Functions with distant fibers and uniform continuity *

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Abstract

The uniformly approachable functions introduced in [DP] are defined by a property stronger than continuity and weaker than uniform continuity, which is preserved under composition. (So they give rise to a category which sits between the category of metric spaces with all continuous functions and the category of metric spaces with all uniformly continuous functions.) Solving a problem left open in [BeDi], we give a complete characterization of the polynomial maps $f: \mathbf{R}^n \to \mathbf{R}$ which are uniformly approachable. They coincide with the polynomial maps f with distant fibers, i.e., such that any two distinct fibers $f^{-1}(x)$ and $f^{-1}(y)$ are at positive distance. The same holds more generally for any real valued function on $\mathbf{R}^n$ whose fibers have finitely many connected components. To prove this we show that every real valued continuous functions with distant fibers on a uniformly locally connected metric space is uniformly approachable, and any (weakly) uniformly approachable function on $\mathbf{R}^n$ has "distant connected components of fibers".

We observe that a bounded continuous function $f: \mathbf{R}^n \to \mathbf{R}$ has distant fibers if and only if it is uniformly continuous. This suggests that for a reasonable metric space X the uniform continuity of a bounded continuous function $f: X \to \mathbf{R}$ depends only on the fibers of f. We show that this is the case when X is connected and locally connected.

A useful tool in the study of uniformly approachable functions on domains more general than $\mathbf{R}^n$ is given by the technique of "truncations" (g is a truncation of f if it is locally constant where it differs from f). On $\mathbf{R}^n$ the functions with many uniformly continuous truncations coincide with the functions with distant connected components of fibers. We improve the technique of the magic set introduced in [BeDi] and studied in [B2, BC1, BC2, CS] showing that every continuous function with "small fibers" on a locally arcwise connected metric space X has a magic set $M \subset X$ (i.e. every continuous $g: X \to \mathbf{R}$ with $g(M) \subset f(M)$ is a truncation of f).

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1 Introduction

The study of closure operators (in the sense of [DT, DP, DTW], but see also [C, B1]) led to the consideration of a class closed under composition of functions between metric spaces (or uniform spaces) which have a property stronger than continuity and weaker than uniform continuity. Such functions were introduced in [DP, Definition 4.1] and were called there "uniformly approachable". Following [BeDi, Definition 2.1] we will call them "**weakly uniformly approachable**" (**WUA**) since in [BeDi] the name "**uniformly approachable**" (**UA**) has been reserved for a stronger notion.

The definition of the WUA functions is brief but quite mysterious: *a function* $f: X \to Y$ *between two metric (or uniform) spaces is* WUA *if for every point* $x \in X$ *and every subset* $M \subseteq X$*, there exists a uniformly continuous* $g: X \to Y$ *such that* $gx = fx$ *and* $g(M) \subseteq f(M)$. It is easy to see that f must necessarily be continuous

[DP, Lemma 4.2]. It is also clear that a uniformly continuous function f is WUA since we can take $g = f$. Besides these obvious remarks, the quantification over arbitrary subsets of the space makes it difficult to develop a proper intuition for WUA-functions. Developing such an intuition is one of the goals of this paper. It was already proved in [BeDi, Theorem 5.2] that each proper function $f: \mathbf{R}^n \to \mathbf{R}$ is WUA (even UA). So the WUA functions generalize at the same time the proper functions and the uniformly continuous ones. The UA functions are defined similarly to the WUA functions except that g is required to coincide with f on a compact set K rather than just at a point x [BeDi, Definition 2.1]. The motivation for this strengthening is that UA functions are better behaved under some kinds of unions (see [BeDi, Theorem 11.1]) and are somehow easier to study. We do not know whether $WUA = UA$ for all functions $f: \mathbf{R}^n \to \mathbf{R}$.

The study of the uniformly approachable functions has proven to be a fruitful source of problems both of topological and of set-theoretic character, mainly concerning the possible behaviour of the fibers of a continuous real valued function. Some of these questions, and in particular the first question in [BeDi], have been proved to be independent of the axioms ZFC of set theory [BC1, CS]. To discuss these developments let us consider more closely the definition of WUA-function. At first sight, $f \in WUA$ is a very weak condition on f. Given x and M we must look for a uniformly continuous function g with $gx = fx$, which is certainly a very weak requirement on g, and $g(M) \subseteq f(M)$. If $x \in M$, or more generally if $fx \in f(M)$, we could simply take a constant function g with value fx. So the problem is when $fx \notin f(M)$, which rules out the constant functions g. Even in this case, if M is not chosen carefully, the inclusion $g(M) \subseteq f(M)$ may not appear to be a strong condition on g. However by choosing M carefully, we will see that $g(M) \subseteq f(M)$ can give us quite strong restrictions on g. Before establishing this, one cannot even rule out that WUA coincides with the class of all continuous functions. Actually this is indeed the case for functions on $\mathbf{R}$: every continuous $f: \mathbf{R} \to \mathbf{R}$ is WUA [DP] and even UA [BeDi, Proposition 3.5]. The first example of a non-WUA function was found by M. Burke [B]: he showed that the function $f: (x, y) \mapsto xy$ from $\mathbf{R}^2$ to $\mathbf{R}$ is not WUA. This can be witnessed by the subset $M = \bigcup_n f^{-1}(1/n)$ and the point $x = (0, 0)$.

From the above discussion, it is clear that the main problem in understanding the WUA functions is how to choose the set M needed to witness that a given function f is not WUA. The strongest information that we can hope to deduce from $g(M) \subseteq f(M)$ is $f = g$, but this is a bit too much: g could be a constant function. In [BeDi], it is shown that under some assumption on the fibers of f one can construct a set M such that from $g(M) \subseteq f(M)$ one can deduce that g is a 'truncation' of f. The notion of truncation is an important tool introduced in [BeDi, Definition 4.1]: $g \in C(X)$ is a **truncation** of $f \in C(X)$ if the space X can be partitioned in two parts $X = A \cup B$ so that $g = f$ on A and g is constant on each connected component of B. Typical truncations arise when we define a function by cases: so if we set $gx = x^2$ for $|x| < 1$ and $gx = 1$ for $|x| \geq 1$, then g is a truncation of the function x^2. Any function $f \in C(\mathbf{R})$ can be made uniformly continuous by truncating it in various ways, as we just did for the function x^2. However on $\mathbf{R}^2$ there are functions, like $(x, y) \mapsto xy$, which have no non-constant uniformly continuous truncations while some others, like $x^2 + y^2$ have many uniformly continuous truncations. A naive conjecture is that the ones with many uniformly continuous truncations are WUA, while the others are not. To state it more precisely let us say that a function $f \in C(X)$ is **truncation-UA** (briefly **TUA**) if the following holds: *for every compact set $K \subseteq X$ there is a uniformly continuous truncation g of f which coincides with f on K*. It is easy to see that $(x, y) \mapsto xy$ is not TUA. The equality $WUA = UA = TUA$ seems to holds for a large class of functions on $\mathbf{R}^n$, so that one can naturally conjecture that it holds for all functions on $\mathbf{R}^n$. However very recently [CD2] found an example based on the Cantor function that shows that this is not the case (it was known that for functions on some non-separable spaces the equality $UA = TUA$ is false [BDP]). The first important result about truncations is [BeDi, Theorem 8.1]: *Let X be a separable topological space. Then there is a set $M \subseteq X$ such that for every $f, g \in C(X)$, if each fiber of f is countable and $g(M) \subseteq f(M)$, then g is a truncation of f.* This result was used to show that certain functions defined on certain graphs embedded in $\mathbf{R}^2$ are not WUA [BeDi, Proposition 10.1]. (Consequently any extension to the whole of $\mathbf{R}^2$ of such a function is not WUA.) One drawback of this result is that the assumption that f has countable fibers is very strong: it is reasonable if f is a function on $\mathbf{R}$, but it cannot be satisfied if f is a function on $\mathbf{R}^n$ for $n1$ (however we can still try to apply the technique to the restriction of f to a subspace). If one assumes the continuum hypothesis one can get the following nice result where the assumptions on the fibers of f have been weakened [BeDi, Theorem 8.5]: *Let X be a separable Baire space and assume that the continuum hypothesis holds. Then there is a subset $M \subseteq X$ such that for every pair of continuous nowhere constant real valued functions f, g on X, if $g(M) \subseteq f(M)$ then $f = g$.* Note that to get the strong conclusion $f = g$ we make an assumption also on the fibers of g (without this assumption we can only deduce that g is a truncation of f). Such a set M was called a **magic set for** X in [BeDi]. (Note that M depends only on the space X and not on f.) In [BeDi] the question whether the continuum hypothesis was necessary to prove the existence of a magic set was left open. It was noted in [BC1] that sets M with the weaker property

$$f(M) = g(M) \Rightarrow f = g \text{ whenever } f \text{ and } g \text{ are nowhere constant}$$

had been considered in [DPR] for entire functions $f, g : \mathbf{C} \to \mathbf{C}$, and were called **sets of range uniqueness** (briefly, **SRU**). The following curious question is considered in [DPR]: while it is obvious that for a converging sequence $a_n \to 0$ in $\mathbf{C}$, the equalities $f(a_n) = g(a_n)$ (for $n = 1, \ldots, n, \ldots$) yield $f = g$, setting $M := \{a_n : n \in \mathbf{N}\}$, it is not clear when $f(M) = g(M)$ yields $f = g$. In the same paper, it was shown that this occurs when $a_n = 1/n$, but there are examples of converging sequences such that the set M fails to have this property, i.e., M is not SRU for the class of the entire functions. In [BC1, Example 5.17], using a result of [S], it is shown that it is consistent with ZFC to assume that there is a separable Baire space $X \subseteq \mathbf{R}$ with no SRU for the class $C(X)$ of the real valued nowhere constant continuous functions on X, so a fortiori there is no magic set for X. It was left open whether a magic set for $\mathbf{R}$ could be proved to exist in Zermelo Fraenkel set theory (ZFC), i.e. whether the continuum hypothesis could be avoided. That this is not the case was later shown by Ciesielski and Shelah [CS].

In section 6 of this paper, we show that the technique of the magic set can be slightly modified to prove that UA implies TUA for functions $f \in C(X)$ satisfying a reasonable smallness condition on the fibers (Definition 6.2), without assuming the continuum hypothesis. Here X is only assumed to be a locally arcwise connected separable space.

The main contribution of this paper however is that for $X = \mathbf{R}^n$ we can dispense with the magic set, using completely different techniques, based on the "unicoherence" of the topological space $\mathbf{R}^n$ (see Definition 4.7 or [K]), to obtain much stronger results about WUA and UA functions.

Let us start with the following observation (see Theorem 3.7): *A bounded continuous function $f: X \to \mathbf{R}$ on a "uniformly locally connected space" X, is uniformly continuous if and only if it has "**distant fibers**" (**DF**), in the sense that any two fibers $f^{-1}(x)$, $f^{-1}(y)$ of f are at positive distance.* (Without uniform local connectedness there are counterexamples: e.g. the arclength function on the circle minus one point.) So in particular if two bounded functions on a uniformly locally connected space have the same fibers (in the sense that $f^{-1}(fx) = g^{-1}(gx)$ for every x) then one is uniformly continuous iff the other is uniformly continuous. This last statement remains true for real valued functions on a connected and locally connected space (Theorem 3.10), but we cannot assert in this case that the relevant property of the fibers is DF. We will prove that for functions on a uniformly locally connected space, DF implies UA (Theorem 3.15). Clearly, any proper function is DF since it has compact fibers, so proper functions on a uniformly locally connected space are UA (as already proved in [BeDi]). The implication $DF \to UA$ is strict, however we can easily show that a UA function (on any space) necessarily has **"distant connected components of fibers"** (**DCF**) in the sense that any two components of distinct fibers are at positive distance. So

$$DF \to UA \to DCF$$

for functions on a uniformly locally connected space, and we are close to a characterization of UA-functions. For functions whose fibers have finitely many connected components, so for instance for polynomials, $DF = DCF$ coincide, so for such functions $DF = UA = DCF$. An example of a function which is DCF without being DF is $x \mapsto \sin(x^2)$.

The treatment of WUA functions is more complicated. For real valued functions on $X = \mathbf{R}^n$, we prove that $WUA \to DCF$. So over $\mathbf{R}^n$ we have

$$DF \to UA \to WUA \to DCF.$$

Here we use the "unicoherence" [K] of $\mathbf{R}^n$: every pair of closed connected sets whose union is $\mathbf{R}^n$ has a connected intersection. The only other properties that we use are the uniform local connectedness and the fact that every compact set is contained in a compact one.

This yields an enlightening proof of the fact that the polynomial map $(x, y) \mapsto xy$ is not UA: it depends on the fact that it is not DCF. We can show that $DCF = TUA$ over $\mathbf{R}^n$ and (despite the counterexample of [CD2]) we do believe that over $\mathbf{R}^n$ UA is much closer to DCF than to DF.

So far we have only considered real valued functions. However the definitions of WUA and UA functions can be extended in the obvious way so that they apply to arbitrary functions between uniform spaces and in this more general setting they are studied in [CD1] (focusing on complex valued functions). Even the simplest analytic perfect mappings $f : \mathbf{C} \to \mathbf{C}$, e. g. $f(z) = z^2$ (or any complex polynomial functions of degree greater than 1), do not have to be WUA [CD1, Example 4.1]. These authors conjecture that an entire analytic function $f : \mathbf{C} \to \mathbf{C}$ is WUA if and only if f is linear ([CD1, Conjecture 4.3]).

The main problem remains the following. We know that UA and WUA form a class of functions between metric spaces (or uniform spaces) which is closed under composition and thus yield a category which is intermediate between the category of continuous and the category of uniformly continuous functions. Since the definition of UA and WUA is highly non-intuitive, the problem arises whether we can characterize this category in some other way, as we tried with partial success using the properties "distant fibers" (DF) or "distant connected components of fibers" (DCF).

This paper is entirely dedicated to one application of the uniformly approachable functions, namely the description of the relationship between distant fibers and uniform continuity. Further applications will be given in the forthcoming paper [BDP] (see also [BeDi, DP] for applications to the theory of closure operators).

2 Definitions and preliminary results

We deal almost always with continuous functions $f: X \to Y$ on metric spaces X, Y, although most definitions and results generalize immediately to the case of uniform spaces. In most cases, we take $Y = \mathbf{R}$. We write $C(X, Y)$ for the set of continuous functions $f: X \to Y$ and $C(X)$ for $C(X, \mathbf{R})$. We use the abbreviation "**UC**" or "u.c." for uniformly continuous.

Definition 2.1 We say that $f \in C(X, Y)$ is **UA** (**uniformly approachable**), if for every compact set $K \subseteq X$ and every set $M \subseteq X$, there is a UC function $g \in C(X, Y)$ which coincides with f on K and satisfies $g(M) \subseteq f(M)$. We then say that g is a **(K, M)-approximation of f**. If we require in the definition of UA that K consists of a single point we obtain the weaker notion **WUA** (**weakly UA**). Clearly, UA implies WUA.

Lemma 2.2 *([CD1]) WUA and UA functions are closed under composition.*

Corollary 2.3 *If the restriction of f to a subspace is not UA, then f is not UA. Similarly for WUA.*

Proof. If $f \in C(X)$ is UA and L is a subspace of X, then the restriction of f to L is given by the composition of the inclusion map $L \to X$ with f. We can now apply the previous lemma together with the observation that the inclusion map is UA. Similarly for WUA. QED

We recall Katětov's theorem: if X is a uniform space, F is a subset of X, and $[a, b]$ is a compact interval of $\mathbf{R}$, then any u.c. function $f: F \to [a, b]$ can be extended to a u.c. function $g: X \to [a, b]$ (see [P, p. 52]). If X is metric the extension g can be easily constructed using the distance function.

Lemma 2.4 *([BeDi, Proposition 2.3(2)]) If $K, M \subseteq X$, K is compact and $K \cap \overline{M} = \emptyset$, then every $f \in C(X)$ has a (K, M)-approximation.*

Proof. Suppose $K \cap \overline{M} = \emptyset$. If $M \neq \emptyset$ take any point $m \in M$ and set $g_1(\overline{M}) = f(m)$, $g_1(x) = f(x)$ for each $x \in K$. The function $g_1 : K \cup \overline{M} \to \mathbf{R}$ is uniformly continuous. Now Katětov's theorem allows us to extend g_1 to a u.c. function $g : X \to \mathbf{R}$ which is obviously a (K, M)-approximation of f. If $M = \emptyset$ apply Katětov's theorem to $f|_K$. QED

3 Functions with distant fibers are UA

The **distance between two subsets** of a metric space is defined as the infimum of the distances of the pair of points taken one from the first subset and the other from the second subset. Here we are abusing the term "distance" since the triangle inequality does not necessarily hold.

Definition 3.1 We say that $f \in C(X)$ has **distant fibers** (**DF**) if any two distinct fibers $f^{-1}(x)$, $f^{-1}(y)$ of f are at positive distance.

Any function with compact fibers is DF. So in particular any proper function is DF (f is proper if the f-counterimage of any compact set is compact).

In this section we show that every DF function $f \in C(X)$ is UA, provided X is **uniformly locally connected** [HY, 3-2], i.e., for every $\varepsilon 0$ there is $\delta 0$ such that any two points at distance $< \delta$ lie in a connected set of diameter $< \varepsilon$. We actually use the following equivalent form of uniform local connectedness.

Lemma 3.2 *A metric space X is uniformly locally connected iff for every pair of sequences x_n, y_n in X with $d(x_n, y_n) \to 0$ there exist a $n_0 \in \mathbf{N}$ and connected sets $I(x_n, y_n)$ containing x_n and y_n for every $n \geq n_0$ in such a way that the the diameter of $I(x_n, y_n)$ tends to zero (we write diam $I(x_n, y_n) \to 0$).*

Any convex subset of $\mathbf{R}^n$ with the usual metric is uniformly locally connected: we can take the straight segments $[x_n, y_n]$ as connecting sets $I(x_n, y_n)$. An example of a space homeomorphic to $\mathbf{R}$ that is not uniformly locally connected is the circle minus one point with the metric induced by $\mathbf{R}^2$. Further examples of uniformly locally connected spaces are provided by the following remark.

Remark 3.3 If X is a **uniform neighbourhood retract** in $\mathbf{R}^n$ (i.e., there is $\varepsilon 0$ and a uniformly continuous retraction $r: X^\varepsilon \to X$ where $X^\varepsilon = \{y \in \mathbf{R}^n \mid d(y, X) < \varepsilon\}$), then X is uniformly locally connected. Indeed, assume $d(x_n, y_n) \to 0$ where $x_n, y_n \in X$ ($n \in \mathbf{N}$). Using the vector space structure of $\mathbf{R}^n$ let I_n be the segment joining x_n and y_n. Then $\text{diam}(I_n) = d(x_n, y_n) \to 0$ where $\text{diam}(I_n)$ is the diameter of I_n. If X is convex I_n is contained in X and we are done. In the general case, we can find $0 < \varepsilon$ and a uniformly continuous retraction $r: X^\varepsilon \to X$. By choosing a subsequence we can assume $d(x_n, y_n) < \varepsilon$ for every n, so that $I_n \subseteq X^\varepsilon$ and it makes sense to consider the connected set $r(I_n) \subseteq X$. Since r is u.c., diam $(r(I_n))$ can be uniformly bounded in terms of $\text{diam}(I_n)$.

Before proving $DF \to UA$ we introduce the auxiliary notion AP (a weakening of the notion of proper function) and show that $DF = AP$ for functions $f \in C(X)$ on a uniformly locally connected space X.

Definition 3.4 $f \in C(X, Y)$ is AP (**almost proper**) if f is u.c. on the f-counterimage of every compact set.

Note that for bounded functions AP coincides with u.c.

Lemma 3.5 *$DF \to AP$ for functions $f \in C(X)$ on a uniformly locally connected space X.*

Proof. Suppose that f is not AP. Then there is $\delta 0$ and points $x_n, y_n \in X$ $(n \in \mathbf{N})$ such that $d(x_n, y_n) \to 0$, $|fx_n - fy_n|\delta$ and the sequences $f(x_n)$ and $f(y_n)$ are bounded. By taking a subsequence we can assume that fx_n converges to some $a \in \mathbf{R}$ and fy_n converges to some $b \neq a$. Without loss of generality $a = \lim_n fx_n < b = \lim_n fy_n$. Choose $u, v \in \mathbf{R}$ with $a < u < v < b$. Taking subsequences we can assume $fx_n < u < v < fy_n$ for every n. Fix connected sets I_n joining x_n and y_n for each n such that $\mathrm{diam}(I_n) \to 0$. On the connected set I_n, the function f takes a value greater than v (at y_n) and a value smaller than u (at x_n), so it must also take the values u and v. Hence $d(f^{-1}(u), f^{-1}(v)) = 0$ and f is not DF. QED

Lemma 3.6 *$AP \to DF$ on every space. Consequently, $AP = DF$ on uniformly locally connected spaces.*

Proof. If f has two distinct non empty fibers $f^{-1}(y_1)$ and $f^{-1}(y_2)$ at distance zero, then f is not u.c. on the counterimage of any compact set containing y_1, y_2. QED

Since for bounded functions AP coincides with u.c., it follows:

Theorem 3.7 *A bounded function $f \in C(X)$ on a uniformly locally connected space X is u.c. if and only if it is DF.*

It is important in the above result that the range is $\mathbf{R}$: the function $f\colon \mathbf{R} \to \mathbf{R}^2$ defined by $f(x) = (\arctan x, \sin x^2)$ is bounded and injective (so DF) but it is not u.c. The above result may fail if the X is not uniformly locally connected: take the arclength function on the circle minus one point.

Corollary 3.8 *A bounded function $f \in C(\mathbf{R}^n)$ with compact fibers is u.c. In particular, every bounded injective function $f \in C(\mathbf{R})$ is u.c.*

We say that f and g **have the same fibers** if for every x we have $f^{-1}(fx) = g^{-1}(gx)$, i.e., each fiber of f is a fiber of g (possibly with a different value). Theorem 3.7 implies that for bounded functions $f \in C(X)$ on a uniformly locally connected space X the fact that f is u.c. depends only on the fibers of f, i.e., if two such functions have the same fibers, one is u.c. iff the other is u.c. We prove that this is still true replacing the hypothesis that X is uniformly locally connected with the assumption that X is connected and locally connected.

Lemma 3.9 *Let X be a connected and locally connected regular space and let A, B, D be closed subsets of X such that D separates A and B (see Definition 4.8). Then A does not separate B and D.*

Proof. Choose a covering of X by a family of connected open sets such that no set of the family intersects both A and D. Since X is connected there is a finite sequence of sets $U_1, \ldots, U_n$ from the family such that U_1 intersects A, U_n intersects B and each U_i intersects only the U_j's with $|i - j| \leq 1$ ([HY, Theorem 3-4]). Let k be the largest index such that U_k intersects A. Since D separates A and B, the union $U_k \cup U_{k+1} \cup \ldots \cup U_n$ must intersect D. Since U_k cannot intersect both A and D, there is lk with U_l intersecting D. The set $U_l \cup U_{l+1} \cup \ldots \cup U_n$ connects D and B and does not intersect A, so A does not separate B and D. QED

Theorem 3.10 *Let (X, d) be a connected and locally connected metric space. Suppose $f, g \in C(X, [0,1])$ have the same fibers and f is u.c. Then also g is u.c.*

Proof. Suppose g is not u.c. Since g is bounded there are points a_n and b_n in X ($n \in \mathbf{N}$) with $d(a_n, b_n) \to 0$ and $\inf_n g(b_n) \sup_n g(a_n)$. Since f is bounded we can assume that $L = \lim_n f(a_n)$ exists, and since f is u.c. $L = \lim_n f(b_n)$. Choose $d \in X$ such that $\inf_n g(b_n) g(d) \sup_n g(a_n)$. Since f, g have the same fibers, if $g(d) \neq g(d')$, then $f(d) \neq f(d')$. So by changing d we can further assume that $f(d) \neq L$. Fix n so large that $f(d)$ is either smaller than both $f(a_n)$ and $f(b_n)$ or larger than both, and $g(d)$ lies between $g(a_n)$ and $g(b_n)$. Assume for instance $fd < fa_n < fb_n$, the other cases being similar. The set $D = g^{-1}(gd)$ separates $A = g^{-1}(ga_n)$ and $B = g^{-1}(gb_n)$. On the other hand the set $f^{-1}(fa_n)$ separates $f^{-1}(fb_n)$ and $f^{-1}(fd)$. But these three sets coincide with A, B, D in the given order, so A separates B and D, contradicting Lemma 3.9. QED

Remark 3.11 For unbounded functions the theorem is false: take the polynomial functions x and x^3.

Question 3.12 What spaces can be substituted for $[0,1]$ as the range space in Theorem 3.10? What about the circle?

Having proved $DF = AP$, it remains to prove $DF \to UA$ on uniformly locally connected spaces X. So we need to define the relevant (K, M)-approximation of an AP function $f \in C(X)$. We will take, for the desired approximation, a "truncation of f" in the sense of the following definition.

Definition 3.13 Let $f \in C(X)$ and $a, b \in \mathbf{R}$ with $a \leq b$. The (a, b)**-truncation** of f is the bounded function $f_{(a,b)}$ which coincides with f on the f-counterimage of $[a, b]$, has value a whenever f has value $\leq a$, and has value b whenever f has value $\geq b$.

Lemma 3.14 *Let X be a uniformly locally connected space and let $f \in C(X)$ have DF. Then for every $a < b$ in $\mathbf{R}$, the (a, b)-truncation of f has DF (and so is u.c., being a bounded function).*

Proof. Let $g := f_{(a,b)}$. If g is not DF, we can find $u < v$ in $\mathbf{R}$ and two sequences in X with $d(x_n, y_n) \to 0$ and $g(x_n) = u < v = g(y_n)$. Since X is uniformly locally connected, for n large enough x_n and y_n are contained in a connected set I_n with $\text{diam}(I_n) \to 0$. Now $g(I_n)$ is connected, so it contains the whole interval $[u, v]$. So by replacing x_n and y_n with other two points x'_n and y'_n inside I_n (so as to assure that $d(x'_n, y'_n)$ still tends to zero) we can arrange so that u and v are different from a and from b. But then $g = f$ on the new sequences, so that f is not DF, a contradiction. QED

Theorem 3.15 *$DF \to UA$ for functions $f \in C(X)$ on a uniformly locally connected space X.*

Proof. Let $K \subseteq X$ be compact and $M \subseteq X$. We must find a (K, M)-approximation g of f. Let $[a, b]$ be a compact interval containing $f(K)$. If there are points of $f(M)$ smaller than or equal to a, let $a' \leq a$ be such an element, otherwise let $a' = a$. Similarly if there are points of $f(M)$ greater than or equal to b, let $b' \geq b$ be such an element, otherwise let $b' = b$. Take the (a', b')-truncation g of f. Clearly $g_{|K} = f_{|K}$ and $g(M) \subseteq f(M)$. By Lemma 3.14 g is DF, hence u.c. (being bounded). This proves that f is UA. QED

We will later prove (Theorem 6.6) that the assumption that X is uniformly locally connected cannot be omitted.

Remark 3.16 (a,b)-truncations were introduced in [BeDi], where it was proved that (a,b)-truncations of a perfect function $f : \mathbf{R}^n \to \mathbf{R}$ are u.c. [BeDi, Lemma 5.1]. It was also mentioned there that $\mathbf{R}^n$ can be replaced by any uniformly locally connected metric space.

4 UA-functions have distant connected component of fibers

4.1 UA implies DCF

We have seen that if X is uniformly locally connected, any function $f \in C(X)$ with distant fibers is UA. The converse does not hold: any continuous function on $\mathbf{R}$ is WUA [DP] and even UA [BeDi, Proposition 3.5] but does not necessarily have distant fibers (take the function $\sin(x^2)$). We prove however the following weak form of the converse: any UA function has "distant connected components of fibers," as will be explained below.

Definition 4.1 Given $f \in C(X)$ and $x \in X$. The connected component of $f^{-1}(f(x))$ containing x is denoted C^f_x.

Definition 4.2 A function $f \in C(X)$ has **distant connected components of fibers (DCF)** if any two components of distinct fibers are at positive distance, i.e., C^f_a and C^f_b have positive distance whenever $fa \neq fb$.

Clearly $DF \to DCF$. For a function whose fibers have finitely many connected components, $DF = DCF$. In general the equality fails: take $x \mapsto \sin(x^2)$.

Theorem 4.3 *UA implies DCF for functions $f \in C(X)$ on any space X.*

Proof. If f is not DCF we can find $a,b \in X$ such that C^f_a and C^f_b have distance zero and $f(a) \neq f(b)$. Consider the compact set $K = \{a,b\}$ and let $M = C^f_a \cup C^f_b$. If for a contradiction g is a (K,M)-approximation of f then $g(M) \subseteq f(M) = \{f(a), f(b)\}$, and since C^f_a and C^f_b are connected g is constant on each of these two sets. Moreover these two constants are distinct since they coincide with $f(a)$ and $f(b)$ respectively (as $g = f$ on K). Together with the fact that C^f_a and C^f_b are at distance zero, this contradicts the uniform continuity of g. QED

For instance $f: \mathbf{R}^2 \to \mathbf{R}$, $(x,y) \mapsto xy$ is not UA since it is not DCF. Burke [B, §5] proved that f is not even WUA.

So for functions $f \in C(X)$ on a uniformly locally connected space X we have $DF \to UA \to DCF$.

Corollary 4.4 *For a polynomial function $f \in C(\mathbf{R}^n)$ $DF = UA = DCF$.*

Proof. The fibers of a polynomial have finitely many connected components, so $DF = DCF$ for all polynomials. QED

With the same proof the equality $DF = UA = DCF$ holds for every semialgebraic functions $f \in C(\mathbf{R}^n)$, i.e., a function whose graph is a subset of $\mathbf{R}^{n+1}$ definable by a finite boolean combination of sets given by polynomial equations $p = 0$ and inequalities $p0$.

In view of the above corollary the question arises of which polynomials have DF. Clearly they include the polynomials $f \in C(\mathbf{R}^n)$ with compact fibers, but also some other polynomials like $f(x,y) = x^2$ which do not depend on some of the variables. Other examples arise taking compositions with a linear automorphism of $\mathbf{R}^n$. Are these essentially all the DF polynomials?

4.2 WUA implies DCF for functions on $\mathbf{R}^n$

Assuming $X = \mathbf{R}^n$ we can strengthen the results of the previous section showing that weakly UA functions have DCF. We begin with Burke's result since it illustrates in a simple situation the basic technique that we will use in this section.

Theorem 4.5 *(Burke) The function $f: \mathbf{R}^2 \to \mathbf{R}$ sending (x, y) to xy is not WUA.*

Proof. Let H_n be the connected component of the fiber $f^{-1}(1/n)$ contained in the positive quadrant, let $M = \bigcup_n H_n$, and let $p = (0, 0)$. Note that $p \in \overline{M} \setminus M$. We claim that f has no (p, M)-approximation. Suppose for a contradiction there is a uniformly continuous $g \in C(\mathbf{R}^2)$ with $g(M) \subseteq f(M)$ and $g(p) = f(p) = 0$. Since $f(M)$ is countable, so is $g(M)$. It then follows that g is constant on each of the connected sets H_n. Since the various H_n are at distance zero from each other and g is uniformly continuous, g must then be constant on their union M. By continuity g is constant on $\overline{M}$ and since $p \in \overline{M}$, $g = 0$ on M. This contradicts $g(M) \subseteq f(M)$ since the latter set does not contain 0. QED

The next theorem says that if a function f on a separable metric space X has too many connected components of fibers at distance zero from one of them, then f is not WUA.

Theorem 4.6 *Let X be a separable metric space and suppose that there is an uncountable set $Y \subseteq \mathbf{R}$ and for each $y \in Y$ a connected component C^y of $f^{-1}y$ such that for some $z \in Y$ we have $d(C^y, C^z) = 0$ for all $y \in Y$. Then f is not WUA.*

Proof. The idea is to find a countable subfamily $\{y_n \mid n \in \mathbf{N}\} \subseteq Y$ such that the components C^{y_n} will play the role of the sets H_n in Burke's proof. Let $N = \bigcup_{y \in Y} C^y$. Since $N \subseteq X$ and X is a separable metric space, N is separable. Hence at most countably many C^y can be open in N. So there is some $C^y \neq C^z$ which is not open in N (where z is as in the statement of the theorem). We can then find in N a sequence of points x_n not in C^y and converging to a point $x = \lim_n x_n$ in C^y. Let C^{y_n} contain x_n, and let $M = C^z \cup \bigcup_n C^{y_n}$. Suppose for a contradiction that $g \in C(X)$ is a $(\{x\}, M)$-approximation of f. Then $g(M) \subseteq f(M)$ is countable, hence totally disconnected. So g restricted to the connected set C^z is constant, and g restricted to each C^{y_n} is also constant. Since g is u.c. and by assumption $d(C^{y_n}, C^z) = 0$, g must be constant on the whole of M, and therefore also on its closure $\overline{M}$. Since $x \in \overline{M}$, g has the constant value $g(x) = f(x)$ on M. This however contradicts the inclusion $g(M) \subseteq f(M)$ since $f(x)$ does not belong to the latter set. QED

We want to weaken the assumptions of the above theorem by showing that it is enough to require that f has two connected components of distinct fibers at distance zero (rather than an uncountable family). What we need is a property of $\mathbf{R}^n$ called "unicoherence."

Definition 4.7 (See [K, §41, X]) A space U is **unicoherent** if it is connected and for every pair of closed connected subspaces A, B such that $U = A \cup B$, the intersection $A \cap B$ is connected.

$\mathbf{R}^2$ minus one point is not unicoherent. Let S^1 be the boundary of the closed unit disk in $\mathbf{R}^2$. For a space X to be unicoherent it suffices that every $f \in C(X, S^1)$ is homotopic to a constant map (see [K, §52, II]). So every contractible space is unicoherent and in particular $\mathbf{R}^n$ is unicoherent. Unicoherence is equivalent to a certain connectivity property. In order to state it we need:

Definition 4.8 Following [K, §16, V] we say that two subsets of a topological space S are **separated** if the closure of each of them does not meet the other. A subset X of a space S **separates** the nonempty sets H and K if the complement of X can be partitioned in two separated sets, one of which containing H, the other containing K (see [K, §16, VI]).

If X separates H from K, then the same remains true if we replace X with a larger set $X' \supseteq X$ disjoint from $H \cup K$, and we replace H, K with any two smaller sets $H' \subseteq H$ and $K' \subseteq K$.

Remark 4.9 If a subset X of a space S separates a point x from a point y, then it separates any connected subset of $S \setminus X$ containing x from any connected subset of $S \setminus X$ containing y. In particular if X separates two points x, y, then it intersects every connected set containing both x and y.

The converse holds in any connected locally connected regular topological space [HY, Theorem 3-6] but it is false in general. Consider for instance two distinct points a, b in $\mathbf{R}^2$ which are in $\overline{L} \setminus L$ where L is the subspace of $\mathbf{R}^2$ consisting of the family of parallel lines $x = 1/n$. Then in the space $S = L \cup [a, b]$ the points a, b cannot be separated by any singleton $X = \{x\}$ with $x \in (a, b)$ but every connected set of S containing both a and b meets (contains) X.

Theorem 4.10 *(see [HY, p. 359]) For a connected and locally connected space S the following two properties are equivalent:*

1. *S is unicoherent.*

2. *If A and B are disjoint closed subsets of S and x and y are points of S such that neither A nor B separates x from y in S, then $A \cup B$ does not separate x from y in S.*

We will use the unicoherence of $\mathbf{R}^n$ in the form given by point (2) of the above theorem.

Lemma 4.11 *For a continuous function $f \in C(\mathbf{R}^n)$ if a fiber $f^{-1}(r)$ separates two points $x, y \in \mathbf{R}^n$, then also a connected component of $f^{-1}(r)$ separates x from y.*

Proof. Assume that for a continuous function $f \in C(\mathbf{R}^n)$ the fiber $f^{-1}(r)$ separates two points $x, y \in \mathbf{R}^n$. By a theorem of Mazurkiewicz (see [K, §49, Theorem V.3]) every closed separator C between x and y contains a closed **irreducible separator** F between x and y, where a separator F between x and y is irreducible, if it is minimal with respect to inclusion [K, §46, VII]. Every closed irreducible separator F is connected: in fact, if $F = A \cup B$, with A, B disjoint closed subsets of F, then by the unicoherence of $\mathbf{R}^n$ one of the sets A, B must separate x from y, contradicting the minimality of F. Let G be a connected component of $f^{-1}(r)$ that contains a connected irreducible separator $F \subseteq f^{-1}(r)$. Then G separates x from y. QED

Theorem 4.12 *If $f \in C(\mathbf{R}^n)$ has two connected components A, B of distinct fibers at distance zero, then it has a family, of cardinality of the continuum, of connected components of distinct fibers at distance zero from each other and from A and B.*

Before going into the proof, notice that Theorem 4.12 would be obvious if we considered fibers instead of connected components of fibers: if $f^{-1}(a)$ and $f^{-1}(b)$ have distance zero (say $a < b$), then the fibers $f^{-1}(c)$ with $a \leq c \leq b$ have distance zero from each other (since a short segment connecting $f^{-1}(a)$ and $f^{-1}(b)$ intersects all these fibers). If one tries to imitate this argument for the connected components of the fibers, one encounters the following difficulty. Let A be a connected component of $f^{-1}(a)$, let B be a connected component of $f^{-1}(b)$ and suppose $d(A,B)=0$. Given an intermediate value $a<c<b$ we would like to find a connected component C of $f^{-1}(c)$ at distance zero from both A and B. Any segment joining A with B certainly meets $f^{-1}(c)$ in some point x, however the connected component of $f^{-1}(c)$ containing x is not necessarily at distance zero from A and B: for instance x could be a local maximum.

Proof of Theorem 4.12: Recall that C^f_x is the connected component of the fiber $f^{-1}(fx)$ which contains the point x. Suppose there are $x,y\in\mathbf{R}^n$ with $fx<fy$ and $d(C^f_x,C^f_y)=0$. For each w with $fx<w<fy$, the fiber $f^{-1}(w)$ separates x from y. By Lemma 4.11 one connected component C^w of $f^{-1}(w)$ separates x from y. Then C^w also separates C^f_x from C^f_y by Remark 4.9. For each $\varepsilon 0$ the sets C^f_x, C^f_y can be connected by a segment of length $<\varepsilon$, which necessarily intersects C^w. So the members of the family $\langle C^w \mid fx<w<fy\rangle$ have distance zero from each other and from C^f_x, C^f_y. QED.

The above theorem, combined with Theorem 4.6, shows that a function with two connected components of distinct fibers at distance zero is not WUA. In other words we have:

Corollary 4.13 *If a function $f\in C(\mathbf{R}^n)$ is WUA, then it is DCF.*

The corollary fails for functions on $\mathbf{R}$ minus one point (cf Remark 7.5).

5 Truncations

Definition 5.1 ([BeDi, Definition 4.1]) $g\in C(X)$ is a **truncation** of $f\in C(X)$ if g is constant on each connected component of the set $\{x\mid fx\neq gx\}$. We say that g is a K**-truncation** of f if it is a truncation of f with $g=f$ on K.

A special case of truncation is provided by the (a,b)-truncations of Definition 3.13. However truncations are more general than (a,b)-truncations. Any function $f\in C(\mathbf{R})$ can be made uniformly continuous by truncating it in various ways, although an (a,b)-truncation may not suffice (excluding the constant functions). For instance the function $\sin(x^2)$ can be made u.c. by a K-truncation (any K) but not by a non-constant (a,b)-truncation. On $\mathbf{R}^2$ there are functions, like $(x,y)\mapsto xy$, with the property that all its truncations (except the constant ones) are not uniformly continuous. The next definition captures the functions with many uniformly continuous truncations.

Definition 5.2 A function $f\in C(X)$ is **truncation-UA** (**TUA**) if for every compact set $K\subseteq X$ there is a uniformly continuous truncation g of f which coincides with f on K.

Every $f\in C(\mathbf{R})$ is TUA: we can truncate f so that it becomes eventually constant both for $x\to\infty$ and for $x\to-\infty$ (cf. [BeDi, Proposition 3.5] where this construction is used to show that every $f\in C(\mathbf{R})$ is UA). We have already remarked that $(x,y)\mapsto xy$ is not TUA.

We now study the behaviour of truncations in locally connected spaces.

Lemma 5.3 *Let $f\in C(X)$, X locally connected, g a truncation of f, and let $\mathcal{U}$ be a connected component of $\{x\mid f(x)\neq g(x)\}$. Then g is constant on $\overline{\mathcal{U}}$ and $g=f$ on $\partial\mathcal{U}$.*

Proof. The fact that g is constant on $\overline{\mathcal{U}}$ follows from the fact that g is constant on $\mathcal{U}$ and continuity. We prove the second statement. Since X is locally connected and $\mathcal{U}$ is a connected component of an open set, $\mathcal{U}$ is open. Hence $\partial\mathcal{U} \cap \mathcal{U} = \emptyset$. Let $x \in \partial\mathcal{U}$. If for a contradiction $f(x) \neq g(x)$, then $x \in \mathcal{U}_1$ where $\mathcal{U}_1$ is a connected component of $\{x \mid f(x) \neq g(x)\}$ different from $\mathcal{U}$. Now $\mathcal{U}_1$ is, as above, a connected neighborhood of x, so that $\mathcal{U} \cap \mathcal{U}_1 \neq \emptyset$, contradicting the disjointness of the connected components. QED

Recall that C^f_x is the connected component containing x of the fiber $f^{-1}(fx)$.

Lemma 5.4 *Let $f \in C(X)$ with X locally connected. Let $x \in X$ and let g be a truncation of f with $g(x) = f(x)$. Then $g = f$ on C^f_x (so $C^f_x \subseteq C^g_x$).*

Proof. If for a contradiction there is $y \in C^f_x$ with $f(y) \neq g(y)$, consider the connected component $\mathcal{U}$ of $\{u \mid f(u) \neq g(u)\}$ containing y. C^f_x has a point $y \in \mathcal{U}$ and a point $x \notin \mathcal{U}$. Since C^f_x is connected $C^f_x \cap \partial\mathcal{U} \neq \emptyset$. Let $u \in C^f_x \cap \partial\mathcal{U}$. We reach a contradiction via the chain of equalities $f(y) = f(u)$ (as $y, u \in C^f_x$) $= g(u)$ (as $g = f$ on $\partial\mathcal{U}$ by Lemma 5.3) $= g(y)$ (as g is constant on $\overline{\mathcal{U}}$). QED

Definition 5.5 Given $f \in C(X)$, $K \subseteq X$, let $K^f = \bigcup_{x \in K} C^f_x$.

Let us observe that $K \subseteq K^f \subseteq f^{-1}(f(K))$ and $f(K) = f(K^f)$.

Corollary 5.6 *Let $f \in C(X)$ with X locally connected. Every K-truncation g of f is also a K^f-truncation.*

So if f is not u.c. on K^f, then f has no u.c. K-truncations (note that K^f need not be compact even when K is compact). This can be used to show that TUA, like UA, implies DCF. We will later see that $TUA = DCF$ on $\mathbf{R}^n$.

Theorem 5.7 *TUA implies DCF for functions $f \in C(X)$ on any locally connected space X.*

Proof. If f is not DCF we can find $a, b \in X$ such that C^f_a and C^f_b have distance zero and $f(a) \neq f(b)$. Consider the compact set $K = \{a, b\}$. Then $K^f = C^f_a \cup C^f_b$ and f is not u.c. on K^f. So f has no K-truncations. QED

The two concepts UA and TUA seem to coincide for a large class of functions, however counterexample exists on non-separable spaces [BDP] (for some recent developments see §8).

6 The magic set

Under suitable hypothesis and a careful choice of M we show that a (K, M)-approximation is a K-truncation. We use this to show that DF does not imply UA (or even WUA) if the space is not uniformly locally connected.

Definition 6.1 [BD] We say that $M \subseteq X$ is a **magic set** for $f \in C(X)$, provided for every $g \in C(X)$ if $g(M) \subseteq f(M)$ then g is a truncation of f.

Definition 6.2 Let X be locally arcwise connected. We say that $f \in C(X)$ has **small fibers** if for every connected open set $\mathcal{U} \subseteq X$ and every $x, y \in \mathcal{U}$, there is an arc ϕ inside $\mathcal{U}$ connecting x and y and such that each fiber of f has at most countable intersection with ϕ.

Clearly if f has small fibers, then f has nowhere dense fibers. Every non constant polynomial function has small fibers. The proof of the following result is an adaptation of some results in [BD]. The improvement is that we do not assume here that f has countable fibers or that the continuum hypothesis holds.

Theorem 6.3 *Let X be a locally arcwise connected topological space with $|C(X)| \leq 2^{\aleph_0}$ (e.g. X separable). Let $f \in C(X)$ have small fibers. Then there is a magic set $M \subseteq X$ for f.*

In particular every (K, M)-approximation of f is a K-truncation of f.

Moreover we can choose M disjoint from any given fiber of f, and more generally for every subset G of $\mathbf{R}$ of cardinality less than the continuum, we can choose M disjoint from $f^{-1}(G)$.

Proof. Let $\{g_\alpha \mid \alpha < 2^{\aleph_0}\}$ be an enumeration of all $g \in C(X)$ which are not truncations of f. We must prove that there is $M \subseteq X$ such that for every $\alpha < 2^{\aleph_0}$, $g_\alpha(M) \not\subseteq f(M)$. We construct $M \subseteq X$ by stages. At the stage $\alpha < 2^{\aleph_0}$, we will put a new element m_α in M which will "kill" g_α. The definition of $m_\alpha \in X$ is done by transfinite induction on $\alpha < 2^{\aleph_0}$. Suppose that we have already defined $m_\beta \in X$ for each $\beta < \alpha$. We need to define m_α. Consider the function g_α of our enumeration. Since g_α is not a truncation of f, there is a connected component $\mathcal{U}_\alpha$ of $\{x \mid f(x) \neq g_\alpha(x)\}$, such that g_α is non-constant on $\mathcal{U}_\alpha$. The image $g_\alpha(\mathcal{U}_\alpha)$ is a non-trivial connected set of $\mathbf{R}$, so it has the cardinality of the continuum. Choose, if possible, $m_\alpha \in \mathcal{U}_\alpha$ so that the following conditions hold:

- for every $\gamma < \alpha$, $f(m_\alpha) \neq g_\gamma(m_\gamma)$ and $f(m_\alpha) \notin G$ $\quad (1)_\alpha$
- for every $\gamma < \alpha$, $g_\alpha(m_\alpha) \neq f(m_\gamma)$ $\quad (2)_\alpha$

We must prove that there is an element $m_\alpha \in \mathcal{U}_\alpha$ with the desired properties. Let $x, y \in \mathcal{U}_\alpha$ be two points where g_α assumes different values. Since X is locally connected, $\mathcal{U}_\alpha$ is open. Since f has small fibers, there is an arc $T \subseteq \mathcal{U}_\alpha$ connecting x and y and intersecting each fiber of f in countably many points. So T intersects the set $S_\alpha = f^{-1}(G) \cup \bigcup_{\gamma<\alpha} f^{-1}(g_\gamma(m_\gamma))$ in strictly less than $2^{\aleph_0}$ points. $g_\alpha(T)$ is a non-trivial connected subset of $\mathbf{R}$, so it has the cardinality $2^{\aleph_0}$. Let T' be the set of those points of T which are not in the set S_α. Then $g_\alpha(T')$ has still the cardinality of the continuum. In particular, there is a point $z \in T'$ such that $g_\alpha(z) \neq f(m_\gamma)$ for all $\gamma < \alpha$. Set $m_\alpha = z$. Then m_α satisfies the above conditions. This ends the construction.

Let $M = \{m_\alpha \mid \alpha < 2^{\aleph_0}\}$. It is clear that $M \cap f^{-1}(G)) = \emptyset$. To finish the proof, it suffices to show that $g_\alpha(m_\alpha) \notin f(M)$. Suppose for a contradiction that $g_\alpha(m_\alpha) = f(m_\gamma)$. By $(2)_\alpha$, γ cannot be $< \alpha$. By $(1)_\gamma$ (i.e., (1) with the roles of α and γ exchanged), α cannot be $< \gamma$. Since $m_\alpha \in \mathcal{U}_\alpha \subseteq \{x \mid f(x) \neq g_\alpha(x)\}$, α cannot be equal to γ. QED

We know that for reasonable spaces UA is trapped between DF and DCF: $DF \to UA \to DCF$. We can now shrink the gap by showing $UA \to TUA$ for reasonable functions (so $DF \to UA \to TUA \to DCF$ for reasonable functions).

Theorem 6.4 *$UA \to TUA$ for functions $f \in C(X)$ with small fibers on a space X satisfying the hypothesis of Theorem 6.3.*

Proof. If M is a magic set for f then every (K, M)-approximation of f is a K-truncation. QED

Theorem 6.5 *Let X be as in Theorem 6.3 and suppose that f has small fibers and has no non-constant u.c. truncations. Then f is not WUA.*

Proof. Let $x \in X$. Apply Theorem 6.3 choosing M disjoint from the fiber of f passing through x. Then f has no $(\{x\}, M)$-approximations. In fact if g is such an approximation, then on the one hand g is a u.c. truncation of f and on the other hand g is non-constant because it satisfies the two conditions $gx = fx$ and $g(M) \subseteq f(M)$ with M disjoint from the fiber of f trough x. QED

Theorem 6.6 *In the implication $DF \to UA$ the assumption that the space is uniformly locally connected is necessary. There are metric spaces homeomorphic to $\mathbf{R}$ where the implication $DF \to WUA$ does not hold, so a fortiori $DF \to UA$ fails.*

Proof. Let $S^1 \subseteq \mathbf{R}^2$ be the circle and X be S^1 minus a point. Then X is homeomorphic to $\mathbf{R}$ (but it is not uniformly locally connected). Let $f: X \to \mathbf{R}$ be the arclength function. Then f is DF (being injective) but does not have any non-constant u.c. truncation g, hence it is not WUA. To see this it suffices to prove that if g is a truncation of f, then gx tends to two different limits when x approaches the missing point $p \in S^1 \setminus X$ from the two different sides. This follows at once from the fact the connected components of the set where $f \neq g$ (and where g is constant) must have at most one boundary point due to the injectivity of f (recall that $g = f$ on the boundary points). QED

7 $DCF = TUA$ on $\mathbf{R}^n$

Lemma 7.1 *Let $f \in C(\mathbf{R}^n)$, $K \subseteq \mathbf{R}^n$ a connected subset and C a connected component of $\mathbf{R}^n \setminus K^f$ (see Definition 5.5). Then f is constant on ∂C.*

Proof. Suppose $fx < fy$ with $x, y \in \partial C$. The points x and y belong to the closure of K^f. Let F be a fiber of f separating x and y and let S be a connected component of F separating x and y (Lemma 4.11). Then S intersects C (because $C \cup \{x\} \cup \{y\}$ is connected) and also K^f for the same reason. This is absurd, since if S intersects K^f is entirely contained there. QED

Definition 7.2 Given a function $f \in C(\mathbf{R}^n)$ and a connected set $K \subseteq \mathbf{R}^n$, we define the **minimal K-truncation** of f as the function f_K which coincides with f on K^f and on each connected component C of the complement of K^f assumes the constant value that f assumes on ∂C (using Lemma 7.1).

The name "minimal K-truncation" is justified by the fact that every K-truncation is automatically a K^f-truncation.

Theorem 7.3 *Let $f \in C(\mathbf{R}^n)$ be a DCF function and let K be a compact connected set. Then the minimal K-truncation f_K of f is DF (hence also u.c., since it is bounded).*

Proof. Suppose that the minimal K-truncation g of f has two fibers $A = f^{-1}(a)$ and $B = f^{-1}(b)$ at distance zero, say $a < b$. Then the boundaries ∂A and ∂B are also at distance zero (as $\mathbf{R}^n$ is uniformly locally connected) so we can find points $a_n \in \partial A$ and $b_n \in \partial B$ with $d(a_n, b_n) \to 0$. We use the following fact about the minimal K-truncation: the boundary of every fiber of g is contained in the closure of K^f. (If x is not in the closure of K^f then g is locally constant around x so it is not in the boundary of a fiber.) So K^f contains a point x_n arbitrarily close to a_n and a point y_n arbitrarily close to b_n. So we can assume $d(x_n, y_n) \to 0$ and $\lim f(x_n) = a < b = \lim f(y_n)$. Now let p_n be a point of K such that there is a connected component of a fiber of f joining p_n and x_n, and similarly choose $q_n \in K$ corresponding to $y_n \in K^f$. Since K is compact we can assume that p_n converges to $p \in K$ and q_n converges to $q \in K$. Then $p \in A$ and $q \in B$. Choose $a < c < b$. By Lemma 4.11 there is a connected component S of $f^{-1}(c)$ separating p and q, and therefore also any connected neighbourhood of p disjoint from S from any connected neighbourhood of q disjoint from S. So for all large enough n, S separates each p_n from each q_n. Therefore for all large enough n, S separates each x_n from each y_n (by Remark 4.9). With the same argument we find a connected component S' of another fiber of f which also separates x_n and y_n for all large n. So $d(S, S') = 0$ (by uniform local connectedness, or simply because S and S' meet any straight segment joining x_n and y_n). Thus f is not DCF. QED

Corollary 7.4 *DCF implies TUA for functions $f \in C(\mathbf{R}^n)$ (and so $DCF = TUA$ over $\mathbf{R}^n$ by Theorem 5.7).*

Proof. Assume f is DCF. Given a compact set K we must find a uniformly continuous K-truncation of f. Enlarging K, if necessary, we can assume that K is connected. The minimal K-truncation f_K of f is DF by Theorem 7.3 and bounded (since $f_K(\mathbf{R}^n) \subseteq f(K)$), so it is UC. QED

Remark 7.5 By Corollary 4.13 and the above corollary $WUA \to TUA$ for functions $f \in C(\mathbf{R}^n)$, but this implication may fail even for $f \in C(\mathbf{R} \setminus \{0\})$: a locally constant function on $\mathbf{R} \setminus \{0\}$ is WUA but neither DCF nor TUA (unless it is constant).

Example 7.6 [BD, Proposition 10.1 (iii)] DCF does not imply TUA or WUA in more general spaces. We can find a counterexample $f \in C(X)$ where $X \subseteq \mathbf{R}^2$ is a connected set which is a uniform neighbourhood retract in $\mathbf{R}^2$. In fact one can take X to be the square grid consisting of all those points (x, y) in $\mathbf{R}^2$ such that either x or y is an integer. On X one can find a function with countable fibers (hence DCF) and without non-constant u.c. truncations (hence neither TUA nor WUA by Theorem 6.5).

8 Questions

We have proved that

Theorem 8.1 *$AP = DF \to UA \to WUA \to TUA = DCF$ for every $f \in C(\mathbf{R}^n)$. In particular, $AP = DF = WUA = UA = TUA = DCF$ for every $f \in C(\mathbf{R}^n)$ whose fibers have finitely many connected components (e.g. every polynomial function f).*

Clearly $DF \neq DCF$, but our results leave open the question whether we have $WUA = UA = TUA = DCF$ for *every* $f \in C(\mathbf{R}^n)$, more precisely:

Question 8.2 1. Does TUA imply UA or at least WUA for $f \in C(\mathbf{R}^n)$?

2. Does WUA imply UA for $f \in C(\mathbf{R}^n)$? What about $f \in C(\mathbf{R}^2)$?

Question (2) goes back to [BeDi, Question 6], where it was set for connected spaces. In August 1998, a negative answer to (1) was found by Ciesielski and Dikranjan [CD2]. More precisely there exists a TUA function $f \in C(\mathbf{R}^2)$ that is not WUA (hence not UA). The restriction of the function f to an appropriate connected subset A of $\mathbf{R}^2$ is WUA and TUA, but not UA (this answers negatively [BeDi, Question 6]).

We proved that $WUA \to DCF \to TUA$ for $f \in C(\mathbf{R}^n)$, or more generally for functions $f \in C(X)$ on a uniformly locally connected unicoherent space X where every compact set is contained in a connected one. On the other hand, $UA \to TUA$ for functions $f \in C(X)$ with small fibers on a separable space X, according to Theorem 6.4. Nevertheless, WUA need not imply TUA even for $f \in C(\mathbf{R} \setminus \{0\})$ (cf. Remark 7.5), while we have no counter-example to disprove $UA \to TUA$. Hence the following question remains open:

Question 8.3 Does UA imply TUA for functions $f \in C(X)$ on a connected and uniformly locally connected space X?

If two bounded functions on a connected and locally connected space have the same fibers then one is uniformly continuous iff the other is uniformly continuous (Theorem 3.10). This leaves open the question to isolate the explicit property of the fibers of bounded functions that gives u.c. Obviously, this property implies DF, and in the case of uniformly locally connected spaces it coincides with DF by Theorem 3.7. Another question is whether this "fiber-invariance" of u.c. can be extended to UA.

Question 8.4 Suppose $f, g \in C(X)$ have the same fibers and f is UA. Is also g a UA function? Is this true for bounded functions ?

A positive answer to this question may serve as a first step in finding a definite characterization of UA in terms of the behaviour of the fibers. Here another important point has to be taken in consideration: UA and WUA functions are closed under composition while DF and DCF are not. Hence it seems important to isolate a natural property of the fibers (say a variant of DCF) which is preserved under composition.

By Theorem 8.1 DF coincides with all other properties for polynomial functions $\mathbf{R}^n \to \mathbf{R}$. This determines our next question (see also the comment at the end of §4.1).

Problem 8.5 *Characterize the polynomial functions* $\mathbf{R}^n \to \mathbf{R}$ *with* DF.

Clearly all polynomials with compact fibers have this property. In connection with this it was pointed out to us by F. Broglia that the polynomials with compact fibers are precisely those that are proper functions.

Generalizing the well known UC spaces ([A1, A2, Be, BDC]) we study the spaces X with the property that every function $f \in C(X)$ is UA in a forthcoming paper [BDP]. One can produce examples of spaces such that it cannot be determined in Zermelo Fraenkel set theory whether they have this property.

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