

ZERO-GROUPS AND MAXIMAL TORI

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Abstract. We give a presentation of various results on zero-groups in o-minimal structures together with some new observations. In particular we prove that if G is a definably connected definably compact group in an o-minimal expansion of a real closed field, then for any maximal definably connected abelian subgroup T of G , G is the union of the conjugates of T . This can be seen as a generalization of the classical theorem that a compact connected Lie group is the union of the conjugates of any of its maximal tori.

§1. Introduction. We consider groups definable in an o-minimal expansion $\mathbf{M} = (M, <, +, \cdot, \dots)$ of a real closed field $(M, <, +, \cdot)$. Classical examples of such groups are the (real)algebraic subgroups of the general linear group $GL(n, M)$. Identifying the algebraically closed field $C = M[\sqrt{-1}]$ with M^2 in the standard way, we also obtain all the algebraic subgroups of $GL(n, C)$. Less classical examples of definable groups can be found in [18] or in [14]. By [16] each definable group can be equipped with a unique group topology which makes it into a definable manifold (see Definition 3.1). A notion of definable compactness for definable manifolds (and more generally for definable spaces) can be introduced as in [14]. (In the semialgebraic case the definably compact spaces are the complete spaces of [6].) One has also a notion of definable connectedness: a definable group is definably connected if it has no proper definable subgroups of finite index [16]. Unless $M = \mathbb{R}$, definable compactness (or connectedness) does not imply compactness (or connectedness). In this note we prove:

THEOREM 6.12. *If G is a definably connected definably compact group in an o-minimal expansion \mathbf{M} of a real closed field, then for any maximal definable abelian subgroup T of G , G is the union of the conjugates of T .¹*

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¹We learned from a referee that a recent paper of M. Edmundo [8] contains another proof of this result which uses the classification of definable semisimple groups.

This can be seen as a generalization of the classical theorem that a compact connected Lie group is the union of the conjugates of any of its maximal tori. Indeed the classical theorem can be deduced from our theorem taking $M = \mathbb{R}$, after some definability considerations (Corollary 6.13).

We show more generally that if G is a definably connected definably compact group and $H < G$ is a definable subgroup of G such that the (o-minimal) Euler characteristic of G/H is different from zero, then G is the union of the conjugates of H (Theorem 6.5).

Under the hypothesis of Theorem 6.12 it is not difficult to prove that the “Weyl group” $W(G) := N_G(T)/T$ is finite (Theorem 6.14).

From Theorem 6.12 and earlier work on definable abelian groups one can obtain a result of M. Otero, also proved by M. Edmundo in [8] by different methods, stating that every definably compact definably connected group G is divisible (Corollary 6.15).

The proof of Theorem 6.12 makes use of an o-minimal version of Lefschetz fixed point theorem (see [7] or [2, Theorem 3.3]) and of the notion of 0-group introduced by Strzebonski in [18]. Strzebonski generalized many classical results on p -groups to the case of 0-groups. It turns out that, in the definably compact case, the 0-groups are exactly the definably connected abelian groups (Corollary 5.7).

Granted the appropriate background the proof of Theorem 6.12 is rather short, but we have taken this opportunity to give an exposition, with bibliographical references, of the relevant notions and results, and to make some side observations.

§2. Euler characteristic. We assume some familiarity with the basic notions of o-minimality (see [19]). Fix in the sequel an o-minimal structure $\mathbf{M} = (M, <, +, \cdot, \dots)$ expanding a real closed field (the dots represent possible additional structure besides the field structure). Although many of the results we use remain true also for more general o-minimal structures, working over a field simplifies some proofs and moreover the known proofs of the fixed point theorem (Theorem 6.4) do make use of the field structure. A subset $X \subset M^k$ is definable if it is (first order) definable in \mathbf{M} possibly with parameters. For instance, if \mathbf{M} is a real closed field $(M, <, +, \cdot)$ without additional structure, the definable sets are exactly the semialgebraic sets. We give to M the topology generated by the open intervals and to M^n the product topology. To each definable set X one can attach two invariants: its dimension $\dim(X) \in \mathbb{N}$ and its o-minimal Euler characteristic $E(X) \in \mathbb{Z}$.

DEFINITION 2.1 (see [19]). The *dimension* of X is $\geq n$ iff X contains a subset definably homeomorphic to an open subset of M^n . The (o-minimal) *Euler characteristic* $E(X)$ is defined as the number of even dimensional cells in X

minus the number of the odd dimensional cells in X , relative to any given cell decomposition.

The name ‘‘Euler characteristic’’ may be slightly misleading, since $E(X)$ does not always coincide with the classical Euler characteristic $\chi(X)$, even when the underlying o-minimal structure M is based on the real numbers. For instance an open interval $I \subset \mathbb{R}$ has $E(I) = -1$ and $\chi(I) = 1$. To understand why this is the case one needs to remind that o-minimal cells do not include the boundary, so an open interval is a odd dimensional cell. This discrepancy is reflected in the fact that, while χ is invariant under homotopies, E satisfies instead the following:

PROPOSITION 2.2 (see [19, Chapter 4 (1.3), (2.4)]). *$\dim(X)$ and $E(X)$ are invariant under definably bijections, not necessarily continuous.*

Following [18] we can now define $\dim(G/H)$ and $E(G/H)$ where G is a definable group and $H < G$ is a definable subgroup.

DEFINITION 2.3. *A definable group is a definable set $G \subseteq M^k$ together with a definable group operation. We do not require the group operation to be continuous in the topology induced from M^k . Let G be a definable group, and let H be a definable subgroup, not necessarily normal. Let G/H be the set of left cosets of H . By ‘‘definable choice’’ (see [19, Chapter 6 (1.2) p. 94]) there is a definable function ι with domain G such that $\iota(g) = \iota(h)$ iff $gH = hH$ (one can also require $\iota(g) \in gH$). So whenever convenient we can identify G/H with the definable set $\iota(G)$ (identifying $gH \in G/H$ with $\iota(g)$). Different choices of ι give rise to definable bijections, so as a definable set G/H is only defined up to definable bijections. However, since the o-minimal Euler characteristic and the dimension are invariant under definable bijections, $E(G/H)$ and $\dim(G/H)$ are well defined.*

It is possible to define, besides $E(X)$, another invariant $E'(X)$ which more closely resembles the classical Euler characteristic and is invariant under definable homotopies. To this aim it suffices to replace, in the classical definition $\chi(X) = \sum_i (-1)^i \text{rank } H_i(X)$, the classical homology group H_i with the o-minimal homology groups of Woerheide [20] (which are naturally isomorphic the classical ones when the o-minimal structure is based on the reals, see [1, Proposition 3.2]). When X is a closed and bounded subset of M^k one can easily prove using the triangulation theorem that $E(X) = E'(X)$ (see [2, p. 788]), so in particular if $M = \mathbb{R}$ and $X \subset \mathbb{R}^k$ is compact, then $E(X) = E'(X) = \chi(X)$. Thus $E(X)$ is a tool which allows one to use combinatorial arguments typical of finite groups theory (thanks to its invariance under definable bijections), while at the same time permitting to draw conclusions of topological nature (thanks to the fact that it coincides with $E'(X)$ when X is closed and bounded). We will implicitly use these facts in the sequel. In fact the results relying on the fixed point theorem depend on the use of E' .

§3. Definable spaces. The notion of definable space is discussed in [19]. We give here an apparently weaker definition that is equivalent up to isomorphisms.

DEFINITION 3.1. Let X be a definable set and let τ be a topology on X . We say that (X, τ) is a *definable space* if there is a finite cover of X by definable sets U_1, \dots, U_k , open in τ , and a natural number n , such that for each U_i there is a definable function $\varphi_i: U_i \rightarrow V_i$ which is a homeomorphism between U_i (with the τ topology) and a subset V_i of M^n (where M has the topology generated by the open intervals, and M^n has the product topology). If moreover the V_i are open in M^n , we say that (X, τ) is a *definable manifold* of dimension n . The collection of the “*local charts*” (U_i, φ_i, n) is called an *atlas* of (X, τ) . So each definable space or manifold has a finite atlas.

The easiest example of definable space is a *definable subspace* of M^n , namely a definable set $X \subset M^n$ with the topology inherited by the ambient space M^n . A definable space is *affine* if it can be definably embedded in M^n for some n (namely if it is definably homeomorphic to a definable subset of M^n considered as a definable subspace of M^n).

We will later need:

THEOREM 3.2 ([17], see also [19, Chapter 10 (1.8)]). *A necessary and sufficient condition for a definable space to be affine is that it is regular, namely its points are closed and for every point p and closed set C there are open neighbourhoods of p and C which are disjoint.*

Since each open set is a union of definable open sets, it is easy to see that a definable space is regular if and only if it is *definably regular*, namely its points are closed and for every point p and definable closed set C there are definable open neighbourhoods of p and C which are disjoint.

LEMMA 3.3. *Let (X, τ) be a definable manifold of dimension n and let $Y \subset X$ be a subset of X of dimension m , equipped with the subspace topology. Then Y is a definable (sub)manifold of X if and only if every point $p \in Y$ has a definable open neighbourhood $O \subset Y$ definably homeomorphic to an open subset of M^m .*

So although definable manifolds are required to have a finite atlas, for submanifolds it is not necessary to require this finiteness condition as it is always automatically ensured.

PROOF. The lemma was proved in [1, Proposition 4.2] in the case when $(X, \tau) = M^n$. The general case can be reduced to this special case using the fact that X is covered by finitely many open sets definably homeomorphic to open subsets of M^n . \dashv

Our main source of definable spaces, besides the subspaces of M^n , are the definable groups. Recall that for a definable group $G \subseteq M^k$ we do not require the group operation to be continuous in the topology induced from

the ambient space M^k . The good topology on definable groups is not the topology of the ambient space but the one given by the following:

THEOREM 3.4. [16, Proposition 2.5] *If G is a definable group, then there is a (unique) topology τ on G , called the definable manifold topology of G , such that:*

1. (G, τ) is a topological group (i.e. multiplication and inversion are continuous);
2. (G, τ) is a definable manifold of dimension n , where n is the (o-minimal) dimension of G .

The uniqueness of the definable group topology follows from:

LEMMA 3.5. [13, Lemma 1.11] *If $f: H \rightarrow G$ is a definable group homomorphism between definable groups equipped with definable manifold topologies, then f is continuous.*

For definable spaces we have the following notion of definable compactness:

DEFINITION 3.6. [14] An Hausdorff definable space (X, τ) is *definably compact* iff for every definable function $f: I \rightarrow X$, where $I = (a, b) \subset M$ is an open interval, $\lim_{t \rightarrow b^-} f(t)$ exists in (X, τ) .

Since by o-minimality every definable function $f: I \rightarrow (X, \tau)$ is piecewise continuous, without loss of generality we can assume f continuous.

PROPOSITION 3.7. [14, Theorem 2.1] *Let $X \subset M^n$ be a definable set with the topology induced by the ambient space M^n . Then X is definably compact iff it is closed and bounded.*

The closed and bounded subsets of M^n need not be compact if $M \neq \mathbb{R}$, but they behave in many respects like compact sets within the definable category. For instance the image of a closed and bounded set under a continuous definable function is closed and bounded (see [19, Chapter 6 (1.9) p. 95] for o-minimal structures expanding an ordered abelian group, and [14] for arbitrary o-minimal structures).

LEMMA 3.8. *Let $f: (X, \tau) \rightarrow (Y, \mu)$ be a definable continuous surjective function between definable spaces and assume that (X, τ) is definably compact. Then (Y, μ) is definably compact and f is a closed map.*

PROOF. Let $\gamma: I \rightarrow Y$ be a definable function, $I = (a, b)$. By definable choice γ can be lifted to a definable function $\sigma: I \rightarrow X$ with $f \circ \sigma = \gamma$. Since σ has a limit in X (and γ and σ are piecewise continuous by o-minimality), $f \circ \sigma$ has a limit in Y . So Y is definably compact. Since a closed subset of a definably compact set is definably compact, the same argument shows that f is a closed map. \dashv

THEOREM 3.9. [16, Corollary 2.8] *Let G be a definable group and let $H < G$ be a definable subgroup. Then H is closed in the definable group topology of G .*

Moreover we have:

THEOREM 3.10. [15, Lemma 2.6] *Let G be a definable group and $H < G$ be a definable subgroup. Then the definable group topology on H coincides with the topology as a subspace of G*

The proof in [15] is in term of generic elements. The next lemma yields a proof based on Lemma 3.3.

LEMMA 3.11. *Let $f : H \rightarrow G$ be an injective definable morphism of definable groups. Then f is a homeomorphism from H to $f(H)$, where H and G have the definable group topology and $f(H) \subseteq G$ has the subspace topology.*

PROOF. $f : H \rightarrow G$ is continuous by Lemma 3.5. Let $K = f(H) \subseteq G$ and let $m = \dim H = \dim K$.

By the cell decomposition theorem K contains an open subset $V \subset K$ (in the topology inherited from G) definably homeomorphic to an open subset of M^m (take a cell of dimension m in the intersection of K with a local chart of G).

Now K is a subgroup of G and since it has the subspace topology it is also a topological subgroup. This implies that the translates of V in K are definably homeomorphic to each other, and since they cover K , we get that every point of K has a neighbourhood definably homeomorphic to an open subset of M^m . The number of such translates need not be finite, but nevertheless by Lemma 3.3 we obtain that K is a definable submanifold of G . We can then apply Lemma 3.5 to conclude that $f^{-1} : K \rightarrow H$ is continuous, thus finishing the proof. \dashv

If H is definably compact Lemma 3.11 has a shorter proof. Indeed in this case the continuous map $f : H \rightarrow G$ is a closed map (by Lemma 3.8), hence an homeomorphism onto its image.

We will later need:

DEFINITION 3.12. A definable space (X, τ) is *definably connected* if and only if it has no definable proper non-empty clopen subset.

REMARK 3.13. Each definable space (X, τ) is a finite union of maximal definably connected subsets, called its definably connected components.

PROOF. Immediate from the cell decomposition theorem when X is a subspace of M^k . The general case follows working in the local charts. \dashv

REMARK 3.14. A definable space (X, τ) is definably connected if and only if it is definably path connected, i.e. any two points can be joined by a definable continuous path. (See [19, Chapter 6, Proposition 3.2, p. 100] for the case of subspaces of M^k .)

§4. Homogeneous spaces. Besides definable groups, another source of definable manifolds are the definable homogeneous spaces, namely the definable sets on which a definable group acts transitively by a definable action.

THEOREM 4.1. [13, Theorem 2.11] *Let G be a definable group, and let $\alpha: G \times V \rightarrow V$ be a definable transitive action on the definable set V .² There is a topology τ on V such that:*

1. (V, τ) is a definable manifold;
2. the action $\alpha: G \times V \rightarrow V$ is continuous (where G has the definable manifold topology).

COROLLARY 4.2. [13, Corollary 2.14] *Let G be a definable group, and let $H < G$ be a definable subgroup, not necessarily normal. There is a topology τ on G/H such that:*

1. $(G/H, \tau)$ is a definable manifold;
2. the natural action $L: G \times G/H \rightarrow G/H$ given by left multiplication $L(g_1, g_2H) = g_1g_2H$ is continuous (where G has the definable manifold topology).

THEOREM 4.3. *Let G be a definable group and let $H < G$ be a definable subgroup. The following topologies on G/H coincide:*

1. The topology τ on G/H given by Theorem 4.2;
2. The quotient topology ν on G/H (where G has the definable manifold topology);
3. If H is normal in G , the group G/H has its own definable manifold topology given by Theorem 3.4, which also coincides with the quotient topology.

Thanks to this result we can speak of the *definable manifold topology* on G/H without ambiguity.

PROOF. The projection $\pi: G \rightarrow (G/H, \tau)$ is continuous because $\pi(g) = gH = L(g, 1H)$ and L is continuous. Since the kernel is H , passing to the quotient we have an induced continuous map $id: (G/H, \nu) \rightarrow (G/H, \tau)$ where ν is the quotient topology and id is the identity. To prove that $\nu = \tau$ it remains to show that $id: (G/H, \nu) \rightarrow (G/H, \tau)$ is an open map (hence an homeomorphism). By definition of the quotient topology this is equivalent to say that $\pi: G \rightarrow (G/H, \tau)$ is an open map, and this is ensured by Theorem 4.4 below. Finally if $H \triangleleft G$, we know that G/H has a unique group topology which is at the same time a definable manifold topology. Since the quotient topology ν on G/H is a group topology, and we have just shown that it coincides with the definable manifold topology τ , the proof is finished. \dashv

²We recall that the action α is transitive if for every $v, v' \in V$ there is $g \in G$ with $\alpha(g, v) = v'$.

It remains to prove:

THEOREM 4.4. *Let $\alpha: G \times V \rightarrow V$ be a definable transitive action and let τ be the definable manifold topology on V given by Theorem 4.1. Let $v \in V$. The continuous map $\gamma: G \rightarrow (V, \tau)$ defined by $\gamma(g) = \alpha(g, v)$, is an open map.*

PROOF. We say that γ is open at $p \in G$ if for every open neighborhood O of p , $\gamma(O)$ contains $\gamma(p)$ in its interior. Clearly if γ is open at every point then it is an open map. Using the transitivity and the continuity of the action of G (on both G and V), it is easy to see that if γ is open at some point then it is open at every point. So it suffices to prove that γ is open at some point. We will work on local charts and we will use the following easy consequence of the trivialization theorem (see [19, Chapter 9 (1.2) p. 142] for the statement of the trivialization theorem):

CLAIM 4.5. *Let $X, Y \subseteq M^k$ be definable sets with the topology induced by M^k and let $f: X \rightarrow Y$ be a continuous onto map. Then f is open at some point of X .*

To prove the claim recall that $f: X \rightarrow Y$ is definably trivial if there is a definable homeomorphism $\sigma: X \rightarrow Y \times F$ such that $f = \pi_1 \circ \sigma$, where $\pi_1: Y \times F \rightarrow Y$ is the projection. Clearly if f is definably trivial it is an open map, since so are the projections. In the general case, by the trivialization theorem and the surjectivity of f we can find an open set O of Y such that the restriction of f to $f^{-1}(O)$ is a definably trivial map onto O , and the claim follows.

To finish the proof consider a cell decomposition of G compatible with the open sets of a finite atlas of G and with the γ -preimages of the open sets of a finite atlas of V . At least one cell of this decomposition of G is mapped by γ into a subset of V containing a non-empty open set $P \subset V$. Let $X = \gamma^{-1}(P) \subseteq G$. The restriction of γ to X is a surjective continuous map $\gamma|_X: X \rightarrow P$. Moreover by our construction X is contained in a single chart of G , and P is contained in a single chart of V . So using the claim we can conclude that $\gamma|_X$ is open at some point p of X , so a fortiori γ is open at p . \dashv

REMARK 4.6. If G is definably compact the proof of Theorem 4.3 can be simplified. In fact by Lemma 3.8 and the continuity of π , if G is definably compact, also $(G/H, \tau)$ is definably compact and the continuous injective map $id: (G/H, v) \rightarrow (G/H, \tau)$ is a closed map, hence an homeomorphism.

§5. Zero-groups.

DEFINITION 5.1. G is a 0-group if it is a definable group and for every proper definable subgroup $H < G$ we have $E(G/H) = 0$ (we allow H to be the trivial group).

Note that the trivial group is a 0-group since it has no proper subgroups.

THEOREM 5.2. [16, Proposition 2.12] *Let G be a definable group. Put on G the definable manifold topology. Let $G^0 \subset G$ be the definably connected component of the identity of G . Then G^0 is a normal subgroup of G and it is the smallest definable subgroup of G of finite index. Thus G is definably connected if and only if it has no proper definable subgroups of finite index.*

The definition of definable connectedness given above (Definition 3.12) is thus consistent with the following:

DEFINITION 5.3. A definable group G is *definably connected* iff G has no definable proper subgroup of finite index.

REMARK 5.4. If G is a 0-group, then G is definably connected.

PROOF. Let $G^0 < G$ be the definably connected component of the identity. Then $[G : G^0]$ is finite and therefore $E(G/G^0) = [G : G^0] \neq 0$. If $G^0 \neq G$ this would contradict the definition of 0-group. \dashv

THEOREM 5.5. [18, Corollary 5.7] *If G is a 0-group, then G is abelian.*

THEOREM 5.6 ([7] or [2, Theorem 3.3]). *If G is a definably compact infinite group, $E(G) = 0$.*

COROLLARY 5.7. *Let G be a definably compact group. Then G is a 0-group if and only if it is abelian and definably connected.*

PROOF. Suppose G is definably compact, abelian and definably connected. We must prove that G is a 0-group. Let $H < G$ be a proper subgroup (if H does not exist, G is the trivial group, which is a 0-group). Since G is abelian $H \triangleleft G$. The group G/H is definably compact since so is G , and it is infinite since G is definably connected. So $E(G/H) = 0$ by Theorem 5.6. \dashv

Strzebonki gave an example of a definable (even real semialgebraic) 0-group which is not definably compact [18, Example 5.3]. The 0-groups which are definably compact are exactly the Strzebonski tori defined below:

DEFINITION 5.8. G is a Strzebonski torus if and only if G and all definably connected subgroups of G are 0-groups.

Since 0-groups are definably connected, this is equivalent to the definition given by Strzebonski in [18]: G is a (Strzebonski) torus if G is a 0-group and every $H < G$ contains a 0-group $K < H$ with $[H : K]$ finite.

PROPOSITION 5.9. *Let G be a definable group. Then G is a Strzebonski torus iff G is definably compact abelian and definably connected.*

PROOF. Assume G is a Strzebonski torus. Then in particular is a 0-group, so it is abelian and definably connected. It remains to show that G is definably compact. We reason by contradiction using a result of Peterzil and Steinhorn [14]: if a definable group G is not definably compact, then it contains a definable one-dimensional torsion free subgroup $H < G$. We can assume that H is definably connected (as otherwise replace it with its component at the

identity). Since groups with $E = 0$ have elements of every prime order (by [18]), it follows that $E(H) \neq 0$, so G is not a Strzebonski torus.

For the opposite direction suppose G is definably compact abelian and definably connected. We have already proved that G is a 0-group. To show that it is a torus we must show that if $H < G$ is definably connected, then H is a 0-group. This is clear since H satisfies the same assumptions used to show that G is a 0-group. \dashv

§6. 0-Sylow subgroups and maximal tori.

THEOREM 6.1. *Let G be a definably compact group and let $H < G$ be a definable subgroup. If $E(G/H) \neq 0$ and $f : G/H \rightarrow G/H$ is a definable continuous map definably homotopic to the identity (with respect to the definable manifold topology of G/H), then f has a fixed point.*

PROOF. Let us first observe:

CLAIM 6.2. G/H is a regular topological space (we do not need definable compactness here).

Indeed each definable manifold, although it does not need to be Hausdorff, it is certainly T_1 (i.e. its points are closed). So G with the definable manifold topology is T_1 . Now we use the fact that if G is a T_1 topological group and $H < G$ is a closed subgroup, then G/H is regular (see [11, Chapter 1, Theorem 1.6]).

By Theorem 3.2 we can thus identify G/H with a definable submanifold of M^k . For such manifolds a definable version of the singular homology groups has been developed by Woerheide ([20]), and a notion of orientability can then be defined as in [2] in terms of the definable homology groups. A routine argument already used in [2] (for G instead of G/H) shows:

CLAIM 6.3. G/H is definably orientable.

The idea is that (as for classical homogeneous spaces) an orientation on G/H is obtained by choosing a local orientation at a point and extending it consistently to the whole space G/H using the transitivity of the action of G on G/H .

To conclude the proof of Theorem 6.1 it now suffices to invoke Theorem 6.4 below. \dashv

THEOREM 6.4 ([7] or [2, Theorem 3.3]). *Let $X \subseteq M^k$ be a definable set. Suppose that X , with the subspace topology from M^k , is a definably compact definably oriented definable manifold with $E(X) \neq 0$. Let $f : X \rightarrow X$ be a definable continuous map definably homotopic to the identity. Then f has a fixed point.*

We can now prove:

THEOREM 6.5. *Let G be a definably connected definably compact group and let $H < G$ be a definable subgroup with $E(G/H) \neq 0$. Then $G = \bigcup_{g \in G} gHg^{-1}$.*

PROOF. Let $g \in G$. We want to prove that $g \in \bigcup_{x \in G} xHx^{-1}$. Consider the map $L_g: G/H \rightarrow G/H$ defined by $L_g(xH) = \alpha(g, xH) = gxH$. We claim that L_g is definably homotopic to the identity. Granted the claim, by Theorem 6.1 L_g has a fixed point $xH \in G/H$. So $gxH = xH$ and therefore $g \in xHx^{-1}$. To prove the claim recall, by 3.14, that G is definably path connected. So there is a definable continuous function $t \mapsto g_t, t \in [0, 1]^M$, with $g_0 = 1, g_1 = g$. Hence $t \mapsto L_{g_t}$ is a definable homotopy between $id = L_{g_0}$ and $L_g = L_{g_1}$. \dashv

To apply the above results to 0-subgroups we recall some results of Strzembonski.

DEFINITION 6.6. Let G be a definable group. A 0-group $H < G$ is called 0-Sylow if it is a maximal 0-subgroup of G .

REMARK 6.7. [18, Theorem 2.14] If $K < H < G$ are definable groups (no normality assumptions), there is a definable bijection from $G/H \times H/K$ to G/K . So $E(G/H)E(H/K) = E(G/K)$ and $\dim(G/H) + \dim(H/K) = \dim(G/K)$.

PROPOSITION 6.8 (see [18, Remark 2.18]). *If H, G are 0-groups and H is a proper subgroup of G , then $\dim(H) < \dim(G)$.*

PROOF. By Remark 6.7 $\dim(G) = \dim(G/H) + \dim(H)$. Since G is a 0-group, $E(G/H) = 0$. So G/H is infinite (because for a finite set X , $E(X) = \text{Card}(X)$). But then $\dim(G/H) > 0$ (since infinite definable sets have positive dimension) and the result follows. \dashv

COROLLARY 6.9. [18, Remark 2.18] *Every 0-subgroup is contained in a 0-Sylow.*

THEOREM 6.10. [18] *Let G be a definable group and let $H < G$ be a definable subgroup which is a 0-group.*

1. *If $E(G/H) = 0$, then there is a 0-subgroup K of G with $H < K < N_G(H)$ and $K \neq H$.*
2. *If $E(G/H) \neq 0$, then H is 0-Sylow.*
3. *Thus $E(G/H) \neq 0$ if and only if H is 0-Sylow.*

PROOF. Part 1. is ([18, Theorem 2.14]). Part 2. follows immediately from the definitions and Remark 6.7. \dashv

THEOREM 6.11. *Let G be definably compact and definably connected. Let $T < G$ be a 0-Sylow of G . Then $G = \bigcup_{x \in G} xTx^{-1}$. (Moreover by [18, Theorem 2.21] each two 0-Sylow subgroups are conjugates.)*

PROOF. By Theorem 6.5. \dashv

Since in the definably compact case the 0-groups are the definably connected abelian groups the above theorem has the following equivalent formulation:

THEOREM 6.12. *If G is a definably connected definably compact group, then for any maximal definably connected abelian subgroup T of G , G is the union of the conjugates of T .*

If the o-minimal structure is an expansion of \mathbb{R} we obtain the following classical result:

COROLLARY 6.13. *If G is a compact connected Lie group, then for any maximal abelian connected closed subgroup of $H < G$, G is the union of the conjugates of H .*

PROOF. The only thing to observe is that we do not need any definability assumptions. The definability comes for free since any compact Lie group G is isomorphic to a compact subgroup K of $GL(n, \mathbb{R})$ for some n (see [4, Chapter 3, Theorem 4.1, p. 136]) and any such K is a (real)algebraic subgroup of $GL(n, \mathbb{R})$ [5, Proposition 2, p. 230], hence it is definable in the o-minimal structure $(\mathbb{R}, <, +, \cdot)$. \dashv

Let $N_G(T)$ be the normalizer of T in G . In the hypothesis of Theorem 6.12 we have:

THEOREM 6.14. *The “Weyl group” $W(G) := N_G(T)/T$ is finite.*

PROOF. Suppose $W = W(G)$ is infinite. Then so is its definably connected component $W^0 < W$. Since G is definably compact, so is W^0 . Hence $E(W^0) = 0$. Now $E(W) = E(W/W^0)E(W^0) = 0$. By definition this means $E(N_G(T)/T) = 0$. Hence, by Theorem 6.10, T is not a zero-Sylow, a contradiction. \dashv

A compact connected abelian Lie group is isomorphic to a torus, namely a product of finitely many copies of \mathbb{R}/\mathbb{Z} . This suggests that a definably compact definably connected abelian definable group G of (o-minimal) dimension n should resemble an n -dimensional torus $(\mathbb{R}/\mathbb{Z})^n$. This analogy is partially confirmed in [9], where it is shown that such a group G has the same torsion of an n -dimensional torus. Further information on G comes by considering quotients. Assuming the o-minimal structure sufficiently saturated, in [3] it is shown that G has a smallest type-definable subgroup $G^{00} < G$ of bounded index (which is moreover normal and divisible). By [10, Theorem 8.1] G^{00} is torsion free and G/G^{00} isomorphic to the torus $(\mathbb{R}/\mathbb{Z})^n$.

However the analogy with tori has its limitations: for instance there are definably compact definably connected abelian definable group G of dimension $n > 1$ without any proper infinite definable subgroup (see [14]). Hence, given a definably connected group G of dimension > 1 , we have no control on how big a minimal 0-subgroup H is: it is not necessarily the case that $\dim(H) = 1$ (H could be G itself).

COROLLARY 6.15. [12, 8] *If G is a definably connected definably compact group, then G is divisible.*

PROOF. By Corollary 6.12 we can reduce to the case where G is abelian. The divisibility of definably connected abelian groups follows from the results of [18], as observed in [9]. In fact for $k \in \mathbb{N}$ define $p_k: H \rightarrow H$ by $p_k(x) = x^k$. Since H is abelian p_k is a homomorphism. By [18] the torsion subgroups $\text{Ker } p_k$ is finite, and hence $\dim H = \dim \text{Im } p_k$. Since H is definably connected $H = \text{Im } p_k$, hence H is divisible. \dashv

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