Uniform quasi components, thin spaces and compact separation *

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Abstract

We prove that every complete metric space X that is thin (i.e., every closed subspace has connected uniform quasi components) has the compact separation property (for any two disjoint closed connected subspaces A and B of X there is a compact set K disjoint from A and B such that every neighbourhood of K disjoint from A and B separates A and B).

The real line and all compact spaces are obviously thin. We show that a space is thin if and only if it does not contain a certain forbidden configuration. Finally we prove that every metric UA-space (see [BD1]) is thin. The UA-spaces form a class properly including the Atsuji spaces.

1 Introduction

The class of the topological spaces X having connected quasi components is closed under homotopy type, it contains all compact Hausdorff spaces (see [E, Theorem 6.1.23]) and every subset of the real line. Some sufficient conditions are given in [GN] (in terms of existence of Vietoris continuous selections) and in [CMP] (in terms of the quotient space ΔX in which each quasi-component is identified to a point), but an easily-stated description of this class does not seem to be available (see [CMP]). The situation is complicated even in the case when all connected components of X are trivial, i.e., when X is *hereditarily disconnected* [E]. (The term *totally disconnected* is used for spaces having trivial quasi components [E].) In these terms the question is to distinguish between hereditarily disconnected and totally disconnected spaces (examples to this effect go back to Knaster and Kuratowski [KK]).

The connectedness of the quasi component (i.e., the coincidence of the quasi component and the connected component) in topological groups is also a rather hard question. Although a locally compact space does not need to have connected quasi components [E, Example 6.1.24], all locally compact groups have this property (this is an easy consequence of the well known fact that the connected component

*The first and the second author were partially supported by a Research Grant of MURST. The third author was partially supported by the grant GA ČR 201/00/1466.

AMS classification numbers: Primary 54C30, 54F55; Secondary 41A30, 54E35.

of a locally compact group coincides with the intersection of all open subgroups of the group [HR, Theorem 7.8]). Recently all countably compact groups were shown to have this property too ([D3], see also [D2, D4]). Many examples of pseudocompact group where this property strongly fails in different aspects, as well as further information on quasi components in topological groups, can be found in ([D1, D2, D4], see also [U] for a plane group with non-connected quasi component).

Given a uniform space X and a point $x \in X$ we have the following inclusions

$$C_x(X) \subseteq Q_x(X) \subseteq Q^u_x(X),$$

where $C_x(X)$ denotes the connected component of x, $Q_x(X)$ the quasi component of x, and $Q^u_x(X)$ the uniform quasi component of x (see §2).

Definition 1.1 A uniform space X is **thin** if for every closed subset Y of X and every $y \in Y$, the uniform quasi component of y in Y is connected.

By the above mentioned classical result ([E, Theorem 6.1.23]) the compact uniform spaces are thin (since the uniform quasi components coincide with the quasi components in this case). In this paper we study the thin spaces and our main result is establishing a separation property for every complete metric thin space. The relevant separation property is defined as follows.

Definition 1.2 We say that a uniform space X has the **compact separation property** (briefly *CSP*), if for any two disjoint closed connected subspaces A and B there is a compact set K disjoint from A and B such that every neighbourhood of K disjoint from A and B separates A and B (consequently K intersects every closed connected set which meets both A and B, see Definition 2.1).

Note that for a locally compact space X, the CSP can be equivalently expressed in the following simpler form: for any two disjoint closed connected subspaces A and B there is a compact set K disjoint from A and B which separates A and B.

The real line, every zero-dimensional space and every compact space have *CSP*. On the other hand the real plane $\mathbf{R}^2$ does not have *CSP*: take A, B to be two parallel lines. The main result of this paper, which we prove in section 4, is the following:

Theorem A. *Every complete thin metric space has* CSP.

The proof is based on a criterion for thinness given by Proposition 3.7. We show that a space is thin if and only if it does not contain a certain forbidden configuration (that we call **garland**, cf. Definition 3.5).

Another result, given in section 3, is that every metric UA-space (see Definition 1.4) is thin, so every complete metric UA-space has CSP. The UA-spaces are those uniform spaces where every continuous real-valued function can be approximated in a suitable way by uniformly continuous ones. They obviously comprise the compact spaces. The relevant definitions are as follows.

Definition 1.3 ([BD1, Definition 2.1]) A function $f\colon X \to Y$ between two uniform spaces is **uniformly approachable**, briefly UA, if for every compact set $K \subseteq X$ and every subset $M \subseteq X$, there exists a uniformly continuous $g\colon X \to Y$, called a (K, M)-approximation of f, such that $f = g$ on K and $g(M) \subseteq f(M)$.

This is a modification of the notion originally introduced in [DP, Definition 4.1] (with the same name) in connection with the study of closure operators in the sense of [DT, DP]. (The difference is that in [DP] the compact set K was assumed to be a single point.) The UA-functions have also been studied in [BD2, BDP1, BDP2, B1, CD1, CD2], for related topics (such as truncations, magic sets etc.) see also [B2, BC1, BC2, CS].

Every uniformly continuous function is obviously UA, and it is easy to see that a UA function f must necessarily be continuous [DP, Lemma 4.2]. Moreover the UA-functions are closed under composition, so they define a category which sits between the category of uniform spaces with all continuous functions and the category of uniform spaces with all uniformly continuous functions. A characterization of the UA-functions is still missing, although a complete answer exists in the special case of polynomial maps $f: \mathbf{R}^n \to \mathbf{R}$: such an f is UA if and only if any pair of distinct fibers of f are at positive distance [BDP1].

Definition 1.4 ([BD1, Definition 2.4]) A uniform space X is a UA**-space** if every continuous $f: X \to \mathbf{R}$ is UA.

Besides the compact spaces, the UA-spaces include the **Atsuji spaces**, which are those metric spaces X such that every continuous $f: X \to \mathbf{R}$ is uniformly continuous [A1, A2], and also the real line [BD1, Proposition 3.5]. Other examples and general results on UA-spaces are given in [BD1, BD2]. We know for instance that every linear chain of compact sets, each attached to the next by a single point, is UA (provided each subchain is closed, see [BD1, Theorem 11.4]). This can be generalized to certain tree-like unions of compact sets [BDP2]. A useful observation is the following:

Fact 1.5 *If a function is UA, its restriction to a subspace is UA. So if a normal uniform space X contains a non-UA closed subspace, then X is not UA.*

This can be used to prove that $\mathbf{R}^2$ is not UA [BD1] (see also [B1]). In §3 of this paper we prove:

Theorem B. *Every metric UA-space is thin.*

This gives as a corollary:

Theorem C. *Let X be a complete metric UA-space. Then X has CSP.*

In particular, CSP holds for every Atsuji space, since every Atsuji space is UA and complete (one can also reason directly from the following characterization of an Atsuji space X: the set X' of accumulation points of X is compact, and for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$ with $d(x, X') > \varepsilon$ and $d(y, X') > \varepsilon$ one has $d(x, y) \geq \delta$ [H]).

In §5 we give examples showing that the implications in Theorems A and B cannot be inverted:

(A) a complete separable metric space satisfying CSP need not be thin;

(B) a complete (connected) metric thin space need not be UA.

In section 2 we give a self contained exposition of all the needed properties of the quasi components, the uniform quasi components, and the uniformly connected sets.

Notations. *$C(X, Y)$ is the set of all continuous functions from X to Y. We also write $C(X)$ for $C(X, \mathbf{R})$. The abbreviation "u.c." stands for "uniformly continuous".*

ACKNOWLEDGMENTS: The paper enjoyed a very careful and constructive reading by a referee, who observed that Theorem A can be given in a more general form (see §5.1).

2 Quasi components

Definition 2.1 Two subsets A, B of a topological space X are **separated** if the closure of each of them does not meet the other (this is equivalent to say that A and B are clopen in $A \cup B$). So X is connected if and only if it cannot be partitioned in two separated sets.

A subset S of X **separates** the nonempty sets A and B if the complement of S can be partitioned in two separated sets, one of which containing A, the other containing B (see [K, §16, VI]).

We say that S **cuts** between A and B if S intersects every connected set which meets both A and B. (If S is empty this means that there is no connected set which meets both A and B.)

If $S \subseteq X$ separates A and B, then S cuts between A and B. The converse holds in a connected locally connected regular topological space [HY, Theorem 3-6], but it is false in general.

Example 2.2 Consider a subset X of the plane consisting of two points a and b together with countably many parallel lines so that both a and b are at distance $1/n$ from the n-th line. The empty set cuts between a and b in X, but it does not separate a and b.

Note that X is connected if every $f \in C(X, \{0,1\})$ is constant.

Definition 2.3 A uniform space X is **uniformly connected** if every u.c. function $f \in C(X, \{0,1\})$ is constant. A subspace $A \subseteq X$ is uniformly connected if it is uniformly connected with respect to the induced uniformity, i.e. if every u.c. function $f \in C(A, \{0,1\})$ is constant.

Recall that the **quasi component** of a point $x \in X$ is the intersection of all clopen sets containing x. Hence x is in the same quasi component of y in X if x cannot be separated from y, i.e. for every partition $X = A \cup B$ with A, B open, x and y lie both in A or both in B. Therefore, the quasi component of x in X is the set of all points $y \in X$ such that every function $f \in C(X, \{0,1\})$ gives the same value to x and y.

The **uniform quasi component** of x in X is defined in the same way but requiring f to be uniformly continuous.

Definition 2.4 Let $\mathcal{P}$ be an open cover of X. Given $x, y \in X$, a **$\mathcal{P}$-chain from x to y** is a finite sequence $x_0, x_1, \ldots, x_k$ of points in X with $x_0 = x, x_k = y$ and for all $i < k$ there is $P \in \mathcal{P}$ containing x_i and x_{i+1}.

If X is metric space and $\mathcal{P}$ consists of all open balls of diameter ε, then a $\mathcal{P}$-chain is also called an **ε-chain**.

Definition 2.5 A **uniform cover** of a uniform space X is an open cover $\mathcal{P}$ such that for some member $U \subseteq X \times X$ of the uniformity of X, the collection of all the open sets $U[x] = \{y \mid (x,y) \in U\}$ is a refinement of $\mathcal{P}$.[1]

If X is metric this means that there is $\varepsilon > 0$ such that each open set of diameter $< \varepsilon$ is contained in some element P of $\mathcal{P}$.

We need the following easy characterization of the quasi component and the uniform quasi component in terms of $\mathcal{P}$-chains.

Lemma 2.6

1. *Two points x, y of a topological space X belong to the same quasi component if and only if for every open cover $\mathcal{P}$ of X there is a $\mathcal{P}$-chain from x to y.*
2. *Two points x, y of a uniform space X belong to the same uniform quasi component iff for every uniform open cover $\mathcal{P}$ of X there is a $\mathcal{P}$-chain from x to y.*
3. *In particular if X is metric we obtain: x, y belong to the same uniform quasi component iff for every $\varepsilon > 0$, there is ε-chain from x to y.*

[1] An open cover $\mathcal{Q}$ is a **refinement** of an open cover $\mathcal{P}$ if every $Q \in \mathcal{Q}$ is contained in some $P \in \mathcal{P}$.

Proof. Suppose x, y belong to the same (uniform) quasi component and let $\mathcal{P}$ be a (uniform) open cover of X. Given $x \in X$, define $O_x \subseteq X$ as the set of points reachable from x by a $\mathcal{P}$-chain. Suppose for a contradiction that $y \notin O_x$. Clearly there is no $P \in \mathcal{P}$ which intersects both O_x and $X \setminus O_x$. So the function with value 0 on O_x and 1 on $X \setminus O_x$ is (uniformly) continuous and takes different values on x and y, a contradiction.

Conversely suppose that x, y belong to two different (uniform) quasi components. Then there is a (uniformly) continuous function $f \in C(X, \{0,1\})$ with $f(x) = 0$ and $f(y) = 1$. Let $\mathcal{P}$ be the (uniform) open cover consisting of the (uniformly) clopen sets $f^{-1}(0)$ and $f^{-1}(1)$. Then there is no $\mathcal{P}$-chain from x to y. QED

Definition 2.7 Given an open cover $\mathcal{P}$ of a topological space X and $A \subseteq X$ we say that A is **$\mathcal{P}$-connected** if every pair of points of A can be joined by a $\mathcal{P}$-chain contained in A.

For instance every $\mathcal{P}$-chain is $\mathcal{P}$-connected. We characterize connected and uniformly connected sets in terms of $\mathcal{P}$-chains.

Corollary 2.8
1. *A topological space X is connected iff for every open cover $\mathcal{P}$ of X, X is $\mathcal{P}$-connected.*
2. *A uniform space X is uniformly connected iff for every uniform open cover $\mathcal{P}$ of X, X is $\mathcal{P}$-connected.*
3. *In particular if X is metric, then X is uniformly connected iff for every $\varepsilon > 0$ any two points of X can be joined by an ε-chain.*

Proof. X is (uniformly) connected iff every pair of points $x, y \in X$ lie in the same (uniform) quasi component. Now apply Lemma 2.6. QED

Definition 2.9 Let $\mathcal{P}$ be an open cover of a topological space X. We say that a subset S of X **$\mathcal{P}$-cuts** between two subsets A and B of X if S is disjoint from $A \cup B$ and intersects every $\mathcal{P}$-connected set (or equivalently every $\mathcal{P}$-chain) which meets both A and B.

If S $\mathcal{P}$-cuts between A and B, then S cuts between A and B. Actually more is true:

Lemma 2.10 *If S $\mathcal{P}$-cuts between A and B in a topological space X, then S separates A and B in X.*

Proof. Let O_A be the set of points $x \in X \setminus S$ such that there is a $\mathcal{P}$-chain contained in $X \setminus S$ from some point of A to x. Then O_A contains A and is disjoint from S and from B. It suffices to show that O_A is clopen in $X \setminus S$. It is open because if $x \in O_A$, then for every $P \in \mathcal{P}$ containing x, $P \cap (X \setminus S) \subseteq O_A$. It is closed because if $x \in X \setminus S$ and $x \notin O_A$, then for every $P \in \mathcal{P}$ containing x, $P \cap (X \setminus S)$ is disjoint from O_A. QED

So "$\mathcal{P}$-cuts" entails "separates" which implies "cuts". For open separators we have the following partial converse.

Lemma 2.11 *If an open set S separates A and B in a topological space X, then for some open covering $\mathcal{P}$ of X, S $\mathcal{P}$-cuts between A and B*

Proof. The complement of S can be partitioned in two separated sets H and K. Let $\mathcal{P}$ be the covering of X consisting of S together with all the open sets which do not intersect both H and K. QED

3 UA spaces have connected uniform quasi components

Given a uniform space X we give a necessary condition for a function $f \in C(X)$ to be UA in terms of uniform quasi components.

Lemma 3.1 *Let $f \in C(X)$. Suppose there are $y \neq z$ in $\mathbf{R}$ and two points $a \in f^{-1}(y)$ and $b \in f^{-1}(z)$ such that the uniform quasi component of a in $f^{-1}(y) \cup f^{-1}(z)$ contains b. Then f is not UA.*

Proof. Let $M = f^{-1}(y) \cup f^{-1}(z)$. If f is UA, then its restriction $f_{|M}$ is UA. Let $g \in C(M)$ be a $(\{a, b\}, M)$-approximation of $f_{|M}$. Then $g(M) \subseteq \{y, z\}$, $g(a) = y$ and $g(b) = z$. Since g is uniformly continuous, g witnesses the fact that b does not belong to the uniform quasi component of a in M, a contradiction. QED

Lemma 3.2 *Suppose that a metric space X contains two disjoint closed sets H and K and a point $a \in H$ such that the uniform quasi component of a in $H \cup K$ intersects K. Then X is neither thin nor UA.*

Proof. Clearly, X cannot be thin since the closed subspace $Y = H \cup K$ has a uniform quasi component (that of a) which is not connected since it hits both H and K, which are closed and disjoint.

The closed subspace $H \cup K$ is not UA since we can apply Lemma 3.1 to the characteristic function of H. Hence X is not UA by Fact 1.5. QED

Corollary 3.3 *Two disjoint closed uniformly connected subsets A, B of a thin metric space X are at positive distance.*

Proof. Apply Lemma 3.2 with $H = A, K = B$. QED

Clearly, one can prove the above corollary for UA spaces as well, but we do not put it here explicitly since UA spaces will be proved to be thin in the sequel.

Obviously, a closed uniformly connected subspace of a thin space is connected. The counterpart of this fact for UA was proved in [BD1]. Now we obtain a new proof of this result:

Corollary 3.4 *A closed uniformly connected subspace C of a UA metric space X is connected.*

Proof. If S is not connected it can be partitioned in two non-empty closed sets H and K. Apply Lemma 3.2 with this choice of H and K. QED

Definition 3.5 Given two distinct points a, b of a metric space X such that the uniformly connected component of a contains b, there exists for each n a finite set $L_n \subset X$ whose points form a $1/n$-chain from a to b. We say that the sets L_n, together with a and b, form a (**discrete**) **garland**, if there is an open subset V of X which separates a and b and such that $V \cap \bigcup_n L_n$ is closed (and discrete).

Let X be a metric space and let A, B be two disjoint closed uniformly connected subsets of X. A garland from a point a of A to a point b of B may appear in the following situation:

(1) $d(A, B) = 0$. Now fix $a_n \in A$ and $b_n \in B$ with $d(a_n, b_n) < 1/n$ and take L_n to be the union of two $1/n$-paths, one connecting a to a_n in A, the other connecting b_n to b in B. Now $V = X \setminus (A \cup B)$ witnesses that a, b, $\langle L_n \mid n \in \mathbf{N} \rangle$ is a discrete garland in which $V \cap \bigcup_n L_n$ is empty.

(2) $b \in Q^u_a(X)$ and there exists an open set V of X disjoint from $Q^u_a(X)$ that separates A and B (this entails that $Q^u_a(X)$ is not connected, we show in the proof of Proposition 3.7 how to produce a discrete garland from a to b under this assumption). Let us mention that while the garland in (1) has a very particular nature, the one produced here is a generic one (see Proposition 3.7 for more details).

In the following lemma we show that one can shrink an open separator.

Lemma 3.6 *Let X be a metric space and $V \subseteq X$ an open set which separates two subsets A and B of X. Then there is a open set $U \subseteq X$ which separates A and B and such that $\overline{U} \subset V$.*

Proof. The complement of V can be partitioned in two closed sets $H \supseteq A$ and $K \supseteq B$. By normality H and K are included in two disjoint open sets with disjoint closures H' and K'. The complement U of $H' \cup K'$ works. QED

Now we are in position to give a criterion for thinness for metric spaces in terms of existence of garlands in the space.

Proposition 3.7 *For a metric space X the following conditions are equivalent:*

(a) X is thin;

(b) X contains no garlands;

(c) X contains no discrete garlands.

Proof. (a) $\to$ (b) The existence of a garland $a, b, \langle L_n \mid n \in \mathbf{N}\rangle$ in X leads to the existence of an open set V separating a and b such that $D = V \cap \bigcup_n L_n$ is closed. The complement of V can be partitioned in two separated sets $H \ni a$ and $T \ni b$. Then the closed subspace $Y = H \cup T \cup D$ of X and $a \in Y$ witness the non-thinness of X. Indeed, the uniform quasi component Q of a in Y is not connected since the sets H and $T \cup D$ are closed and disjoint, and they both intersect Q.

(b) $\to$ (c) is trivial.

(c) $\to$ (a) Now assume that X is not thin. In order to produce a discrete garland take a closed subspace Y of X and $a \in Y$ witnessing non-thinness of X. Then there exists a pair of closed disjoint non-empty subsets H and K of Y such that $Q = Q^u_a(Y) = H \cup K$ with $a \in H$. Then $U = X \setminus (H \cup K)$ is an open set of X that separates H and K. By Lemma 3.6 there exists an open subset $V \subseteq U$ separating H and K such that $\overline{V} \subset U$. Fix a point $b \in K$. By Lemma 2.6, for each n there is a finite set L_n in Y whose points form a $1/n$-chain from a to b. We will show that $a, b, \langle L_n \mid n \in \mathbf{N}\rangle$ form a discrete garland. So it suffices to show that $D = V \cap \bigcup_n L_n$ is closed and discrete, i.e., the set D has no accumulation points in X. In fact if x is such an accumulation point, then any neighbourhood of x meets infinitely many L_n, and therefore for every n there is a $1/n$-chain from a to x, showing that x belongs to $Q^u_a = H \cup K$. But clearly x belongs also to the closure $\overline{V} \subseteq U$, which is disjoint from $H \cup K$, and we have a contradiction. QED

A more careful analysis of the above proof shows that if X admits a garland $a, b, \langle L_n \mid n \in \mathbf{N}\rangle$, then the closed set and the uniform quasi component witnessing the fact that X is not thin can be obtained by taking simply the closure Z of the subset $\{a, b\} \cup \bigcup_{n \in \mathbf{N}} L_n$ and the uniform quasi component of a in Z.

The following proposition is the last step in the proof of Theorem B.

Proposition 3.8 *If a metric space X contains a garland, then X is not UA.*

Proof. Given a garland $a, b, \langle L_n \mid n \in \mathbf{N}\rangle$ in X, let V be as required in Definition 3.5. So $D = V \cap \bigcup_n L_n$ is closed and the complement of V can be partitioned in two separated sets $H \ni a$ and $T \ni b$. The sets H and K are closed in the complement of V, hence they are closed also in X (as V is open). It follows that the set $K = T \cup D$ is closed as well. Since $\bigcup_n L_n$ is contained in $H \cup K$, the uniform quasi component of the point a in $H \cup K$ contains b. Hence by Lemma 3.2 X is not UA. QED

This proves that every metric UA space is thin, i.e., Theorem B.

4 The compact separation theorem

Definition 4.1 Given an open cover $\mathcal{P}$ of X and $x \in X$ the **star** $St(x, \mathcal{P})$ is the union of all $P \in \mathcal{P}$ containing x. If $S \subseteq X$, $St(S, \mathcal{P})$ is the union of all the sets $St(x, \mathcal{P})$ with $x \in S$, i.e. the union of all the $P \in \mathcal{P}$ which intersect S.

Definition 4.2 Let $\mathcal{P}$ be an open cover of a topological space X and let $V \subseteq X$. We say that two sets $L_1, L_2 \subseteq X$ are **$(\mathcal{P}, V)$-apart** if for each $P \in \mathcal{P}$ which intersects both sets L_1 and L_2 misses V.

If $V = X$, this intuitively means that the sets L_1 and L_2 are far away from each other (by an amount measured by $\mathcal{P}$). If $V \subseteq X$ we obtain a relative notion: the portions of L_1 and L_2 which are close to V are far away from each other.

Lemma 4.3 *Let $\mathcal{P}$ be an open cover of a topological space X, let $V \subseteq X$ and let $\{L_i \mid i \in I\}$ be a family of finite subsets of X which are pairwise $(\mathcal{P}, V)$-apart. Then $V \cap \bigcup_i L_i$ has no accumulation point in X.*

Proof. If $x \in X$ is an accumulation point of $V \cap \bigcup_i L_i$, then any $P \in \mathcal{P}$ containing x intersects V and contains infinitely many points of $\bigcup_i L_i$. Since each L_i is a finite set, P must intersect infinitely many L_i, contradicting the $(\mathcal{P}, V)$-apartness. QED

The next lemma is the main step in the proof of the compact separation theorem. It permits to shrink an open separator to a smaller open one that is covered a finitely many balls of a given radius $\varepsilon > 0$.

Lemma 4.4 *Let X be a thin metric space. Let A, B be disjoint closed connected subsets of X, and let V be an open set which separates A and B. Then for every $\varepsilon > 0$ there is an open set S, with $\overline{S} \subset V$, which separates A and B and is contained in the union of finitely many balls of radius ε.*

Proof. Fix two points $a \in A$ and $b \in B$. By Lemma 3.6 there is an open set U, with closure included in V, which separates A and B. Since U is an open separator, by Lemma 2.11 there is an open covering $\mathcal{P}$ of X such that U $\mathcal{P}$-cuts between A and B. By refining $\mathcal{P}$ if necessary, we can assume that $\overline{St(U, \mathcal{P})} \subset V$ and each member of $\mathcal{P}$ is contained in a ball of radius ε. Fix a sequence $(\mathcal{P}_n \mid n \in \mathbf{N})$ of open coverings of X, each refining $\mathcal{P}$, and such that every member of $\mathcal{P}_n$ is contained in a ball of radius $1/n$. Consider a finite sequence $L_0, L_1, L_2, \ldots, L_k$ such that each L_i is a finite set whose points form a $\mathcal{P}_i$-chain from some point of A to some point of B and the sets L_n are pairwise $(\mathcal{P}, U)$-apart. We claim that there is a maximal such sequence (possibly empty) $L_0, L_1, \ldots, L_k$. If this is not so, we would obtain a countable sequence $(L_n \mid n \in \mathbf{N})$. Now $U \cap \bigcup_n L_n$ is closed by Lemma 4.3. By Lemma 2.8 we can adjoin to each L_n a portion of A and B to obtain a $1/n$-chain from a to b, thus obtaining a garland which contradicts (by Proposition 3.7) the fact that X is thin. Thus we can fix a maximal sequence $L_0, L_1, \ldots, L_k$ as above and define S as the union of all $P \in \mathcal{P}_{k+1}$ which intersect both U and

$L_0 \cup \ldots \cup L_k$. Then S misses $A \cup B$ since $St(U, \mathcal{P}_{k+1}) \subset V$. Moreover, S $\mathcal{P}_{k+1}$-cuts between A and B because otherwise a chain C witnessing the opposite would contradict the maximality of $L_0, \ldots, L_k$ (for we could set $L_{k+1} = C$). By construction $S \subseteq St(U, \mathcal{P}_{k+1})$ and since $\mathcal{P}_{k+1}$ refines $\mathcal{P}$ we have have $\overline{S} \subseteq V$. The rest is clear. QED

This lemma gives the following immediate corollary.

Corollary 4.5 *Let X be a thin metric space and let A, B be disjoint closed connected subsets of X such that $X \neq A \cup B$. Then there is a collection $\{H_n \mid n \in \mathbf{N}\}$ of nonempty closed subsets of X such that for every n,*

1. *$H_{n+1} \subseteq H_n$,*
2. *H_n separates A and B,*
3. *H_n is contained in a finite union of balls of diameter $< 1/n$.*

Now Theorem A follows from the following more precise result:

Theorem 4.6 *Let X be a complete thin metric space and let A, B be disjoint closed connected subsets of X. Then:*

1. *there is a compact set K such that each neighbourhood of K disjoint from $A \cup B$ separates A and B;*
2. *hence K intersects every closed connected set which meets A and B;*
3. *if X is also locally compact, there is a compact set K' which separates A and B.*

Proof. To prove (1) let H_n be as in Corollary 4.5 and let $K = \bigcap_n H_n$. It suffices to prove that every open neighbourhood U of K contains some H_n. (This shows in particular that K is non-empty, and the compactness of K follows from the fact that K is closed and totally bounded.) We reason by contradiction. So assume that for each n there is $x_n \in H_n \setminus U$. We can then easily extract from $\{x_n\}$ a subsequence $\{y_i\}$ such that for each n all but finitely many of the points y_i lie in only one of the finitely many open balls of the fixed cover of H_n. It follows then that $\{y_i\}$ is a Cauchy sequence, hence it converges to a point y which must lie in K and also outside of U, a contradiction.

To prove (2) consider a closed set C avoiding K and intersecting both A and B. Then the complement of $C \cup A \cup B$ in X is an open neighbourhood of K disjoint from $A \cup B$. Hence C cannot be connected by the separation property established in part (1).

To prove (3) assume that X is locally compact. Then we can find a compact neighbourhood K' of K disjoint from $A \cup B$ and we apply (1). QED

5 Examples and questions

We have seen that a complete thin metric space has CSP. This suggests the following

Question 5.1 Is it true that a complete thin *uniform* space has CSP? What about a complete UA uniform space?

Our next question is about how much we use the fact that **uniform** quasi components are connected.

Question 5.2 Is it true that every complete metric space X such that every closed subspace of X has connected quasi components has necessarily CSP ?

Our next examples show that the implications in Theorems A and B cannot be inverted.

5.1 CSP vs thin and complete

There exist many examples of separable metric space with CSP that are not thin:

(i) the circle minus a point (has two closed connected subsets at distance zero, so it cannot be thin by Corollary 3.3);

(ii) the rationals $\mathbf{Q}$ (uniformly connected non-connected, hence not thin).

None of the above examples is complete. Here we offer an example of a complete separable metric space with CSP that is not thin.

Example 5.3 Let H_1 and H_2 be the branches of hyperbolas $\{(x, y) \in \mathbf{R}^2 : xy = 1\}$ and $\{(x, y) \in \mathbf{R}^2 : xy = 2\}$, respectively, contained in the first quadrant. Then the space $X = H_1 \cup H_2$ with the metric induced from $\mathbf{R}^2$ is a complete separable space. Since H_1 and H_2 are connected and at distance zero, it follows from Corollary 3.3 that X is not thin. On the other hand, the empty set separates the closed connected sets H_1 and H_2. So if A and B are closed connected disjoint sets in X, it remains to consider only the case when both A and B are contained in the same component H_i $(i = 1, 2)$. Now A and B can be separated by a point.

As the referee kindly observed, Theorem A can be given the following more general form. A metrizable space X with compatible metrics d_1, d_2 such that (X, d_1) is complete (i.e. X is Čech-complete) and (X, d_2) is thin admits also a compatible metric d such that (X, d) is complete and thin (namely, $d = max\{d_1, d_2\}$). Hence every Čech-complete metrizable space that admits a compatible thin metric has CSP. This explains why the spaces in (i) and Example 5.3 have CSP. Although we essentially used completeness in the proof of Theorem A do not know whether it is necessary, in other words:

Question 5.4 Are there examples of thin spaces that do not have CSP? What about UA spaces?

As the following example shows, neither thinness nor UA-ness is preserved by passage to completions, thus an immediate application of Theorem A (via passage to completions) cannot help in trying to answer Question 5.4.

Example 5.5 There is a UA metric space whose completion is not thin (hence not UA). Let $X = \bigcup_{n\in\mathbf{N}}\{1/n\} \times I$, where I is the unit interval $[0, 1] \subset \mathbf{R}$, let $a = (0, 0)$, $b = (0, 1)$ and $Y = X \cup \{a, b\}$. We put on Y the following metric. The distance between two points (x_1, y_1) and (x_2, y_2) is $|y_1 - y_2|$ if $x_1 = x_2$. Otherwise the distance is the minimum between $y_1 + y_2 + |x_1 - x_2|$ and $(1 - y_1) + (1 - y_2) + |x_1 - x_2|$. With this metric Y is the completion of X and the two points a, b are the limits for $n \to \infty$ of $(1/n, 0)$ and $(1/n, 1)$ respectively. The space Y is not thin since there is a garland consisting of a, b and $\langle L_n \mid n \in \mathbf{N}\rangle$ where L_n is a $1/n$-chain between a and b in $\{1/n\} \times I$. The space X is UA since X is a union of a chain of compact sets, each attached to the next by at most one point (see [BD1, Theorem 11.4] and the introduction).

5.2 Thin does not imply UA for complete metric spaces

We give an example of a complete connected thin metric space that is not UA.

Example 5.6 For a cardinal α denote by $J(\alpha)$ be the hedgehog of α spikes (see [E, Example 4.1.5]). Let us see that $J(\alpha)$ is thin. By Proposition 3.7 if $J(\alpha)$ is not thin, it contains a discrete garland $a, b, \langle L_n \mid n \in \mathbf{N}\rangle$. Let V be an open set separating a, b such that $V \cap \bigcup_n L_n$ is closed and discrete. The minimal connected set C containing a, b must non-trivially intersect V, so it contains an open interval I on one of the spikes. Now whenever $1/n$ is less than the diameter of I, L_n must intersect I, so $V \cap \bigcup_n L_n$ has an accumulation point, which is a contradiction.

This gives the following immediate corollary based on one of the main results of [BDP2] (see [vD] for the definition of the cardinal number $\mathbf{b}$):

Corollary 5.7 *For every $\alpha \geq \mathbf{b}$ the hedgehog $J(\alpha)$ is thin (so has the property CSP), but not UA.*

The space $J(\alpha)$ is not separable for $\alpha > \omega$. On the other hand, $\mathbf{b} > \omega$ ([vD]), hence the above examples are not separable. It was proved in [BDP2] that the hedgehogs $J(\alpha)$ are UA for all $\alpha < \mathbf{b}$. Hence one cannot hope to get in this way an example of a separable space with the above properties.

Question 5.8 Is it true that every (complete) metric thin separable space is UA?

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