Local connectedness and extension of uniformly continuous functions*

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Abstract

A metric space X is straight if for each finite cover of X by closed sets, and for each real valued function f on X, if f is uniformly continuous on each set of the cover, then f is uniformly continuous on the whole of X. The straight spaces have been studied in [BDP1], which contains characterization of the straight spaces within the class of the locally connected spaces (they are the uniformly locally connected ones) and the class of the totally disconnected spaces (they coincide with the totally disconnected Atsuji spaces). We show that the completion of a straight space is straight and we characterize the dense straight subspaces of a straight space. In order to clarify further the relation between straightness and the level of local connectedness of the space we introduce two more intermediate properties between straightness and uniform local connectedness and we give various examples to distinguish them. One of these properties coincides with straightness for complete spaces and provides in this way a useful characterization of complete straight spaces in terms of the behaviour of the quasi-components of the space.

1 Introduction

All spaces in the sequel are metric. Given a space X, $C(X)$ denotes the set of all continuous functions $f\colon X \to \mathbb{R}$. The following notion, already studied in [BDP1], will be the object of investigation of this paper.

Definition 1.1. A space X is called **straight** if whenever X is the union of finitely many closed sets, then $f \in C(X)$ is uniformly continuous (briefly, u.c.) if and only if its restriction to each of the closed sets is u.c.

Example 1.2. Every compact space is obviously straight. For the same reason every UC space is straight (a space X is UC, or Atsuji, if every $f \in C(X)$ is uniformly continuous).

The property of being straight is strictly related to the connectivity properties of the space.

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Definition 1.3. A space X is **locally connected** at $x \in X$ if and only if every neighbourhood of X contains a connected neighbourhood. X is locally connected if and only if it is locally connected at every point.

Definition 1.4. ([HY, 3-2]) A metric space X is **uniformly locally connected** (ULC), if for every $\varepsilon > 0$ there is $\delta > 0$ such that any two points at distance $< \delta$ lie in a connected set of diameter $< \varepsilon$.

A locally connected space is straight if and only if it is ULC ([BDP1, Theorem 3.9]). In particular $\mathbb{R}$ is straight and every topological vector space is straight. The circle minus a point is not straight (it is locally connected but not uniformly so).

$\mathbb{Z}$ is straight. $\mathbb{Q}$ is not straight. More generally a totally disconnected space is straight if and only if it is a UC-space ([BDP1, Theorem 4.6]).

It is a non-trivial fact, proved in [BDP1], that in the definition of "straight" it suffices to consider binary unions:

Theorem 1.5. *([BDP1]) A space X is straight if and only if whenever X is the union of two closed sets, then $f \in C(X)$ is u.c. if and only if its restriction to each of the closed sets is u.c.*

Using this characterization one can prove the following necessary and sufficient condition for straightness, which will be often used in the sequel. First a definition.

Definition 1.6. Let (X, d) be a metric space. A pair C^+, C^- of closed sets of X is *u-placed* if $d(C^+_\varepsilon, C^-_\varepsilon) > 0$ holds for every $\varepsilon > 0$, where $C^+_\varepsilon = \{x \in C^+ : d(x, C^+ \cap C^-) \geq \varepsilon\}$ and $C^-_\varepsilon = \{x \in C^- : d(x, C^+ \cap C^-) \geq \varepsilon\}$.

In other words C^+, C^- is u-placed if for every pair of sequences $x_n \in C^+$ and $y_n \in C^-$ with $d(x_n, y_n) \to 0$, we have $d(x_n, C^+ \cap C^-) \to 0$ (for $n \to \infty$).

Theorem 1.7. *([BDP1, Corollary 2.10]) A metric space (X, d) is straight if and only if every pair of closed subsets, which form a cover of X, is u-placed.*

We will also use the following:

Proposition 1.8. *If a metric space (X, d) is straight and a metric space (Y, ρ) contains X as a dense subspace, then Y is straight.*

Proof. Immediate from the definition of straight space, using the fact that a continuous function on Y which is u.c. on X is u.c. on Y. ☐

dense in Y,

In particular, the completion of a straight space is straight. More generally, we characterize the dense straight subspaces of a straight space (Theorem 2.2).

In order to better understand the class of straight spaces it is convenient to introduce, besides the ULC spaces already mentioned, two further classes of spaces related to the connectivity properties of the space. A metric space X is *weakly uniformly locally connected* (shortly WULC), if for every pair of sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ in X with $d(x_n, y_n) \to 0$ and without accumulations points, there exist $n_0 \in \mathbb{N}$ and connected sets C_n in X for each $n \geq n_0$, such that $x_n, y_n \in C_n$ and diam $C_n \to 0$. The larger class of ALC space is defined analogously, using a weaker notion of connectivity (see Definition 4.7). It is important to keep in mind that we have the implications:

$$ULC \Rightarrow WULC \Rightarrow ALC \Rightarrow \text{ straight} \tag{1}$$

and that these conditions are all equivalent for locally connected spaces (by Theorem 3.4). The situation changes when local connectedness is traded for completeness. For complete spaces ALC is equivalent to straight (Theorem 4.11) and yields a useful characterization in terms of the properties of the quasi-components of the space (Corollary 4.12). On the other hand, UC implies WULC (Lemma 4.15), but does not imply ULC, so the first implication cannot be inverted even for UC spaces (recall that UC spaces are complete). A rather involved counterexample shows that WULC is properly stronger than ALC even for complete spaces (Theorem 5.13).

On the other hand, there are many properties of WULC spaces and ALC ones, which are quite similar. This similarity stems from Lemma 4.17 which characterizes both properties in the style of the well-known description of UC spaces.

Since ALC coincides with straightness for complete spaces, it makes sense to compare these two properties for spaces having the strongest *complementary* property, namely, precompact spaces. It turns out that the property ALC is properly stronger than straightness even in the class of locally compact precompact spaces (see §5), or in the framework of precompact topological groups (see [DP]).

The class of straight spaces is not closed under finite products and the problem to establish when straightness is available for finite products of spaces is extremely hard. In a forthcoming paper [BDP2] dedicated entirely to this problem we apply the above mentioned criterion for straightness (and its stronger versions, ALC, WULC, ULC) of dense subspaces to obtain a complete solution. More precisely, we show that the product $X \times Y$ of two metric spaces is straight if and only if both X and Y are straight and one of the following conditions holds: (a) both X and Y are precompact; (b) both X and Y are ULC; (c) one of the spaces is both precompact and ULC.

Notations.

1. Unless otherwise stated, the metric $d(x,y)$ on a product $\Pi_{i=1}^{n} X_i$ of finitely many metric spaces (X_i, d_i) is defined as the sum $\Sigma_i d_i(x_i, y_i)$, where x_i, y_i are the coordinates of x, y.

2. We will frequently use subscripts like C_ε^+, C_ε^- and variants of it (e.g. $A_\varepsilon, B_\varepsilon$ where A, B is a given binary cover of a space). Such notations refers to Definition 1.6.

3. The ball of center x and radius ε in a metric space (X,d) is denoted by $B_\varepsilon(x)$. If the metric is not clear from the context we also use the notation $B_\varepsilon^d(x)$. For a metric space M, we use also $B_\varepsilon^M(x)$; it can be convenient when we deal with a space and its subspaces.

2 Tight extensions

Our first result is about preservation of straightness under extensions. The following property of extensions will be needed.

Definition 2.1. An extension $X \subseteq Y$ of topological spaces is called **tight** if for every closed binary cover $X = F^+ \cup F^-$ one has

$$\overline{F^+}^Y \cap \overline{F^-}^Y = \overline{F^+ \cap F^-}^Y. \tag{2}$$

Let us note that even a one-point extension can easily fail to be tight: take $X = \{1/n : n \in \mathbb{N}\}$, $Y = X \cup \{0\}$ and as F^+, F^- the subsequences with even and odd indices respectively.

Theorem 2.2. *Let X, Y be metric spaces, $X \subseteq Y$ and let X be dense in Y. Then X is straight if and only if Y is straight and the extension $X \subseteq Y$ is tight.*

Proof. (a) If X is straight then Y is straight as well by Proposition 1.8.

(b) Now we prove that if the extension $X \subseteq Y$ is tight and Y is straight, then also X is straight. Consider a closed binary cover $X = F^+ \cup F^-$. Then $\overline{F^+} \cup \overline{F^-} = Y$ is a closed binary cover of Y. Let $\varepsilon > 0$. Then

$$d((\overline{F^+})_\varepsilon, (\overline{F^-})_\varepsilon) > 0 \tag{3}$$

by straightness of Y and Theorem 1.7. On the other hand, it can be easily seen using the tigthness of the extension that $F^+_\varepsilon \subseteq (\overline{F^+})_\varepsilon$ and $F^-_\varepsilon \subseteq (\overline{F^-})_\varepsilon$ (in fact if $d(x, F^+ \cap F^-) > \varepsilon$, then clearly $d(x, \overline{F^+ \cap F^-}) > \varepsilon$ and by the tightness of the extension $d(x, \overline{F^+} \cap \overline{F^-}) > \varepsilon$). Thus (3) yields now $d(F^+_\varepsilon, F^-_\varepsilon) > 0$. This proves that X is straight.

(c) It remains to prove that if X is straight, then the extension $X \subseteq Y$ is tight. Assume for a contradiction that the extension $X \subseteq Y$ is not tight. Then there exists a closed binary cover $X = F^+ \cup F^-$ such that (2) fails. Then there exists some $y \in \overline{F^+} \cap \overline{F^-}$ such that $y \notin \overline{F^+ \cap F^-}$. Clearly, then $y \notin X$. By the assumption $y \notin \overline{F^+ \cap F^-}$ we can find $\varepsilon > 0$ such that $d(y, F^+ \cap F^-) > 2\varepsilon$. Pick sequences $z^+_n \in F^+$ and $z^-_n \in F^-$ with $z^+_n \to y$, and $z^-_n \to y$ and $d(z^+_n, F^+ \cap F^-) > \varepsilon$, $d(z^-_n, F^+ \cap F^-) > \varepsilon$. Therefore $z^+_n \in F^+_\varepsilon$ and $z^-_n \in F^-_\varepsilon$, witnessing $d(F^+_\varepsilon, F^-_\varepsilon) = 0$. This shows that X cannot be straight by Theorem 1.7. ☐

Corollary 2.3. *Let X be a metric space. Then X is straight if and only if its completion $\widetilde{X}$ is straight and $\widetilde{X}$ is a tight extension of X.*

Proposition 2.4. *Let Y be a metric space and $Y = K \cup X$, where K is a compact subspace of Y and X is a closed subset of X. Then Y is straight if and only if X is straight.*

Proof. Suppose X is not straight. Then $X = X^+ \cup X^-$ for some pair of closed sets X^+, X^- which is not u-placed. We claim that $K \cup X^+$ and $K \cup X^-$ are not u-placed in Y, witnessing the fact that Y is not straight. In fact by the assumptions there is $\varepsilon > 0$ and sequences $x_n \in X^+$ and $y_n \cup X^-$ with $d(x_n, y_n) \to 0$ and $d(x_n, B^+ \cap B^-) > \varepsilon$. Clearly $(x_n)_{n \in \mathbb{N}}$ cannot have converging subsequences as otherwise the limit point would be in $X^+ \cap X^-$. It then follows, since K is compact, that $d(\{x_n \mid n \in \mathbb{N}\}, K) > 0$. Thus the same pair of sequences witness that $K \cup X^+$ and $K \cup X^-$ are not u-placed in Y.

For the converse suppose that Y is not straight. Then $Y = Y^+ \cup Y^-$ for some pair of closed sets Y^+, Y^- which is not u-placed. We claim that $X^+ = Y^+ \cap X$ and $X^- = Y^- \cap X$ form a pair which is not u-placed in X, and therefore X is not straight. In fact by the assumptions there is $\varepsilon > 0$ and sequences $x_n \in Y^+$ and $y_n \in Y^-$ such that $d(x_n, y_n) \to 0$ and $d(x_n, Y^+ \cap Y^-) > \varepsilon$. As above these sequences cannot have converging subsequences and therefore they are at positive distance from K. It then follows that these sequences are actually in X, witnessing the fact that the pair X^+, X^- is not u-placed. ☐

It is important to underline that the subset Y in the above corollary is *closed*. Indeed, if X is the circle, and K is just a singleton in X, then $Y = X \setminus K$ is not straight. An alternative proof of the above result, kindly proposed by the referee, follows directly from the definition of straight space and the observation that, since $K \subset Y$ is compact, then $f \in C(Y)$ is u.c. iff its restriction to $Y \setminus K$ is u.c. (plus the observation that any continuous function on a closed subset X of Y extends to a continuous function on Y).

3 Uniform local connectedness

Let us start be an equivalent description of ULC spaces.

Lemma 3.1. *A metric space (X, d) is ULC if and only if for each $\varepsilon > 0$ there is a positive δ such that for each $x \in X$, there is a connected set W_x such that*

$$B_\delta(x) \subseteq W_x \subseteq B_\varepsilon(x). \tag{4}$$

Proof. Assume (X, d) is ULC. Given $\varepsilon > 0$ let $\delta > 0$ be such that any two points at distance $< \delta$ lie in a connected set of diameter $< \varepsilon$. Fix $x \in X$. Given $y \in B_\delta(x)$, we can then find a connected set C_y containing x and y and contained in $B_\varepsilon(x)$. The union $\bigcup_y C_y$ is the desired connected set W_x. The opposite implication is obvious. ☐

As a corollary we obtain:

Proposition 3.2. *A ULC metric space is locally connected.*

We will however also need the more informative version given by Lemma 3.1.

Proposition 3.3. *([BDP1, Lemma 3.1]) Uniformly locally connected spaces are straight.*

Theorem 3.4. *([BDP1, Theorem 3.9]) A locally connected metric space is straight if and only if it is uniformly locally connected.*

Definition 3.5. Given a metric space (X, d) and $x, y \in X$ we define $d^*(x, y)$ as the minimum between 1 and the infimum of the diameters of the connected subsets I of X containing x and y. So if there is no such a set I, $d^*(x, y) = 1$.

It is an easy and well known fact that d^* is a metric (see for instance [HS, Thm. 2.4] or [BDP1, Lemma 3.6]). Using this metric we can characterize the uniform local connectedness as follows:

Remark 3.6. A space X is uniformly locally connected if and only if for each pair of sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in X with $d(x_n, y_n) \to 0$, we have $d^*(x_n, y_n) \to 0$.

We shall use the following simple facts:

Lemma 3.7. *Let (X, d) be ULC. If (X, d) is dense in (Y, ρ) then (Y, ρ) is ULC as well.*

Proof. Fix $\varepsilon > 0$ and $y \in Y$. Since X is ULC there is a positive $\delta < \varepsilon$ such that for every $x \in X$ we have

$$B^d_{2\delta}(x) \subset W_x \subset B^d_{\varepsilon/2}(x)$$

for some connected W_x. Now choose $x_0 \in X$ with $\rho(x_0, y) < \delta$. Then

$$B^\rho_\delta(y) \subset B^\rho_{2\delta}(x_0) \subset \overline{W_{x_0}} \subset \overline{B^\rho_{\varepsilon/2}(x_0)} \subset B^\rho_\varepsilon(y)$$

where $\overline{W_x}$ is the closure in Y of the connected set W_x and therefore it is connected. Since ε and δ do not depend on y, it follows that Y is ULC by Lemma 3.1. □

Lemma 3.7 gives immediately:

Corollary 3.8. *The completion of a ULC space is ULC.*

Recalling also Theorem 2.2 and Proposition 3.3, we obtain:

Corollary 3.9. *Let (X, d) be ULC. If (X, d) is dense in (Y, ρ) then (Y, ρ) is a tight extension of X.*

For locally connected metric spaces, straightness is equivalent to the ULC property (Theorem 3.4). The next proposition says that the same is true for dense subspaces of uniformly connected spaces.

Proposition 3.10. *Let $X \subseteq Y$ be dense in Y. If Y is ULC then the following three conditions are equivalent:*

1) *X is ULC;*

2) *X is straight;*

3) *Y is a tight extension of X.*

Proof. By Theorem 2.2, properties in 2) and 3) are equivalent. The implication 1) $\Rightarrow$ 2) is true for metric spaces in general (Proposition 3.3). It remains to show 3) $\Rightarrow$ 1).

Assume 3), i.e., Y is a tight extension of X. We would like to show that X must be locally connected. By Theorem 3.4, this would prove that X is uniformly locally connected. So striving for a contradiction, assume X is not locally connected, i.e. there is $\varepsilon > 0$ such that $B^X_\varepsilon(x)$ does not contain any connected neighbourhood of x. Consider $B^Y_\varepsilon(x)$ and its components (in Y, of course). As Y is locally connected, these components are open. Denote the component, containing x, as C. Hence C is a clopen set in $B^Y_\varepsilon(x)$. Put $D = C \cap X$. Notice that D is a clopen set in $B^X_\varepsilon(x)$. D is also an open neighbourhood of x, contained in $B^X_\varepsilon(x)$, hence D cannot be connected by our assumption. So $D = E \cup F$ where E and F are disjoint relatively open subsets of D. Hence both E and F are clopen sets in $B^X_\varepsilon(x)$. Observe that connectedness of C enforces:

$$\overline{E}^Y \cap \overline{F}^Y \cap C \neq \emptyset \tag{5}$$

Define:

$$G^+ = E \cup \left(X \setminus B^X_\varepsilon(x)\right)$$
$$G^- = X \setminus E.$$

As Y is a tight extension of X, it holds true: $\overline{G^+}^Y \cap \overline{G^-}^Y = \overline{G^+ \cap G^-}^Y$.
As $\overline{X \setminus B^X_\varepsilon(x)}^Y \cap B^Y_\varepsilon(x) = \emptyset$, (5) yields a contradiction as E and F are disjoint clopen sets in $B^X_\varepsilon(x)$. □

Remark 3.11. The assumptions of Proposition 3.10 cannot be weakened. Any compact space is straight, however not all of them are locally connected (= ULC in the compact case). Local connectedness cannot be added to the equivalent properties in Proposition 3.10. It is enough to consider $\mathbb{R} \setminus \{0\}$. This subspace of the real line $\mathbb{R}$ is dense, locally connected without being straight - in contrast with $\mathbb{R}$.

It will be proved in [BDP2] that tight extensions are preserved by products under suitable assumptions. More precisely, if X, Y are ULC metric spaces and X is dense in Y, then for any metric space Z, $Y \times Z$ is a tight extension of $X \times Z$.) This should be compared with Lemma 5.6 and Lemma 5.9 where other examples of tight extensions will be presented.

4 Quasi components

4.1 Various notions of connectedness

Here we introduce two notions of connectedness weaker than ULC but strong enough to imply straightness.

Definition 4.1. The **quasi-component** of a point $x \in X$ is the intersection of all clopen sets containing x.

Hence x is in the same quasi-component of y in X if x cannot be separated from y, i.e. for every partition $X = A \cup B$ with A, B open, x and y lie both in A or both in B. Therefore, the quasi component of x in X is the set of all points $y \in X$ such that every $f \in C(X, \{0,1\})$ gives the same value to x and y.

Let us make an easy and well-known illustration:

Example 4.2. Consider a set $X \subset \mathbb{R}^2$ consisting of two points a, b on the x-axis together with a family of parallel lines L_n of equation $x = 1/n$. Then a, b lie in the same quasi-component of X but in different connected components (each consisting of a single point).

In analogy with the definition of d^* we can define a metric $\hat{d}$ as follows:

Definition 4.3. Given a metric space (X, d) and $x, y \in X$ we say that $I \subset X$ **quasi-connects** x and y if x, y belong to I and are in the same quasi-component of I. We define $\hat{d}(x, y)$ as the minimum between 1 and the infimum of the diameters of the subsets I of X which quasi-connect x and y. So if there is no such a set I, $\hat{d}(x, y) = 1$.

Lemma 4.4. *$\hat{d}$ is a metric.*

Proof. The verification of the triangle inequality is based on the observation that if x, y are in the same quasi-component of $I \subset X$ and y, z are in the same quasi-component of $J \subset X$, then x, z are in the same quasi-component of $I \cup J$. (In fact, given a continuous function $f \colon I \cup J \to \{0,1\}$ its restriction to I must map x, y to the same point, and its restriction to J must map y, z to the same point, so f maps x, z to the same point.) □

Remark 4.5. (a) Note that $\hat{d}(x, y)$ can only increase passing to a subspace, because if a set quasi-connects two points in a subspace, it does so in the space. (A good example is given by the reals $\mathbb{R}$ and the rationals $\mathbb{Q}$ endowed with the usual metric d. While $\hat{d}(x, y)$ computed in $\mathbb{R}$ coincides with $\inf\{d(x, y), 1\}$, for $x, y \in \mathbb{Q}$, $\hat{d}(x, y)$ computed in it, is 1 whenever $x \neq y$. This remains true in every zero-dimensional space.)

(b) If a space is locally connected, the connected components are open (and obviously also closed), and therefore coincide with quasi components. So in the locally connected case $d^* = \hat{d}$.

Definition 4.6. A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (X, d) is **discrete** if it has no accumulation points in X, and it is **uniformly discrete** if there is a non-zero lower bound to the set of all the distances $d(x_n, x_m)$ for $n \neq m$. Two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are **adjacent** if $d(a_n, b_n) \to 0$ for $n \to \infty$.

Definition 4.7. We use the metrics d^* and $\hat{d}$ to give the following definitions (the first one agrees with the definition of ULC already given):

i) (X, d) is **uniformly locally connected** (ULC) if for each pair of adjacent sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, $\lim_{n\to\infty} d^*(a_n, b_n) = 0$,

ii) (X, d) is **weakly uniformly locally connected** (WULC) if for each pair of *discrete* adjacent sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, $\lim_{n\to\infty} d^*(a_n, b_n) = 0$,

iii) (X, d) is **approximatively locally connected** (ALC) if for each pair of *discrete* adjacent sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, $\lim_{n\to\infty} \hat{d}(a_n, b_n) = 0$.

Clearly (i) implies (ii) which implies (iii). Every compact space satisfies (ii) and (iii) since it does not contain discrete sequences.

The definition of *ALC* can be equivalently stated as follows: *for every pair of discrete adjacent sequences (x_n) and (y_n) and for every ε, there is $m \in \mathbb{N}$ such that $\hat{d}(x_m, y_m) < \varepsilon$.*

It is necessary to restrict ourselves to *discrete* sequences in the definition of *ALC* because of the following:

Proposition 4.8. *Let (X, d) be a metric space such that for each pair of adjacent sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, $\lim_{n\to\infty} \hat{d}(a_n, b_n) = 0$. Then X is locally connected (and therefore $d^* = \hat{d}$).*

Proof. Take $x \in X$ and consider an open neighbourhood U of x. We need to find a connected neighbourhood of x contained in U. Let Q be a quasi-component of B which contains x. Clearly Q is closed in B. To finish the proof it suffices to show that Q is open (in that case Q would be a minimal clopen set containing x, hence connected). Assume for a contradiction that Q is not a neighbourhood of some $a \in Q$. Then there exists a sequence $a_n \to a$ in U such that $a_n \notin Q$ for every $n \in \omega$. Since Q is also the quasi-component of a, this implies that a_n and a cannot be quasi-connected by a set contained in U. It follows that there is $\delta > 0$ such that $\hat{d}(a_n, a) > \delta$ for every n (it suffices to take δ smaller than the distance between a and the complement of U). This contradicts our assumptions. ☐

Thus we see that although the ALC property is strictly weaker than WULC, these two properties generalize ULC in a very similar way. See also the remarks following (1) in the introduction (page 3).

Theorem 4.9. *If X is ALC, then X is straight.*

Proof. Assume X is ALC. If X is not straight then, by Theorem 1.7, there are closed sets $F_0,\ F_1 \subseteq X$, some $\eta > 0$ and two adjacent sequences $(a_n)_{n\in\mathbb{N}},\ (b_n)_{n\in\mathbb{N}}$ such that $d(a_n, F_0 \cap F_1) > \eta$ and $d(b_n, F_0 \cap F_1) > \eta$ for all n. Clearly the two sequences must be discrete (because any accumulation point must be in the intersection $F_0 \cap F_1$ and at the same time at distance $\geq \eta$ from this intersection). Since X is ALC there is $n \in \mathbb{N}$ and a set $I \subset X$ of diameter $< \frac{\eta}{4}$ which quasi-connects a_{n_0} and b_{n_0}. But then I is disjoint from $F_1 \cap F_2$ and F_0 and F_1 form a clopen partition of I separating a_n and b_n - a contradiction. □

In general, the property ALC is properly stronger than straightness even in the class of locally compact precompact spaces (see §5). However for complete spaces ALC is equivalent to straight (Theorem 4.11). We also show that WULC is stronger than ALC even for complete spaces (Theorem 5.13).

4.2 A characterization of the complete straight spaces

We will give a characterization of the complete spaces which are straight.

Lemma 4.10. *Let (X, d) be a metric space containing a pair of uniformly discrete adjacent sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ such that for some $\varepsilon > 0$ we have $\forall n\ \hat{d}(a_n, b_n) > \varepsilon$. Then X is not straight.*

Proof. By considering a tail of the sequence we can assume without loss of generality that there is a family of disjoint closed balls O_n of diameter ε with $a_n, b_n \in O_n$ for each $n \in \mathbb{N}$. Moreover we can assume that the points a_n, b_n are at distance $> \varepsilon/2$ from the complement of O_n. Since $\hat{d}(a_n, b_n) > \varepsilon$, O_n can be partitioned in two closed sets A_n and B_n containing a_n and b_n respectively. Now let C be the closure of the complement of $\bigcup_n A_n$, and let D be the closure of the complement of $\bigcup_n B_n$. Then $C \cup D = X$, $a_n \in D$, $b_n \in C$, and a_n, b_n are at distance $\geq \varepsilon/2$ from $C \cap D$. Since $d(a_n, b_n) \to 0$, this implies that the pair C, D is not u-placed and therefore X is not straight by Theorem 1.7. □

Theorem 4.11. *Let (X, d) be a complete metric space. If (X, d) is straight then it is ALC.*

Proof. Assume (X, d) is complete. If X is not ALC there are discrete adjacent sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ and $\varepsilon > 0$ with $\hat{d}(a_n, b_n) > \varepsilon$ for every n. As (X, ρ) is complete $(a_n)_{n\in\mathbb{N}},\ (b_n)_{n\in\mathbb{N}}$ are uniformly discrete. But then X is not straight by Lemma 4.10. □

Corollary 4.12. *A complete space X is straight if and only if it is ALC.*

Now, in analogy with Corollary 2.3 and Corollary 3.8 we obtain

Corollary 4.13. *The completion of an ALC space is ALC.*

Actually, one can prove the following stronger property (compare with Lemma 3.7):

Lemma 4.14. *Let (X, d) be ALC. If (X, d) is dense in (Y, ρ) then (Y, ρ) is ALC as well.*

Proof. Assume Y is not ALC.

Let (y'_n) and (y''_n) be adjacent discrete sequences in Y, witnessing Y is not ALC, i.e.

$$\lim_{n\to\infty} \hat{\rho}(y'_n, y''_n) \neq 0.$$

So there is $\delta > 0$ such that $\delta < \limsup \hat{\rho}(y'_n, y''_n)$. Without loss of generality we may suppose

- for each $n \in \mathbb{N}$, $\hat{\rho}(y'_n, y''_n) > \delta$,
- for each $n \in \mathbb{N}$, $\rho(y'_n, y''_n) < \frac{\delta}{8}$.

For each $n \in \mathbb{N}$, define $O_n = B_{\frac{\delta}{4}}(y'_n) \cup B_{\frac{\delta}{4}}(y''_n)$. As $\hat{\rho}(y'_n, y''_n) > \delta$, O_n cannot quasi-connect points y'_n, y''_n. Hence for each $n \in \mathbb{N}$, $O_n = P'_n \cup P''_n$ where $y'_n \in P'_n$, $y''_n \in P''_n$, moreover P'_n and P''_n are disjoint and relatively clopen in O_n. By the density of X, we can find adjacent discrete sequences (x'_n) and (x''_n) in X such that $\rho(x'_n, y'_n) \to 0$, $\rho(x''_n, y''_n) \to 0$ and $x'_n \in P'_n$ and $x''_n \in P''_n$ for each $n \in \mathbb{N}$. As (y'_n) and (y''_n) are adjacent discrete sequences in Y, we may conclude that (x'_n) and (x''_n) are adjacent discrete sequences in X. At first, sequences (x'_n) and (x''_n) are clearly adjacent. For their discreteness, it is enough to realize that if sequences (x'_n) and (x''_n) were not discrete, they would have a cluster point in X. This point would be also a cluster point of sequences (y'_n) and (y''_n) in Y which would create a contradiction.

Since X is ALC, $\hat{d}(x'_n, x''_n) \to 0$ in X. Take $i \in \mathbb{N}$ such that $\hat{d}(x'_i, x''_i) < \frac{\delta}{8}$ and a set Q of diameter at most $\frac{\delta}{8}$ which quasi-connects points x'_i and x''_i. However, $Q \subseteq O_i$ and so the sets P'_i, P''_i induce a non-trivial separation of Q between points x'_i and x''_i in X - a contradiction. □

Lemma 4.14, along with Corollary 4.12, allows us to see that a dense subspace X of an ALC space Y need not be ALC even if Y is a tight extension of X (unlike the case of the properties ULC and straightness, compare with Proposition 3.10 and Theorem 2.2). The needed counter-example is obtained by taking a non-complete straight space that is not ALC (see §5 for such examples in the class of locally compact precompact spaces).

To complete a picture, we present also Corollary 4.18 which reformulates Lemma 4.14 for WULC space. It will be a consequence of Lemma 4.17.

4.3 UC spaces

Here we study the connection of our new notions to UC spaces.

Lemma 4.15. *A UC space X does not contain any pair of discrete adjacent sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ with $a_n \neq b_n$ for all n. So a UC space is WULC and ALC is a trivial way.*

Proof. According to known characterization, X is UC if and only if the subspace X' of the non-isolated points of X is compact and for every $\varepsilon > 0$ the space $X \setminus B_\varepsilon(X')$ is uniformly discrete. So a pair of discrete adjacent sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ with $a_n \neq b_n$ cannot lie in $B_\varepsilon(X')$. Since this holds for every $\varepsilon > 0$, $d(a_n, X') \to 0$. Thus X' contains a sequence adjacent to (a_n), which necessarily has a cluster point since X' is compact. But then also (a_n) has a cluster point, contradicting the assumptions. □

Theorem 4.16. *Let X be a UC space and let Y be a compact ULC space. Then $X \times Y$ is WULC.*

Proof. Let $(x_n, y_n) \in X \times Y$ and $(x'_n, y'_n) \in X \times Y$ be two discrete adjacent sequences. Since Y is compact it follows that $(x_n)_{n\in\mathbb{N}}$ and $(x'_n)_{n\in\mathbb{N}}$ are discrete adjacent sequence in X. Since X is UC, by the previous lemma we have $x_n = x'_n$ for all but finitely many n. Now using the assumption that Y is ULC we get $d^*(y_n, y'_n) \to 0$. Since $x_n = x'_n$ for all big enough n we can conclude that $d^*((x_n, y_n), (x'_n, y'_n)) \to 0$ □

The following proposition points out an analogy between UC spaces and WULC/ALC spaces.

Lemma 4.17. *Let (X,d) be a metric space. Then X is WULC if and only if*

($*$) for each $\varepsilon > 0$, there is a compact set $L \subseteq X$ such that for each $\delta_1 > 0$ there is $\delta_2 > 0$ such that

$$\Big(\{x,y\} \subseteq X \setminus B_{\delta_1}(L) \ \&\ d(x,y) < \delta_2\Big) \Rightarrow d^*(x,y) < \varepsilon.$$

Moreover X is ALC if and only if:

($**$) for each $\varepsilon > 0$, there is a compact set $L \subseteq X$ such that for each $\delta_1 > 0$ there is $\delta_2 > 0$ such that

$$\Big(\{x,y\} \subseteq X \setminus B_{\delta_1}(L) \ \&\ d(x,y) < \delta_2\Big) \Rightarrow \hat{d}(x,y) < \varepsilon.$$

Proof. We prove the first part, the second is similar. Assume that X is weakly uniformly locally connected and choose $\varepsilon > 0$. Let L_ε be the set consisting of those points $z \in X$ such that there are two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ converging to z and such that $d^*(x_n,y_n) \geq \varepsilon$ for every $n \in \mathbb{N}$.

We claim that L_ε is compact. As $\emptyset$ is compact we may suppose $L_\varepsilon \neq \emptyset$. If $\{a_n\} \subseteq L_\varepsilon$ is a sequence without any cluster point then, using the definition of L_ε, one can easily find two sequences $\{x_n\}_{n\in\omega}$, $\{y_n\}_{n\in\omega}$ adjacent to $\{a_n\}$ such that $\forall n \in \omega :\ d^*(x_n,y_n) \geq \varepsilon$ which contradicts the fact that X is WULC and finishes the proof of the claim.

Now it suffices to show that the compact set L_ε satisfies ($*$) for every $\delta_1 > 0$. Assume that ($*$) fails for some $\delta_1 > 0$, i.e., for every $n \in \mathbb{N}$ there exist a pair of points x_n, y_n such that $d^*(x_n,y_n) \geq \varepsilon$, $\{x_n,y_n\} \subseteq X \setminus \mathcal{B}_{\delta_1}(L_\varepsilon)$ and $d(x_n,y_n) < 1/n$. This will produce discrete adjacent sequences $\{x_n\}_{n\in\omega}$, $\{y_n\}_{n\in\omega}$ in $X \setminus \mathcal{B}_{\delta_1}(L_\varepsilon)$, contradicting the WULC property. Indeed, if z were a cluster point of x_n, then $z \in L_\varepsilon$ and this contradicts the choice of x_n.

Assume that X is not weakly uniformly locally connected . Then there are discrete adjacent sequences $\{x_n\}_{n\in\omega}$, $\{y_n\}_{n\in\omega}$ in X and $\varepsilon > 0$ such that $\forall\ n \in \omega\ :\ d^*(x_n,y_n) \geq \varepsilon$. However both

$$A = \{x_n \ :\ n \in \omega\}, \text{ and } B = \{y_n \ :\ n \in \omega\}$$

are closed discrete sets, hence the intersections $A \cap L$ and $B \cap L$ are finite for every compact set L. Then $\delta_1 = d(A \setminus L, L) = d(B \setminus L, L) > 0$ again by the compactness of L. It is clear now that $\{x_n,y_n\} \subseteq X \setminus \mathcal{B}_{\delta_1}(L)$ for all but finitely many $n \in \omega$. So ($*$) fails. □

The following corollary of Lemma 4.17 gives a property of WULC spaces analogous to the corresponding results already proved for straight, ULC and ALC spaces (see Proposition 1.8, Lemma 3.7 and Lemma 4.14 respectively).

Corollary 4.18. *Let (X,d) be WULC. If (X,d) is dense in (Y,ρ) then (Y,ρ) is WULC as well.*

Proof. We start similarly as in the proof of Lemma 4.14.

Assume Y is not WULC.

Let (y'_n) and (y''_n) be adjacent discrete sequences in Y, witnessing Y is not WULC, i.e.

$$\lim_{n\to\infty} \rho^*(y'_n, y''_n) \neq 0.$$

So there is $\delta > 0$ such that $\delta < \limsup \rho^*(y'_n, y''_n)$. Without loss of generality we may suppose

- for each $n \in \mathbb{N}$, $\rho^*(y'_n, y''_n) > \delta$.

We use now Lemma 4.17 for X. Put $\varepsilon = \frac{\delta}{4}$. Take a compact set $L \subseteq X$ according to $(*)$ of Lemma 4.17. As X is a subspace of Y, L is compact also in Y. Sequences (y'_n) and (y''_n) are adjacent discrete in Y, so there is $\delta_1 > 0$ such that

$$\{y'_n, y''_n\} \subseteq Y \setminus B^Y_{2\delta_1}(L) \tag{6}$$

for all but finitely many $n \in \mathbb{N}$. Hence we may assume without loss of generality that (6) occurs for all $n \in \mathbb{N}$. Take now $\delta_2 > 0$, corresponding to δ_1 by $(*)$ of Lemma 4.17. Observe that $(*)$ forces that $\delta_2 \leq \varepsilon$. Consider a couple $\{y'_m, y''_m\}$ such that $\rho(y'_m, y''_m) < \frac{\delta_2}{2}$. As X is dense in Y, there are sequences (x'_n) and (x''_n) such that

- $\{x'_n\}_{n\in\mathbb{N}} \cup \{x''_n\}_{n\in\mathbb{N}} \subseteq X \setminus B^X_{\delta_1}(L)$,
- $x'_n \to y'_m$,
- $x''_n \to y''_m$,
- $d(x'_i, x''_j) < \delta_2$ for any $i, j \in \mathbb{N}$.

Hence for any $i, j \in \mathbb{N}$, there is a connected set $Q(i,j)$ such that $\{x'_i, x''_j\} \subseteq Q(i,j) \subseteq X$ of diameter at most ε. For $i \in \mathbb{N}$ put $Q_i = \bigcup\{Q(i,j) \ : \ j \in \mathbb{N}\}$ and $Q = \bigcup\{Q_i \ : \ i \in \mathbb{N}\}$. Then $\mathrm{diam}_X Q_i \leq 2\varepsilon = \delta/2$ and consequently, $\mathrm{diam}_X Q_i \cup Q_{i'} \leq 4\varepsilon = \delta$ for every pair of distinct $i, i' \in \mathbb{N}$, as $Q_i \cap Q_{i'} \neq \emptyset$. Hence $\mathrm{diam}_X Q \leq \delta$. However, Q is also connected in X, since each Q_i is connected (as a union of the connected sets $Q(i,j)$ with common point x'_i) and $x''_0 \in Q_i$ for all $i \in \mathbb{N}$. Then $\overline{Q}^Y$ is a connected set of diameter smaller than δ, which contains $\{y'_m, y''_m\}$ - a contradiction. □

5 Examples

5.1 Straight spaces which are not ALC

In this section we show that straight locally compact metric spaces need not be ALC and that the product of two straight spaces need not be straight.

Given two metric spaces (X, d_1) and (Y, d_2) we consider on $X \times Y$ the metric $d((x,y),(x',y')) = d_1(x,x') + d_2(y,y')$ unless otherwise stated. The following easy lemma will be necessary.

Lemma 5.1. *Let X, Y be metric spaces. If $X \times Y$ is straight, then both X and Y are straight. (The converse is not always true, see for instance Example 5.4.)*

Proof. Let $f : Z = X \times Y \to Y$ be the canonical projection. Assume $Y = F^+ \cup F^-$ is a closed binary cover of Y. Then $Z = f^{-1}(F^+) \cup f^{-1}(F^-)$ is a closed binary cover of Z. Now if $g : Y \to \mathbb{R}$ is a continuous function such that $g|_{F^+}$ and $g|_{F^-}$ are u.c., then $f_1 = g \circ f : Z \to \mathbb{R}$ is continuous and $f_1|_{f^{-1}(F^+)}$ and $f_1|_{f^{-1}(F^-)}$ are u.c. as compositions of u.c. functions. Then f_1 is u.c. since Z is straight. To see that g is u.c. note that a function $g : Y \to \mathbb{R}$ is u.c. if and only if the function $g \circ f$ is u.c. □

Definition 5.2. The ordinal numbers ω and $\omega+1$ will be considered as metric spaces with the following metric. On $\omega+1$ consider the metric which makes it isometric to $\{1/n \mid n \in \mathbb{N}\} \cup \{0\}$ as a subspace of $\mathbb{R}$ (with $n \in \omega$ going to $1/n$ and ω going to 0). On ω we put the induced metric as a subspace of $\omega+1$.

Remark 5.3. Notice that with the above metric $\omega+1$ is compact and ω is precompact. Also notice that, although ω is topologically homeomorphic to $\mathbb{N}$, $\mathbb{N}$ is straight, while ω is not. Recall that $\mathbb{N}$ is a uniformly discrete metric space.

Example 5.4. 1. $(0,1) \times (\omega+1)$ is straight, locally compact, precompact, not ALC;

2. The complete metric space $\mathbb{R} \times (\omega+1)$ is not straight;
3. $\mathbb{R} \times \mathbb{N}$ and $(0,1) \times \mathbb{N}$ are straight;
4. Let $X = [0,1] \times (\omega+1)$ and $p = (0,\omega) \in X$. Then $X \setminus \{p\}$ is straight, locally compact, precompact, not ALC.
5. Let $Y = [0,1] \times \omega$, $p = (0,\omega) \in [0,1] \times (\omega+1)$ and $q = (1,\omega) \in [0,1] \times (\omega+1)$. Then $Y \cup \{p,q\}$ is not straight.
6. However the topological space in (5.) has another metric which induces the same topology and makes it complete and WULC (but not ULC), hence straight.

Proof. The proof will use some results proved in the sequel and serves as a motivation for those results.

(1.) $(0,1)$ is precompact and *ULC* and $(\omega+1)$ is straight (being compact). So the space $X = (0,1) \times (\omega+1)$ is straight by Theorem 5.12. Moreover X is clearly locally compact and precompact. To show that it is not ALC apply Corollary 5.11.

(2.) The product $Z = \mathbb{R} \times (\omega+1)$ is complete, so by Theorem 4.11 it suffices to see that Z is not ALC. Consider the discrete adjacent sequences $\{x_n\}$ and $\{x_n\}$, where $x_n = (0,2n)$ and $y_n = (0,2n+1)$. Then $\widehat{d}(x_n,y_n) \not\to 0$ since x_n and y_n belong to distinct clopen quasi-components of the whole space Z, so they cannot be quasi-connected.

(3.) Both spaces are ULC, hence straight by Theorem 3.4.

(4.) $X \setminus \{p\}$ is straight by Theorem 5.12 below. Lemma 5.10 implies that $X \setminus \{p\}$ is not ALC.

(5.) The fact that $Y \cup \{p,q\}$ is not straight is easy: the sets $\bigcup_n (0,1) \times \{2n\} \cup \{p,q\}$ and $\bigcup_n (0,1) \times \{2n+1\} \cup \{p,q\}$ define a cover by closed sets which are not u-placed. Now use Theorem 1.7.

(6.) Put on $Y \cup \{p,q\}$ the following metric. On each interval $[0,1] \times \{n\} \subset Y$ with $n \in \omega$ we consider the obvious metric which makes the interval isometric with $[0,1]$. We stipulate that the distance between $(0,n)$ and $(0,\omega)$ is $1/n$. Similarly the distance between $(1,n)$ and $(1,\omega)$ is $1/n$. So the end-points of the intervals $[0,1] \times \{n\}$ converge to $(0,\omega)$ and $(1,\omega)$. The distance between any other pair of points is the maximal possible distance, compatible with the given conditions. It is easy to see that with this metric $Y \cup \{p,q\}$ is complete.

The proof that it is WULC is based on the fact that any discrete sequence must eventually lie outside of some neighbourhood U of $\{p, q\}$, together with the observation that any two points outside of U at sufficiently small distance must lie in the same interval.

Examples (1.) and (4.) share the same collection of properties. Nevertheless, we have preferred to include them both as their structure and the reasoning involved are different. □

Remark 5.5. (a) K. Yamada [Y] noticed the straightness of (1.) and announced that if $X \times (\omega+1)$ is straight, then X is precompact. This follows also from our results in [BDP2] mentioned at the end of Introduction.

(b) Let X be the metric discrete sum of $(0,1)$ and $(\omega+1)$. Then X is precompact and ALC (locally compact, too), but $X \times X$ is straight and it is not ALC anymore (use example (1.)). More general statements will be given in [BDP2].

In the rest of this section we prove the results needed for point (4.) and we produce similar examples. The following lemma produces examples of tight extensions.

Lemma 5.6. *Let $X \subseteq Y$ be a tight extension by a locally connected space Y such that for every open connected set W of Y also $X \cap W$ is connected in Y. Then also $X \times (\omega+1) \subseteq Y \times (\omega+1)$ is a tight extension.*

Proof. Set $Z = Y \times (\omega+1)$ and consider a closed binary cover $X \times (\omega+1) = F^+ \cup F^-$. Take $z \in \overline{F^+}^Z \cap \overline{F^-}^Z$. We must prove that $z \in \overline{F^+ \cap F^-}^Z$. If $z \in X \times (\omega+1)$ this is clear, so assume that $z \notin X \times (\omega+1)$.

Case A. $z = (y, \omega)$ for some $y \in Y$. Fix a connected open neighborhood W of y in Y. Then for every $n \in \omega$ consider the connected open subset $W \times \{n\}$ of $Y \times \{n\}$. By our hypothesis we get a connected subset

$$(X \cap W) \times \{n\} = (X \times \{n\}) \cap (W \times \{n\}) \subseteq F^+ \cup F^-.$$

Since $X \cap W \times \{n\}$ is connected there are, for each n, only three possibilities:

(a_n^+) $(W \times \{n\}) \cap F^- = \emptyset$ (and consequently $X \cap (W \times \{n\}) \subseteq F^+$);

(a_n^-) $(W \times \{n\}) \cap F^+ = \emptyset$ (and consequently $X \cap (W \times \{n\}) \subseteq F^-$);

(b_n) $D_n = (W \times \{n\}) \cap F^+ \cap F^- \neq \emptyset$.

Case A.1. There exists n_0 such that (b_n) fails for all $n > n_0$. So either (a_n^+) or (a_n^-) occur for every $n > n_0$.

Case A.1.1. There exists $n_1 \geq n_0$ such that (a_n^+) holds for all $n > n_1$. Let $V = W \times (n_1, \omega]$, this is an open neighbourhood of z. Then the density of $(X \cap W) \times \{n\}$ in $W \times \{n\}$ implies

$$W \times \{\omega\} \subseteq \overline{F^+}. \tag{7}$$

(Note that for this conclusion also the weaker assumption, "(a_n^+) holds for infinitely many n", suffices.) Moreover, by our assumption on z we have also $z \in \overline{F^-}$. Since $V \cap F^- \subseteq W \times \{\omega\}$ by our assumption a_n^+), we conclude that $z \in \overline{F^- \cap (W \times \{\omega\})}$. Then there exists a sequence

$x_n \in F^- \cap (W \times \{\omega\})$ converging to z. On the other hand, by (7) $x_n \in \overline{F^+}$. Since $x_n \in X$, this gives $x_n \in F^+ = X \cap \overline{F^+}$ by the closedness of F^+. Thus $x_n \in F^- \cap F^+$ and this proves $z \in \overline{F^+ \cap F^-}^Z$ in this case.

Case A.1.2. There exists $n_2 \geq n_0$ such (a_n^-) holds for all $n > n_2$. Now we argue as in Case A.1.1.

Case A.1.3. There are infinitely many n such that (a_n^+) holds and there are infinitely many n such that (a_n^-) holds. Now instead of only (7) we can prove both (7) and[1]

$$W \times \{\omega\} \subseteq \overline{F^-}. \tag{8}$$

Now take any sequence $x_n \in (W \cap X) \times \{\omega\}$ converging to z (since X is dense in Y this is possible). Now (7) and (8) will ensure, as above, that $x_n \in F^- \cap F^+$. This proves $z \in \overline{F^+ \cap F^-}^Z$ in this case.

Case A.2. For every connected open neighbourhood W of y there exist infinitely many n_k such that (b_{n_k}) holds. Now to prove that $z = (y, \omega) \in \overline{F^+ \cap F^-}^Z$ take a neighbourhood U of z in Z. Since by our hypothesis we have a base of connected open neighbourhoods W of y in Y as above, we may assume without loss of generality that $U = W \times (n_0, \omega]$ for some $n \in \omega$. By our assumption there exists $n > n_0$ and a point $x \in W \cap D_n \subseteq U \cap F^+ \cap F^-$. This proves $z \in \overline{F^+ \cap F^-}^Z$.

Case B. $z = (x, n)$ for some $n \in \omega$. Now $O = Y \times \{n\}$ is a clopen set in Z with an obvious homeomorphism $O \to Y$ that sends $O \cap (X \times \{n\})$ onto X. Hence $z \in \overline{F^+ \cap F^-}^Z$ follows from the fact that $X \subseteq Y$ is a tight extension. $\square$

Remark 5.7. One can replace in the above lemma $\omega + 1$ be any space that is locally homeomorphic to $\omega + 1$.

Example 5.8. As an application of Lemma 5.6 we give a second proof of the fact that $(0,1) \times (\omega + 1)$ is straight. Indeed, $(0,1)$ is straight, being ULC, hence $[0,1]$ is a tight extension of $(0,1)$ by Theorem 2.2. Now by Lemma 5.6 also $[0,1] \times (\omega + 1)$ is a tight extension of $(0,1) \times (\omega + 1)$. So, again by Theorem 2.2, $(0,1) \times (\omega + 1)$ is straight.

The next lemma can be proved arguing as in the proof of the previous one. This lemma has the advantage to have a simpler proof due to the following slight difference: now the assumption that for every open connected set W of Y also $X \cap W$ is connected in Y is no more needed and Case B does not appear.

Lemma 5.9. *Let $X \subseteq Y$ be a tight extension by a locally connected space Y. Then also $X \times (\omega + 1) \cup Y \times (\omega) \subseteq Y \times (\omega + 1)$ is a tight extension.*

Lemma 5.10. *Let Y be a metric space and let X be a dense proper subset of Y. Then for every $y \in Y \setminus X$ no subspace Z of $Y \times (\omega + 1)$ such that*

$$X \times (\omega + 1) \subseteq Z \subseteq Y \times (\omega + 1) \setminus \{(y, \omega)\} \tag{$*$}$$

can be ALC.

[1] see the observation after (7).

Proof. By the density of X in Y the point $y \in Y$ is not isolated. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X converging to y. Then

$$u_n = (x_{2n}, 2n) \in X \times (\omega + 1) \subseteq Z \text{ and } v_n = (x_{2n+1}, 2n+1) \in X \times (\omega + 1) \subseteq Z$$

define two adjacent discrete sequences in Z such that u_n, v_n belong to distinct clopen sets of Z (as they belong to distinct clopen sets of $Y \times (\omega+1)$). Such sequences cannot be quasi-connected by any set in Z, hence Z is not *ALC*. □

The lemma gives the following immediate corollary.

Corollary 5.11. *If $X \times (\omega + 1)$ is ALC, then X is complete.*

We shall see in [BDP2] that X from Corollary 5.11 must be even compact. Results from [BDP2], mentioned at the end of Introduction, will give this fact very quickly.

Now we are ready to give two series of examples of straight spaces that are not ALC.

Theorem 5.12. *Let Y be a locally connected compact space and let X be a dense proper subset of Y.*

(a) the space $Z_1 = X \times (\omega + 1) \cup Y \times (\omega)$ is straight;

(b) if for every open connected subset C of Y also $C \cap X$ is connected, then the space $Z_2 = X \times (\omega + 1)$ is straight;

(c) both spaces Z_1, Z_2 are not ALC.

Proof. Note that $Y \times (\omega + 1)$ is a compact, hence straight space.

(a) By Lemma 5.9 the space $Y \times (\omega+1)$ is a tight extension of its subspace Z_1. So by Theorem 2.2 also Z_1 is straight.

(b) By Lemma 5.6 the space $Y \times (\omega+1)$ is a tight extension of its subspace Z_2. So by Theorem 2.2 also Z_2 is straight.

(c) By our hypothesis there exists a point $y \in Y \setminus X$. Clearly, both Z_1, Z_2 satisfy $(*)$, so Lemma 5.10 can be applied to claim that these spaces are not ALC. □

Let us observe that, by Corollary 4.12, these examples are *never* complete (but can be chosen locally compact). Indeed, take $X = (0, 1]$ and $Y = [0, 1]$. Then Z_1 is just $[0, 1] \times (\omega+1) \setminus \{(0, \omega)\}$, i.e., a compact set minus a point.

The choice of precompact spaces in the above examples is necessary because, as already remarked, if X is not precompact, then $X \times (\omega + 1)$ is not straight (cf. [BDP2]).

5.2 A complete straight space which is not WULC

Recall that a complete space is straight iff it is ALC.

Theorem 5.13. *There is a complete metric space (Z, d) which is ALC (equivalently straight) but not WULC.*

Fix $\varepsilon > 0$. We first describe a preliminary space (G_ε, d) which will be a building block of the final space (Z, d).

Definition 5.14. Let

$$G_\varepsilon = I \cup O \cup \bigcup_{k,n\in\mathbb{N}} I_{k,n}$$

where:

1. $I = \{i_n \mid n \in \mathbb{N}\} \cup \{i\} \subset G_\varepsilon$ is a countable set of points, called *input nodes*, with $d(i_n, i) = \varepsilon/n$. So $\lim_n i_n = i$ and we call i the *limit input node*.

2. $O = \{o_n \mid n \in \mathbb{N}\} \cup \{o\} \subset G_\varepsilon$ is a countable set of points (disjoint from I), called *output nodes*, with $d(o_n, o) = \varepsilon/n$. So $\lim_n o_n = o$ and we call o the *limit output node*.

3. $I_{k,n}$ is a subspace of G_ε isometric to the interval $[0, \varepsilon] \subset \mathbb{R}$. The end points of the interval $I_{k,n}$ are the input node $i_k \in I_{k,n}$ and another point $o_n^k \in I_{k,n}$ with $d(o_n^k, o_n) = \varepsilon/k$. So $d(i_k, o_n^k) = \varepsilon$ and $\lim_k o_n^k = o_n$.

4. The intersection between two different intervals is either empty or an input node. More precisely: $I_{k,n} \cap I_{k,n'} = \{i_k\}$ and $I_{k,n} \cap I_{k',n'} = \emptyset$ if $k \neq k'$. We also require $I_{k,n} \cap (O \cup I) = \{i_k\}$.

5. The metric d on G_ε is the biggest possible compatible with the above constrains. For instance the distance between the input node i_k and the output node o_n is $\varepsilon + \varepsilon/k$ because we need to go from i_k to o_n^k (distance ε) and then from o_n^k to o_n (distance ε/k).

Note that the distance between an input node and an output node is $> \varepsilon$ and $\leq 2\varepsilon$. The total diameter of G_ε is slightly larger ($2\varepsilon + 1/2\varepsilon$), because we need to take into account also the points in the middle of the intervals.

Lemma 5.15. *G_ε is complete.*

Proof. Consider a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in G_ε. The subset $K = I \cup O$ of G_ε is compact (being the union of two converging sequences together with their limit points). So if (x_n) does not converge, there must be some $\delta > 0$ such that all but finitely many x_n are outside $B_\delta(K)$ (the set of points at distance $< \delta$ from K). For $\delta < \varepsilon$ the set $G_\varepsilon - B_\delta(K)$ is a disjoint union of closed intervals (subsets of the various $I_{n,k}$) at distance $\geq \delta$ from one another. It follows that $G_\varepsilon - B_\delta(K)$ is complete and (x_n) converges. □

Lemma 5.16. *G_ε is WULC.*

Proof. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be a pair of adjacent discrete sequences. Since these sequences are discrete, there is a neighborhood U of the limit output node o and a neighborhood W of the limit input node i such that for all big enough n we have $a_n, b_n \notin U \cup W$. The set $(O \cup I) \setminus (U \cup W)$ is finite, so there is a positive lower bound $\delta > 0$ to the distances of two points in this set. If $d(a_n, b_n) < \delta$, then by definition of the metric, $\{a_n, b_n\}$ is contained in at most two intersecting intervals, whose sum of lengths is bounded by the distance between the points a_n and b_n. Since these intervals connect the points a_n and b_n the desired result follows. □

In the next lemma we describe the connected component and the quasi-components of the space G_ε.

Lemma 5.17. *(a) The connected component of the limit input node $i \in I$ is the singleton $\{i\}$.*

(b) *The connected component of the input node $i_k \in I$ is the union $C_k = \bigcup_n I_{k,n}$, of all the intervals emanating from i_k. Each component C_k is a open, hence it is also the quasi-component of i_k.*

(c) *Let $p \in O = \{o_n \mid n \in \mathbb{N}\} \cup \{o\}$ be an output node. Then*

  (c_1) *the connected component of p is the singleton $\{p\}$.*

  (c_2) *the quasi-component of p is the set $\{i\} \cup O$.*

Proof. (a) is obvious.

(b) The first half is obvious. C_k is open, because it has distance $\geq \varepsilon/k$ from its complement in G_ε. Hence C_k is clopen, and consequently a quasi-component.

The singleton components are not open, since they are limits of sequences.

(c_1) is obvious.

(c_2) Assume that A is a clopen set containing o_n. Then $o_n^k \in A$ for all sufficiently large k, $\lim_k \varepsilon/k = 0$. Since C_k is connected, it follows that also C_k is contained in A for all sufficiently large k. Then $i = \lim_n i_n \in A$ too. This proves that every o_n belongs to the quasi-component of i for every n. Then $o = \lim_n o_n \in O$ must be in A. Since $\{i\} \cup O$ is the complement of the union of the remaining quasi-components C_k, we are done. □

Lemma 5.18. *If we adjoin to G_ε a connected set containing all the input nodes, then G_ε becomes connected.*

Proof. By Lemma 5.17 $G_\varepsilon \setminus O$ is the union of the connected components of the input nodes. So if all the input nodes are in the same component, then $G_\varepsilon \setminus O$ is connected. But O is in the closure of $G_\varepsilon \setminus O$. Now use the fact that the closure of a connected set is connected. □

Now we define a new space Z_n obtained by putting together n copies of G_ε with $\varepsilon = 1/n$:

Definition 5.19. For each positive integer n we define a space Z_n by:

$$Z_n = G^1_{1/n} \cup \ldots \cup G^n_{1/n}$$

where each $G^i_{1/n}$ with $i \neq 1$ is an isomorphic copy of $G_{1/n}$ and $G^1_{1/n}$ is a copy of $G_{1/n}$ modified by adding, for each pair of its input nodes $p, q \in I$, a interval of length $d(p,q)$ connecting them (for later reference we call it a *special interval*). The union defining Z_n is disjoint except for the fact that the output nodes of $G^i_{1/n}$ are identified (if $i < n$) with the input nodes of $G^{i+1}_{1/n}$ (i.e. the output node with index k of $G^i_{1/n}$ coincides with the input node with the same index of $G^{i+1}_{1/n}$, and the limit output node of $G^i_{1/n}$ coincides with the limit input node of $G^{i+1}_{1/n}$).

The distance function d on Z_n is the maximal possible compatible with these requirements. So in particular

$$d(G^i_{1/n}, G^j_{1/n}) \geq (|i-j|-1) \cdot \frac{1}{n}, \text{ if } i \neq j \qquad (*)$$

(and obviously $d(G^i_{1/n}, G^{i+1}_{1/n}) = 0$ since there is a non-empty intersection).

Our final space Z is the disjoint union

$$Z = \bigcup_n Z_n,$$

where for $n \neq m$ the distance between a point in Z_n and a point in Z_m is 3.

Remark 5.20. The diameter of $G_{1/n}$ tends to 0 for $n \to \infty$, but the diameter of Z_n does not. In fact $G^1_{1/n}$ and $G^n_{1/n}$ are subsets of Z_n, with $d(G^1_{1/n}, G^n_{1/n}) \geq \frac{n-2}{n} \geq \frac{1}{2}$ for $n > 3$.

Lemma 5.21. *Z_n and Z are complete.*

Proof. Z_n is complete because it is a finite union of complete spaces (by Lemma 5.15). Z is complete because a Cauchy sequence must eventually lie inside a single Z_n. □

Lemma 5.22. *Given $1 \leq i < j \leq n$ consider the subset X of $Z_n = G^1_{1/n} \cup \ldots \cup G^n_{1/n}$ defined by $X = \bigcup_{i \leq s \leq j} G^s_{1/n}$.*

(a) If $i = 1$, X is connected.

(b) If $i > 1$,

(b_1) all points of X lie in the same quasi-component;

(b_2) any pair of points $a, b \in X$ is quasi-connected by $X' = \bigcup_{i-1 \leq s \leq j} G^s_{1/n}$;

(b_3) $diam(X') \leq diam(X) + (2 + 1/2) \cdot 1/n$.

Proof. (a) can be proved by induction on $j - i$ using Lemma 5.18.

(b) To prove (b_1) apply Lemma 5.17, while (b_2) follows from the fact that each point of X is in the same connected component of some input node or some output node. For (b_3) notice that if $\text{diam}(X) < \eta$, then $\text{diam}(X') < \eta + \text{diam}(G_{1/n}) = \eta + (2 + 1/2) \cdot 1/n$. □

Lemma 5.23. *Z is ALC.*

Proof. Consider a pair of discrete adjacent sequences $(a_t)_{t \in \mathbb{N}}$ and $(b_t)_{t \in \mathbb{N}}$. We must prove that $\hat{d}(a_t, b_t) \to 0$.

If a_t, b_t remain inside a finite union $\bigcup_{n \leq n_0} Z_n$, then since each Z_n is a finite union of copies of $G_{1/n}$ (possibly with the addition of the special intervals), we may assume that all the a_t, b_t lie inside a single copy of $G_{1/n}$. In this case we are done since $G_{1/n}$ is *WULC* (even with the added intervals).

We should deal now with the case when $a_t, b_t \in Z_{n_t}$, with $n_t \to \infty$. Let $\varepsilon_t = 1/n_t$ and choose i_t, j_t such that $a_t \in G^{i_t}_{\varepsilon_t}$ and $b_t \in G^{j_t}_{\varepsilon_t}$. We can assume $i_t \leq j_t$ so that both a_t and b_t belong to $X_t = \bigcup_{i_t \leq s \leq j_t} G^s_{\varepsilon_t} \subset Z_{n_t}$. Since $d(a_t, b_t) \to 0$, by the definition of the metric we get $\text{diam}(X_t) \to 0$. By Lemma 5.22 a_t, b_t are quasi-connected by $X'_t = \bigcup_{i_t - 1 \leq s \leq j_t} G^s_{\varepsilon_t}$ ("-1" is not needed if $i_t = 1$), and $\text{diam}(X'_t) \to 0$. □

Lemma 5.24. *Z is not WULC.*

Proof. Consider, for each n, two points $a_n, b_n \in Z_n = G^1_{1/n} \cup \ldots \cup G^n_{1/n}$ which are output nodes of $G^n_{1/n}$. Let T_n be a connected subset of Z containing a_n, b_n. Then by Lemma 5.22 T_n must contain at least a point of $G^1_{1/n}$, and since $d(G^1_{1/n}, G^n_{1/n}) \geq 1/2$ for $n > 3$ (see Remark 5.20), $\text{diam}(T_n)$ does not tend to zero. □

Theorem 5.13 is thus proved.

Remark 5.25. A more symmetric example of a complete metric space which is *ALC* but not *WULC* is obtained by modifying the definition of

$$Z_n = G^1_{1/n} \cup \ldots \cup G^n_{1/n}$$

in such a way that now all the $G^i_{1/n}$ are copies of $G_{1/n}$ (namely we do not treat $G^1_{1/n}$ in a special way) but this time the output nodes of $G^n_{1/n}$ are identified with the input nodes of $G^1_{1/n}$ (and the output nodes of $G^i_{1/n}$, for $i < n$, are identified as usual with the input nodes of $G^{i+1}_{1/n}$). The resulting space Z_n is connected, but it can be disconnected removing any $G^i_{1/n}$. As above $Z = \bigcup_n Z_n$ is complete, ALC, but not WULC.

6 Open questions

Theorem 2.2 gives a criterion for straightness of dense subspaces of straight spaces. The counterpart of this question for *closed* subspaces is still open:

Question 6.1. Give a description of the closed subspaces of a straight space that are straight.

Direct summands of straight space (cf. Lemma 5.1), as well as uniform retracts (in particular, clopen subspaces) are always straight (cf. [BDP2]). On the other hand, the spaces X in which *every* closed subspace is straight are precisely the UC spaces [BDP1]. Hence every non-UC straight space has a plenty of closed non-straight subspaces.

A rather specialized problem reads as follows:

Question 6.2. Is there a space with properties described in Theorem 5.13, i.e. complete straight and not WULC, which would be moreover *locally compact*?

The construction from Theorem 5.13 cannot give a locally compact example. At the moment, we do not know whether one could modify this construction to answer Question 6.2 in the affirmative. We conjecture though that there is no such a space.

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