An additivity theorem for uniformly continuous functions *

Alessandro Berarducci

Dipartimento di Matematica, Università di Pisa

Via Buonarroti 2,56127 Pisa, Italy

berardu@dm.unipi.it

Dikran Dikranjan

Dipartimento di Matematica e Informatica, Università di Udine

Via Delle Scienze 206, 33100 Udine, Italy

dikranja@dimi.uniud.it

Jan Pelant

Institute of Mathematics, Czech Academy of Sciences

Žitna 25 11567 Praha 1, Czech Republic

pelant@cesnet.cz

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Abstract

We consider metric spaces X with the nice property that any continuous function $f\colon X \to \mathbb{R}$ which is uniformly continuous on each set of a finite cover of X by closed sets, is itself uniformly continuous. We characterize the spaces with this property within the ample class of all locally connected metric spaces. It turns out that they coincide with the uniformly locally connected spaces, so they include for instance all topological vector spaces. On the other hand, in the class of all totally disconnected spaces, these spaces coincide with the UC spaces.

1 Introduction

All spaces in the sequel are metric and $C(X)$ denotes the set of all continuous real-valued functions of a space X.

The main goal of this paper is to study the following natural notion:

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Definition 1.1. A space X is called **straight** if whenever X is the union of finitely many closed sets, then $f \in C(X)$ is uniformly continuous (briefly, u.c.) iff its restriction to each of the closed sets is u.c.

The following apparently weaker notion is easier to deal with:

Definition 1.2. A space X is called **2-straight** if whenever X is the union of two closed sets, then $f \in C(X)$ is u.c. iff its restriction to each of the closed sets is u.c.

We will prove that the two notions are equivalent (Theorem 2.9).

Example 1.3. The unit circle in $\mathbb{R}^2$ is compact, hence straight. The circle minus one point is not straight: to see this one identifies the circle with the set of complex numbers of absolute value 1 and considers the function $f(\theta) = e^{i\theta}$ defined for $0 < \theta < 2\pi$. Then the inverse function f^{-1} on the unit circle minus one point is not u.c. but its restrictions on $\{e^{i\theta} : 0 < \theta \leq \pi\}$ and on $\{e^{i\theta} : \pi \leq \theta < 2\pi\}$ are u.c.

The following well known notion [A1, A2] is obviously related to straightness.

Definition 1.4. A metric space X is called *UC* provided each $f \in C(X)$ is u.c.

UC spaces are often called Atsuji spaces. Each *UC* space is clearly straight and the converse is true for totally disconnected spaces. This is one of the main results of the paper (Theorem 4.6).

In the presence of local connectedness, things change radically (cf. Example 1.6). Let us recall first the following stronger version of local connectedness:

Definition 1.5. ([HY, 3-2]) A metric space X is **uniformly locally connected**, if for every $\varepsilon > 0$ there is $\delta > 0$ such that any two points at distance $< \delta$ lie in a connected set of diameter $< \varepsilon$.

It is relatively easy to see that uniformly locally connected spaces are straight (Lemma 3.1). For a locally connected metric space X we will prove that X is straight iff it is uniformly locally connected (Theorem 3.9). So we have a complete characterization of straight metric spaces both in the totally disconnected and in the locally connected case.

Example 1.6. Let D be a closed unit disk in the Euclidean plane. Put $X = D \setminus \{c\}$ where c is the center of D. Then X is straight by Theorem 3.9 but it is neither UC nor complete.

In §5 we show that the following two stronger versions of straight coincide with UC:

(a) a space is UC iff all its closed subspaces are straight (Proposition 5.1);

(b) a space X is UC iff whenever X is the union of a countable locally finite family of closed sets, then $f \in C(X)$ is u.c. iff its restriction to each of the closed sets is u.c. (Proposition 5.4).

2 General properties of straight spaces

We start by a well known characterization of UC spaces. For a metric space (X, d), $M \subseteq (X, d)$ and $\varepsilon > 0$, $\mathcal{B}_\varepsilon(M) = \{x \in M \ : \ d(x, M) < \varepsilon\}$; we write just $\mathcal{B}_\varepsilon(x)$ for $M = \{x\}$. It is well-known that each $f \in C(X)$ is uniformly continuous (u.c.) if X is compact or uniformly discrete. Recall that X is *uniformly discrete* if there is $\delta > 0$ such that any two distinct points of X are at distance at least δ.

Theorem 2.1. *A metric space (X, d) is UC iff there is a compact $K \subset X$ such that for any $\varepsilon > 0$, $X \setminus \mathcal{B}_\varepsilon(K)$ is uniformly discrete.*

Clearly, a compact set K as in the above theorem contains all non-isolated points of the space X.

Definition 2.2. Let X be a metric space. Two sequences x_n, y_n in X with $d(x_n, y_n) \to 0$ will be called *adjacent* sequences. If moreover x_n (or equivalently y_n) form a closed discrete set then two sequences x_n, y_n are called *discrete adjacent* sequences. If moreover x_n (or equivalently y_n) is Cauchy (resp. uniformly discrete) then these two sequences x_n, y_n are called *Cauchy discrete adjacent* (*uniformly discrete adjacent*, respectively) sequences. When a particular metric d is needed we will say d-adjacent.

Discrete adjacent sequences allow for an alternative characterization of the UC space:

Theorem 2.3. [Hu] *A metric space X is UC iff X contains no pair of discrete adjacent sequences.*

Definition 2.4. Let (X, d) be a metric space. A pair C^+, C^- of closed sets of X is *u-placed* if $d(C^+_\varepsilon, C^-_\varepsilon) > 0$ holds for every $\varepsilon > 0$, where $C^+_\varepsilon = \{x \in C^+ : d(x, C^+ \cap C^-) \geq \varepsilon\}$ and $C^-_\varepsilon = \{x \in C^- : d(x, C^+ \cap C^-) \geq \varepsilon\}$.

Remark 2.5. Note that $C^+_\varepsilon = C^+$ and $C^-_\varepsilon = C^-$ when $C^+ \cap C^- = \emptyset$ in Definition 2.4. Hence a partition $X = C^+ \cup C^-$ of X into clopen sets is u-placed iff C^+, C^- are uniformly clopen (a subset U of a space X is *uniformly clopen* if its characteristic function $\chi_U \colon X \to \{0, 1\}$ is uniformly continuous where $\{0, 1\}$ is discrete).

Lemma 2.6. *For a metric space (X, d) and a pair C^+, C^- of closed subsets of X the following conditions are equivalent:*

(1) the pair C^+, C^- is u-placed;

(2) a continuous function $f : C^+ \cup C^- \to \mathbb{R}$ is u.c. whenever $f|_{C^+}$ and $f|_{C^-}$ are u.c.

(3) same as (2) with $\mathbb{R}$ replaced by a general metric space (M, ρ)

Proof. We begin with the implication (1) $\Rightarrow$ (3). Assume (1) and let $f : C^+ \cup C^- \to (M, \rho)$ be a continuous function such that $f|_{C^+}$ and $f|_{C^-}$ are u.c. If either $C^+ = C^+ \cup C^-$ or $C^- = C^+ \cup C^-$ the proof is over, so assume $C^+ \neq C^+ \cup C^-$ and $C^- \neq C^+ \cup C^-$.

Note that if $C^+ \cap C^- = \emptyset$, then by Remark 2.5 C^+, C^- are uniformly clopen, i.e., $d(C^+, C^-) > 0$. Then (3) is obviously fulfilled. This is why we assume also $C^+ \cap C^- \neq \emptyset$ from now on.

Let $\varepsilon > 0$. By the uniform continuity of $f|_{C^+}$ and $f|_{C^-}$ there exists $\delta > 0$ such that if $d(x, y) < \delta$ and either $x, y \in C^+$ or $x, y \in C^-$ then

$$\rho(f(x), f(y)) < \varepsilon/2. \tag{$*$}$$

Since the pair C^+, C^- is u-placed we have $\delta_1 = \text{dist}(C^+_{\delta/2}, C^-_{\delta/2}) > 0$. Set $\delta_2 = \frac{1}{2}\min\{\delta, \delta_1\}$. Let us see now that $d(x, y) < \delta_2$ yields $\rho(f(x), f(y)) < \varepsilon$ for any pair $x, y \in C^+ \cup C^-$. If either $x, y \in C^+$ or $x, y \in C^-$ then this is obvious by $(*)$. Suppose $x \in C^+$ and $y \in C^-$. By the definition of δ_1 it follows that for some $z \in C^+ \cap C^-$ one of the following holds

$$\text{either } d(x, z) < \delta/2 \text{ or } d(y, z) < \delta/2.$$

Since also $d(x, y) < \delta/2$ by the choice of δ_2, we conclude that in each case we have

$$d(x, z) < \delta \text{ and } d(y, z) < \delta.$$

Now $(*)$ applied to the pair x and z first, and then to the pair y, z gives $\rho(f(x), f(y)) < \varepsilon$. This concludes the proof of (1)$\Rightarrow$ (3).

Since the implication (3) $\Rightarrow$ (2) is trivial, to finish the proof it suffices to prove the implication (2) $\Rightarrow$ (1). Assume (2). If $C^+ \cap C^- = \emptyset$, then by the previous remark the pair C^+, C^- is u-placed iff C^+, C^- are uniformly clopen. To see that this occurs, consider the characteristic function $\chi : C^+ \cup C^- \to \mathbb{R}$ of the set C^+. It is continuous and (2) yields that χ is u.c. as well. Consequently C^+, C^- are uniformly clopen. Now assume that $C^+ \cap C^- \neq \emptyset$ and consider the function $f : C^+ \cup C^- \to \mathbb{R}$ defined by $f(x) := d(x, C^+ \cap C^-)$, for $x \in C^+$, and by $f(x) := -d(x, C^+ \cap C^-)$, for $x \in C^-$. Obviously the restrictions of f on C^+ and C^- are u.c. so that (2) yields that f is u.c. This implies $d(C^+_\varepsilon, C^-_\varepsilon) > 0$ for every $\varepsilon > 0$. In fact, assume the contrary. Then for some $\varepsilon > 0$ we have $d(C^+_\varepsilon, C^-_\varepsilon) = 0$. This means that there exist two adjacent sequences $(x_n) \subseteq C^+_\varepsilon$ and $(y_n) \subseteq C^-_\varepsilon$. On the other hand, the definition of f gives $f(x_n) \geq \varepsilon$ and $f(y_n) \leq -\varepsilon$, contradicting the uniform continuity of f. □

Corollary 2.7. *A metric space (X, d) is 2-straight iff every pair of closed subsets, which form a cover of X, is u-placed.*

The following easy combinatorial property of adjacent sequences witnessing distance zero for two closed sets covering a straight space will be essentially used in the proof of Theorem 2.9.

Corollary 2.8. *Let X be a 2-straight space and $X = A \cup B$ be a closed cover of X. If $(a_n) \subset A$ and $(b_n) \subset B$ are adjacent sequences, then there is a sequence $(c_n) \subset A \cap B$ adjacent with (a_n).*

Proof. By the above corollary $d(a_n, A \cap B) \to 0$. Now it suffices to take any sequence $(c_n) \subset A \cap B$ witnessing that fact. □

Theorem 2.9. *If a metric space X is 2-straight, then X is straight.*

Proof. Assume X is 2-straight. Let X be the union of finitely many closed sets $C_1, \ldots, C_n$, and let $f \in C(X)$ be such that each restriction $f|_{C_k}$ $(k = 1, 2, \ldots, n)$ of f is u.c. We must prove that f is u.c. The difficulty is that the union of a subfamily of $\{C_1, \ldots, C_n\}$ is not necessarily 2-straight, so the obvious induction on n fails. We define an equivalence relation between sequences (x_n)

of elements of X as follows: $(x_n) \sim (y_n)$ iff both $(x_n), (y_n)$ and $(f(x_n)), (f(y_n))$ are couples of adjacent sequences. If (x_n) and (y_n) are adjacent sequences contained in the same set C_i, then $(x_n) \sim (y_n)$ because $f_{|C_i}$ is u.c. Assume for a contradiction that f is not uniformly continuous. Then there are two sequences $(a_n), (b_n)$ which are adjacent and yet for some $\delta > 0$ we have $d(f(a_n), f(b_n)) > \delta$ for every n, hence in particular $(a_n) \not\sim (b_n)$.

Taking a subsequence if necessary, we can assume that there is a set, say C_1, containing all the elements a_n, and another set C_t containing all the elements b_n. Clearly for every $\alpha \colon \mathbb{N} \to \mathbb{N}$, the subsequences $(a_{\alpha(n)})$ and $(b_{\alpha(n)})$ are not $\sim$-equivalent. To such a function α we associate the set σ_α of those $i \in \{1, \ldots, n\}$ such that the set C_i contains a sequence $\sim$-equivalent to $(a_{\alpha(n)})$. Then $1 \in \sigma_\alpha$ and $t \notin \sigma_\alpha$. Now let us fix α such that $\sigma = \sigma_\alpha$ is maximal with respect to inclusion, and let us argue for a contradiction. Let $A = \bigcup_{i \in \sigma} C_i$ and $B = \bigcup_{k \notin \sigma} C_k$. Since X is 2-straight, by Corollary 2.8 applied to the sequences $(a_{\alpha(n)})$ and $(b_{\alpha(n)})$, there is a sequence $(c_n) \subset A \cap B$ adjacent to $(a_{\alpha(n)})$. We can then find a subsequence $(c_{r(n)})$ contained in some of the intersections $C_i \cap C_k$ for $i \in \sigma$ and $k \notin \sigma$. Define $\gamma \colon \mathbb{N} \to \mathbb{N}$ by $\gamma(n) = \alpha(r(n))$. Then $\sigma_\gamma \supseteq \sigma \cup \{k\}$, contradicting the maximality of σ. □

Corollary 2.10. *A metric space (X, d) is straight iff every pair of closed subsets which cover X is u-placed.*

The above criterion gives a complete characterization of straight spaces. In the case of locally connected spaces we will obtain a better characterization.

We formulate now several easy corollaries of Lemma 2.6. They will be useful in Section 4.

Corollary 2.11. *Let (X, d) be straight. For any clopen $Z \subseteq X$, $dist(Z, X \setminus Z) > 0$.*

Notation 2.12. *For a topological space X, let $Isol(X) = \{x \in X \ : \ x \text{ is isolated in } X\}$ and $Der\, X = X \setminus Isol\, X$. The symbol $Clopen(X)$ denotes the Boolean algebra of all clopen subsets of X. The completion of a metric space (X, d) will be denoted by $(\widetilde{X}, \widetilde{d})$.*

Corollary 2.13. *Let (X, d) be straight. The Boolean algebras $Clopen(X)$ and $Clopen(\widetilde{X})$ are isomorphic. The isomorphism takes $U \in Clopen(X)$ to its closure $\overline{U}$ in $\widetilde{X}$. The inverse is the restriction: $Z \in Clopen(\widetilde{X})$ is mapped to $Z \cap X$. In particular X is connected iff $\widetilde{X}$ is connected.*

Lemma 2.14. *Let (X, d) be straight. Then $\overline{Isol(X)}^X$ is complete.*

Proof. Put $Z = \overline{Isol(X)}^X$. Suppose there is a Cauchy sequence $(x_n)_{n \in \omega}$ in Z without any cluster point. As $Isol(X)$ is dense in Z, we may assume $(x_n)_{n \in \omega} \subset Isol(X)$. Define $D = \{x_{2n+1} : n \in \omega\}$. Then D is a clopen set with $d(D, X \setminus D) = 0$ which contradicts straightness of X, see Corollary 2.11. □

3 A characterization of straight spaces in the locally connected case

Lemma 3.1. *If (X, d) is uniformly locally connected, then (X, d) is straight.*

Proof. Let $X = \bigcup_{i=1}^{k} X_i$ be a finite cover of X by closed sets. Let $f \in C(X)$ and assume that the restriction of X to each X_i is u.c. If f is not u.c. there is $\varepsilon > 0$ and two sequences (x_n) and (y_n) in X such that $d(x_n, y_n) \to 0$ and $|f(x_n) - f(y_n)| > \varepsilon$. (Note that (x_n) and (y_n) cannot have accumulation points in X.) Since X is uniformly locally connected, for n large enough there are connected sets I_n joining x_n and y_n whose diameters tend to 0. On the other hand the diameter of $f(I_n)$ is $> \varepsilon$. Since $f(I_n) = \bigcup_{i=1}^{k} f(I_n \cap X_i)$ is connected, there is some $i \in \{1, \ldots, k\}$ such that $\limsup_n diam(f(I_n \cap X_i)) > \varepsilon/k$, which is absurd since f is u.c. on X_i. □

Remark 3.2. A metric space (X, d) is uniformly locally connected iff for any pair of adjacent sequences (x_n) and (y_n) there are, for n large enough, connected sets I_n joining x_n and y_n, with $diam(I_n) \to 0$. The hypothesis that (X, d) is uniformly locally connected in Lemma 3.1 can be weakened by requiring the stated condition only for those adjacent sequences (x_n) and (y_n) which have no accumulation points in X (cf. Proposition 5.7).

Before stating the famous Yefremovich lemma, we mention another well-known result on metric spaces which may be proved applying the Ramsey theorem.

Lemma 3.3. *Let (X, d) be a metric space and $M \subset X$ be infinite. Then M contains a non-trivial Cauchy sequence or M contains an infinite uniformly discrete subset.*

Lemma 3.4. *(Yefremovich) Let (X, ρ) be a metric space, and let (x_n) and (y_n) be two sequences in X with $\rho(x_n, y_n) > \varepsilon$ for all n. Then there is an infinite set $J \subset \mathbb{N}$ such that $\rho(x_n, y_m) > \varepsilon/4$ for all $n, m \in J$.*

Proof. We say that two pairs (x_n, y_n) and (x_m, y_m) are separated if $d(x_n, y_m) > \varepsilon/4$ and $d(x_m, y_n) > \varepsilon/4$. We must prove that there is an infinite set $J \subset \mathbb{N}$ such that any distinct pairs with indexes in J are separated. If this is not the case, then by Ramsey theorem there is an infinite set $J \subset \mathbb{N}$ such that any distinct pairs with indexes in J are not separated. Hence

$$d(x_n, x_m) \geq 3\varepsilon/4 \text{ and } d(y_n, y_m) \geq 3\varepsilon/4 \text{ for } n < m \text{ in } J. \tag{1}$$

Taking $i(1) < i(2) < i(3) < i(4)$ in J we argue as follows. Since the pairs $(x_{i(1)}, y_{i(1)})$ and $(x_{i(4)}, y_{i(4)})$ are not separated, one has either $d(x_{i(1)}, y_{i(4)}) \leq \varepsilon/4$ or $d(x_{i(4)}, y_{i(1)}) \leq \varepsilon/4$. In the first case (1) gives $d(x_{i(2)}, y_{i(4)}) \geq \varepsilon/2$ and $d(x_{i(3)}, y_{i(4)}) \geq \varepsilon/2$. Then non-separatedness of the pairs $(x_{i(2)}, y_{i(2)})$ and $(x_{i(4)}, y_{i(4)})$ yields $d(x_{i(4)}, y_{i(2)}) \leq \varepsilon/4$. Analogously we get $d(x_{i(4)}, y_{i(3)}) \leq \varepsilon/4$. This yields $d(y_{i(2)}, y_{i(3)}) \leq \varepsilon/2$ which contradicts (1). The case $d(x_{i(4)}, y_{i(1)}) \leq \varepsilon/4$ is handled analogously. □

We will apply Lemma 3.4 to the metric $\rho = d^*$ defined below.

Definition 3.5. Given a metric space (X, d) and $x, y \in X$ we define

$$d^*(x, y) = \min\{1, \inf\{\varepsilon \mid \text{ there is a connected set of diameter} \leq \varepsilon \text{ containing } x \text{ and } y\}\}$$

(so $d^*(x, y) = 1$ if there is no connected set containing x and y).

Lemma 3.6. *1. The map d^* induced by (X, d) is a metric on X.*

2. If d is bounded by 1, $d^* \geq d$.

3. If (X,d) is locally connected, d and d^* induce the same topology on X, and $d^{**} = d^*$.

4. If (X,d) is totally disconnected, (X,d^*) is a uniformly discrete space.

Proof. 1. To prove that d^* is a metric it is enough to observe that $d^*(x,y) < \varepsilon_1$ and $d^*(y,z) < \varepsilon_2$ implies $d^*(x,z) < \varepsilon_1 + \varepsilon_2$.

2. Is clear.

3. Let $\varepsilon < 1$. The obvious implication $d^* < \varepsilon \to d < \varepsilon$ shows that the open sets of (X,d) are open in (X,d^*). For the converse let $B^*(x,\varepsilon) = \{y \mid d^*(x,y) < \varepsilon\}$. It suffices to show that x is in the interior of $B^*(x,\varepsilon)$ in the topology of (X,d). By local connectedness there is a connected open set O containing x and of diameter $< \varepsilon$. Then clearly $x \in O \subseteq B^*(x,\varepsilon)$, so x is in the interior of $B^*(x,\varepsilon)$ as desired. We have thus proved that d and d^* induce the same topology. To see that $d^{**} = d^*$ note that the inequality $d^{**} \geq d^*$ holds by part 2. For the converse suppose $d^*(x,y) < \varepsilon$. Then x, y are contained in a connected set C of d-diameter $< \varepsilon$. For a connected set the d^*-diameter coincides with the d-diameter. So x, y are contained in a connected set of d^*-diameter $< \varepsilon$. Hence $d^{**}(x,y) < \varepsilon$. Therefore $d^{**} \leq d^*$.

4. Is clear, since d^* takes only values 0 and 1 in this case. □

Remark 3.7. (X,d) is uniformly locally connected iff d and d^* are uniformly equivalent, namely every pair of d-adjacent sequences (x_n) and (y_n) is also d^*-adjacent.

Lemma 3.8. *Let (X,d) be a locally connected metric space. Suppose there is $\varepsilon > 0$ and two d-adjacent sequences (x_n) and (y_n) in X with $d^*(x_n, y_n) > \varepsilon$ for every n. Then (X,d) is not straight.*

Proof. Since d^* is a metric (Lemma 3.6) taking a subsequence if necessary we can assume by Lemma 3.4 that $d^*(x_n, y_m) > \varepsilon/4$ for all n, m. According to Lemma 3.3, taking a subsequence we can assume that one of the following two cases holds: 1) (x_n) is a Cauchy sequence; 2) there is $\delta > 0$ such that $d(x_n, x_m) > \delta$ for every n, m.

In case 1) there is a closed ball M_1 of diameter $\varepsilon/8$ containing x_n and y_n for all but finitely many n, say for all $n \geq n_0$. Let $M \supset M_1$ be a ball of diameter $\varepsilon/4$ with the same center as M_1 and let $H = X \setminus int(M)$. For $n \geq n_0$ the distances of x_n and y_n from H are bounded away from 0 (they are $\geq \varepsilon/8$). We claim that the interior of M can be partitioned in two relatively closed sets F and G, one containing all the x_n with $n \geq n_0$, the other containing all the y_n with $n \geq n_0$. Granted this we can write X as the union of the two closed sets $H \cup F$ and $H \cup G$ intersecting in H, and since these sets are not u-placed (as witnessed by our two sequences), we conclude that X is not straight. To prove the claim note first that M has been chosen so small that it cannot contain a connected set joining a point in $\{x_n \mid n \in \mathbb{N}\}$ to a point in $\{y_n \mid n \in \mathbb{N}\}$. Let $A \subset M$ be the closure of the union of the connected components of the points x_n belonging to M, and let $B \subset M$ be defined similarly with respect to the points y_n. Finally let $C \subset M$ be the closure of the union of all components of M which are disjoint from $\{x_n \mid n \in \mathbb{N}\}$ and $\{y_n \mid n \in \mathbb{N}\}$. Since X is locally connected, a point in the interior of M belongs to exactly one of the sets A, B, C. So the claim is proved setting $F = A \cap int(M)$ and $G = (B \cup C) \cap int(M)$.

Consider now case 2) and fix $\delta > 0$ such that $d(x_n, x_m) > \delta$ for every n, m. Define now M as the union of a family of closed balls M_n of the same radius $\lambda < \min\{\delta/2, \varepsilon/4\}$, with M_n centered in x_n. Note that the balls M_n are disjoint, and each one is so small that it cannot contain a connected set joining a point from $\{x_n \mid n \in \mathbb{N}\}$ to a point from $\{y_n \mid n \in \mathbb{N}\}$. Since

$d(x_n, y_n) \to 0$, there is n_0 such that for all $n \geq n_0$ the points x_n, y_n lie in M and their distance from $H = X \setminus int(M)$ is bounded away from 0 (we can arrange so that it $\geq \lambda/2$). Reasoning as in case 1) we can partition $int(M)$ in two disjoint relatively closed sets F, G, with F containing all the x_n with $n \geq n_0$ and G containing all the y_n with $n \geq n_0$. To finish the proof we write X as the union of the two closed sets $H \cup F$ and $H \cup G$, and since these sets are not u-placed (as witnessed by our two sequences), we conclude that X is not straight. □

Theorem 3.9. *Let (X, d) be locally connected. Then (X, d) is straight iff it is uniformly locally connected.*

Proof. By Remark 3.7, Lemma 3.8 and Lemma 3.1. □

4 Totally disconnected straight spaces

We recall that the *quasi-component* of a point x is the intersection of all the clopen sets containing x. Following the terminology from Engelking [E], we call a space *totally disconnected* if all quasi-components are trivial, and *hereditarily disconnected* if all connected components are trivial. Obviously, UC spaces are straight, but UC is a much stronger property than straightness. Our main aim in this sections is to show that straightness coincides with UC for totally disconnected spaces:

As a start we show that at least the *complete* totally disconnected straight spaces are UC spaces. Given a metric space (X, d) we denote by $(\widetilde{X}, \widetilde{d})$ the completion of X.

Lemma 4.1. *Let (X, d) be a totally disconnected metric space. If X is straight then $\widetilde{X}$ is UC.*

Proof. By Theorem 2.1 we have to prove that

$$\operatorname{Der} \widetilde{X} = \widetilde{X} \setminus Isol(\widetilde{X}) = \{x \in \widetilde{X} \ : x \text{ is not isolated in } \widetilde{X}\}$$

is compact and for each $\varepsilon > 0$, the set $\widetilde{X} \setminus \mathcal{B}_\varepsilon(\operatorname{Der} \widetilde{X})$ is uniformly discrete. Recall that $Isol(\widetilde{X}) = Isol(X)$.

Claim. X cannot contain two uniformly discrete adjacent sequences (see Definition 2.2).

We use Corollary 2.7. Suppose $(x_n)_{n\in\omega}$ and $(y_n)_{n\in\omega}$ are uniformly discrete adjacent sequences in X. Take $\varepsilon > 0$ such that $d(x_n, x_m) > \varepsilon$ whenever $n \neq m$. Without loss of generality, we may and shall suppose that $d(x_n, y_n) < \frac{\varepsilon}{4}$ for each n. Put $H = X \setminus \bigcup_{n\in\omega} \mathcal{B}_{\frac{\varepsilon}{2}}(x_n)$. Hence H is closed and the distance of any point x_n or y_n from H is at least $\frac{\varepsilon}{4}$. For each n, take a clopen set C_n containing x_n and avoiding y_n (total disconnectedness is used). Put

$$C^+ = H \cup \bigcup_{n\in\omega} C_n \cap \overline{\mathcal{B}_{\frac{\varepsilon}{2}}(x_n)}$$

and

$$C^- = H \cup \bigcup_{n\in\omega} (\overline{\mathcal{B}_{\frac{\varepsilon}{2}}(x_n)} \setminus C_n).$$

So Corollary 2.7 finishes the proof of the claim.

We have therefore proved that each pair of discrete adjacent sequences in X must be Cauchy.

Our next goal is to prove that $\operatorname{Der}\widetilde{X}$ is compact. Assume the contrary, i.e. there is a sequence $(x_n)_{n\in\omega}$ in $\operatorname{Der}\widetilde{X}$ which is ε-discrete for $\varepsilon > 0$. Since $\operatorname{Der}\widetilde{X}$ consists of non-isolated points, this produces a pair of uniformly discrete adjacent sequences in X which contradicts the above claim.

It remains to prove that for any $\varepsilon > 0$, $\widetilde{X} \setminus \mathcal{B}_\varepsilon(\operatorname{Der}\widetilde{X})$ is uniformly discrete.

Put $D = \widetilde{X} \setminus \mathcal{B}_\varepsilon(\operatorname{Der}\widetilde{X})$. As X is dense in $\widetilde{X}$ we obtain that $D \subset X$. Assume that D is not uniformly discrete, i.e. for each positive integer n there are $x_n, y_n \in D$ with $d(x_n, y_n) < \frac{1}{n}$. As these points are at least ε-distant from $\operatorname{Der}\widetilde{X}$, using Lemma 3.3, we may assume without loss of generality that the sequences (x_n) and (y_n) are uniformly discrete adjacent sequences which contradicts the claim. □

Corollary 4.2. *Let (X, d) be a totally disconnected metric space. If X is straight then $\operatorname{Der}\widetilde{X}$ is compact.*

Proof. By Lemma 4.1 $\widetilde{X}$ is UC, hence by the characterization of UC spaces (Theorem 2.1) $\operatorname{Der}\widetilde{X}$ is compact. □

Lemma 4.3. *Let (X, d) be a totally disconnected and straight metric space. Then its completion $\widetilde{X}$ is totally disconnected.*

Proof. Let $z \in \widetilde{X}$ and let Q_z be the quasi-component of z. We must prove $|Q_z| = 1$. Assume for a contradiction $|Q_z| > 1$.

We claim that $|Q_z \cap X| \leq 1$. In fact, if x, y are distinct points in $Q_z \cap X$, then since X is totally disconnected there exists a clopen set Z in X separating x and y in X. Then its closure in $\widetilde{X}$ separates x and y in $\widetilde{X}$ (Corollary 2.11) contradicting the fact that x and y belong to the same quasi component in $\widetilde{X}$.

Since $|Q_z| > 1$, the claim implies that Q_z contains at least one point z' in $\widetilde{X} \setminus X$. Since $Q_z = Q_{z'}$, changing the point if necessary we can assume that $z \in \widetilde{X} \setminus X$.

Let $P = Q_z \cap X$. By the claim P is either empty or consists of a single point. So since $z \notin X$, we can fix an open set $B \supseteq P$ so small that $z \notin \overline{B}$ (all closures are taken in $\widetilde{X}$).

By Lemma 2.14 $\overline{Isol(X)} \subseteq X$, so every point of $\widetilde{X} \setminus X$ belongs to the interior of $\operatorname{Der}\widetilde{X} = \widetilde{X} \setminus Isol(X)$. In particular z is in the interior of $\operatorname{Der}\widetilde{X}$.

Clearly Q_z contains no isolated points of $\widetilde{X}$, so $Q_z \subseteq \operatorname{Der}\widetilde{X}$. By Corollary 4.2 $\operatorname{Der}\widetilde{X}$ is compact. So Q_z is compact, too. Note also that Q_z has an empty interior in $\widetilde{X}$, as otherwise by the density of X in $\widetilde{X}$ and the fact that Q_z contains no isolated points, Q_z would contain infinitely many points of X.

Now Q_z is the intersection of all clopen sets of $\widetilde{X}$ that contain Q_z, so using the compactness of $\operatorname{Der}\widetilde{X}$ we can find for every n a clopen set U_n of $\widetilde{X}$ containing Q_z such that $U_n \cap \operatorname{Der}\widetilde{X}$ is contained in the open neighbourhood $\mathcal{B}_{1/n}(Q_z)$. We can arrange so that $U_n \supseteq U_{n+1}$. Observe that

$$U_n \cap \operatorname{Der}\widetilde{X} \not\subseteq Q_z \cup \overline{B} \tag{1}$$

for every n. Indeed a neighbourhood of z lies in the former set and not in the latter.

Define $W_n = U_n \setminus U_{n+1}$. By (1) for infinitely many n we have

$$W_n \cap \operatorname{Der}\widetilde{X} \not\subseteq Q_z \cup \overline{B}. \tag{2}$$

We have thus produced an infinite family (W_n) with the following properties:

(i) Each W_n is clopen:

(ii) The sets W_n are pairwise disjoint;

(iii) $W_n \cap \operatorname{Der} \widetilde{X}$ is not included in $\overline{B}$ for every n;

(iv) $W_n \cap \operatorname{Der} \widetilde{X} \subseteq \mathcal{B}_{1/n}(Q_z)$ for every n.

Taking a subsequence we can also assume that (2) holds for every n. So for each n we can choose a point $w_n \in \widetilde{X}$ in the difference $(W_n \cap \operatorname{Der} \widetilde{X}) \setminus (Q_z \cup \overline{B})$. Since all the points w_n lie in the compact set $\operatorname{Der} \widetilde{X}$, taking a subsequence we can assume that $d(w_n, w_{n+1}) \to 0$. Hence $d(W_n \setminus \overline{B}, W_{n+1} \setminus \overline{B}) \to 0$. Since $W_n \setminus \overline{B}$ is open in $\widetilde{X}$,

$$d\big((W_n \setminus \overline{B}) \cap X, (W_{n+1} \setminus \overline{B}) \cap X\big) \to 0. \tag{3}$$

Define

- $V = \bigcup_n W_{2n}$,
- $C^+ = \overline{Isol(X)} \cup (V \cap X) \cup P$,
- $C^- = \overline{Isol(X)} \cup (X \setminus V)$.

By (iv) above, $\overline{V \cap X} = (V \cap X) \cup P$, so C^+ is closed in X. Obviously C^- is also closed in X. We have $C^+ \cup C^- = X$ and $C^+ \cap C^- = \overline{Isol(X)} \cup P$. We will show that the pair C^+, C^- is not u-placed, contradicting the straightness of X. To this aim we must ensure that $d(C^+_\varepsilon, C^-_\varepsilon) = 0$ for some appropriate $\varepsilon > 0$. Recall that $C^+_\varepsilon = C^+ \setminus \mathcal{B}_\varepsilon(C^+ \cap C^-)$ and $C^-_\varepsilon = C^- \setminus \mathcal{B}_\varepsilon(C^+ \cap C^-)$.

Now $Q_z \setminus B$ is included in $\widetilde{X} \setminus X$, so it is disjoint from $\overline{Isol(X)}$, and since $Q_z \setminus B$ is compact, it has a positive distance r from $Isol(X)$. Choose $\varepsilon < r/2$ and such that $\mathcal{B}_\varepsilon(P) \subseteq B$. By (iv) for n sufficiently large $W_n \setminus B$ is disjoint from $\mathcal{B}_\varepsilon(C^+ \cap C^-)$. The desired conclusion follows from (3) and the fact that $C^+_\varepsilon \supseteq (W_{2n} \setminus \overline{B}) \cap X, C^-_\varepsilon \supseteq (W_{2n+1} \setminus \overline{B}) \cap X$ for n sufficiently large. ☐

Lemma 4.4. *Let (X, d) be totally disconnected and straight. Then (X, d) is complete.*

Proof. Assume for a contradiction that $X \neq \widetilde{X}$. Take $x \in \widetilde{X} \setminus X$. By Lemma 2.14, there is $\varepsilon > 0$ such that $\mathcal{B}_\varepsilon(x) \cap Isol(X) = \emptyset$. By Corollary 4.2 $\operatorname{Der} \widetilde{X}$ is compact. Hence $\widetilde{X}$ is locally compact at x. By Lemma 4.3 $Q_x = \{x\}$, so $\{x\}$ is an intersection of clopen sets in $\widetilde{X}$. By local compactness of $\widetilde{X}$ it then follows that $\widetilde{X}$ has a local base $(W_n)_n$ at x of clopen sets. We can arrange so that $W_n \supseteq W_{n+1}$. Define $V = \bigcup_n W_{2n} \setminus W_{2n+1}$. Note that $X \cap V$ is a clopen subset of X at distance zero from its complement. This contradicts Corollary 2.11. ☐

Remark 4.5. This Lemma should be compared with Example 1.6. It could be interesting to know conditions implying completeness for straight spaces.

Theorem 4.6. *Let (X, d) be a totally disconnected metric space. Then X is straight iff X is UC.*

Proof. By Lemmas 4.1 and 4.4. ☐

5 Further results on straight spaces

We show in this section that two very natural stronger versions of straightness coincide with UC.

A closed subspace of a straight space need not be straight (just take two branches of the hyperbola in the plane). Since closed subspaces of UC spaces are UC, every closed subspace of a UC space is straight. Surprisingly, the converse is also true:

Proposition 5.1. *For every metric space X the following are equivalent:*

1. *X is UC;*
2. *every closed subspace of X is straight, i.e. X is hereditarily straight.*
3. *every pair of closed subsets is u-placed.*

Proof. We only need to prove the implication $2 \Rightarrow 1$. Assume X is not UC. Then one can find a pair A, B of disjoint closed countable sets with $d(A,B)=0$. Now the closed subspace $Y = A \cup B$ of X is not straight being a union of two disjoint clopen sets A, B with $d(A,B)=0$. □

Proposition 5.2. *For a straight space X TFAE:*

1. *X is complete*
2. *every closed precompact subspace of X is straight*
3. *every closed precompact subspace of X is compact*

Proof. Obviously, 1. $\Rightarrow$ 3. $\Rightarrow$ 2. To prove 2. $\Rightarrow$ 1., it suffices to note that every non-convergent Cauchy sequence in X is a closed precompact subset that is not straight (by Lemma 4.4). □

By the above argument we obtain an example of a space X containing a straight subspace Y and a compact subspace K such that their *intersection* is not straight (take any straight non-complete space Y, let X be its completion and let K be any convergent sequence (y_n) such that (y_n) is in Y, but the limit $y \notin Y$.) This should be compared with the following result, whose proof will be given in [BDP2].

Theorem 5.3. *Let X be a metric space and $X = K \cup Y$, where K is a compact subspace of X and Y is a closed subset of X. Then X is straight iff Y is straight.*

Clearly every UC space permits gluing of arbitrary family of u.c. functions, whereas straight spaces permit gluing of *finite* families of u.c. This raises the question whether it is possible to strengthen Theorem 2.9 to *arbitrary* families of u.c. functions. Again, this stronger version of straight does not give anything new, since this notion coincides with UC:

Proposition 5.4. *For every metric space X the following are equivalent:*

1. *X is UC;*
2. *whenever X can be written as a union of a locally finite family $\{C_i\}_{i \in I}$ of closed sets we have that $f \in C(X)$ is u.c. iff each restriction $f|_{C_i}$ of f, $i \in I$, is u.c.;*

3. if $X = \bigcup_{n=1}^{\infty} C_n$, where $\{C_n\}_{n\in\mathbb{N}}$ is a locally finite family of open (closed) sets, we have that $f \in C(X)$ is u.c. iff each restriction $f|_{C_n}$ of f, $n \in \mathbb{N}$, is u.c.

Proof. Obviously, $1 \to 2 \to 3$.

To prove the implication $3 \to 1$ assume that X is not UC. Then there exists a pair A, B of closed disjoint countable discrete sets in X with $d(A, B) = 0$. Let $A = \{a_n\}_n$ and $B = \{b_n\}_n$ be one-to-one numeration of A and B with $d(a_n, b_n) \to 0$. Choose a sequence ε_n of positive real numbers such that:

1) the open ε_n-balls U_n with center a_n are pairwise disjoint;
2) the open ε_n-balls V_n with center b_n are pairwise disjoint;
3) $U_n \cap V_n = \emptyset$ for each n.

For every n let B_n (resp. D_n) be the closed $\varepsilon_n/2$-ball with center a_n (resp., b_n). Set $W = X \setminus \bigcup_n (B_n \cup D_n)$. Since the family $\{U_n\}_n \cup \{V_n\}_n$ is locally finite, $\bigcup_n (B_n \cup D_n)$ is closed, hence W is an open set in X. Therefore, $X = W \cup \bigcup_n (U_n \cup V_n)$ is a countable locally finite open cover of X. Now define a function $f : X \to \mathbb{R}$ as follows. If $x \in U_n$ for some $n \in \mathbb{N}$, let $f(x) = \frac{d(x, X\setminus U_n)}{\varepsilon_n}$, otherwise set $f(x) = 0$. Since for every n the restriction $f|_{U_n}$ is a $\frac{1}{\varepsilon_n}$-Lipschitzian function, clearly all restrictions $f|_{U_n}$, $f|_{V_n}$ and $f|_W$ are u.c., so that in particular f is continuous. Since $d(a_n, b_n) \to 0$ with $f(a_n) \geq 1$, $f(b_n) = 0$, the function f is not u.c. This contradicts the hypothesis 3). Replacing this open cover with a closed one with the same properties one obtains a similar proof for the version of 3 with closed sets. □

In connection with item 3 of the above proposition let us mention that if X is a straight space and $X = \bigcup_{n=1}^{m} C_n$ is an arbitrary finite cover of X, then $f \in C(X)$ is u.c. iff each restriction $f|_{C_n}$ of f, $1 \leq n \leq m$, is u.c. (it suffices to observe that uniform continuity of $f|_{C_n}$ implies uniform continuity of $f|_{\overline{C_n}}$ and apply straightness to the closed cover $X = \bigcup_{n=1}^{m} \overline{C_n}$). In other words, the restraint to consider only *closed* covers can be relaxed in Definition 1.1 and 1.2.

5.1 Open questions

Problem 5.5. *Find a characterization of the straight spaces in the category of uniform spaces, both in the locally connected and in the totally disconnected case.*

Clearly, one can try to solve this problem by introducing an appropriate version of uniform local connectedness for uniform spaces. In a forthcoming paper [BDP2] we study straightness for some uniform spaces (as topological groups), as well as the connection between a stronger version of straightness and uniform local connectedness. Some preservation properties of straight spaces can also be found in [BDP2] (in particular preservation straightness under taking extensions).

The following questions is suggested by Theorem 4.6:

Question 5.6. Are hereditarily disconnected straight metric spaces UC?

A positive answer holds in the totally disconnected case. One can easily see that hereditarily disconnected UC spaces are totally disconnected (actually, zero-dimensional). Therefore, to answer negatively this question it suffices to find straight hereditarily disconnected space that is not totally disconnected.

We have given complete characterization of the straight metric spaces both in the locally connected and in the totally disconnected case, but we still do not know a characterization for general metric spaces. A sufficient condition is given by a weakening of uniform local connectedness, as the following proposition shows (see Remark 3.2).

Proposition 5.7. [BDP2] *Suppose that for any pair of discrete adjacent sequences* (x_n) *and* (y_n) *in* (X, d)*, there are, for* n *large enough, connected sets* I_n *joining* x_n *and* y_n*, with* $diam(I_n) \to 0$. *Then* (X, d) *is straight.*

Question 5.8. Is the hypothesis in the above proposition a necessary and sufficient condition for a complete space to be straight? (In the non-complete case there are counterexamples [BDP2]).

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References

[A1] Masahiko Atsuji, *Uniform continuity of continuous functions on metric spaces*, Pac. J. Math. 8 (1958) 11-16.

[A2] Masahiko Atsuji, *Uniform continuity of continuous functions on metric spaces*, Can. J. Math. 13 (1961) 657-663.

[BD] A. Berarducci, D. Dikranjan, *Uniformly approachable functions and UA spaces*, in: International Conference on Topology, Trieste 1993, Rend. Its. Matematico Univ.di Trieste, **25** (1993) 23–56.

[BDP1] A. Berarducci, D. Dikranjan, J. Pelant, *Functions with distant fibers and uniform continuity*, To appear in Topology and Its Applications **121** (2002), no. 1-2, 3–23.

[BDP2] A. Berarducci, D. Dikranjan, J. Pelant, *Locally connectedness and extension of uniformly continuous functions*, work in progress.

[E] R. Engelking, *General Topology, Revised and Completed Edition*, Sigma Series in Pure Mathematics, vol. **6**, Heldermann Verlag, Berlin 1989.

[HY] J. G. Hocking, G. S. Young, *Topology*, New York, Dover Publications 1988 (originally published: Reading, Massachusetts, Addison – Wesley 1961).

[Hu] H. Hueber, *On uniform continuity and compactness in metric spaces*, Amer. Math. Monthly **88** (1981), no. 3, 204–205.

[P] W. Page, *Topological Uniform Structures*, Dover Publications, Inc., New York (1988).