O-minimal cohomology: finiteness and invariance results*

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Abstract

The topology of definable sets in an o-minimal expansion of a group is not fully understood due to the lack of a triangulation theorem. Despite the general validity of the cell decomposition theorem, we do not know whether any definable set is a definable CW-complex. Moreover the closure of an o-minimal cell can have arbitrarily high Betti numbers. Nevertheless we prove that the cohomology groups of a definably compact set over an o-minimal expansion of a group are finitely generated and invariant under elementary extensions and expansions of the language.

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1 Introduction

Delfs [De85] considered a sheaf cohomology theory for (abstract) semialgebraic sets over arbitrary closed fields and proved a semialgebraic version of homotopy invariance. In [EdJoPe06] this was generalized to definable sets and maps over an o-minimal expansion of a group. If one further assumes that the o-minimal structure expands a field, then one can use the triangulation theorem (see [vdD98]) and the o-minimal version of the Eilenberg-Steenrod axioms ([EdWo08] or [EdJoPe06]) to show that the cohomology groups (with integer coefficients) of a definable set X coincide

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with the simplicial cohomology groups of the finite simplicial complex associated to a triangulation, and therefore are finitely generated and invariant under both elementary extensions and expansions of the language (see [BeOt02, EdWo08]). We prove that this continues to hold for arbitrary o-minimal expansions of groups, provided we restrict ourselves to definably compact sets (§7 and §8).

Working without the field assumption entails various difficulties. To begin with one cannot make use of the apparatus of singular cohomology. So we work, as in [EdJoPe06], with sheaf cohomology. More precisely, given a definable set $X \subset M^n$, the set of types $\widetilde{X}$ of X with the "spectral topology" is a quasi-compactification of X, and we define $\mathrm{H}^i(X; \mathcal{F}) := \mathrm{H}^i(\widetilde{X}; \mathcal{F})$, where $\mathcal{F}$ is a sheaf of Abelian groups on $\widetilde{X}$.

Now consider the case when G is an Abelian group and $\mathcal{F}$ is the constant sheaf G (i.e. the sheaf generated by the presheaf with constant value G). Assuming that X is *definably compact* (i.e. closed and bounded) and G is is finitetely generated, we prove that $\mathrm{H}^i(X; G)$ is finitely generated, and invariant under both elementary extensions $N \succ M$ (i.e. $\mathrm{H}^i(X; G) = \mathrm{H}^i(X(N); G)$) and expansions of the language of M (note that expanding the language leaves X invariant but alters $\widetilde{X}$).

As already remarked this would be easy to prove if M expands a field. If M does not expand a field we do not have the triangulation theorem but we still have the cell decomposition theorem (see [vdD98]) and the o-minimal version of the Eilenberg-Steenrod axioms ([EdJoPe06]). One could then be tempted to invoke, with some adaptations, the uniqueness theorem for cohomology functors satisfying the the axioms. However, despite the general validity of the cell decomposition theorem, we do not know whether a definable set in an o-minimal expansion of a group is a definable CW-complex (with the obvious definition obtained by relativizing the notion of CW-complex to the definable category), so we cannot apply the uniqueness theorem. The problem is that in the definition of a CW-complex one requires that the cells come equipped with an attaching map that extends continuously to the boundary, while for the o-minimal cells we do not have any such control of the boundary. Moreover we will show that the closure of a single cell can have arbitrarily high Betti numbers (§4).

As standard references on sheaf cohomology and Čech cohomology we use [Go73] and [Br97].

2 Cohomology of definable sets

Let $M = (M, <, \ldots)$ be an o-minimal structure expanding a dense linear order $(M, <)$. We put on M the topology generated by the open intervals and on M^n the product topology. If X is a subset of M^n we put on X the induced topology from M^n. By a *definable* set we mean a first-order definable (with parameters) subset of M^n for some n. Let $X \subset M^n$ be a definable set. Let $\widetilde{X}$ be the set of types $p(x) \in S_n(M)$ that contain a defining formula for X. A type $p \in \widetilde{X}$ can be naturally identified with an ultrafilter of definable sets with $X \in p$. We endow $\widetilde{X}$ with the *spectral topology*: as a basis of open sets we take the sets of the form $\widetilde{U}$ with U a definable open subset of X. Notice that the usual Stone topology on $\widetilde{X}$ is a refinement of the spectral topology; hence, $\widetilde{X}$ (with the spectral topology) is quasi-compact, but, in general, not Hausdorff. Moreover, the natural embedding of X into $\widetilde{X}$ is continuous.

In the sequel we make the further assumption that M is an o-minimal expansion of an ordered group. In this case $\widetilde{X}$ is *normal* [EdJoPe06, Theorem 2.12], namely every two disjoint closed subsets can be separated by disjoint open sets (but points need not be closed since $\widetilde{X}$ is not Hausdorff). Normality is important for the development of a good cohomology theory and in particular for the results of §6.

Given a definable set $X \subset M^n$ and a sheaf $\mathcal{F}$ of Abelian groups (or R-modules) on $\widetilde{X}$, we define,

$$\mathrm{H}^i(X;\mathcal{F}) := \mathrm{H}^i(\widetilde{X};\mathcal{F}), \tag{2.1}$$

where $\mathrm{H}^i(\widetilde{X};\mathcal{F})$ is the i-th sheaf cohomology group of $\mathcal{F}$ as defined in [Go73] or [Br97]. Given a definable subset $A \subset X$, we write $\mathrm{H}^*(A;\mathcal{F})$ for $\mathrm{H}^*(\widetilde{A};\mathcal{F}\restriction\widetilde{A})$. Finally we define $\mathrm{H}^i(X, A;\mathcal{F})$ as $\mathrm{H}^i(\widetilde{X},\widetilde{A};\mathcal{F})$.

Equivalently one could work with sheaves directly on X rather than $\widetilde{X}$ by considering X not as a topological space but as a site in the sense of Groethendieck (see [EdJoPe06, §3]), but we will not need this fact.

If $f\colon X \to Y$ is a definable function, then f induces a function $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ by $f(p) = \{Z \mid f^{-1}(Z) \in p\}$ where Z ranges over the definable subsets of Y and we identify p with an ultrafilter of definable sets. We have $\widetilde{f(X)} = \widetilde{f}(\widetilde{X})$ and $\widetilde{f}^{-1}(\widetilde{Z}) = \widetilde{f^{-1}(Z)}$. It follows that if f is continuous, then $\widetilde{f}$ is continuous. So if f is an homeomorphism, then $\widetilde{f}$ is an homeomorphism.

If G is an Abelian group and X is a topological space, the constant sheaf on X with stalk G will also be denoted G.

A *definable homotopy* between two definable functions $f\colon X \to Y$ and $g\colon X \to Y$ is a definable continuous function $F\colon I \times X \to Y$, where $I = [a, b]$ is some closed bounded interval in M, such that $F(a, x) = f(x)$ and $F(b, x) = g(x)$ for every $x \in X$. Note that F induces a map $\widetilde{F}\colon \widetilde{I \times X} \to \widetilde{Y}$, but in general $\widetilde{I \times X} \neq \widetilde{I} \times \widetilde{X}$, so we cannot consider $\widetilde{F}$ as a sort of "homotopy" parametrised by $\widetilde{I}$. Nevertheless we have the following definable version of the homotopy axiom:

Fact 2.1. ([EdJoPe06]) *If X, Y are definable sets and $f, g\colon X \to Y$ are definably homotopic definable maps, then $\widetilde{f}, \widetilde{g}\colon \widetilde{X} \to \widetilde{Y}$ (although not necessarily homotopic) induce the same homomorphism in cohomology, namely* *for every Abelian group G we have*

$$f^* = g^* \colon \mathrm{H}^i(Y;G) \to \mathrm{H}^i(X;G).$$

Similarly for maps $f, g \colon (X, A) \to (Y, B)$ of pairs.

This has been proved by [De85] in the semialgebraic case but using a different definition of $\widetilde{X}$ via the real spectrum. Jones [Jo06] extended it to the case of definable sets and maps in an o-minimal expansion M of an ordered field. In [EdJoPe06] it is shown that it suffices that M is an o-minimal expansion of an ordered group.

Corollary 2.2. ([EdJoPe06]) *The o-minimal version of the Eilenberg-Steernod axioms hold for* H^* *(with constant coefficients).*

Remark 2.3. As observed in [EdJoPe06], the only difficulty is the homotopy axiom (in the form given in Fact 2.1). Note that, since we have defined the cohomology of a definable set X using sheafs on the topological space $\widetilde{X}$, we will be able to apply the excision theorem, the exactness axiom, and the Mayer-Vietoris theorem, directly in the form given in [Br97], without worrying whether the particular form employed is formally deducible from the o-minimal version of the Eilenberg-Steenrod axioms. (Note that [Br97] uses almost always family of supports in his formulae, but this is irrelevant since working without supports is the same as taking the family of all closed subsets as support.)

3 Contractibility of cells

Let M be an o-minimal expansion of a group.

Lemma 3.1. *Let I be a bounded interval in M (closed, half-closed, or open). Then I is definably contractible to a point.*

Proof. Let us consider the case $I = (a, b)$. Given $0 < t \leq \frac{b-a}{2}$, there is a (unique) definable continuous function $f_t \colon (a, b) \to (a, b)$ such that f_t is the identity on $[a + t, b - t]$ and it is constant on both $[a, a + t]$ and $[b - t, b]$, with values $a + t$ and $b - t$ respectively (so the image of f_t is $[a + t, b - t]$). Define f_0 to be the identity. Then $(f_t)_{0 \leq t \leq \frac{b-a}{2}}$ is a deformation retract of (a, b) to the point $\frac{a+b}{2}$. The other cases are similar. □

We will employ the following notation: given $B \subseteq M^{n-1}$ and $f, g : B \to M$,

$$(f, g)_B := \big\{(x, y) \in M^{n-1} \times M : x \in B \;\&\; f(x) < y < g(x)\big\},$$
$$[f, g]_B := \big\{(x, y) \in M^{n-1} \times M : x \in B \;\&\; f(x) \leq y \leq g(x)\big\},$$
$$\Gamma(f) := \big\{(x, y) \in M^{n-1} \times M : x \in B \;\&\; f(x) = y\big\}, \text{ the graph of } f.$$

Lemma 3.2. *If C is a bounded cell of dimension $m > 0$ in M^n then there is a deformation retract of C onto a cell of strictly lower dimension. So by induction every bounded cell is definably contractible to a point.*

Proof. If C is the graph of a function we can reason by induction on the dimension of the ambient space. So the only interesting case is when $m > 1$ and $C = (f, g)_B$. Let $h = \frac{f+g}{2}$. We will define a deformation retract from C to $\Gamma(h)$. We can assume that h is a constant function, since we can reduce to this case by a definable homeomorphism which fixes all but the last coordinate (just take any constant function $h_1 \colon B \to M$, and define f_1, g_1 so that they differ from h_1 by the same amount in which f, g differ from h). Since C is bounded, there are constants $a, b \in M$ such that h is the constant function $\frac{a+b}{2}$ and $(f, g)_B \subset B \times (a, b)$. By (the proof of) Lemma 3.1 there is deformation retract of (a, b) onto $\{\frac{a+b}{2}\}$, which induces a deformation retract of $B \times (a, b)$ onto $B \times \{\frac{a+b}{2}\}$, namely onto the graph of h. □

By the homotopy axiom (Fact 2.1) we obtain:

Corollary 3.3. *If C is a bounded cell of dimension m in M^n then $\mathrm{H}^p(C; G) = 0$ for all $p > 0$ and every Abelian group G.*

If we generalize slightly the definition of definable homotopy and allow the parameter of a homotopy to vary in the interval $[-\infty, +\infty]$, we get that Fact 2.1 is still true, and therefore in Lemmas 3.1 and 3.2 we can drop the "boundedness" hypothesis. Thus, Corollary 3.3 is true also for unbounded cells.

4 Cells with non-acyclic closure

Let M be an o-minimal expansion of a group and let $X \subset M^n$ be a definably compact set. We will prove (Theorem 7.4) that the cohomology groups $\mathrm{H}^p(X; \mathbb{Z})$ of X are finitely generated. An important special case is when X is the closure $\overline{C}$ of a bounded cell C. One may be tempted to conjecture that $\mathrm{H}^i(\overline{C}; \mathbb{Z}) = 0$ in dimension $i > 0$, but Theorem 4.1 shows that in general this is false. Indeed $\mathrm{H}^1(\overline{C}; \mathbb{Z})$ can have arbitrarily large finite rank.

Theorem 4.1. *Let $M = (\mathbb{R}, <, +, \cdot)$. Given $n \in \mathbb{N}$, there is a bounded cell C of dimension 2 in $\mathbb{R}^4$ whose closure $\overline{C}$ is definably homotopy equivalent to a disc with n "holes" (so $\mathrm{H}^1(\overline{C}, \mathbb{Z}) \cong \mathbb{Z}^n$).*

Proof. Let $D \subset \mathbb{R}^2$ be the open unit disc $\{(x, y) \mid x^2 + y^2 < 1\}$, and let $R \subset D$ be one of its rays. For concreteness let us say that R is the intersection of D with the positive x-axis.

Step 1. We claim that there is a definable function $g\colon D \setminus R \to \mathbb{R}$ such that the closure $\overline{\Gamma(g)}$ of the graph of g is definably homeomorphic to a closed disc with n holes. Indeed the case with two holes is illustrated in Figure 1 and the generalization to n holes is straightforward.

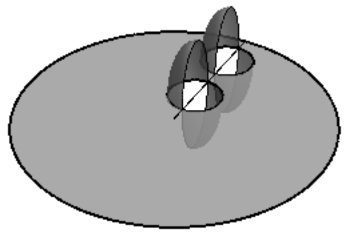


Figure 1: The graph of g

Step 2. We claim that there is a two dimensional cell $F \subset \mathbb{R}^3$ that is definably homeomorphic to $D \setminus R$ via a definable homeomorphism $\psi\colon F \to D \setminus R$ that extends to a homeomorphism $\overline{\psi}\colon \overline{F} \to \overline{D}$ of the respective closures. To prove the claim we define F as the graph of a suitable definable function $f\colon S \to \mathbb{R}$ where $S := (0,1) \times (0,1) \subset \mathbb{R}^2$ is the open unit square. To this aim subdivide S into 5 triangles $T_1, \ldots, T_5$ with a common vertex in $p_0 := (1/2, 1)$ as in Figure 2.

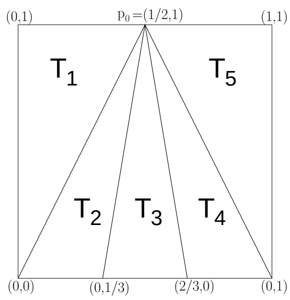


Figure 2: The square S

Now define $f\colon S \to \mathbb{R}$ as the unique continuous function such that:

1. f is the constant function 0 on T_3;
2. f is the constant function 1 on $T_1 \cup T_5$;
3. on T_2, the restriction of f to each horizontal segment $\{(x, y) : y = y_0\}$ is an affine transformation, and similarly for T_4.

Notice that f extend continuously to all points of $\overline{S}$, except the point p_0. The closure $\overline{F}$ of the graph F of f contains the vertical segment $p_0 \times [0,1] \subset \mathbb{R}^3$ (the dashed line in Figure 3). Clearly $\overline{F}$ is definably homeomorphic to a closed disc $\overline{D}$. (Indeed it suffices to show that it is definably homeomorphic to a closed exagon, and this follows by considering the upper part of Figure 3 as a plane figure rather than the projection of a three-dimensional one.) Moreover there is a homeomorphism $\overline{\psi}\colon \overline{F} \cong \overline{D}$ that sends the vertical segment $p_0 \times [0,1] \subset \overline{F}$ to one of the rays of the disc $\overline{D}$, say $\overline{R}$. The restriction of $\overline{\psi}$ to F is the required homeomophism $\psi\colon F \to D \setminus R$.

Step 3. By composition we obtain a definable function $g \circ \psi\colon F \to \mathbb{R}$ whose graph $C = \Gamma(g \circ \psi)$ is the desired 2-dimensional cell. Indeed C is definably homeomorphic to $\Gamma(g)$ via the map $(x, (g \circ \psi)(x)) \mapsto (\psi(x), (g \circ \psi)(x))$ and this map can be extended to a homeomorphism $\overline{C} \cong \overline{\Gamma(g)}$ replacing ψ with its extension $\overline{\psi}$ in the formula. $\square$

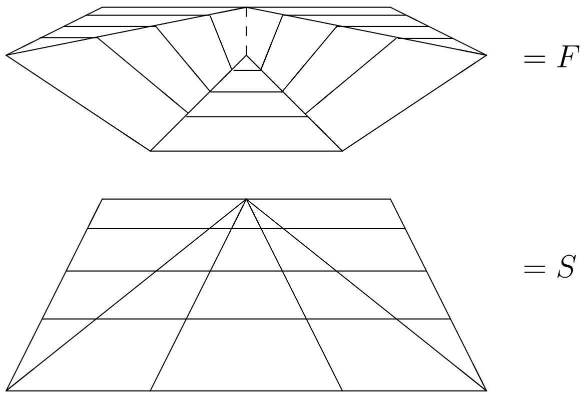


Figure 3: A two-dimensional cell in $\mathbb{R}^3$

Remark 4.2. The number of cells of a cell decomposition does not entail any bound on the Betti numbers: there is a definable set $X \subset \mathbb{R}^3$ with a cell decomposition $\{C, J\}$ consisting of a 2-cell C and a 1-cell J (so $X = C \cup J$) such that $H^1(X, \mathbb{Z})$ has arbitrarily high rank.

Proof. Let S be the open unit square and let I be a 1-dimensional cell contained in its boundary. Let $g\colon I \to \mathbb{R}$ be the constant function zero, and J be its graph; let $f\colon S \to \mathbb{R}$ be definable continuous function that extends to a continuous function $\overline{f}\colon \overline{S} \to \mathbb{R}$ whose restriction to I has many oscillations above and below zero, and C be its graph. Then $X = C \cup J$ is as desired. □

Despite the above remark, it is not however excluded that by a refined version of the cell decomposition theorem one could obtain decompositions that do determine bounds on the cohomology groups (see Question 5.3 below).

5 Acyclic coverings

In this section we give a sufficient condition, not based on deformation retracts, to prove that an inclusion $X \subset Y$ of topological spaces induces an isomorphism in cohomology (see Lemma 5.5).

Given an open cover $\mathcal{U} = \{U_i \mid i \in I\}$ of a topological space X, and a subset $J \subset I$, we write U_J for the intersection $\bigcap_{i \in J} U_i$. The following theorem of Leray says that, given an "acyclic covering", the cohomology of a sheaf can be computed as the Čech cohomology of the covering. We denote by $\check{\mathrm{H}}^*$ the Čech cohomology functor.

Fact 5.1. ([Br97, Thm. 4.13, p. 193], [Go73, §5.4, p. 213]) *Let $\mathcal{F}$ be a sheaf on a topological space X and let $\mathcal{U} = \{U_i \mid i \in I\}$ be an open covering of X having the property that $\mathrm{H}^p(U_J; \mathcal{F}) = 0$ for every $p > 0$ and every finite $J \subset I$. Then the canonical homomorphism $\check{\mathrm{H}}^*(\mathcal{U}; \mathcal{F}) \to \mathrm{H}^*(X; \mathcal{F})$ is an isomorphism.*

If moreover we assume that $\mathcal{F}$ is the constant sheaf G and $\mathrm{H}^0(U_J; \mathcal{F}) = G$ for every J with U_J non-empty, then from the definition of Čech cohomology it follows that $\check{\mathrm{H}}^*(\mathcal{U}; G)$ coincides with the i-th simplicial cohomology group with coefficients in G of the nerve $N(\mathcal{U})$ of $\mathcal{U}$. So we have:

Corollary 5.2. *Let X be a topological space. If there exists a finite open cover $\mathcal{U} = \{U_i \mid i \in I\}$ of X with $\mathrm{H}^p(U_J; G) = 0$ for every $p > 0$ and $\mathrm{H}^0(U_J; G) = G$ for every finite $J \subset I$ with U_J non-empty, then $\mathrm{H}^i(X; G)$ is isomorphic to the i-th simplicial cohomology group of the nerve $N(\mathcal{U})$ of the covering, so in particular it is finitely generated if G is finitely generated.*

Unfortunately we do not know the answer to the following:

Question 5.3. *Let X be a definably compact set in an o-minimal expansion M of a group. Does the topological space $\widetilde{X}$ have a cover as in Corollary 5.2?*

The answer is positive if M expands a field, as a simple application of the triangulation theorem shows.

Remark 5.4. Let $f \colon X \to Y$ be a continuous function, let $\mathcal{V} = \{V_i \mid i \in I\}$ be an (indexed) open cover of Y and consider the (indexed) open cover $f^{-1}(\mathcal{V}) := \{f^{-1}(V_i) \mid i \in I\}$ of X. It follows easily from the definition of the induced homomorphism $\check{f}$ in Čech cohomology (see [EiSt52, IX, §4] or [Br97, III.4.1.4]) that there is a commutative diagram:

$$\begin{array}{ccc}
\check{\mathrm{H}}^p(\mathcal{V}; G) & \xrightarrow{a} & \check{\mathrm{H}}^p(f^{-1}(\mathcal{V}); G) \\
\downarrow b & & \downarrow c \\
\check{\mathrm{H}}^p(Y; G) & \xrightarrow{\check{f}} & \check{\mathrm{H}}^p(X; G)
\end{array}$$

where b, c are the natural morphisms (see [Go73]) and a is induced by the simplicial map on the nerves of the indexed coverings sending $f^{-1}(V_i)$ to V_i.

Lemma 5.5. *Let $X \subset Y$ be topological spaces and let G be an Abelian group. Suppose that there are coverings $\mathcal{U} = \{U_i \mid i \in I\}$ of X and $\mathcal{V} = \{V_i \mid i \in I\}$ of Y indexed by the same finite set I such that:*

1. *$U_i \subset V_i$ for all $i, j \in I$.*

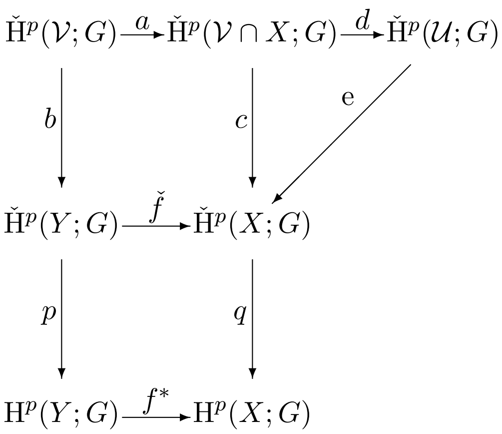
2. *For all finite $F \subset I$, $U_F = \bigcap_{i \in F} U_i$ is non-empty iff $V_F = \bigcap_{i \in F} V_i$ is non-empty (i.e. the natural map among the nerves of the coverings is an isomorphism).*

3. *For each finite $F \subset I$ the sets U_F and V_F are either empty or connected, and for all $q > 0$, $\mathrm{H}^q(U_F; G) = \mathrm{H}^q(V_F; G) = 0$.*

Then the inclusion map $X \subset Y$ induces an isomorphism $\mathrm{H}^*(Y; G) \to \mathrm{H}^*(X; G)$.

Note that we do not require that $V_i \cap X = U_i$.

Proof. We are going to apply Remark 5.4 to the case when $X \subset Y$ and f is the inclusion map. In this case $f^{-1}(\mathcal{V}) = \mathcal{V} \cap X$ (by definition). Consider the following commutative diagram, where a, b, c are as in Remark 5.4, d is induced by the natural simplicial isomorphism on the nerves of the coverings, e is the natural morphism form the Čech cohomology of a covering to the Čech cohomology of the space, and p, q are the natural maps from Čech to sheaf cohomology.



By our assumptions on the coverings, a and d are isomorphisms. By Fact 5.1 $p \circ b$ and $q \circ e$ are isomorphisms. So $f^* : \mathrm{H}^*(Y; G) \to \mathrm{H}^*(X; G)$ is an isomorphism. $\square$

6 Limits

We collect in this section some facts about limits that will shall need later.

Definition 6.1. (See [Br97]) Let X be a topological space, $A \subseteq X$, and $\mathcal{F}$ a sheaf on X. For every neighbourhood U of A in X, the inclusion map $\lambda_U^A : A \to U$ induces a map in cohomology $\mathrm{H}^*(\lambda_U^A) : \mathrm{H}^*(U; \mathcal{F}) \to \mathrm{H}^*(A; \mathcal{F})$.

If $U \subseteq V$, the maps $\mathrm{H}^*(\lambda_V^U)$, $\mathrm{H}^*(\lambda_U^A)$, and $\mathrm{H}^*(\lambda_V^A)$ commute. Hence, we have a canonical map

$$\varinjlim_{A \subseteq U} \mathrm{H}^*(U; \mathcal{F}) \to \mathrm{H}^*(A; \mathcal{F}).$$

We say that A is *taut in X* if, for every sheaf, the above map is an isomorphism.

Fact 6.2. *Let X be an arbitrary topological space and let $A \subseteq X$. Assume that A has a basis of neighbourhoods which are normal and paracompact (but not necessarily Hausdorff) as topological spaces. Then A is taut in X.*

A similar result can be found in [Go73, Thm. II.4.11.1] where it is proved that every closed subset of a paracompact Hausdorff space is taut. It can be checked that the same proof, with minor modifications, also yields the above fact. In any case we only need Corollary 6.3 below, which has been proved in the semialgebraic case in [De85, Thm. 3.1] (but with $\widetilde{X}$ replaced by a subset of the real spectrum of a ring) and in the o-minimal case in [Jo06, Prop. 5.3.1]. In the last paper it is assumed that M is an o-minimal expansion of a field, but the argument works for o-minimal expansions of groups, as observed in [EdJoPe06] (see comments preceeding Thm. 4.3 in that paper).

Corollary 6.3. *Let X be a definable set in an o-minimal expansion M of a group. Let A be a quasi-compact subset of $\widetilde{X}$. Then A is taut in $\widetilde{X}$.*

Lemma 6.4. *Let X be a topological space and let $\big(Y_i\big)_{i \in I}$ be a downward directed family of subsets of X. Let $A \subset \bigcap_{i \in I} Y_i$. Assume that A and each Y_i are taut in X, and that, for every neighbourhood V of A, there exists $i \in I$, such that $Y_i \subseteq V$. Then for every sheaf $\mathcal{F}$ on X,*

$$\mathrm{H}^*(A; \mathcal{F}) = \varinjlim_{i \in I} \mathrm{H}^*(Y_i; \mathcal{F}).$$

Proof. Since A and each Y_i are taut in X,

$$\mathrm{H}^*(A; \mathcal{F}) = \varinjlim_{A \subseteq U} \mathrm{H}^*(U; \mathcal{F}) = \varinjlim_{i \in I} \varinjlim_{Y_i \subseteq U} \mathrm{H}^*(U; \mathcal{F}) = \varinjlim_{i \in I} \mathrm{H}^*(Y_i; \mathcal{F}). \qquad \square$$

Lemma 6.5. *Let $\big(Y_t\big)_{t>0}$ be a definable family of definably compact subsets of some definable set Y, such that $Y_{t'} \subseteq Y_t$ for every $0 < t' < t$. Then, for every sheaf $\mathcal{F}$ on $\widetilde{Y}$, the natural homomorphism*

$$\varinjlim_{t \to 0} \mathrm{H}^*\big(Y_t; \mathcal{F}\big) \to \mathrm{H}^*\big(\bigcap_{t>0} Y_t; \mathcal{F}\big),$$

is an isomorphism.

Proof. Since the Y_t are definably compact, for every definable neighbourhood U of A there exists $t > 0$ such that $Y_t \subseteq U$. Moreover A and Y_t are taut by Corollary 6.3. Hence we can apply Lemma 6.4. $\square$

Remark 6.6. Let R be the limit of a direct system $(R_i \mid f_{i,j})_{i,j\in I}$ of Abelian groups and morphisms $f_{i,j}\colon R_i \to R_j$. Suppose that each $f_{i,j}$ is an isomorphism. Then for each i the natural morphism $R_i \to R$ is an isomorphism.

Lemma 6.7. *Let $\big(Y_t\big)_{t>0}$ be a definable family of definably compact subsets of some definable set Y. Let $\mathcal{F}$ be a sheaf on $\widetilde{Y}$. Suppose that for every $0 < t' < t$ the inclusion $Y_{t'} \subseteq Y_t$ induces an isomorphism*

$$\mathrm{H}^p(Y_t;\mathcal{F}) \cong \mathrm{H}^p(Y_{t'};\mathcal{F}).$$

Then for every fixed $s > 0$ the inclusion $\bigcap_{t>0} Y_t \subset Y_s$ induces an isomorphism

$$\mathrm{H}^*\big(Y_s;\mathcal{F}\big) \cong \mathrm{H}^*\big(\bigcap_{t>0} Y_t;\mathcal{F}\big).$$

7 Finiteness results for cohomology

As usual let M be an o-minimal expansion of group. Given a definably compact set X we want to prove that $\mathrm{H}^i(X;G)$ is finitely generated for every i and every Abelian group G. An important special case is when X is the closure $\overline{C}$ of a bounded cell C. We will show (as a consequence of Corollary 7.3) that there is a point $a \in C$ such that $\mathrm{H}^p(\overline{C} \setminus \{a\};G) \cong \mathrm{H}^p(\partial C;G)$, where $\partial C := \overline{C} \setminus C$ is the boundary of C. Granted this, since ∂C has smaller dimension than $\overline{C}$, by induction on the dimension we can assume that the cohomology groups of ∂C are finitely generated and carry on with the inductive proof. At first sight the fact that $\mathrm{H}^p(\overline{C} \setminus \{a\};G) \cong \mathrm{H}^p(\partial C;G)$ looks rather intuitive: one may even be tempted to conjecture that ∂C is a definable deformation retract of $\overline{C} \setminus \{a\}$ (as it would be the case for the cells of a CW-complex), or at least that these two spaces are definably homotopy equivalent. However we are not able to prove this fact. We proceed instead in a different manner, with the role of definable deformation retracts being taken by Lemma 5.5. There is however a further complication. We are not able to apply Lemma 5.5 directly to the pair of sets $(X,Y) = (\overline{C} \setminus \{a\}, \partial C)$, but only to pairs of sets of the form $(C \setminus \{a\}, C \setminus C_t)$, where $(C_t)_{t>0}$ is a suitable definable collection of sets with $\bigcup_{t>0} C_t = C$, and consequently $\bigcap_{t>0}(\overline{C} \setminus C_t) = \partial C$ (the singleton $\{a\}$ is one of the C_t). So we first prove $\mathrm{H}^p(C \setminus \{a\}) \cong \mathrm{H}^p(C \setminus C_t)$. Then we deduce $\mathrm{H}^p(\overline{C} \setminus \{a\}) \cong \mathrm{H}^p(\overline{C} \setminus C_t)$ by the excision theorem. Finally we let $t \to 0$ to obtain $\mathrm{H}^p(\overline{C} \setminus \{a\}) \cong \mathrm{H}^p(\partial C)$. This is the idea. Let us now come to the details.

Lemma 7.1. *Let $C \subset M^n$ be a bounded cell of dimension m. There is a definable family $\{C_t \mid t > 0\}$ of definably compact sets $C_t \subset C$ such that, for every Abelian group G:*

1. *$C = \bigcup_{t>0} C_t$.*
2. *If $0 < t' < t$, then $C_t \subset C_{t'}$ and the inclusion $C \setminus C_{t'} \subset C \setminus C_t$ induces an isomorphism $\mathrm{H}^p(C \setminus C_t; G) \to \mathrm{H}^p(C \setminus C_{t'}; G)$.*
3. *$C \setminus C_t$ has the same cohomology groups of an $m-1$ dimensional sphere, namely $\mathrm{H}^p(C \setminus C_t; G) = 0$ for $p \notin \{0, m-1\}$, and $\mathrm{H}^p(C \setminus C_t; G) = G$ for $p \in \{0, m-1\}$.*

Note that the result is obvious if M expands a field, since in that case C is definably homeomorphic to an open ball, and each open ball is the increasing union of its concentric closed sub-balls.
Proof. We define C_t as follows.

1. If $n = 1$ and $C = (a, b)$, then $C_t = [a+\gamma_t, b-\gamma_t]$ where $\gamma_t = \min\{\frac{a+b}{2}, t\}$ (so C_t is non-empty).
2. If $n = 1$ and C is a singleton in M, $C_t = C$.
3. Let $n > 1$ and $C = \Gamma(f)$, where $f \colon B \to M$. By induction B_t is defined and we set $C_t = \Gamma(f{\restriction}B_t)$.
4. Let $n > 1$ and $C = (f, g)_B$. By induction B_t is defined. We put $C_t = [f + \gamma_t, g - \gamma_t]_{B_t}$, where $\gamma_t := \min(\frac{f-g}{2}, t)$.

With this definition we have:

Claim 1. *For each $t > 0$ there is a covering $\mathcal{U} = \{U_i \mid i \in I\}$ of $C \setminus C_t$ such that:*

1. *The index set I is the family of the closed faces of an m-dimensional cube, where $m = \dim(C)$. (So $|I| = 2m$).*
2. *If $F \subset I$, then $U_F := \bigcap_{i \in F} U_i$ is either empty or a cell. (So in particular $\mathrm{H}^p(U_F; G) = 0$ for all $p > 0$ and, if $U_F \neq \emptyset$, $\mathrm{H}^0(U_F; G) = G$.)*
3. *For $F \subset I$, $U_F \neq \emptyset$ if and only if $\bigcap F \neq \emptyset$. (So the nerve of $\mathcal{U}$ is isomorphic to the nerve of a covering of an m-cube by its closed faces.)*

For example for $m = 2$ we have four open sets U_i which intersect each other as in Figure 4. Note that the claim implies, by Corollary 5.2, that $C \setminus C_t$ has the same cohomology groups of an $m-1$ dimensional sphere. To prove the claim we define $\mathcal{U}$ by induction on the dimension n of the ambient space. We distinguish four cases according to the definition of C_t.

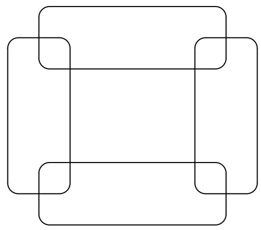


Figure 4

1. If $n = 1$ and C is a singleton, then $\mathcal{U}$ is the covering consisting of one open set (given by the whole space C).

2. If $n = 1$ and $C = (a, b)$, then $C \setminus C_t$ is the union of the two open subsets $(a, a + \gamma_t)$ and $(b - \gamma_t, b)$, and we define $\mathcal{U}$ as the covering consisting of these two sets.

3. Let $n > 1$ and $C = \Gamma(f)$, where $f : B \to M$. By induction we have a covering $\mathcal{V}$ of $B \setminus B_t$ with the stated properties, and we define $\mathcal{U}$ to be the covering of $C \setminus C_t$ induced by the natural homeomorphism between the graph of f and its domain.

4. Let $n > 1$ and $C = (f, g)_B$. By definition $C_t = (f + \gamma_t, g - \gamma_t)_{B_t}$. By induction $B \setminus B_t$ has a covering $\mathcal{V} = \{V_j \mid j \in J\}$ with the stated properties, where J is the set of closed faces of the cube $[0, 1]^{m-1}$. Define a covering $\mathcal{U} = \{U_i \mid i \in I\}$ of $C \setminus C_t$ as follows. As index set I we take the closed faces of the cube $[0, 1]^m$. Thus $|I| = |J| + 2$, the two extra faces corresponding to the "top" and "bottom" face of $[0, 1]^m$. We associate to the top face the open set $(g - \gamma_t, g)_B \subset C \setminus C_t$ and to the bottom face the open set $(f, f + \gamma_t)_B \subset C \setminus C_t$. The other open sets of the covering are the preimages of the sets V_j under the projection $M^n \to M^{n-1}$. This defines a covering of $C \setminus C_t$ with the stated properties.

It remains to show that the inclusion map $C \setminus C_{t'} \subset C \setminus C_t$ induces an isomorphism $\mathrm{H}^p(C \setminus C_t; G) \to \mathrm{H}^p(C \setminus C_{t'}; G)$. To this aim it suffices to observe that by (the proof of) Claim 1 there are coverings $\mathcal{U}$ of $C \setminus C_{t'}$ and $\mathcal{V}$ of $C \setminus C_t$ satisfying the assumptions of Lemma 5.5. □

Lemma 7.2. *Let X be a definably compact set, C be a cell of maximal dimension in X, and G be an Abelian group. Then for each $0 < t' < t$ the inclusion map $X \setminus C_{t'} \subset X \setminus C_t$ induces an isomorphism*

$$\mathrm{H}^*(X \setminus C_t; G) \cong \mathrm{H}^*(X \setminus C_{t'}; G).$$

Proof. Since C is a cell of maximal dimension, $X \setminus C$ is closed in X. Moreover $X \setminus C$ is contained in $X \setminus C_{t'}$, which is open in X. So by the excision theorem ([Br97, Thm. 12.9] and Remark 2.3) we conclude (exciding the complement of C) that the inclusion of pairs $(C \setminus C_t, C \setminus C_{t'}) \to (X \setminus C_t, X \setminus C_{t'})$ induces an isomorphism

$$\mathrm{H}^*(X \setminus C_t, X \setminus C_{t'}) \cong \mathrm{H}^*(C \setminus C_t, C \setminus C_{t'}),$$

where we have omitted the coefficients G in the notation. The right-hand side $\mathrm{H}^*(C \setminus C_t, C \setminus C_{t'})$ is zero by Lemma 7.1 and the long exact cohomology sequence

$$\ldots \to \mathrm{H}^p(C \setminus C_t, C \setminus C_{t'}) \to \mathrm{H}^p(C \setminus C_t) \to \mathrm{H}^p(C \setminus C_{t'}) \to \mathrm{H}^{p+1}(C \setminus C_t, C \setminus C_{t'}) \to \ldots$$

of the pair $(C \setminus C_t, C \setminus C_{t'})$ (see [Br97, II-12-(22)] and Remark 2.3). So the left-hand side $\mathrm{H}^*(X \setminus C_t, X \setminus C_{t'})$ is also zero and therefore, by the long cohomology sequence of the pair $(X \setminus C_t, X \setminus C_{t'})$, the natural homomorphism $\mathrm{H}^*(X \setminus C_t) \to \mathrm{H}^*(X \setminus C_{t'})$ is an isomorphism. □

In the above lemma it is not necessary that X is definably compact: it suffices that C is bounded.

Corollary 7.3. *Suppose that X is a definably compact set such that C is a cell of maximal dimension of X (for instance $X = \overline{C}$). Then for every $t > 0$ the inclusion map $X \setminus C \subset X \setminus C_t$ induces an isomorphism*

$$\mathrm{H}^p(X \setminus C_t; G) \cong \mathrm{H}^p(X \setminus C; G).$$

Proof. By Lemmas 7.2 and 6.7. □

Note that for t big enough, C_t is a singleton. So in particular, applying the theorem to $X = \overline{C}$, we have proved that there is a point $a \in C$ such that:

$$\mathrm{H}^p(\overline{C} \setminus \{a\}; G) \cong \mathrm{H}^p(\partial C; G). \tag{7.1}$$

We do not know however whether ∂C is a definable deformation retract of $\overline{C} \setminus \{a\}$.

Theorem 7.4. *Let $X \subset M^n$ be a definably compact set, R a Noetherian ring, and G be a finitely generated R-module. Then, for each p, $\mathrm{H}^p(X; G)$ is a finitely generated R-module. Moreover, $\mathrm{H}^p(X; G) = 0$ for $p > \dim(X)$. In particular, if G is a finitely generated Abelian group, then $\mathrm{H}^p(X; G)$ is a finitely generated Abelian group.*

Proof. By a Mayer-Vietoris argument. Decompose X into cells. Let C be a cell of X of maximal dimension. Given $t > 0$ in M, write X as the union of $X \setminus C_t$ and C. Consider the Mayer-Vietoris sequence (see [Br97, II-13-(32)] and Fact 2.3) associated to this union:

$$\ldots \to \mathrm{H}^{p-1}(C \setminus C_t) \to \mathrm{H}^p(X) \to \mathrm{H}^p(X \setminus C_t) \oplus \mathrm{H}^p(C) \to \mathrm{H}^p(C \setminus C_t) \to \ldots \tag{7.2}$$

where we have omitted the coefficients G in the notation. By Corollary 7.3 the inclusion $X \setminus C_t \subset X \setminus C$ induces an isomorphism in cohomology, so composing with this isomorphism we obtain

$$\ldots \to \mathrm{H}^{p-1}(C \setminus C_t) \to \mathrm{H}^p(X) \to \mathrm{H}^p(X \setminus C) \oplus \mathrm{H}^p(C) \to \mathrm{H}^p(C \setminus C_t) \to \ldots \tag{7.3}$$

Now C has the cohomology of a point and $C \setminus C_t$ has the cohomology of an $(m-1)$-dimensional sphere. So the displayed part of the sequence above has the form:

$$
\begin{aligned}
& G \to \mathrm{H}^p(X) \to \mathrm{H}^p(X \setminus C) \oplus \mathrm{H}^p(C) \to 0, && (7.4)\\
\text{or}\quad & 0 \to \mathrm{H}^p(X) \to \mathrm{H}^p(X \setminus C) \oplus \mathrm{H}^p(C) \to G, && (7.5)\\
\text{or}\quad & 0 \to \mathrm{H}^p(X) \to \mathrm{H}^p(X \setminus C) \oplus \mathrm{H}^p(C) \to 0; && (7.6)
\end{aligned}
$$

where (7.6) applies for $p \in \{m, m-1\}$. By induction on the number of cells, $\mathrm{H}^p(X \setminus C)$ is finitely generated, and vanishes for $p \geq m$. From the above sequences it then follows that the same holds for $\mathrm{H}^p(X)$. (We use the fact that R is Noetherian in (7.5): in fact, by inductive hypothesis, $\mathrm{H}^p(X \setminus C) \oplus \mathrm{H}^p(C)$ is finitely generated, and a submodule of a finitely generated R-module is finitely generated since R is Noetherian.) □

8 Elementary extensions and change of language

Let M be an o-minimal expansion of a group and let $X \subset M^n$ be a definable set. We have seen that we can associate to X the spectral space $\widetilde{X}$ of all its types over M (such a type can be identified with an ultrafilter of M-definable sets that contains X). The cohomology of X has been defined as the cohomology of $\widetilde{X}$. If $N \succ M$ is an elementary extension we may also associate to X the spectral space $\widetilde{X(N)}$ (ultrafilters of N-definable sets that contains $X(N)$).

Theorem 8.1. *Let $\theta\colon \widetilde{X(N)} \to \widetilde{X}$ be the map that sends a type over N to its restriction over M. If X is definably compact then θ induces an isomorphism $\mathrm{H}^*(\widetilde{X}; G) \to \mathrm{H}^*(\widetilde{X(N)}; G)$ for any Abelian group G.*

Proof. For both X and $X(N)$ we have an exact sequence as in equation (7.2) above. More precisely, decompose X into cells. Let C be a cell of X of maximal dimension. Let $t > 0$ and write X as the union of $X \setminus C_t$ and C. Consider the corresponding Mayer-Vietoris sequences for X and $X(N)$:

$$
\begin{array}{ccccccccc}
\cdots & \mathrm{H}^{p-1}(C \setminus C_t) & \longrightarrow & \mathrm{H}^p(X) & \longrightarrow & \mathrm{H}^p(X \setminus C_t) \oplus \mathrm{H}^p(C) & \longrightarrow & \mathrm{H}^p(C \setminus C_t) & \cdots \\
& \big\downarrow \alpha^{p-1} & & \big\downarrow \beta^p & & \big\downarrow \gamma^p & & \big\downarrow \alpha^p & \\
\cdots & \mathrm{H}^{p-1}(C(N) \setminus C_t(N)) & \longrightarrow & \mathrm{H}^p(X(N)) & \longrightarrow & \mathrm{H}^p(X(N) \setminus C_t(N)) \oplus \mathrm{H}^p(C(N)) & \longrightarrow & \mathrm{H}^p(C(N) \setminus C_t(N)) & \cdots
\end{array}
$$

The vertical arrows are the homomorphisms induced by θ, and the diagram commutes. Again, $\mathrm{H}^*(X \setminus C_t)$ is canonically isomorphic to $\mathrm{H}^*(X \setminus C)$, and the same for N. Thus, by induction on the number of cells, γ^* is an isomorphism. Now C has the cohomology of a point and $C \setminus C_t$ has the cohomology of an $(m-1)$-dimensional sphere, and similarly for N; thus, α^* is also an isomorphism. Hence, β^* is also an isomorphism. □

We now extend the above result to certain type-definable sets. As above let $N \succ M$ and let θ be the restriction map on types.

Theorem 8.2. *Let $X \subset M^n$ be a definably compact set. Let $A \subset \widetilde{X}$ be a closed subset and let $A(N) := \theta^{-1}(A) \subset \widetilde{X(N)}$. Then θ induces an isomorphism $\mathrm{H}^*(A;G) \to \mathrm{H}^*(A(N);G)$. In particular if p is a closed type in $\widetilde{X}$, then the set $\theta^{-1}(p)$ of all types of $\widetilde{X(N)}$ which restrict to p has the same cohomology of a point (so in particular it is connected).*

Proof. Each closed set $A \subset \widetilde{X}$ can be written as an intersection $\bigcap_{i\in I} \widetilde{X}_i$, where each X_i is a definably compact subsets of M^n. Now observe that $A(N) = \bigcap_{i\in I} \widetilde{X_i(N)}$. By Fact 6.3, Lemma 6.4 and Theorem 8.1, $\mathrm{H}^*(A(N);G) = \varinjlim_{i\in I} \mathrm{H}^*(\widetilde{X_i(N)};G) = \varinjlim_{i\in I} \mathrm{H}^*(X_i;G) = \mathrm{H}^*(A;G)$. □

In a similar way we can prove:

Theorem 8.3. *If M_1 is an o-minimal expansion of M to a bigger language and $X \subset M^n$ is a definably compact set in M, then the map $\widetilde{X(M_1)} \to \widetilde{X}$ sending each type in the language L_1 to its restriction to L induces an isomorphism $\mathrm{H}^*(\widetilde{X};G) \to \mathrm{H}^*(\widetilde{X(M_1)};G)$ for any Abelian group G.*

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