value and we can apply remark 5.1 and the preimage theorem 3.2 to the appropriate restriction of $h \circ f$. (See [GP] p. 60 for details). QED

Proposition 5.3 *Under the hypothesis of Proposition 5.2, for each $x \in f^{-1}(Z)$, $d_x f$ sends every direct summand H of $T_x(f^{-1}(Z))$ in $T_x X$, isomorphically onto a direct summand of $T_{fx} Z$ in $T_{fx} Y$.*

Proof. Since $f \pitchfork Z$, $d_x f(T_x X) + T_{fx} Z = T_{fx} Y$. Now $d_x f(T_x X)$ is the contribution of two parts: one is $d_x f(H)$ and the other is the image under $d_x f$ of $T_x(f^{-1}(Z))$, which is included in $T_{fx} Z$. This shows that $d_x f(H) + T_{fx} Z = T_{fx} Y$, and the rest follows by counting dimensions. QED

Proposition 5.4 *Let $p > 1$. Let $f: X \to Y$ be a definable C^p-map between definable C^p-manifolds with $Y \subset R^l$ boundaryless and definably compact. Let S be the unit open ball in R^l. Then there is a definable C^{p-1}-map $F: X \times S \to Y$ such that $F(x, 0) = f(x)$ and for fixed $x \in X$ the map from S to Y sending s to $F(x, s)$ is a submersion.*

Proof. Let σ be a definable C^{p-1}-submersion of an definable open neighbourhood U of Y onto Y given by Theorem 4.4. We can assume that U is an ε-neighbourhood since Y is definably compact. Define $F(x, s) = \sigma(f(x) + \varepsilon s)$. QED

Theorem 5.5 (Transversality theorem) *Let $F: X \times S \to Y$ be a definable C^p-map, with X, Y, S definable C^p-manifolds, Y and S boundaryless. For $s \in S$, let $f_s: X \to Y$ be defined as $f_s(x) = F(x, s)$. Let Z be a definable boundaryless submanifold of Y. If both F and $F_{|\partial(X \times S)}$ are transversal to Z, then for all $s \in S$ outside of a definable set of dimension $< \dim S$, f_s and $f_{s|\partial X}$ are transversal to Z.*

Proof. Since S is boundaryless $X \times S$ is a definable C^p-manifold. Let $\pi: X \times S \to S$ be the projection. By Sard's theorem, it is enough to show that if s is a regular value of $\pi_{|F^{-1}(Z)}$ and $\pi_{|\partial(F^{-1}Z)}$, then $f_s \pitchfork Z$ and $f_{s|\partial X} \pitchfork Z$. So let s be a regular value of $\pi_{|F^{-1}(Z)}$. We show that $d_x f_s(T_x X) + T_z Z = T_z Y$ whenever $f_s(x) = z \in Z$. Note that $d_x f_s(T_x X) = d_{(x,s)} F(H)$ where $H = T_x(X \times \{s\})$. Since s is a regular value of $\pi_{|F^{-1}(Z)}$, H is a direct summand of $T_{(x,s)}(F^{-1}Z)$ in $T_{(x,s)}(X \times S)$, so by Proposition 5.3, its image under $d_{(x,s)} F$ is a direct summand of $T_z Z$ in $T_z Y$, as desired. The argument for $\pi_{|\partial F^{-1}(Z)}$ is the same. (See [GP], §2.3 for details.) QED

We say that $f: X \to Y$ is **definably C^p-homotopic** to $g: X \to Y$, if there is a definable C^p-map $F: I \times X \to Y$ with $I = [0, 1]$, such that $f(x) = F(0, x)$ and $g(x) = F(1, x)$ for all $x \in X$. Note that if X is a boundaryless definable C^p-manifold, then $I \times X$ is a definable C^p-manifold with boundary $\{0\} \times X \cup \{1\} \times X$.

Theorem 5.6 (Transversality homotopy theorem) *Let $p > 1$. Let X, Y be definable C^p-manifolds with Y boundaryless and definably compact. Then for every definable C^p-map $f: X \to Y$ and for every definable boundaryless submanifold Z of Y, there is a definable C^{p-1}-map $g: X \to Y$ definably C^{p-1}-homotopic to f such that $g \pitchfork Z$ and $g_{|\partial X} \pitchfork Z$.*

Proof. By Proposition 5.4 there is a definable C^{p-1}-map $F: X \times S \to Y$ with F and $F_{|\partial(X\times S)}$ transversal to Z (where S is the open unit ball in some R^l). By Theorem 5.5 there is $s \in S$ with $f_s: x \mapsto F(x, s)$ transversal to Z and $f_{s|\partial X}$ transversal to Z. Clearly f_s is definably C^{p-1}-homotopic to f. QED

Lemma 5.7 *Let A be a definable closed subset of R^k. Then there is a definable C^p-map $f: R^k \to R$ whose zero set is A.*

Proof. [DM] Theorem C 11. QED

Corollary 5.8 *Let $X \subset R^k$ be a definable C^p-manifold. Given two disjoint definable sets $F_0, F_1 \subset X$ closed in X, there is a definable C^p-map $\delta: X \to R$ which is 0 exactly on F_0, 1 exactly on F_1 and $0 \le \delta \le 1$.*

Proof. For $i = 0, 1$ let G_i be a closed set in R^k with $F_i = G_i \cap X$. By Lemma 5.7 there are definable C^p-maps $g_i: R^k \to R$ with $Z(g_i) = G_i$. Let $\delta_i(x) = \frac{g_i^2(x)}{1+{g_i}^2(x)}$ and $\delta(x) = \frac{\delta_0(x)+\delta_0(x)\delta_1(x)}{\delta_0(x)+\delta_1(x)}$. Then $\delta_{|X}$ works. QED

Theorem 5.9 (Extension theorem) *Let $p > 1$. Let $f: X \to Y$ be a definable C^p-map between definable C^p-manifolds with Y definably compact and boundaryless. Let Z be a definable boundaryless submanifold of Y. If $f_{|\partial X} \pitchfork Z$, then there is a definable C^{p-1}-map $g: X \to Y$ which is definably C^{p-1}-homotopic to f, $f = g$ on ∂X, and $g \pitchfork Z$.*

Proof. By Corollary 5.8 there is a definable C^p-map $\delta: X \to [0, 1]$ which is 0 exactly on ∂X. Consider the definable C^{p-1}-map $F: X \times S \to Y$ given by Proposition 5.4. Define $G: X \times S \to Y$ by $G(x, s) = F(x, \delta^2(x)s)$. We claim that G and $G_{|\partial(X\times S)}$ are transversal to Z. Granted this by the transversality theorem 5.5 there is $s \in S$ such that the map $g(x) = G(x, s)$ is transversal to Z, and this g is the required map. To prove the claim let $x \in X$ and consider first the case $\delta(x) \neq 0$. Then the map $s \mapsto G(x, s)$ is the composition of the two submersions $s \mapsto \delta^2(x)s$ from S to S and $r \mapsto F(x,r)$ from S to Y, so G is a submersion at (x,s). On the other hand if $\delta(x) = 0$, then $x \in \partial X$ and by the chain rule the differential $d_{(x,s)}G: T_xX \times T_sS \to T_{G(x,s)}Y$ is given by $d_{(x,s)}G(v,w) = d_{(x,\delta^2(x)s)}F(v, \delta^2(x) \cdot w + 2\delta(x)d_x\delta(v) \cdot s) = d_{(x,0)}F(v,0) = d_xf(v)$, so $d_{(x,s)}G$ has the same image of d_xf, which contains the image of $d_x(f_{|\partial X})$. The claim follows. QED

6 One dimensional manifolds

Lemma 6.1 *Let C be a bounded one-dimensional cell of a C^p-cell decomposition of R^k. Then there is a projection $\theta: R^k \to R$ such that θ maps C C^p-diffeomorphically onto a bounded open interval (a,b) and the closure of C homeomorphically onto $[a,b]$. In particular $\overline{C} \setminus C$ consists of exactly two points.*

Proof. Let $\pi: R^k \to R^{k-1}$ be the projection onto the first $k-1$ coordinates.

Case 1. If $\pi(C)$ is a point, then let θ be the projection on the last coordinate.

Case 2. C is the graph $\Gamma(f)$ of a definable C^p-map $f: \pi(C) \to R$. Then $\pi(C)$ is a one-cell in R^{k-1} and by induction we can assume that there is a projection $\sigma: R^{k-1} \to R$ which maps $\pi(C)$ C^p-diffeomorphically onto an open bounded interval (a,b) and its closure homeomorphically onto $[a,b]$. Clearly π maps C C^p-diffeomorphically onto $\pi(C)$. So it is enough to show that π maps $\overline{C}$ homeomorphically onto $\overline{\pi(C)}$, because then we can take $\theta = \sigma \circ \pi$. Since π is continuous and $\overline{C}$ is definably compact, it suffices to show that $\pi_{|\overline{C}}$ is a bijection onto $\overline{\pi C}$. It is onto because its image $\pi\overline{C}$ is definably compact, so closed, hence contains $\overline{\pi C}$. To prove that it is injective, we show that the composition $\sigma \circ \pi: \overline{C} \to [a,b]$ is injective. Note first that if $x \in \overline{C}$, and $\pi(x) \in \pi(C)$, then $x \in C$. So if $x \in \overline{C}$ and $\sigma(\pi(x)) = \theta(x) \in (a,b)$, then $x \in C$. Now suppose $\theta(x) = \theta(y) = c$ with x,y distinct elements in $\overline{C}$. Then c must be a or b because θ is injective on C. So we may assume $\theta(x) = \theta(y) = a$. Take U and V disjoint definable open neighbourhoods of x and y in R^k. Then $a \in \overline{\theta(U \cap C)} \cap \overline{\theta(V \cap C)}$. This is absurd since by o-minimality a cannot be in the closure of two disjoint definable open subsets of (a,b). QED

Lemma 6.2 *Given $v > 0, w > 0, a < b, c < d$ in R, there is a definable C^1-diffeomorphism $\eta: [a,b] \to [c,d]$ with $\eta'(a) = v$ and $\eta'(b) = w$.*

Proof. We may define $\eta(t) = c + \int_a^t f$ where $f: [a, b] \to R$ is a piecewise linear function taking positive values with $f(a) = v, f(b) = w$ and $\int_a^b f = d - c$. (The integral is meant in the sense of antiderivative. It exists since f is piecewise linear.) QED

Lemma 6.3 *If a definable map $f: [0, 1] \to R^k$ is differentiable at every point of its domain, then it is C^1.*

Proof. We can assume $k = 1$. We just check the continuity of f' at 0. By the mean value theorem there is $0 < \xi_t < t$ such that $f'(\xi_t)$ tends to $f'(0)$ when $t \to 0$. On the other hand by o-minimality $\lim_{x \to 0} f'(x)$ exists in $R \cup \{+\infty, -\infty\}$, and obviously must coincide with $\lim_{t \to 0} f'(\xi_t)$. QED

Theorem 6.4 *Let C be a bounded one-dimensional cell of a C^1-cell decomposition of R^k. Then $\overline{C}$ is a definable C^1-manifold definably C^1-diffeomorphic to $[0, 1]$ with boundary $\partial \overline{C} = \overline{C} \setminus C$.*

Proof. By Lemma 6.1 there is a definable homeomorphism $\phi: [0, 1] \to \overline{C}$ whose restriction to $(0, 1)$ is a C^1-diffeomorphism onto C. We shall define a reparametrization $\gamma: [0, 1] \to [0, 1]$ so that $\theta = \phi \circ \gamma: [0, 1] \to \overline{C}$ is the required C^1-diffeomorphism. In order to ensure that the limit $\theta'(t) \in R^k$ exists when t tends to 0 or 1, we shall arrange so that:

(1) $|\theta(t) - \theta(0)| = t$ for t close to 0;

(2) $|\theta(t) - \theta(1)| = 1 - t$ for t close to 1.

For t sufficiently close to zero, by continuity there is $t' \in [0, 1]$ with $|\phi(t') - \phi(0)| = t$. By o-minimality there is $\varepsilon > 0$ and a definable C^1-map $\gamma_0: (0, \varepsilon] \to [0, 1]$ satisfying $|\phi(\gamma_0(t)) - \phi(0)| = t$. Similarly, taking a smaller ε if necessary, there is a definable C^1-map $\gamma_1: [1 - \varepsilon, 1) \to [0, 1]$ satisfying $|\phi(\gamma_1(t)) - \phi(1)| = 1 - t$. We may also assume $\varepsilon < 1/2$, $\gamma_0(\varepsilon) < \gamma_1(1 - \varepsilon)$ and γ_0 and γ_1 have strictly positive derivative on their domains of definition. By Lemma 6.2 there is a C^1-diffeomorphism $\mu: [\varepsilon, 1 - \varepsilon] \to [\gamma_0(\varepsilon), \gamma_1(1 - \varepsilon)]$ with derivatives at the end points matching those of γ_0 and γ_1, so that the union $\gamma = \gamma_0 \cup \mu \cup \gamma_1$ is a C_1-diffeomorphims from $(0, 1)$ onto itself. We extend γ by continuity to $[0, 1]$ setting $\gamma(0) = 0, \gamma(1) = 1$. With this choice of γ, (1) and (2) hold for $\theta = \phi \circ \gamma$. Therefore the right and left derivatives of θ at 0 and 1 respectively, which exist by o-minimality, must be finite and different from zero. So by Lemma 6.3, θ is C^1. We have thus constructed a C^1 homeomorphism $\theta: [0,1] \to \overline{C}$ with non-zero derivative at every point. By Lemma 2.4 θ is the required diffeomorphism. QED

Theorem 6.5 *Let $X \subset R^k$ be a one dimensional definably connected definably compact C^1-manifold. If X has non-empty boundary, then X is definably C^1-diffeomorphic to $[0,1]$.*

Proof. Fix a C^1-cell-decomposition of R^k compatible with X and ∂X. The proof is by induction on the number l of 1-cells contained in X. If $l = 1$, then X is the closure of a 1-cell and we apply Theorem 6.4. Assume $l > 1$ and let $x \in \partial X$. Then there is a unique 1-cell C with x in its closure, because otherwise every sufficiently small definable neighbourhood of x in X can be disconnected by removing x, contradicting $x \in \partial X$. Let $\overline{C} = C \cup \{x\} \cup \{x'\}$. The point x' must be in the closure of a 1-cell $C' \neq C$ otherwise $\overline{C}$ will be clopen in X, and it cannot be in the closure of any other 1-cell of X because otherwise every sufficiently small definable neighbourhood of x' in X would have more than two definably connected components after removing x', contradicting the fact that X is a one dimensional manifold. Let $X' = X \setminus (C \cup \{x\})$. Then X' is a one dimensional definable C^1-manifold with x' in its boundary, since a definable open neighbourhood around x' is given by $C' \cup \{x'\}$ which by Theorem 6.4 it is definably C^1-diffeomorphic to $[0,1)$. It is also clear that X' is definably compact, since X' is closed in X. Finally X' is definably connected because if X' is the disjoint union of two closed sets A and B with $x' \in A$, then X would be disconnected by $A \cup \overline{C}$ and B. By induction there is a definable C^1-diffeomorphism $f: [0,1] \to X'$ with $f(1) = x'$, and by Theorem 6.4 there is a definable C^1-diffeomorphism $g: [1,2] \to \overline{C}$ with $g(1) = x'$. In the special case when $f'(1) = g'(1) \in T_{x'}X$, then the union $f \cup g: [0,2] \to X$ is the required C^1-diffeomorphism (after rescaling). In the general case we note that there is $\lambda \in R$ with $f'(1) = \lambda g'(1)$ since $T_{x'}X$ is one dimensional, and moreover $\lambda > 0$ because $g'(1)$ and $f'(1)$ are both inward vectors of $T_{x'}\overline{C} \subset T_{x'}X$ (see Definition 2.1). We can apply Lemma 6.2 to modify g in order to reduce to the special case. QED

7 Orientation

7.1 Definable orientation on a manifold

An **orientation of a (non-trivial) finite dimensional vector space** V over R is a function which assigns to each ordered basis $(b_1, \dots, b_n)$ either the sign $+1$ (positive basis) or the sign -1 (negative basis), and which assigns the same sign to two ordered basis iff the matrix of the change of basis has positive determinant. An **orientation of a zero dimensional vector space** is given by the vector space itself with one of the two signs $+1$ or -1 assigned to it (or to its empty basis). Each vector space has exactly two orientations.

The **standard orientation of** R^m assigns a positive sign to standard basis $((1,0,\dots,0),\ (0,1,0,\dots,0),\ \dots,\ (0,0,\dots,0,1))$.

A linear map $L: V \to W$ between oriented vector spaces is **orientation preserving** if it sends a basis to a basis with the same sign.

If U is an open subset of either H^m or R^m and $y \in U$, then $T_yU = R^m$, so it carries the standard orientation.

Let X be a definable C^p-manifold of dimension m and let $\mathbf{B}(X) \subset (TX)^m$ be the union $\bigcup_{x \in X} \{x\} \times \mathbf{B}_x$ where $\mathbf{B}_x \subset (T_xX)^m$ is the set of all bases of T_xX. An **definable orientation on** X is an orientation of each tangent space T_xX given by a definable function $\mathsf{sign}: \mathbf{B}(X) \to \{+1, -1\}$ and such that the orientation is locally constant in the sense that for every $x \in X$ there is a definable chart (W, h) around x such that for each $y \in W$ the isomorphism $d_yh: T_yX \to R^m$ is orientation preserving.

Note that a zero dimensional definably oriented C^p-manifold is just a finite set of points with a sign attached to each point called the **orientation number** of that point.

If X is a definably oriented C^p-manifold, then we denote by $-X$ the same manifold with the **opposite orientation** obtained by changing the sign of each basis of each tangent space.

Lemma 7.1 *A definably connected definable C^p-manifold X with a definable orientation has exactly two definable orientations.*

Proof. By definable choice there is a definable function which assigns to each $x \in X$ a basis b_x of T_xX. Given two orientations, say sign_1 and sign_2, the set $\{x \in X \mid \mathsf{sign}_1(b_x) = \mathsf{sign}_2(b_x)\}$ where the two orientations agree is open and definable, and similarly the set where they disagree. Since X is definably connected one of the two sets must be empty. QED

Corollary 7.2 *A definable C^p-diffeomorphism between definably connected oriented C^p-manifolds, either preserves the orientation at each point, or it reverses the orientation at each point.*

Corollary 7.3 *A definably orientable C^p-manifold with n definably connected components, has 2^n definable orientations.*

7.2 Product orientation

Given two oriented vector spaces V, W we define the **product orientation** on $V \oplus W$ as follows. An ordered basis of $V \oplus W$ consisting of an ordered basis of V followed by an ordered basis of W has a sign given by the product of the signs of the given bases of V and W. If W is zero dimensional, then $V \oplus W = V$ and the sign of a basis of $V \oplus W$ equals the sign it has as a basis of V times the sign of W. Similarly if V is zero dimensional.

If X, Y are definable C^p-manifolds, then $T_{(x,y)}(X \times Y) = T_x X \oplus T_y Y$, so if X, Y are oriented we can use this equation to orient $X \times Y$ via the product orientation on each tangent space.

We write $-(X \times Y)$ or $X \times (-Y)$ or $(-X) \times Y$ to indicate the opposite of the product orientation.

7.3 Orientation of the boundary

Let X be a definably oriented C^p-manifold. The **induced orientation of the boundary** ∂X is defined as follows. Let $x \in \partial X$, let $v \in T_x X$ be an outward vector (see Definition 2.1) and let $\langle v \rangle$ be the linear span of v. Then

$$\langle v \rangle \oplus T_x \partial X = T_x X.$$

We orient ∂X so that this equation holds as oriented vector spaces, where $\langle v \rangle$ has the orientation which makes v a positive basis and $\langle v \rangle \oplus T_x \partial X$ has the product orientation. In other words a basis $(b_1, \dots, b_{m-1})$ of $T_x \partial X$ is positive iff $(v, b_1, \dots, b_{m-1})$ is a positive basis of $T_x X$.

Theorem 7.4 *Let X be a definably compact oriented one-dimensional C^p-manifold. Then the sum of the orientation numbers at the boundary points of X (with the induced orientation) is zero.*

Proof. By Theorem 6.5, each definably connected component of X with non-empty boundary, is definably C^1-diffeomorphic to $[0, 1]$. So we can assume $X = [0, 1]$, $\partial X = \{0, 1\}$. Therefore all the tangent spaces $T_x X$ can be canonically identified with R. Hence X has two obvious orientations: the first orients each $T_x X$ as R with the standard orientation, the second orients each $T_x X$ as $-R$. Since X is definably connected there are no other definable orientations on X, and the result follows by the definition of the induced orientation of the boundary. QED

7.4 Preimage orientation

Let $f: X \to Y$ be a definable C^p-map between definably oriented C^p-manifolds with f and $f_{|\partial X}$ transversal to a definably oriented boundaryless submanifold $Z \subset Y$. By Proposition 5.2 $f^{-1}(Z)$ is a definable submanifold of X with boundary

$$\partial(f^{-1}Z) = (f^{-1}Z) \cap \partial X$$

and tangent spaces

$$T_x(f^{-1}Z) = (d_x f)^{-1}(T_{fx}Z).$$

Choose H so that

$$H \oplus T_x(f^{-1}Z) = T_x X \qquad (1)$$

By Proposition 5.3 $d_x f: T_x X \to T_{fx}Y$ sends $H \subset T_x X$ injectively onto a direct summand of $T_{fx}Z$:

$$d_x f(H) \oplus T_{fx}Z = T_{fx}Y \qquad (2)$$

We orient $H, d_x f(H)$ and $T_x(f^{-1}Z)$ so that (1) and (2) hold as equations between oriented spaces and $d_x f: H \to d_x f(H)$ is orientation preserving (first orient $d_x f(H)$ so that (2) holds, then H so that $d_x f_{|H}$ is orientation preserving, then $T_x(f^{-1}Z)$ so that (1) holds). This orientation on $T_x(f^{-1}Z)$ is independent of the choice of H, so we have defined an orientation on $f^{-1}Z$, called the **preimage orientation**.

Remark 7.5 In the special case when $Z = \{y\}$ consists of a single positively oriented point $y \in Y$, then the above two equations become $H \oplus T_x f^{-1}(y) = T_x X$ and $d_x f(H) = T_y Y$ (transversality in this case means that y is a regular value). If moreover we assume $\dim X = \dim Y$, then $f^{-1}(y)$ is a finite set of points (being a zero dimensional definable manifold) and f is a definable diffeomorphism on a neighbourhood of each $x \in f^{-1}(y)$. In this case the preimage orientation has the following simple description: each definably connected component of $f^{-1}y$ is just a single point $x \in f^{-1}(y)$ which is oriented with a plus sign iff $d_x f: T_x X \to T_y Y$ preserves the orientation.

8 Intersection theory

8.1 Intersection numbers

Let $f: X \to Y$ be a definable C^p-map between definable C^p-manifolds, with f transversal to a definable submanifold Z of Y. Assume that X, Y, Z are boundaryless, and $\dim X + \dim Z = \dim Y$. Then $f^{-1}(Z)$ is a definable zero dimensional manifold, hence (by o-minimality) a finite set of points, whose cardinality modulo 2 is called the **intersection number mod 2** of f and Z, written $I_2(f, Z)$. Suppose furthermore that X, Y and Z are definably oriented. Then each point in $f^{-1}(Z)$ has an orientation number ± 1 given by the preimage orientation, i.e. if $x \in f^{-1}(Z)$, then the orientation number $I(f, Z)_x$ is $+1$ if the equation $d_x f(T_x X) \oplus T_{fx} Z = T_{fx} Y$ holds as an equation between oriented spaces and -1 otherwise. The **intersection number** $I(f, Z)$ is the sum of these orientation numbers: $I(f, Z) := \Sigma_{x \in f^{-1}Z} I(f, Z)_x$. Later we shall extend the definition of $I(f, Z)$ to the case when f is not necessarily transversal to Z. The results below are stated for the oriented intersection theory. They remain true with the obvious modifications in the modulo 2 case.

Theorem 8.1 *Let X, Y, Z be definably oriented and definably compact C^p-manifolds with $p > 1$, Z a submanifold of Y, Z and Y boundaryless. Let* $\dim \partial X + \dim Z = \dim Y$. *If $f: \partial X \to Y$ is transversal to Z and extends to a definable C^p-map $F: X \to Y$, then $I(f, Z) = 0$.*

Proof. By the extension theorem 5.9 there is a definable C^{p-1}-map $G: X \to Y$ transversal to Z which is definably C^{p-1}-homotopic to F and $G_{|\partial X} = f$. By Proposition 5.2 $G^{-1}(Z)$ is a one-dimensional definable C^{p-1}-manifold, and being a closed submanifold of X it is definably compact. We give to $G^{-1}(Z)$ the preimage orientation and to $\partial(G^{-1}Z)$ the induced boundary orientation. By Theorem 7.4, the sum of the orientation numbers of $\partial(G^{-1}Z)$ is zero. Still by Proposition 5.2 we have $\partial(G^{-1}Z) = G^{-1}(Z) \cap \partial X = f^{-1}(Z)$, however this equation does not take into account the orientation. We must show that the sum of the orientation numbers of $f^{-1}(Z)$ is zero when this set is given the preimage orientation. Hence to finish the proof it suffices to show the following.

Claim 8.2

$$\partial(G^{-1}Z) = (-1)^{\operatorname{codim} Z}(f^{-1}Z)$$

We shall prove this equation in general, namely without assuming that $\dim \partial X + \dim Z = \dim Y$. So $f^{-1}(Z)$ can have an arbitrary dimension. To this aim let $x \in f^{-1}(Z) = \partial(G^{-1}Z)$ and fix H such that

$$H \oplus T_x(f^{-1}Z) = T_x(\partial X) \tag{1}$$

Then we also have

$$H \oplus T_x(G^{-1}Z) = T_xX \tag{2}$$

since the dimensions add up and the intersection $H \cap T_x(G^{-1}Z)$ is zero because it is included in $T_x(\partial X) \cap T_x(G^{-1}Z) = T_x(f^{-1}Z)$ (which has zero intersection with H). Moreover by Proposition 5.3, $d_xf(H) \oplus T_{fx}Z = d_xG(H) \oplus T_{fx}Z = T_{fx}Y$, which allows us to orient first the space $d_xf(H) = d_xG(H)$, and then H requiring that the map $d_xf_{|H} = d_xG_{|H}$ preserves the orientation. By definition of the preimage orientation (1) and (2) then hold as equations between oriented spaces. Since $x \in \partial(G^{-1}Z)$, we can consider an outward vector $v \in T_x(G^{-1}Z)$ which clearly is also an outward vector of T_xX. By definition of the boundary orientation, we have the oriented equations:

$$\langle v \rangle \oplus T_x(\partial(F^{-1}Z)) = T_x(F^{-1}Z) \tag{3}$$

and

$$\langle v \rangle \oplus T_x(\partial X) = T_xX \tag{4}$$

Adding $\langle v \rangle$ to (1) and H to (3) we obtain

$$\langle v \rangle \oplus H \oplus T_x(f^{-1}Z) = \langle v \rangle \oplus T_x(\partial X)$$

$$H \oplus \langle v \rangle \oplus T_x(\partial(G^{-1}Z)) = H \oplus T_x(G^{-1}Z)$$

where the right hand sides are both equal to T_xX as oriented spaces by (2) and (4). Since $H \oplus \langle v \rangle = (-1)^{\dim H} \langle v \rangle \oplus H$, the orientations of $T_x(f^{-1}Z)$ and $T_x(\partial(G^{-1}Z))$ differ by a sign $(-1)^{\dim H} = (-1)^{\mathrm{codim} Z}$. QED

Corollary 8.3 *Let $p > 1$, and let X, Y, Z be definably oriented definably compact boundaryless C^p-manifolds, with Z a submanifold of Y and* $\dim X + \dim Z = \dim Y$. *Suppose that $f, g \colon X \to Y$ be definably C^p-homotopic maps both transversal to Z. Then $I(f, Z) = I(g, Z)$.*

Proof. Let $I = [0, 1]$ and suppose that $F \colon I \times X \to Y$ is a C^p-homotopy between f and g. If we orient I in the positive direction $\partial(I \times X)$ is the union of $X_1 := \{1\} \times X$ and $X_0 := \{0\} \times X$ where the latter has the opposite of the product orientation ($X \times I$ is a definable manifold since X is boundaryless). Therefore, by Theorem 8.1, $0 = I(\partial F, Z) = I(F_{|X_1}, Z) - I(F_{|X_0}, Z) = I(g, Z) - I(f, Z)$ (identifying both X_0 and X_1 with X via orientation preserving diffeomorphisms $(0, x) \mapsto x$ and $(1, x) \mapsto x$) . QED

Definition 8.4 We can now **extend the definition of intersection number to non-transversal maps**. Let $p > 2$, and let X, Y, Z be definably oriented definably compact boundaryless C^p-manifolds, with Z a submanifold of Y and $\dim X + \dim Z = \dim Y$. Given $f: X \to Y$, by the transversality-homotopy theorem 5.6 there is a map g, C^{p-1}-homotopic to f and transversal to $Z \subset Y$. We define $I(f, Z) = I(g, Z)$ and by Corollary 8.3 this definition is independent of the choice of g.

Theorem 8.1 and Corollary 8.3 remain true in the non-transversal case if $p > 2$ (use Theorem 5.6).

9 Applications

9.1 Brouwer fixed point theorem

As a first application of the o-minimal version of Sard's theorem 3.5, the Preimage theorem 3.2, and the classification of definable one-manifolds (Theorem 6.5) we have, with essentially the same proof as in [GP]:

Theorem 9.1 *Any definable C^p-map ($p > 0$) $f: B^n \to B^n$, where $B^n \subset R^n$ is the closed unit ball, has a fixed point.*

Woerheide proves (see [Wo]) a stronger version of this fixed point theorem with "continuous" instead of "C^p".

9.2 Brouwer degree and winding numbers

As an application of the intersection theory we define the Brouwer degree and prove its invariance under definable C^p-homotopy for $p > 2$.

Proposition 9.2 (Stack of record's theorem) *Suppose that y is a regular value of a definable C^p-map $f: X \to Y$, between definable boundaryless C^p-manifolds of the same dimension with X definably compact. Then*

$f^{-1}(y)$ is a finite set $\{x_1, \ldots, x_n\}$ of cardinality n and there is a definably connected open neighbourhood V of y in Y such that $f^{-1}V$ is a disjoint union $U_1 \cup \ldots \cup U_n$ where U_i is an open definable neighbourhood of x_i and f maps each U_i C^p-diffeomorphically onto V.

Proof. Pick disjoint definable open neighbourhoods W_i of x_i which are mapped C^p-diffeomorphically onto open neighbourhoods $f(W_i)$ of y. Then $f(X \setminus (W_1 \cup \ldots \cup W_n))$ is definably compact, hence closed, and it does not contain y. Its complement V has the desired properties except that it may not be definably connected. If it is not, replace it with the definably connected component containing y. QED

Corollary 9.3 *Let $p > 2$. Let $f: X \to Y$ be a definable C^p-map between definably oriented boundaryless C^p-manifolds of the same dimension with Y definably connected and X and Y definably compact. Let each $y \in Y$ be positively oriented. Then the function $y \mapsto I(f, y)$, $y \in Y$, is constant.*

Proof. It suffices to show that $I(f, y)$ is a locally constant function of y (because then it must be globally constant since Y is definably connected). This follows from Remark 7.5, Proposition 9.2, and Corollary 7.2. QED

We define the **degree** $\deg(f)$ as the constant value of the function f given in the corollary. By Corollary 8.3 $\deg(f)$ is invariant under C^p-homotopies. By theorem 8.1 if f extends to a definable C^p-map on a definably compact definably oriented C^p-manifold whose boundary is X, then $\deg(f) = 0$.

If X is not oriented one can define the **degree mod 2** of f, written $\deg_2(f)$, in a similar way using intersection theory modulo 2, i.e. as the number modulo 2 of the counterimages of a regular value of f.

Definition 9.4 Let $p > 2$. Let $X \subset R^k$ be a boundaryless definably compact definable C^p-manifold and $z \in R^k \setminus X$. Let $f^z: X \to S^{k-1}$ be defined as $f^z(x) = \frac{x-z}{|x-z|}$. We define $W_2(X, z)$, the **winding number mod 2 of X around** z, as $\deg_2(f^z)$.

The winding number in the oriented case can be defined similarly.

9.3 Jordan-Brouwer separation theorem

Woerheide [Wo] developed o-minimal homology and proved the following version of the Jordan-Brouwer separation theorem.

Theorem 9.5 *If s is a definable subspace of S^n that is definably homeomorphic to S^{n-1}, then $S^n \setminus s$ has exactly two definably connected components and s is their common topological boundary.*

Assuming some differentiability the developments so far allows us to prove the following.

Theorem 9.6 *Let $p > 2$. The complement of a definably compact definably connected boundaryless C^p-manifold $X \subset R^k$ of codimension one, consists of exactly two definably connected open sets: D_0 (the outside) and D_1 (the inside). Moreover $\overline{D_0}$ and $\overline{D_1}$ are definable C^p-manifolds with boundary $\partial\overline{D_0} = \partial\overline{D_1} = X$ and $\overline{D_1}$ is definably compact.*

Proof. Let $D_0 = \{z \in R^k \setminus X \mid W_2(X,z) = 0\}$ and $D_1 = \{z \in R^k \setminus X \mid W_2(X,z) = 1\}$. Since the winding number is a locally constant function of z, D_0, D_1 are open. We first show that D_0 and D_1 are non-empty. Given $z \notin X$ and $v \in S^{k-1}$ let $r(z,v)$ be the ray emanating from z in direction v. Then $r(z,v) \pitchfork X$ if and only if v is a regular value of $f^z: X \to S^{k-1}$, $x \mapsto \frac{x-z}{|x-z|}$. Moreover the ray has non-empty intersection with X if and only if v is in the image of f^z. Since X has codimension one, for some z the image of f^z has non-empty interior. To see this, take $c \in X$, $z \notin c + T_cX$. Then by the inverse function theorem f^z maps an open definable neighbourhood of c in X onto an open definable set in S^{k-1}. Now by Sard's theorem there is a regular value v in the image of f^z. Hence $r(z,v)$ has a non-empty transversal intersection with X, necessarily finite. Now take two points z_0, z_1 on the ray and outside of X, such that the segment from z_0 to z_1 meets X in exactly one point. It is then easy to see that $W_2(X,z_0) \neq W_2(X,z_1)$ proving the non-emptiness of D_0 and D_1.

We next show that $R^k \setminus X$ has exactly two definably connected components, necessarily coinciding with D_0 and D_1. Consider a definably connected component C of $R^k \setminus X$. Then $\overline{C} \cap X$ is non-empty. By the Corollary 3.4 around any given point of $\overline{C} \cap X$ there is an open definable neighbourhood U in R^k such that $U \setminus X$ has exactly two definably connected components with $U \cap X$ as their common boundary. But then $U \cap X \subset \overline{C} \cap X$, showing that $\overline{C} \cap X$ is open in X, and therefore coincides with X since X is definably connected. Since this holds for every definably connected component of the complement of X, the same argument shows that there are at most two such components.

It only remains to show that $\overline{D_1}$ is definably compact. To see this it is enough to observe that if z is large enough, namely outside of a ball containing X, then $f^z\colon X \to S^{k-1}$ is not onto, and therefore has degree zero (being definably C^p-homotopic to a constant). QED

9.4 Lefschetz numbers

In the next proposition the assumption that X, Y, Z are boundaryless is needed to ensure that the relevant products are definable C^p-manifolds.

Proposition 9.7 *Let $f\colon X \to Y$ and $g\colon Z \to Y$ be definable C^p-maps between definable boundaryless C^p-manifolds with $\dim X + \dim Z = \dim Y$. Then $f \pitchfork g$ if and only if $f \times g\colon X \times Z \to Y \times Y$ is transversal to the diagonal $\Delta \subset Y \times Y$.*

Proof. Let $fx = gz = y \in Y$, let $A = d_x f(T_x X), B = d_z g(T_z Z), C = T_y Y$ and let $D \subset T_y Y \times T_y Y$ be the diagonal. The transversality of f and g (at x, z) is expressed by the equation $A \oplus B = C$. By linear algebra this is equivalent to $(A \times B) \oplus D = C \times C$, which in turn expresses the transversality condition $f \times g \pitchfork \Delta$ (at x, z). The result follows. (See [G] §3.3 for details.) QED

Let $f\colon X \to Y$ and $g\colon Z \to Y$ be transversal definable C^p-maps between definably oriented definably compact boundaryless C^p-manifolds with $\dim X + \dim Z = \dim Y$. Then

$$d_x f(T_x X) \oplus d_z g(T_z Z) = T_y Y \tag{1}$$

whenever $fx = gz = y$. Moreover $d_x f$ and $d_z g$ are injective, so they induce an orientation on $d_x f(T_x X)$ and $d_z g(T_z Z)$. If (1) holds as an equation between oriented spaces, we define $I(f,g)_{(x,z)} = +1$, otherwise $I(f,g)_{(x,z)} = -1$. We then define the **intersection number** $I(f,g)$ as the sum of all these local contributions $I(f,g)_{(x,z)}$. The sum is finite by Proposition 9.7 and Proposition 5.2. The next proposition allows us to extend the definition of $I(f,g)$ to the non-transversal case.

Proposition 9.8 *Let $f\colon X \to Y$ and $g\colon Z \to Y$ be transversal definable C^p-maps between definably oriented boundaryless C^p-manifolds with $\dim X + \dim Z = \dim Y$. Then $I(f,g) = (-1)^{\dim Z} I(f \times g, \Delta_Y)$ where $\Delta_Y \subset Y \times Y$ is the diagonal oriented so as to make the map $y \mapsto (y,y)$ orientation preserving.*

Proof. Note that $I(f \times g, \Delta_Y)$ is well defined by Proposition 9.7. In the notation of the proof of Proposition 9.7, we orient the diagonal $D \subset C \times C$ via its natural isomorphism with C. Taking into account the orientations we have: $A \oplus B = C$ if and only if $(-1)^{\dim B}(A \times B \oplus D) = C \times C$. The result follows. (See [G] §3.3 for details.) QED

Definition 9.9 Let $p > 2$, and let $f: X \to Y$ and $g: Z \to Y$ be definable C^p-maps between definably oriented definably compact boundaryless C^p-manifolds with $\dim X + \dim Z = \dim Y$. We **extend the definition of** $I(f,g)$ **to the non-transversal case** by setting $I(f,g) := (-1)^{\dim Z} I(f \times g, \Delta_Y)$.

This is well defined and invariant under definable C^p-homotopies. If g is the inclusion we have $I(f,g) = I(f,Z)$ and if f and g are inclusion maps we also write $I(X,Z)$ instead of $I(f,g)$.

Remark 9.10 In the hypothesis of Definition 9.9

$$I(f,g) = (-1)^{dimZ \cdot dimX} I(g,f).$$

Definition 9.11 Let $p > 2$. Given a definably oriented definably compact boundaryless C^p-manifold X and a definable C^p-map $f: X \to X$. We define the **(global) Lefschetz number of** f by $L(f) = I(\Delta_X, graph(f))$ with Δ_X and $graph(f)$ as definable subspaces of $X \times X$. Also we define the **self-intersection number of** X by $\Xi(X) = L(id_X)$.

Proposition 9.12 *Let $p > 2$. Let X be a definably compact definably oriented boundaryless C^p-manifold. Then the self-intersection number of X does not depend on the orientation on X. So $\Xi(X)$ is well defined for definably compact definably orientable C^p-manifolds.*

Proof. By definition $\Xi(X) = I(\Delta_X, \Delta_X) = I(f, \Delta_X)$ where $f: \Delta_X \to X \times X$ is a definable C^p-map homotopic to the inclusion $\iota: \Delta_X \to X \times X$ and transversal to Δ_X. Now $I(f, \Delta_X) = \Sigma_{(x,x) \in f^{-1}\Delta_X} I(f, \Delta_X)_{(x,x)}$ where $I(f, \Delta_X)_{(x,x)}$ is the sign needed to make the equation

$$d_{(x,x)} f T_{(x,x)} \Delta_X \oplus T_{(y,y)} \Delta_X = T_y X \times T_y X$$

hold as equation between oriented spaces, where $(y,y) = f(x,x)$.

Since f is homotopic to ι, x and y are in the same definably connected component of X. So if we change the orientation on X, the orientation on $T_{(x,x)}\Delta_X$ and $T_{(y,y)}\Delta_X$ either both change or both remain the same. The product orientation on $T_yX \times T_yX$ will not change. The result then follows. QED

In the classical case, namely when R is an expansion of the field of real numbers, $\Xi(X) = I(\Delta_X, \Delta_X)$ is the Euler characteristic of X (See [GP]). We conjecture that in the general case $\Xi(X)$ coincides with the model-theoretic Euler characteristic of X. As above, if $p > 2$, $L(f)$ is invariant under definable C^p-homotopies.

With the above definitions we trivially have:

Theorem 9.13 (Lefschetz Fix Point Theorem) *Let $p > 2$ and let X be a definably compact boundaryless definably oriented C^p-manifold. Let $f\colon X \to X$ be a definable C^p-map. If $L(f) \neq 0$, then f has a fixed point.*

Corollary 9.14 *Let $p > 2$ and let X be a definably compact boundaryless definably oriented C^p-manifold. If X admits a definable C^p-map $f\colon X \to X$ definably C^p-homotopic to the identity and without fixed points, then $\Xi(X) = 0$*

10 Embedding abstract-definable manifolds

In the previous sections we have considered manifolds that are submanifolds of some R^k. Here we give a more general definition (following [Pi,PeSt,PPS]).

10.1 Abstract-definable C^p-manifolds

An **abstract-definable C^p-manifold** (with respect to R) is a set M together with an abstract-definable C^p-atlas on M. An **abstract-definable C^p-atlas on M**, in turn, is a finite set (of **C^p-charts**) $\{(W_1, h_1), \ldots, (W_s, h_s)\}$ satisfying three conditions.

(i) $M = \bigcup_{i=1}^{s} W_i$;

(ii) there is an $m \in \mathbf{N}$ (**the dimension of** M), such that each h_i is a bijection from W_i onto a definable open subset of R^m; and

(iii) for each $i, j \in \{1, \ldots, s\}$, if $W_i \cap W_j \neq \emptyset$ then $U := h_i(W_i \cap W_j)$ is a definable open set and $(h_j \circ h_i^{-1})_{|U}$ is a definable C^p-diffeomorphism onto its image.

Two abstract-definable C^p-manifolds are **equal** if they have the same underlying set, M say, and the union of their C^p-atlas is again a C^p-atlas on M.

An abstract-definable C^p-manifold M can be equipped with a topology (**the manifold topology**) so that $W \subset M$ is open if and only if for every $i \in \{1, \ldots, s\}$, $h_i(W \cap W_i)$ is open in R^m. Hence $F \subset M$ is closed in M if for each $i \in \{1, \ldots, s\}$, $h_i(F \cap W_i)$ is closed in $h_i(W_i)$.

A subset $A \subset M$ is **abstract-definable** if for each C^p-chart (W_i, h_i), $h_i(A \cap W_i)$ is a definable subset of R^m.

Let M and N be abstract-definable C^p-manifolds, $f: M \to N$ a map. We say that f **is an abstract-definable map** (resp. **abstract-definable C^p-map, abstract-definable C^p immersion, abstract-definable C^p diffeomorphism**) if for any $x \in M$ and any charts (W, g) on M and (V, h) on N with $x \in W$ and $f(x) \in V$ the map $(h \circ f \circ g^{-1})_{|g(W) \cap f^{-1}(V)}$ is a definable map (resp. definable C^p-map, definable C^p-immersion, definable C^p-diffeomorphism). An abstract-immersion that is also a homeomorphism onto its image is called an **abstract-embedding** .

Remark 10.1 Note that we did not assume that M and the h_i's are definable. If they are, then an abstract-definable subset of M is a definable subset, similarly for abstract-definable maps. Moreover it is no loss of generality to assume that M and the h_i's are definable, since M is abstract-definable C^p-diffeomorphic to some M' satisfying these additional conditions:

Take as M' a definable quotient of $\bigcup_{i=1}^{s} \{i\} \times h_i(W_i)$ identifying $\langle i, u \rangle$ with $\langle j, h_j h_i^{-1}(u) \rangle$ for each $u \in h_i(W_i \cap W_j)$. Then there is a natural bijection h from M to M' and we give an abstract-definable manifold structure to M' so that h is an abstract-definable C^p-diffeomorphism.

10.2 The embedding theorem

For the rest of this section we fix an abstract-definable C^p-manifold M of dimension m with a finite abstract-definable C^p-atlas $\{(W_i, h_i)\}_{i \in I}$.

If $X \subset R^k$ is a definable boundaryless C^p-manifold, then X is an abstract-definable C^p-manifold (since reasoning as in [BCR] section 9.3, one can show that X has a finite family of definable local coordinate systems whose domains cover X), and its manifold topology coincides with the topology induced from R^k. We shall prove a partial converse. For this we need some preliminaries. First we observe that an abstract-definable C^p-manifold X

is $T1$ but, unlike definable C^p-manifolds, does not need to be a Hausdorff space: consider two copies of R with the strictly negative points identified.

We recall that M is **definably normal** if any two abstract-definable closed disjoint sets can be separated by abstract-definable open sets. The following notion was introduced in [PeSt].

Definition 10.2 We say that M **is definably compact** if for every $a, b \in R \cup \{-\infty\} \cup \{\infty\}$ with $a < b$, and for every abstract-definable map $\sigma\colon (a,b) \to M$, both $\lim_{x \to a^+} \sigma(x)$ and $\lim_{x \to b^-} \sigma(x)$ (with respect to the manifold topology) exist in M.

Lemma 10.3 *Let $a, b \in R \cup \{-\infty\} \cup \{\infty\}$ with $a < b$. Then any abstract-definable map $\gamma\colon (a,b) \to M$ is continuous with respect to the manifold topology on M, except at finitely many points.*

Proof. Consider the finite covering $M = \bigcup_{i \in I} W_i$ given by the atlas. We have $\gamma\colon (a,b) \to \bigcup_{i \in I} W_i$. By o-minimality, for each $i \in I$, $h_i \circ \gamma\colon \gamma^{-1}(W_i) \to h_i(W_i) \subset R^m$ is continuous outside a finite set $F_i \subset \gamma^{-1}(W_i)$, and by the definition of the manifold topology $\gamma\colon (a,b) \to M$ is continuous outside the finite set $F = \bigcup_{i \in I} F_i$. QED

Lemma 10.4 *Let M be definably compact and Hausdorff. Then M is definably normal.*

Proof. For each $i \in I$ and $x, y \in W_i$, let $d_i(x,y) := |h_i(x) - h_i(y)| \in R$. Let K_1, K_2 be disjoint abstract-definable closed sets in M. For $\varepsilon \in R$, $\varepsilon > 0$, we define K_1^ε as the set of points x such that for all $j \in I$ with $x \in W_j$ there is a point p in $K_1 \cap W_j$ with $d_i(x,p) < \varepsilon$. Since the index set I is finite, it is easy to see that K_1^ε is open (and contains K_1). Similarly we define K_2^ε. If for some $\varepsilon > 0$ K_1^ε and K_2^ε are disjoint we are done. Otherwise by definable choice and Lemma 10.3, there is a definable continuous function $a\colon (0,\varepsilon) \to X$ which assigns to each $\varepsilon > 0$ a point $a(\varepsilon)$ lying in $K_1^\varepsilon \cap K_2^\varepsilon$. Since M is definably compact the limit $a_0 = \lim_{\varepsilon \to 0} a(\varepsilon)$ exists and is unique since M is Hausdorff. We reach a contradiction showing that $a_0 \in K_1 \cap K_2$. Choose i such that $a_0 \in W_i$. Then for all sufficiently small $\varepsilon > 0$ we have $a(\varepsilon) \in W_i$ so $d_i(a(\varepsilon), K_1 \cap W_i)$ is well defined and must be less than ε since $a(\varepsilon)$ belongs to the ε-neighbourhood of K_1. This implies that $\lim_{\varepsilon \to 0} d_i(a(\varepsilon), K_1 \cap W_i) = 0$. On the other hand since $a(\varepsilon) \to a_0$ this limit must coincide with $d_i(a_0, K_1 \cap$

W_i), so $d_i(a_0, K_1 \cap W_i) = 0$, and therefore $a_0 \in K_1$. A symmetrical argument shows $a_0 \in K_2$ contradicting the disjointness of K_1 and K_2. QED

Lemma 10.5 *Let M be definably normal. Then there are abstract-definable open sets $O_i \subset W_i$ such that $M = \bigcup_{i=1}^{s} O_i$ and $\overline{O_i} \subset W_i$.*

Proof. By definable normality there is an abstract-definable open set O_1 with $M \setminus (W_2 \cup \ldots \cup W_s) \subset O_1 \subset \overline{O_1} \subset W_1$. So $M = W_2 \cup \ldots \cup W_s \cup O_1$. Repeating the procedure starting with this new open covering we arrive at the desired covering $M = O_1 \cup O_2 \cup \ldots \cup O_s$. QED

Lemma 10.6 *Let M be definably compact and $f: M \to R^N$ an injective abstract-definable map. If f is continuous (M with the manifold topology) then f is a homeomorphism on its image.*

Proof. By the way the manifold topology is defined, any open set in M is a union of abstract-definable open sets, because it is so in R^m. Therefore it suffices to prove that if C is an abstract-definable closed subset of M then $f(C)$ is closed in R^N. Suppose this is not the case and let $c \in \overline{f(C)} \setminus f(C)$. Let $\gamma: (0,1) \to f(C)$ be a definable continuous map with $\lim_{t \to 0} \gamma(t) = c$. Then $\sigma = f^{-1} \circ \gamma: (0,1) \to M$ has no left limit point in C, contradicting either the definable compactness of M or the closedness of C. QED

Theorem 10.7 (The Embedding Theorem) *Let M be a definably compact Hausdorff abstract-definable C^p-manifold of dimension m. Let the set $\{(W_1, h_1), \ldots, (W_s, h_s)\}$ be its atlas. Then M is abstract-definable C^p-diffeomorphic to a definable C^p-submanifold of R^{ms+s}.*

Proof. By Remark 10.1 we can assume that M and the h_i's are definable. By Lemma 10.6 it suffices to define an injective abstract-definable C^p-immersion between abstract-definable manifolds from M to R^{ms+s}. By Lemma 10.5 we can choose abstract-definable open sets O_i with $\overline{O_i} \subset W_i$ such that $M = \bigcup_i O_i$. By Lemma 10.6, $h_i(\overline{O_i})$ is definably compact in R^m, hence closed. Therefore we can apply Corollary 5.8 to get a definable C^p-map $\delta_i: R^m \to R$ with $0 \leq \delta \leq 1$, $\delta_i(z) = 1$ if $z \in h_i(\overline{O_i})$ and $\delta_i(z) = 0$ if $z \notin h_i(W_i)$. Then we define $\psi_i: M \to [0,1]$ by $\psi_i = \delta_i \circ h_i$ on W_i and 0 outside W_i. We prove that the function $f := (\psi_1 \cdot h_1, \ldots, \psi_s \cdot h_s, \psi_1, \ldots, \psi_s): M \to R^{ms+s}$ is is the required embedding. Note that f is well defined: although the h_i are only defined on W_i , the products $\psi_i \cdot h_i$ are defined on the whole of M because $\psi_i = 0$ outside W_i. We show that f is injective. Suppose $f(x) = f(y)$ and $x \in O_i$. Then $1 = \psi_i(x) = \psi_i(y)$, so $y \in \overline{O_i}$. Moreover $\psi_i(x)h_i(x) = \psi_i(y)h_i(y)$, so $x = y$ and f is injective. To prove that f is an immersion take $x \in M$, $x \in O_1$ say. We must show that $f \circ h_1^{-1}: h_1(O_1) \to R^{ms+s}$ is an immersion, and this is clear since this map sends an element $z \in h(O_1) \subset R^m$ into an element of R^{ms+s} whose first m coordinates are those of z. QED

The notions of tangent space and differential of a map can be defined for abstract-definable C^p-manifolds as in [PPS], so one can define in the obvious way the concept of orientation for abstract-definable C^p-manifolds.

Definition 10.8 Let M be a Hausdorff definably compact abstract-definable C^p-manifold. By Theorem 10.7 there is an abstract-definable C^p-diffeomorphism $f: M \to X$ where X is a definable C^p-manifold in some R^k. Then M is **abstract-orientable** if and only if $X = f(M)$ is definably orientable. In this case we define $\Xi(M)$ as $\Xi(X) = I(\Delta_X, \Delta_X)$. This definition does not depend on the choice of the definable diffeomorphism f and the orientation we put on X.

11 Applications to definable groups

A definable group is a definable set $G \subset R^k$ equipped with a definable group operation.

Theorem 11.1 *Let G be a definable group. Then, for each $p \in \mathbf{N}$, G can be equipped with abstract-definable C^p-atlas making G an abstract-definable C^p-manifold with a C^p definable group operation. Moreover any two such atlases on G give the same abstract definable C^p-manifold.*

Proof. See [Pi] and also [OPP]. QED

Proposition 11.2 *Let X be an infinite definable C^p-manifold which has a definable C^p group operation. Then X is definably orientable and boundaryless.*

Proof. Let e be the neutral element. Fix an orientation of T_eX. Given $x \in X$ we orient T_xX as follows. Consider the definable C^p-diffeomorphism

$\lambda^x: X \to X$ sending y to xy. Orient T_xX so that $d_e\lambda^x: T_eX \to T_xX$ is orientation preserving. This gives an orientation on X. In fact, to check that the orientation is locally constant, consider a local chart (W, h) around x and a positive basis B_e on T_eX. Let B_x be the basis of R^m image of B_e under $d_e(h \circ \lambda^x)$. Then the function $x \mapsto B_x \in R^{m^2}$ is continuous, so for y sufficiently close to x the linear map sending B_x to B_y is orientation preserving. To check that X is boundaryless just note that X admits a definable C^p-diffeomorphism sending a given point to any other given point. QED

Definition 11.3 Given a group G definable in an o-minimal estructure, we say that G is a **definably compact group** if it is so with respect to the abstract-definable manifold topology given by the previous theorem.

Theorem 11.4 *Let G be an infinite definable group which is also definably compact. Then $\Xi(G) = 0$.*

Proof. By Theorem 11.1 G is an abstract-definable C^p-manifold with a C^p group operation, $p > 2$. Being an abstract-definable manifold and a group, G is Hausdorff. Since G is definably compact, by Theorem 10.7, G is abstract-definably C^p-diffeomorphic to a definably compact C^p-manifold X is some R^k, which has a definable C^p group operation. By Proposition 11.2 X is definably orientable and boundaryless. Being a non-trivial group X admits a definable C^p-map onto itself definably C^p-homotopic to the identity and without fixed points (multiplication by an element $x \neq e$ in the same definably connected component of e). By Corollary 9.14 $\Xi(X) = 0$, hence $\Xi(G) = 0$. QED

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