

ON THE HOMOTOPY TYPE OF DEFINABLE GROUPS IN AN O-MINIMAL STRUCTURE

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ABSTRACT. We consider definably compact groups in an o-minimal expansion of a real closed field. It is known that to each such group G is associated a natural exact sequence $1 \rightarrow G^{00} \rightarrow G \rightarrow G/G^{00} \rightarrow 1$ where G^{00} is the “infinitesimal subgroup” of G and G/G^{00} is a compact real Lie group. We show that given a connected open subset U of G/G^{00} there is a canonical isomorphism between the fundamental group of U and the o-minimal fundamental group of its preimage under the projection $\mathfrak{p}: G \rightarrow G/G^{00}$. We apply this result to show that there is a natural exact sequence $1 \rightarrow G^{00} \rightarrow \widetilde{G} \rightarrow \widetilde{G/G^{00}} \rightarrow 1$ where \widetilde{G} is the (o-minimal) universal cover of G , and $\widetilde{G/G^{00}}$ is the universal cover of the real Lie group G/G^{00} . We also prove that, up to isomorphism, each finite covering $\mathbf{H} \rightarrow G/G^{00}$, with \mathbf{H} a connected Lie group, is of the form $H/H^{00} \rightarrow G/G^{00}$ for some definable group extension $H \rightarrow G$. Finally we prove that the (Lie-)isomorphism type of G/G^{00} determines the definable homotopy type of G . In the semisimple case a stronger result holds: G/G^{00} determines G up to definable isomorphism. Our results depend on the study of the o-minimal fundamental groupoid of G and the homotopy properties of the projection $G \rightarrow G/G^{00}$.

1. INTRODUCTION

Definable groups in o-minimal expansions of a real closed field have been studied by several authors (see [Ot:08] for a survey). The class of such groups includes all semialgebraic groups over a real closed field, which in turn includes all algebraic groups over an algebraically closed field of characteristic zero (with a fixed maximal real closed subfield). Starting with [Pi:88], the main line of research on definable groups has been guided by the analogy with real Lie groups. However there are also some striking differences: the correspondence between Lie groups and Lie algebras works well in the simple and semisimple case (see [PePiSt:00]), but fails in the abelian case due to the possible absence of one-parameter subgroups (see [St:94, PeSt:99]). To remedy this, there have been two lines of attack in the study of definable groups. One through the study of generic subsets, a kind of substitute for the Haar measure (see [Ke:87, BeOt:04, PePi:07, HrPePi:08, HrPi:07]). The other through the study of the o-minimal Euler characteristic ([St:94]) and of the definable homotopy invariants of a definable group (see [BeOt:02, EdOt:04]). The two lines of research are highly intertwined and advances in each side have been possible through the advances on the other side. By taking a quotient by the

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“infinitesimal subgroup” one can associate in a canonical way to every definably compact group G a compact real Lie group G/G^{00} ([Pi:04, BeOtPePi:05]), giving rise to a well behaved functor $F: G \rightarrow G/G^{00}$ from definably compact groups to compact real Lie groups. A combination of the above mentioned approaches has lead to the determination of the dimension of the associated Lie group ([HrPePi:08]), to the proof of “the compact domination conjecture” in [HrPi:07] (extended in [HrPePi:08b] to the non-abelian case), and to various comparison theorems between the homotopy invariants of a definable group and those of the associated Lie group ([Be:07, Be:09, BeMaOt:09]).

An important tool in these investigations has been the study of the definable fundamental group $\pi_1^{\text{def}}(G)$. If G is definably compact abelian and definably connected, $\pi_1^{\text{def}}(G) \cong \mathbb{Z}^n$ where $n = \dim G$ ([EdOt:04]). In general if G is definably compact and definably connected, $\pi_1^{\text{def}}(G) \cong \pi_1(G/G^{00})$ ([BeMaOt:09]). We will establish a local version of this result, namely we show that for each open connected subset U of G/G^{00} there is an isomorphism $\pi_1^{\text{def}}(\mathfrak{p}^{-1}(U)) \cong \pi_1(U)$ where $\mathfrak{p}: G \rightarrow G/G^{00}$ is the projection. So in particular if $U \subset G$ is an open simply connected set, its preimage in G/G^{00} is a (\vee -definable) definably simply connected set. To prove this, it is convenient to first establish a similar result for the “fundamental groupoid”, namely we show that, for each open subset U of G/G^{00} , there is a natural homomorphism of groupoids $\mathfrak{p}_*: \pi^{\text{def}}(\mathfrak{p}^{-1}(U), \mathfrak{p}^{-1}(U)) \rightarrow \pi(U, U)$ (Theorem 4.4). Note that there is no obvious way to define the required homomorphisms. The difficulty is the following. If γ is a definable path in G and $\mathfrak{p}: G \rightarrow G/G^{00}$ is the natural map, then $\mathfrak{p} \circ \gamma$ is *not* a path in G/G^{00} because we are working in different categories: definable paths in G are parametrized by intervals of the o-minimal structure, while paths in G/G^{00} are parametrized by intervals in \mathbb{R} . We will however show that $\mathfrak{p} \circ \gamma$ can be approximated, in a suitable sense, by a path in G/G^{00} (see Definition 4.11). The process of approximation depends on the choice of suitable coverings of G and G/G^{00} , but the resulting homomorphisms are independent of the coverings and enjoy good functorial properties.

In §8 we show that there is a natural homomorphism $\tilde{\mathfrak{p}}: \widetilde{G} \rightarrow \widetilde{G/G^{00}}$ from the o-minimal universal cover \widetilde{G} of G ([EdEl:07]) to the classical universal cover $\widetilde{G/G^{00}}$ of G/G^{00} . This is obtained embedding \widetilde{G} and $\widetilde{G/G^{00}}$ into $\pi^{\text{def}}(G, G)$ and $\pi(G/G^{00}, G/G^{00})$ respectively, and defining $\tilde{\mathfrak{p}}$ as the restriction of $\mathfrak{p}_*: \pi^{\text{def}}(G, G) \rightarrow \pi(G/G^{00}, G/G^{00})$. Moreover we show that the kernel of $\tilde{\mathfrak{p}}$ is isomorphic to G^{00} (Theorem 8.2). We also show that for every finite extension of real Lie groups $\mathbf{H} \rightarrow G/G^{00}$, with \mathbf{H} connected, there is a finite extension of definable groups $H \rightarrow G$ with $H/H^{00} = \mathbf{H}$ (Theorem 8.3).

In the last part of the paper we apply the results on the fundamental groupoid to try to understand up to which extent G/G^{00} determines G , where G is definably compact and definably connected. In [HrPePi:08b] it is proved that in the group language G is elementary equivalent to G/G^{00} . In the same paper it is proved that the commutator subgroup $[G, G]$ is definable (and semisimple) and G is the almost direct product of $[G, G]$ and the identity component $Z^0(G)$ of its center (an abelian definably connected group). The study of definably compact definably connected groups can thus be reduced to a large extent to the abelian case and the semisimple case. So let us first consider these two cases separately.

The study of semisimple definable groups can be essentially reduced to the study of groups defined in the real field $(\mathbb{R}, <, +, \cdot)$. This depends on the fact that any

o-minimal expansion of a field contains an isomorphic copy of the field \mathbb{R}^{alg} of the real algebraic numbers, and any definably connected semisimple definable group is definably isomorphic to a semialgebraic group defined over \mathbb{R}^{alg} ([EdJoPe:07] or [HrPePi:08b, Theorem 4.4]). Using this fact we show that, in the semisimple case, any Lie isomorphism from G_1/G_1^{00} to G_2/G_2^{00} induces a definable isomorphism from G_1 to G_2 (see Theorem 9.5 for the full statement).

The abelian case is in this respect more complicated. Recall that any compact connected abelian real Lie group of dimension n is Lie isomorphic to the n -dimensional torus. The corresponding result fails for definable groups due to the possible lack of definable one-dimensional subgroups. However in [BeMaOt:09] it is proved that any two definably compact definably connected abelian groups G_1 and G_2 of the same dimension are definably homotopy equivalent. Using the work on the fundamental groupoid we strengthen this result as follows. Given a finite subgroup Γ_1 of G_1 there is a definable homotopy equivalence $f: G_1 \rightarrow G_2$ that restricted to Γ_1 is a group isomorphism onto its image Γ_2 and moreover $f(cx) = f(c)f(x)$ for all c is in Γ_1 and all x in G_1 (for the full statement see Theorem 10.4).

Combining the results on the abelian case and the semisimple case, we obtain that given two definably compact definably connected groups G_1 and G_2 with $G_1/G_1^{00} \cong G_2/G_2^{00}$, then G_1 and G_2 are definably homotopy equivalent¹. Moreover, given a finite subgroup Γ_1 of G_1 , we can choose the homotopy equivalence to be “ Γ_1 -equivariant”, namely to respect the action of Γ_1 (for the full statement see Theorem 11.7). In general Γ -equivariant maps are a useful tool to deal with almost direct products (see Lemma 11.2).

Having proved that the homotopy type of a definably compact group G is determined by G/G^{00} , let us observe that if G is not definably compact, the study of its homotopy type can be reduced to the compact case by the results contained in [Co:09].

Our proofs make essential use of the fact that G^{00} is a decreasing intersection of a countable family of definably simply connected open subsets of G (actually open cells). This was established in [Be:09] making use of the “compact domination conjecture” proved in [HrPi:07] (extended in [HrPePi:08b] to the non-abelian case). To a large extent our analysis remains valid replacing $\mathfrak{p}: G \rightarrow G/G^{00}$ with any other “logical quotient” $f: X \rightarrow Y$ (as in [Pi:04, §2]) where X is a \forall -definable set and Y is a locally simply connected second countable locally compact space, provided that each fiber $f^{-1}(y)$ of f is a decreasing intersection $\bigcap_{i \in \mathbb{N}} C_i$ of a countable family of definably simply connected open subsets of X . Under these assumptions Theorem 4.4 gives us an isomorphism $\pi_1^{\text{def}}(X) \cong \pi_1(Y)$. In the group situation, namely when f is the map $\mathfrak{p}: G \rightarrow G/G^{00}$, we obtain stronger results (e.g. Theorem 11.7). Essentially this depends on the fact (established in [BeMaOt:09]) that all the higher homotopy groups of a definably compact abelian group are zero, so the fundamental group already gives us all the relevant homotopic information.

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¹This also follows by the results of Elias Baro in [Ba:09, §4] obtained independently at the same time and by different methods (private communication).

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2. TOPOLOGICAL NOTIONS

Fix an o-minimal structure M expanding a field. By “definable” we mean “definable in M ”. As usual we put on M the order topology and on M^n the product topology. Unless otherwise stated subsets of M^n will be endowed with the induced topology. We say that a set $X \subset M^n$ is *definably path-connected* if each pair of points of X can be connected by a definable (continuous) path in X . (When X is definable, this is equivalent to say that X is *definably connected*, namely if it cannot be partitioned in two non-empty open definable subsets.) We say that X is *definably simply connected* if it is definably path connected and any two definable paths in X with the same endpoints are definably homotopic relative to the endpoints.

Dropping the definability conditions one obtains the corresponding classical notions. The reader should however be warned that definably connected spaces are in general not connected in the classical sense (unless the o-minimal structure is based on the reals), and similarly for the other notions. Let us also recall that a topological space X is *locally simply connected* if for each open set O in X and each $x \in O$, there is an open neighbourhood $V \subset O$ of x which is simply connected. There are several possible definable versions of this notion, but we only need the classical concept.

By [Pi:88] any definable group G in M has a natural group topology making it into a (regular) definable manifold, namely G admits a finite atlas where each chart is definably homeomorphic to an open subsets of M^d , with $d = \dim(G)$. When speaking about a definable group G we will always assume that G has the group topology rather than the topology induced by the ambient space M^n . So in particular a definable path in G will always be assumed to be continuous with respect to the group topology of G . Note that since M expands a field, by [vdD:98, §10, Theorem 1.8] we can always reduce to a situation in which the two topologies coincide, namely we can definably embed G in M^n as a subspace. In this case G is *definably compact* in the sense of [PeSt:99] if and only if G is closed and bounded in M^n .

3. TYPE-DEFINABILITY

We recall that a type-definable set is a set that can be presented as an infinite intersection $\bigwedge_{i \in I} X_i$ where each X_i is definable and I is a possibly infinite index set. Type-definable sets come equipped with a presentation, so it makes sense to interpret them in elementary extensions. Unlike what happens for definable sets, an equality $\bigwedge_{i \in I} X_i = \bigwedge_{j \in J} Y_j$ can hold in some model M and fail in an elementary extension. To have a notion of equality not dependent on the model, we must restrict ourselves to models that are sufficiently saturated. For this reason it is convenient to identify a type-definable set with the set it defines in some big saturated “monster model” and insist that the index sets in the infinite conjunctions should be “small” (with respect to the saturation of the monster model). With these conventions one has for instance that infinite conjunctions commute with the existential quantifier, namely $\exists x \bigwedge_{i \in I} (x \in X_i) \equiv \bigwedge_{i \in I} \exists x (x \in X_i)$, provided the family of definable sets $\{X_i\}_{i \in I}$ is downward directed (e.g. it is closed under finite intersections). It is common practice to say that such equalities hold “by

saturation". An \forall -definable (or-definable) set is a set presented as a union $\bigvee_{i \in I} X_i$, where each X_i definable and the index set I is small. By saturation if a type-definable set is included in an \forall -definable set, there is some definable set between them.

Starting with [Pi:04], type-definability plays an important role in the study of definable groups in an o-minimal structure. Given a definable group G and a type-definable subgroup $H < G$, one says that H has *bounded index* if the index $[G : H]$ is smaller than the amount of saturation of the monster model. Equivalently $[G : H] < 2^{|T|+|A|}$ where $|T|$ is the cardinality of the language and A is the set of parameters over which G and H are defined ([Sh:08, HrPi:07]). If H has bounded index, then G/H does not depend on the model, in the sense that if M is sufficiently saturated and $M' \succ M$, then the natural map $G(M) \rightarrow G(M')/H(M')$ has kernel $H(M)$ and therefore $G(M)/H(M) \cong G(M')/H(M')$ ([Pi:04]). Each definably compact group G in an o-minimal expansion of a field, has a smallest type-definable subgroup $G^{00} < G$ of bounded index, necessarily normal, and G/G^{00} with the "logic topology", is a compact real Lie group ([BeOtPePi:05]) whose dimension equals the o-minimal dimension of G ([HrPePi:08]). Recall that a set $X \subset G/G^{00}$ is open in the *logic topology* if and only if its preimage in G under the natural map $\mathfrak{p}^G : G \rightarrow G/G^{00}$ is \forall -definable. It then follows that the preimage of a closed subset of G/G^{00} is type-definable and that the image of a type-definable subset of G is a closed subset of G/G^{00} .

4. TOPOLOGICAL CONSEQUENCES OF COMPACT DOMINATION

One of the aims of this paper is to explore the topological consequences of the compact domination conjecture proved in [HrPi:07] (and extended in [HrPePi:08b] to the non-abelian case). One of its equivalent formulations says that given a definably compact group G , the image in G/G^{00} of a nowhere dense definable subset of G has Haar measure zero. We will only use the following consequence of compact domination, proved in [Be:09]: the subgroup G^{00} is a decreasing intersection $\bigcap_{i \in \mathbb{N}} C_i$ of definably simply connected (actually even definably contractible) open definable subsets C_i of G . We will explore this situation in the following more general setting:

Assumption 4.1. Let X be a \forall -definable set, let Y be a locally simply connected second countable locally compact Hausdorff space, and let $f : X \rightarrow Y$ be a surjective function with the following properties:

- (1) The preimage of every open subset of Y is \forall -definable (so in particular X is \forall -definable).
- (2) The preimage of every compact subset of Y is type-definable.
- (3) For all $y \in Y$ the type-definable set $f^{-1}(y)$ is a decreasing intersection $\bigcap_{i \in \mathbb{N}} C_i$ of definably simply connected definable open subsets $\{C_i\}_i$ of X .

Note that the second countability assumption ensures that $|Y| \leq 2^{\aleph_0}$, so in particular Y is small.

Example 4.2. The natural projection map $\mathfrak{p} : G \rightarrow G/G^{00}$ satisfies assumption 4.1. More generally let O be an open subset of G/G^{00} , then $\mathfrak{p}|_{\mathfrak{p}^{-1}(O)}$ satisfies Assumption 4.1. It can be verified that the homomorphism $\tilde{\mathfrak{p}} : \tilde{G} \rightarrow \widetilde{G/G^{00}}$ of Definition 8.1 also satisfies Assumption 4.1 (we do not include a proof since we do not need this result).

Definition 4.3 (Definable fundamental groupoid). Given an \mathcal{V} -definable set X , and a subset Γ of X , let $P^{\text{def}}(X, \Gamma)$ be the set of definable paths in X with endpoints in Γ . Let $\pi^{\text{def}}(X, \Gamma)$ be the quotient of $P^{\text{def}}(X, \Gamma)$ modulo definable homotopy of paths (relative to the endpoints). We define an operation $+$ on $\pi^{\text{def}}(X, \Gamma)$ by $[\alpha] + [\beta] = [\alpha + \beta]$ where $\alpha + \beta$ is the concatenation of the paths α and β . Clearly, this is defined only when the final point of α coincides with the initial point of β . With this operation $\pi^{\text{def}}(X, \Gamma)$ is a groupoid, namely a category in which every morphism is an isomorphism (the objects are the elements of Γ , the morphisms are the homotopy classes of paths, and the composition is the concatenation). In particular when $\Gamma = X$ we obtain the *definable fundamental groupoid* $\pi^{\text{def}}(X, X)$ of X . When Γ is a singleton we obtain the definable fundamental group $\pi^{\text{def}}(X, x_0) := \pi^{\text{def}}(X, \{x_0\})$, which will also be written as $\pi_1^{\text{def}}(X)$ when the base point is clear from the context or irrelevant. Dropping “def” one obtains the corresponding classical notions.

In this section we will prove the following theorem.

Theorem 4.4. *Let $f: X \rightarrow Y$ be as in assumption 4.1. Then f is continuous and there is a unique morphism of groupoids $f_*: \pi^{\text{def}}(X, X) \rightarrow \pi(Y, Y)$ with the following properties:*

- (1) $f_* = f$ on the object part of the groupoids, i.e. f_* maps the definable homotopy class of a path with endpoints x and y , to the homotopy class of a path with endpoints $f(x)$ and $f(y)$ respectively.
- (2) For any open $O \subseteq Y$, and for any $[a] \in \pi^{\text{def}}(X, X)$ such that $\text{Im } a \subseteq f^{-1}(O)$, there is a path b in Y such that $\text{Im } b \subseteq O$ and $f_*([a]) = [b]$.

Moreover for this unique morphism f_* we have:

- (3) If Γ is a (possibly infinite) subset of X such that $f|_{\Gamma}$ is injective, then the restriction of f_* to $\pi^{\text{def}}(X, \Gamma)$ is an isomorphism onto $\pi(Y, f(\Gamma))$. In particular if Γ is a singleton, we obtain an isomorphism of the respective fundamental groups.
- (4) If $f: X \rightarrow Y$ satisfies assumption 4.1 and O is an open subset of Y , then the restriction $f^O: f^{-1}(O) \rightarrow O$ of f continues to satisfy the assumption (we have restricted also the codomain to make the map surjective). In this situation we have a commutative diagram

$$\begin{array}{ccc} \pi^{\text{def}}(f^{-1}(O), f^{-1}(O)) & \xrightarrow{f_*^O} & \pi(O, O) \\ \downarrow & & \downarrow \\ \pi^{\text{def}}(X, X) & \xrightarrow{f_*} & \pi(Y, Y) \end{array}$$

where the vertical arrows are the homomorphisms induced by the inclusion maps.

Corollary 4.5. *Let G be a definably compact definably connected definable group. Then the projection $\mathfrak{p}: G \rightarrow G/G^{00}$ induces an isomorphism*

$$\mathfrak{p}_*: \pi_1^{\text{def}}(G) \xrightarrow{\cong} \pi_1(G/G^{00})$$

Moreover if O is an open simply connected subset of G/G^{00} , then $\mathfrak{p}^{-1}(O)$ is an \mathcal{V} -definable definably simply connected open subset of G

Proof. Apply Theorem 4.4(3) with Γ a single point in $\mathfrak{p}^{-1}(O)$. \square

Remark 4.6. Let G be a definably compact group. By [HrPePi:08] the subgroup G^{00} is (divisible and) torsion free (see [Be:07] for the non-abelian case), so if Γ is a finite subgroup of G , then \mathfrak{p}^G maps Γ isomorphically onto its image in G/G^{00} .

Corollary 4.7. *Let G be a definably compact group and let $\mathfrak{p}^G: G \rightarrow G/G^{00}$ be the natural map.*

- (1) *If Γ is a finite subgroup of G , the restriction of \mathfrak{p}_*^G to $\pi^{\text{def}}(G, \Gamma)$ is an isomorphism onto $\pi(G/G^{00}, \mathfrak{p}^G(\Gamma))$.*
- (2) *If $\Gamma \subset G$ is a subset containing one and only one representative for each coset of G^{00} , then the restriction of \mathfrak{p}_*^G to $\pi^{\text{def}}(G, \Gamma)$ is an isomorphism onto $\pi(G/G^{00}, G/G^{00})$.*

Proof. By Theorem 4.4(3) (using Remark 4.6 for the first point). \square

Proof of Theorem 4.4. The proof is split into a number of claims and definitions, and will be completed at the end of this section. The idea is the following. Given an open cover \mathcal{U} of a topological space, the nerve of \mathcal{U} is the simplicial complex whose n -simplexes are the n -tuples of open sets from \mathcal{U} which have a non-empty intersection. We will show that, if \mathcal{U} satisfies suitable assumptions (each element of \mathcal{U} is path connected and the intersection of two elements of \mathcal{U} is contained in a simply connected set), then the fundamental groupoid of the space is determined by the nerve of \mathcal{U} (and the knowledge of which points belong to which sets of the nerve). We will apply this result to the topological space Y . A corresponding result holds also in the o-minimal category (considering definably path-connected and definably simply connected sets), so we can apply it to X . We will show (Claim 4) that X and Y admit two open covers \mathcal{V} and \mathcal{U} which satisfy the required assumptions in the respective categories and have isomorphic nerves (in fact \mathcal{V} will consist of the f -preimages of the sets in \mathcal{U} : the difficulty here is in choosing \mathcal{U} so that all the conditions are met on both sides *simultaneously*). This will essentially prove the result. Guided by the above idea, we will actually develop the proof without explicit mention of the nerves. Let us now come to the formal details.

Claim 1. *f is continuous.*

Proof of claim. We adapt the argument in [Pi:04, Lemma 3.2]. By our assumptions, for every $y \in Y$, the preimage $f^{-1}(y)$ is a small directed intersection of definable open sets. So it suffices to show that (in a saturated model) the intersection $\bigcap_{i \in I} O_i$ of a small directed family of definable open sets O_i is open. So take $x \in \bigcap_{i \in I} O_i$ and let us show that x is in the interior. To this aim let $B_t(x)$ be the ball of center x and positive radius $t \in M$. Since O_i is open in X , there is $t_i > 0$ such that for all $0 < t < t_i$ we have $B_t(x) \cap X \subset O_i$. By saturation there is a single positive $t^* \in M$ which is smaller than each t_i . But then $B_{t^*}(x) \cap X \subset \bigcap_{i \in I} O_i$ so x is in the interior. \square

Claim 2. *Let D be a definable subset of X . Then $f(D)$ is a compact subset of Y . (It follows that the same conclusion holds if D is only assumed to be type-definable.)*

Proof of claim. Let $(O_i \mid i \in I)$ be an open cover of $f(D)$. We must find a finite subcover. The definable set D is included in $\bigcup_{i \in I} f^{-1}(O_i)$ and each $f^{-1}(O_i)$ is ∇ -definable. By saturation there is a finite $J \subset I$ such that $D \subset \bigcup_{i \in J} f^{-1}(O_i)$. It then follows that $f(D) \subset \bigcup_{i \in J} O_i$. \square

- Claim 3.** (1) *Let Z be a compact connected subset of Y . Then the type-definable set $f^{-1}(Z)$ is definably connected.*
 (2) *Let U be an open connected subset of Y . Then the \forall -definable set $f^{-1}(U)$ is definably path-connected.*

Proof of claim. (1) By [BeOtPePi:05, Lemma 2.2] if a type-definable set is the intersection of a filtered family of definably connected sets, then it is itself definably connected (i.e. it is not the union of two non-empty relatively definable open subsets). It then follows from Assumption 4.1 that for each $y \in Y$, the type-definable set $f^{-1}(y)$ is definably connected. Now let Z be a compact connected subset of Y , and suppose for a contradiction that $f^{-1}(Z)$ is the union of two relatively definable disjoint non-empty open sets A and B . Being relatively definable in a type-definable set, A and B are in fact type-definable. So their images $f(A)$ and $f(B)$ are compact by Claim 2. Since $Z = f(A) \cup f(B)$ and Z is connected, $f(A)$ and $f(B)$ have a non-empty intersection. Take $y \in f(A) \cap f(B)$. Then $f^{-1}(y)$ meets both A and B , contradicting the fact that $f^{-1}(y)$ is definably connected.

(2) Let $x, y \in f^{-1}(U)$. Since Y is locally simply connected, it is in particular locally path connected, so its connected open subsets are path-connected. We can thus choose a path a in U connecting $f(x)$ to $f(y)$. Its image $Z := \text{Im}(a)$ is a compact connected subset of U . So by (1) the type-definable set $f^{-1}(Z)$ is definably connected. Since this set is contained in the \forall -definable set $f^{-1}(U)$, by saturation there is a definable set D with $f^{-1}(Z) \subset D \subset f^{-1}(U)$. The definably connected component D' of D containing x must contain also y (since it contains $f^{-1}(Z)$). Now it suffices to recall that a definable set is definably connected if and only if it is definably path connected. \square

Definition 4.8. An open cover \mathcal{P} of a topological space is a *star refinement* of a cover \mathcal{Q} , if for every $P \in \mathcal{P}$, there is a $Q \in \mathcal{Q}$ such that if $P' \in \mathcal{P}$ has a non-empty intersection with P then $P' \subset Q$. In a metric space, and more generally in a uniform space, every open cover has a star refinement. Every Tychonoff space admits a compatible uniform structure, so the existence of star refinements applies to Tychonoff spaces. In particular it applies to any open subset of a locally compact Hausdorff space.

Claim 4. *There are open covers \mathcal{U} of Y and \mathcal{V} of X having the following properties.*

- (1) $\mathcal{V} := \{f^{-1}(U)\}_{U \in \mathcal{U}}$ (so \mathcal{V} is determined by \mathcal{U}).
- (2) *Each element of \mathcal{U} is path connected, and whenever two elements of \mathcal{U} have a non-empty intersection, their union is contained in some simply connected subset of Y .*
- (3) *Each element of \mathcal{V} is definably path connected (and \forall -definable), and whenever two elements of \mathcal{V} have a non-empty intersection, their union is contained in some definably simply connected subset of X .*

Note that it is easy to find a cover \mathcal{U} satisfying (2) (take a star-refinement of a cover by simply connected sets). Moreover it will turn out that if \mathcal{U} satisfies (2) and we define \mathcal{V} as in (1), then \mathcal{V} satisfies (3). However for the moment we cannot assume this fact, so we need to do some more work to find the appropriate \mathcal{U} .

Proof of Claim 4. By assumption 4.1 we can choose, for each $y \in Y$, a definably simply connected open definable set $C_y \subset X$ containing $f^{-1}(y)$. Since Y is locally simply connected and second countable, we can find a decreasing sequence $\{O_n\}_{n \in \mathbb{N}}$

of simply connected open neighborhoods of y with $\bigcap_n O_n = \{y\}$. Moreover since Y is locally compact, we can arrange so that for each n the set $\overline{O_n}$ is compact and $\overline{O_{n+1}} \subset O_n$. (Indeed, given O_n , by local compactness we can find a compact neighbourhood $W_n \subset O_n$ and by local simple connectedness we can find a simply connected open neighbourhood $O_{n+1} \subset W_n$.) By our assumptions $f^{-1}(\overline{O_{n+1}})$ is type-definable. Since $\bigcap_{n \in \mathbb{N}} f^{-1}(\overline{O_n}) = f^{-1}(y) \subset C_y$, by saturation there is some n such that $f^{-1}(\overline{O_n}) \subset C_y$. Fix such an n and let $Z_y = O_n$. So $f^{-1}(Z_y) \subset C_y$. We have thus found a cover $\mathcal{Z} := \{Z_y\}_{y \in Y}$ of Y , such that the preimage of any element of the cover is contained in a definably simply connected set. Now let \mathcal{U} be a star-refinement of \mathcal{Z} . We can assume that each element of \mathcal{U} is path-connected, as otherwise we could replace it by its connected components. So \mathcal{U} satisfies (2). Finally let $\mathcal{V} = \{f^{-1}(U)\}_{U \in \mathcal{U}}$. Then \mathcal{V} is a star-refinement of $\{f^{-1}(U)\}_{U \in \mathcal{Z}}$. We must prove that \mathcal{V} satisfies (3). By Claim 3 each member of \mathcal{V} is definably path-connected. By construction whenever two members of \mathcal{V} intersect, their union is contained in a set of the form C_y , which is a definably simply connected set. \square

Claim 5. *Any \mathcal{V} -definable open subset V of X is a small union of definable open sets.*

Proof of claim. We can write V as a small directed union $\bigcup_{i \in I} D_i$ of definable sets D_i . We claim that V is the union $\bigcup_{i \in I} \text{Int}(D_i)$ of the interiors of the sets D_i . To this aim, let $x \in V$. Since V is open there is a definable open neighbourhood U of x contained in $\bigcup_{i \in I} D_i$. Hence by saturation there is some $i \in I$ such that $U \subset D_i$. But then $x \in \text{Int}(D_i)$. \square

Definition 4.9. Let \mathcal{P} be a family of subsets of a given set. A set is \mathcal{P} -small if it is contained in some member of \mathcal{P} . A function is \mathcal{P} -small if its image is \mathcal{P} -small.

Claim 6. (1) *Given a definable subset D of X , there are finitely many sets in \mathcal{V} whose union covers D . Moreover there is a finite partition of D into definable sets whose closures are \mathcal{V} -small.*

(2) *Given a definable path a in X there is a subdivision $a = a_1 + \dots + a_n$ of a such that each a_i is \mathcal{V} -small.*

Proof of claim. (1) Since $f(D)$ is a compact subset of Y , it can be covered by finitely many members U_1, \dots, U_n of \mathcal{U} . It follows that $D \subset V_1 \cup \dots \cup V_n$ where $V_i := f^{-1}(U_i) \in \mathcal{V}$. Each V_i is \mathcal{V} -definable and open, so by saturation there are definable open subsets O_i of V_i with $D \subset O_1 \cup \dots \cup O_n$. By shrinking each O_i if necessary, we may assume that the closure of O_i is included in V_i for all i . It is now sufficient to take a cell decomposition of D compatible with O_1, \dots, O_n .

(2) Take a finite partition \mathcal{P} of $\text{Im}(a)$ into definable sets whose closures are \mathcal{V} -small. Then take a cell decomposition of $I = \text{dom}(a)$ compatible with $f^{-1}(D)$ for all $D \in \mathcal{P}$. The endpoints of the decomposition yield the desired subdivision of a . \square

Definition 4.10. (1) Given two definable paths a and b in X with the same endpoints, we say that they are \mathcal{V} -contiguous (written $a \sim_{\mathcal{V}} b$) if there are definable paths u, v, a', b' in X such that $a = u + a' + v$ and $b = u + b' + v$ and $\text{Im}(a') \cup \text{Im}(b') \subset V$ for some $V \in \mathcal{V}$. Let $\sim_{\mathcal{V}}^*$ be the transitive closure of $\sim_{\mathcal{V}}$. Two paths in the $\sim_{\mathcal{V}}^*$ relation are said to be \mathcal{V} -equivalent.

(2) Similarly, dropping all definability conditions, one defines the relation of \mathcal{U} -contiguity and \mathcal{U} -equivalence between paths in Y .

- Claim 7.** (1) *Two paths in Y are \mathcal{U} -equivalent if and only if they are homotopic (all the homotopies we consider are relative to the endpoints).*
 (2) *Two definable paths in X are \mathcal{V} -equivalent if and only if they are definably homotopic.*

Proof of claim. One direction is trivial (equivalent implies homotopic). We prove the other direction.

(1) Implicit in the proof of the van Kampen theorem in [Br:68]. One argues as follows. Given an homotopy $F: I \times I \rightarrow Y$ from a to b , we can subdivide the homotopy square $I \times I$ into small squares so that the image of each of them under F is \mathcal{U} -small. If n is the number of squares in the subdivision of $I \times I$, it is easy to see that a is \mathcal{U} -equivalent to b by a sequence of n contiguity moves. The converse is trivial, since two \mathcal{U} -contiguous paths differ for a subpath contained in a simply connected set.

(2) Let $F: I \times I \rightarrow X$ be a definable homotopy between definable paths $a = F(0, -)$ and $b = F(1, -)$ in X . By Claim 6 the image of F can be partitioned into finitely many definable sets D_1, \dots, D_n whose closures are \mathcal{V} -small. Consider a cell decomposition of the homotopy square $I \times I$ compatible with $F^{-1}(D_i)$ for all i . We can then reason as above with the role of the small squares replaced by the cells of the decomposition. For the details see the proof of the o-minimal van Kampen theorem in [BeOt:02]. \square

We are now ready to define $f_*: \pi^{\text{def}}(X, X) \rightarrow \pi(Y, Y)$.

Definition 4.11. On the object part of the groupoids we set $f_* = f$. Given a definable path a in X and a path b in Y we say that a corresponds to b if there is a subdivision $a = a_1 + \dots + a_n$ into \mathcal{V} -small definable paths a_i , and a subdivision $b = b_1 + \dots + b_n$ into \mathcal{U} -small paths such that, for all $i \leq n$, the endpoints of a_i are mapped by $f: X \rightarrow Y$ to the respective endpoints of b_i . If a corresponds to b , then we define $f_*([a]) = [b]$.

The proof of the following claim depends only on Claims 4 and 7, so it is rather symmetric in X and Y . This observation will be needed later.

Claim 8. f_* is well defined.

Proof of claim.

Step 1. First note that two \mathcal{V} -small definable paths a and a' in X with the same endpoints are definably homotopic. In fact if $V \in \mathcal{V}$ and $V' \in \mathcal{V}$ contain the images of a and a' respectively, then by Claim 4 we have that $V \cup V'$ is contained in a definably simply connected set, and therefore a is definably homotopic to a' . Similarly two \mathcal{U} -small paths b and b' in Y with the same endpoints are homotopic. So $f_*([a]) = [b]$ is well defined at least when a and b are \mathcal{V} -small.

Step 2. We next show that if a corresponds to b , then the homotopy class of b is determined by a . So suppose that $a = a_1 + \dots + a_n$ is a subdivision of a into \mathcal{V} -small definable paths, let a_i correspond to b_i (a path in Y), and let $b = b_1 + \dots + b_n$. By Step 1, the homotopy class of each b_j is determined by the corresponding a_j , but we must prove that the homotopy class of b does not depend on the chosen subdivision of a . Since any two subdivisions have a common refinement, it suffices to consider the case in which one of the a_i is further subdivided into \mathcal{V} -small paths. So without loss of generality suppose $i = 1$ and let $a_1 = a_1^1 + \dots + a_1^k$ be a subdivision of a_1 into \mathcal{V} -small paths. We must show that b_1 is homotopic to $b_1^1 + \dots + b_1^k$

where each b_1^j is such that a_1^j corresponds to b_1^j . To this aim let $U \in \mathcal{U}$ be such that $\text{Im}(a_1) \subset f^{-1}(U)$. Then in particular the endpoints of a_1 and of each a_1^j are in $f^{-1}(U)$. Therefore the endpoints of b_1 and of each b_1^j are in U . Reasoning as in Step 1 we can then assume that the image of b_1 and of each b_1^j is entirely contained in U since we can reduce to this case replacing each of these paths by a homotopic path. After these reductions, b_1 and $b_1^1 + \cdots + b_1^k$ are two paths in U with the same endpoints, and therefore they are homotopic (since U is contained in a simply connected set).

Step 3. Finally suppose that a is definably homotopic to a' , and let us show that b is homotopic to b' , where a corresponds to b and a' to b' . By Claim 7 we can assume that a' is \mathcal{V} -contiguous to a . So we can write $a = u + z + v$ and $a' = u + z' + v$ with $\text{Im}(z) \cup \text{Im}(z') \subset V$ for some $V \in \mathcal{V}$. Choose subdivisions of a and a' such that z and z' are segments of the chosen subdivisions. By Step 2 we may assume that b and b' are obtained from a and a' using these subdivisions. So we can assume that b and b' are \mathcal{U} -contiguous. Therefore, by Claim 7, they are homotopic. \square

Claim 9. *The definition of f_* does not depend on the particular choice of the cover \mathcal{U} in Claim 4.*

Proof of claim. Suppose \mathcal{U}' is a refinement of \mathcal{U} still satisfying the conditions in Claim 4. Then clearly if we define $f_*([a])$ using \mathcal{U}' instead of \mathcal{U} we get the same function (since a subdivision of a compatible with the preimages of the sets in \mathcal{U}' is also compatible with the preimages of the sets in \mathcal{U}). Now it suffices to observe that for any two coverings \mathcal{U} and \mathcal{U}' satisfying the conditions in Claim 4 there is a common refinement which still satisfies the conditions (take the connected components of the pairwise intersections of an element of \mathcal{U} and an element of \mathcal{U}'). \square

Claim 10. *f_* is a morphism and satisfies points 1 and 2 in Theorem 4.4.*

Proof of claim. Note that in Definition 4.11, if $a = a_0 + a_1$ and a_0, a_1 correspond to b_0 and b_1 respectively, then a corresponds to $b := b_0 + b_1$. Hence f_* is a morphism. By construction it satisfies 1. We prove 2. Because of Claim 9, by enlarging \mathcal{U} we can suppose it to be a base of the topology of Y . Given an open subset O of Y , we can then express O as the union of a subfamily \mathcal{U}' of \mathcal{U} . Consider $\mathcal{V}' := \{f^{-1}(U)\}_{U \in \mathcal{U}'}$. Given a path a in $f^{-1}(O)$, the construction of $[b] = f_*([a])$ can be carried out subdividing a into \mathcal{V}' -small paths and associating to each of them a \mathcal{U}' -small path in Y . The image of b is clearly a subset of O . \square

Claim 11. *f_* is the unique morphism satisfying 1 and 2 in Theorem 4.4.*

Proof of claim. Let O be a simply connected open subset of Y . If a is a definable path in X with $\text{Im } a \subseteq f^{-1}(O)$, condition 2 in Theorem 4.4 forces $f_*([a])$ to be of the form $[b]$ for some path b with $\text{Im } b \subseteq O$. Since O is simply connected, and the endpoints of b are the images of the endpoints of a , then $[b]$ is completely determined. So f_* is determined on the paths satisfying $\text{Im}(a) \subseteq f^{-1}(O)$ for some open simply connected subset $O \subset Y$. By Claim 6 we can reduce to this situation by subdividing the paths. \square

Claim 12. *Let $\Gamma \subset X$ be such that $f|_{\Gamma}: \Gamma \rightarrow Y$ is injective. Then the restriction of f_* to $\pi^{\text{def}}(X, \Gamma)$ is an isomorphism onto $\pi(Y, f(\Gamma))$.*

Proof of claim. Choose a right inverse $\psi: Y \rightarrow X$ of $f: X \rightarrow Y$ extending $(f \upharpoonright_{\Gamma})^{-1}$. We define an inverse $\psi_*: \pi(Y, f(\Gamma)) \rightarrow \pi^{\text{def}}(X, \Gamma)$ of f_* as follows. Given $[b] \in \pi(Y, f(\Gamma))$, consider a subdivision $b = b_1 + \cdots + b_m$ of b into \mathcal{U} -small paths. For each i , let y_{i-1} and y_i be the endpoints of b_i , and let a_i be a \mathcal{V} -small path in X from $x_{i-1} := \psi(y_{i-1})$ to $x_i := \psi(y_i)$. Then $f_*([a_i]) = [b_i]$. Finally define $\psi_*([b]) = [a_1 + \cdots + a_m]$. Note that ψ_* is well defined by the same argument that proves that f_* is well defined. Indeed in that proof we only used Claims 4 and 7, so we can repeat the argument with the roles of X and Y interchanged. We also claim that $\psi_* = (f_* \upharpoonright_{\pi^{\text{def}}(X, \Gamma)})^{-1}$. In fact, by inspection of the definitions, the same pair of subdivisions $a = a_1 + \cdots + a_m$ and $b = b_1 + \cdots + b_m$ witnesses both $f_*([a]) = [b]$ and $\psi_*([b]) = [a]$ simultaneously. \square

The proof of Theorem 4.4 is now complete, except for point 4, which is clear from the construction and left to the reader. \square

5. EQUIVARIANCE

We now specialize the results of the previous section to the case of spaces that carry a group structure. So let G be a definably compact group. Let $\mathfrak{p}^G: G \rightarrow G/G^{00}$ be the projection map, and let

$$\mathfrak{p}_*^G: \pi^{\text{def}}(G, G) \rightarrow \pi(G/G^{00}, G/G^{00})$$

be the induced morphism of groupoids as in Theorem 4.4.

Definition 5.1. The group G acts on $\pi^{\text{def}}(G, G)$ by $x \cdot [a] = [x \cdot a]$, where $x \in G$ and a is a definable path in G . Similarly we have an action of G/G^{00} on $\pi(G/G^{00}, G/G^{00})$ given by $y \cdot [b] = [y \cdot b]$. Finally we also have an action of G on $\pi(G/G^{00}, G/G^{00})$ sending $(x, [b])$ to $\mathfrak{p}^G(x) \cdot [b]$, where $x \in G$ and b is a path in G/G^{00} .

Theorem 5.2. *The map $\mathfrak{p}_*^G: \pi^{\text{def}}(G, G) \rightarrow \pi(G/G^{00}, G/G^{00})$ is equivariant under the action of G , namely for each $x \in G$ and $[a] \in \pi^{\text{def}}(G, G)$, we have*

$$\mathfrak{p}_*^G(x \cdot [a]) = \mathfrak{p}^G(x) \cdot \mathfrak{p}_*^G([a])$$

Proof. Let $x \in G$ and consider the map

$$\begin{aligned} \mathfrak{p}_x: \pi^{\text{def}}(G, G) &\rightarrow \pi(G/G^{00}, G/G^{00}) \\ [a] &\mapsto \mathfrak{p}^G(x^{-1}) \cdot \mathfrak{p}_*^G(x \cdot [a]) \end{aligned}$$

It is easy to check that \mathfrak{p}_x is a groupoid morphism satisfying the conditions 1 and 2 of Theorem 4.4. Hence, by uniqueness, $\mathfrak{p}_x = \mathfrak{p}_*^G$ for all $x \in G$. It follows that \mathfrak{p}_*^G is equivariant. \square

6. FUNCTORS AND NATURAL TRANSFORMATIONS

In this section we establish the functoriality properties of the morphisms \mathfrak{p}_*^G . We regard the correspondence $G \mapsto \pi^{\text{def}}(G, G)$ as the object part of a functor π^{def} from the category of definably compact groups (and definable group homomorphisms) to the category of groupoids (and groupoid homomorphisms). Similarly we have a functor $\pi_1^{\text{def}}: G \mapsto \pi_1^{\text{def}}(G)$ from definably compact groups to groups. Finally we have a functor $F: G \rightarrow G/G^{00}$ from definably compact groups to compact

Lie groups (and Lie homomorphisms). By Theorem 4.4, for G a definably compact group, the projection $\mathfrak{p}^G: G \rightarrow G/G^{00}$ induces a morphism

$$\mathfrak{p}_*^G: \pi^{\text{def}}(G, G) \rightarrow \pi(G/G^{00}, G/G^{00})$$

Theorem 6.1. *Let G be a definably compact group. The family $\mathfrak{p}_* = (\mathfrak{p}_*^G)_G$ is a natural transformation of the functor*

$$\pi^{\text{def}}: G \mapsto \pi^{\text{def}}(G, G)$$

to the functor

$$\pi \circ F: G \mapsto \pi(G/G^{00}, G/G^{00})$$

In other words, given a definable morphism $f: G_1 \rightarrow G_2$ we have a commutative diagram in the category of groupoids:

$$\begin{array}{ccc} \pi^{\text{def}}(G_1, G_1) & \xrightarrow{\pi^{\text{def}}(f)} & \pi^{\text{def}}(G_2, G_2) \\ \downarrow \mathfrak{p}_*^{G_1} & & \downarrow \mathfrak{p}_*^{G_2} \\ \pi(G_1/G_1^{00}, G_1/G_1^{00}) & \xrightarrow{\pi(F(f))} & \pi(G_2/G_2^{00}, G_2/G_2^{00}) \end{array}$$

where $F(f): G_1/G_1^{00} \rightarrow G_2/G_2^{00}$ is the induced Lie homomorphism.

Proof. Consider an open cover \mathcal{U}_2 of G_2/G_2^{00} by simply connected sets. Let \mathcal{U}_1 be an open cover of G_1/G_1^{00} by simply connected sets which refines $\{F(f)^{-1}(U)\}_{U \in \mathcal{U}_2}$. Using these covers in Definition 4.11, the commutativity of the diagram follows immediately. \square

7. LOCALLY DEFINABLE GROUPS

A *locally definable group* is a countable union of definable sets, equipped with a group operation whose restriction to each definable set is definable. This definition is equivalent to the one in [Ed:06], and it is slightly more restrictive than the notion of \forall -definable group in [PeSt:99] where only the cardinality of the set of parameters is assumed to be countable (the two notions coincide if the language is countable). The restriction to countable unions, is useful in Proposition 7.1 below. As usual we assume that the underlying o-minimal structure M is sufficiently saturated (ω_1 -saturated will suffice here). A *locally definable function* is a function between \forall -definable sets whose restriction to each definable set is definable. A *locally definable homomorphism* between locally definable groups is a homomorphism which is locally definable.

Proposition 7.1 ([Ed:06, Theorem 4.2]). *Let $f: B \rightarrow C$ be a surjective locally definable homomorphism of locally definable groups. Then there is a locally definable section $s: C \rightarrow B$, namely a locally definable function such that $f \circ s$ is the identity on C .*

Following [Ed:06, Definition 3.1] we say that a subgroup A of a locally definable group B is *compatible* if it intersects every definable subset X of B into a definable set. (In the cited paper it is actually required that X be open in the topology induced by the group structure, but this yields an equivalent definition since every definable subset is contained in an open definable subset.)

Proposition 7.2 ([Ed:06, Lemma 3.3 and Theorem 4.2]). *A normal subgroup $A < B$ of a locally definable group B is compatible if and only if it is the kernel of a locally definable surjective homomorphism $f: B \rightarrow C$ between locally definable groups.*

If $1 \rightarrow A \rightarrow B \rightarrow C \rightarrow 1$ is an exact sequence of locally definable groups and locally definable homomorphisms, we denote C by B/A : our notation is unambiguous, since by Proposition 7.1 the quotient C is unique up to locally definable isomorphisms. By [Ed:06, Remark 4.7], the third isomorphism theorem $B/A \cong (B/L)/(A/L)$ holds in the category of locally definable groups and locally definable homomorphisms. More precisely we have:

Proposition 7.3 ([Ed:06, Remark 4.7]). *Consider an exact sequence*

$$1 \rightarrow A \rightarrow B \rightarrow C \rightarrow 1$$

of locally definable groups and locally definable homomorphisms and suppose that the map $A \rightarrow B$ is the inclusion. Let L be a normal subgroup of B contained in A as a compatible subgroup (it follows that L is compatible also in B). Then there is an induced exact sequence of locally definable groups

$$1 \rightarrow A/L \rightarrow B/L \rightarrow C \rightarrow 1$$

The proposition is contained in the cited result of [Ed:06], however for the reader's convenience we sketch a proof. The homomorphism $A \rightarrow A/L$ admits a locally definable section $s: A/L \rightarrow A$. The homomorphism $A/L \rightarrow B/L$ can be obtained as the composition

$$A/L \xrightarrow{s} A \rightarrow B \rightarrow B/L$$

hence it is locally definable. Similarly we obtain the locally definable homomorphism $B/L \rightarrow C$.

8. UNIVERSAL COVER

Let G be a definable group. The (o-minimal) universal cover \tilde{G} of G has been studied in [EdEl:07], where in particular it is shown that there is a locally definable surjective homomorphism $f: \tilde{G} \rightarrow G$ of locally definable groups whose kernel is isomorphic to $\pi_1^{\text{def}}(G)$. We can injectively embed \tilde{G} into the fundamental groupoid $\pi^{\text{def}}(G, G)$ in the following way. Given a definable path a in G starting at the identity, let \tilde{a} be its unique lifting to \tilde{G} starting at the identity, and let $\tilde{a}(1) \in \tilde{G}$ be its endpoint. Any element x of \tilde{G} is of the form $\tilde{a}(1)$, and since $\tilde{a}(1)$ depends only on the definable homotopy class $[a]$, we have an injective function $\iota: \tilde{G} \rightarrow \pi^{\text{def}}(G, G)$ sending x to $[a]$. So we have $\iota(\tilde{G}) \subset \pi^{\text{def}}(G, G)$. Notationally we can suppose that ι is the inclusion, namely we can define

$$\tilde{G} \subset \pi^{\text{def}}(G, G)$$

as the subset consisting of all the definable homotopy classes of paths starting at the identity e_G . This identification is only a matter of notational convenience, since literally \tilde{G} is a subset of some cartesian product of M (being a locally definable group), while $\iota(\tilde{G}) \subset \pi^{\text{def}}(G, G)$ is not. With our identification the group operation of $\tilde{G} \subset \pi^{\text{def}}(G, G)$ is defined by $[a] \cdot [b] = [a] + [a(1) \cdot b]$ where $a(1)$ is the endpoint of the definable path a and $+$ denotes the operation of the groupoid (induced by

concatenation of paths). The covering homomorphism $\tilde{G} \rightarrow G$ sends $[a]$ to $a(1)$, so we see that its kernel is indeed $\pi_1^{\text{def}}(G)$. The topology on $\tilde{G} \subset \pi^{\text{def}}(G, G)$ can be described as follows: if U is a definably simply connected open subset of G , then the set U_0 of all $[a] \in \tilde{G}$ such that $\text{Im}(a) \subset U$ is a basic open neighbourhood of $e_{\tilde{G}}$ in \tilde{G} . (Literally U_0 is a subset of $\iota(\tilde{G})$, but under the identification $\iota(\tilde{G}) = \tilde{G}$, it corresponds to one of the (path-)connected components of the preimage of U in \tilde{G} , which is a basic open neighbourhood of the identity in \tilde{G} .) By left translation we obtain the basic open neighbourhoods around the other points.

With analogous definitions the universal cover of the real Lie group G/G^{00} can be identified with the subset

$$\widetilde{G/G^{00}} \subset \pi(G/G^{00}, G/G^{00})$$

consisting of all the homotopy classes of paths starting at $e_{G/G^{00}} \in G/G^{00}$.

The purpose of these identifications is to be able to define an homomorphism from the universal cover of G to the universal cover of G/G^{00} .

Definition 8.1. The morphism of groupoids

$$\mathbf{p}_*^G: \pi^{\text{def}}(G, G) \rightarrow \pi(G/G^{00}, G/G^{00})$$

given by Theorem 4.4 induces a map

$$\tilde{\mathbf{p}}^G: \tilde{G} \rightarrow \widetilde{G/G^{00}}$$

by restriction, namely $\tilde{\mathbf{p}}^G([a]) := \mathbf{p}_*^G([a])$. Since \mathbf{p}_*^G is equivariant (Theorem 5.2), the map $\tilde{\mathbf{p}}^G$ is a morphism of groups. Moreover $\tilde{\mathbf{p}}^G$ is surjective by Corollary 4.7(2).

Theorem 8.2. *Given a definably compact definably connected definable group G , the kernel $\widetilde{G^{00}}$ of $\tilde{\mathbf{p}}^G$ is isomorphic to G^{00} via the map $[a] \mapsto a(1)$. So we have a commutative diagram*

$$\begin{array}{ccccccc} 1 & \longrightarrow & \widetilde{G^{00}} & \longrightarrow & \tilde{G} & \xrightarrow{\tilde{\mathbf{p}}^G} & \widetilde{G/G^{00}} & \longrightarrow & 1 \\ & & \cong \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & G^{00} & \longrightarrow & G & \longrightarrow & G/G^{00} & \longrightarrow & 1 \end{array}$$

Moreover $\tilde{\mathbf{p}}^G$ sends the kernel $\pi_1^{\text{def}}(G)$ of $\tilde{G} \rightarrow G$ isomorphically onto the kernel $\pi_1(G/G^{00})$ of $\widetilde{G/G^{00}} \rightarrow G/G^{00}$.

Proof. The “moreover” part is given by Corollary 4.5. We prove the first part. Let $[a] \in \tilde{G}$ be in $\ker(\tilde{\mathbf{p}}^G)$ (where a is a definable path in G starting at the identity). Then $\mathbf{p}_*^G([a]) = [b]$ where b is a contractible loop at $e_{G/G^{00}}$. Since \mathbf{p}_*^G must send the endpoints of a to the endpoints of b (namely to the identity of G/G^{00}), it follows that $a(1) \in G^{00}$. So we have a well defined function $[a] \mapsto a(1)$ from $\ker(\tilde{\mathbf{p}}^G)$ to G^{00} which moreover is a group homomorphism. We must prove that it is an isomorphism.

(Surjectivity) Given $x \in G^{00}$ we must find $[a] \in \ker(\tilde{\mathbf{p}}^G)$ with $a(1) = x$. To this aim let U be a simply connected open neighbourhood of $e_{G/G^{00}}$ in G/G^{00} . By Corollary 4.5, its preimage V in G is a (\vee -definable) definably simply connected subset of G . In particular V is definably path connected, so there is a definable path a in V from e_G to x . By Theorem 4.4 there is a path b in U with $\mathbf{p}_*^G([a]) = [b]$. Since $x \in G^{00}$, the endpoint $\mathbf{p}_*^G(x)$ of b is the identity of G/G^{00} , namely b is a loop.

Moreover since the image of b is contained in the simply connected set U , we know that $[b]$ is the identity of $\widetilde{G/G^{00}}$, and therefore $[a] \in \ker(\widetilde{\mathfrak{p}}^G)$.

(Injectivity) Let $[a] \in \ker(\widetilde{\mathfrak{p}}^G)$ and suppose $a(1) = e_G$ (namely a is a loop). By Corollary 4.5, the projection \mathfrak{p}_*^G sends $\pi_1^{\text{def}}(G)$ bijectively to $\pi_1(G/G^{00})$. Since under this map $[a]$ goes to the identity, it follows that a is definably contractible, namely $[a]$ is the identity of \widetilde{G} . \square

We will obtain similar results for the finite extensions of G/G^{00} .

Theorem 8.3. *Let G be a definably compact definably connected definable group. Given an extension of connected Lie groups $f: \mathbf{H} \rightarrow G/G^{00}$ with a finite kernel, there is a definable group extension $\pi: H \rightarrow G$ of G such that $H/H^{00} \cong \mathbf{H}$ (as coverings of G/G^{00}). We thus obtain a commutative diagram:*

$$\begin{array}{ccc} H & \xrightarrow{\varphi} & \mathbf{H} \\ \downarrow \pi & & \downarrow f \\ G & \xrightarrow{\mathfrak{p}} & G/G^{00} \end{array}$$

where $\varphi: H \rightarrow \mathbf{H}$ is the composition of the projection $H \rightarrow H/H^{00}$ with the isomorphism $H/H^{00} \cong \mathbf{H}$.

Proof. Let L be the image of the homomorphism $\pi_1(f): \pi_1(\mathbf{H}) \rightarrow \pi_1(G/G^{00})$. Then by classical results (see [Ha:02, Proposition 1.36 and 1.37]) we have an exact sequence

$$1 \rightarrow \pi_1(G/G^{00})/L \rightarrow \mathbf{H} \rightarrow G/G^{00} \rightarrow 1$$

Let $L' < \pi_1^{\text{def}}(G)$ be the preimage of L under the isomorphism between $\pi_1^{\text{def}}(G)$ and $\pi_1(G/G^{00})$ of Corollary 4.5. By [EdEl:07] we have an exact sequence of locally definable groups

$$1 \rightarrow \pi_1^{\text{def}}(G) \rightarrow \widetilde{G} \rightarrow G \rightarrow 1$$

so we can identify $\pi_1^{\text{def}}(G)$ with a compatible subgroup of \widetilde{G} . The identification sends $[\gamma] \in \pi_1^{\text{def}}(G)$ to $\tilde{\gamma}(1) \in \widetilde{G}$, where $\tilde{\gamma}$ is the lifting of γ at the identity in \widetilde{G} , and $\tilde{\gamma}(1)$ is its endpoint. Since $\pi_1^{\text{def}}(G)$ is discrete, the compatibility condition implies that $\pi_1^{\text{def}}(G)$ (hence also its subgroup L') intersects every definable subset of \widetilde{G} into a finite set. Moreover since $L' < \pi_1^{\text{def}}(G)$ and $\pi_1^{\text{def}}(G)$ is contained in the center of \widetilde{G} ([Ed:06, Corollary 3.16]), L' is normal in \widetilde{G} . By the third isomorphism theorem (Proposition 7.3) we obtain an exact sequence

$$1 \rightarrow \pi_1^{\text{def}}(G)/L' \rightarrow \widetilde{G}/L' \xrightarrow{p} G \rightarrow 1$$

of locally definable groups. Since the kernel $\pi_1^{\text{def}}(G)/L' \cong \pi_1(G/G^{00})/L$ is finite and G is definable, the locally definable group $H := \widetilde{G}/L'$ is actually definable, and $p: H \rightarrow G$ is a locally definable covering map ([Ed:05, Theorem 3.6]). By [Ed:05, Remark 4.4], the image of $\pi_1(p): \pi_1^{\text{def}}(H) \rightarrow \pi_1^{\text{def}}(G)$ is L' . We have an induced homomorphism $F(p): H/H^{00} \rightarrow G/G^{00}$, and by Theorem 6.1 the image of $\pi_1(F(p)): \pi_1(H/H^{00}) \rightarrow \pi_1(G/G^{00})$ is L , namely it coincides with the image of $\pi_1(f): \pi_1(\mathbf{H}) \rightarrow \pi_1(G/G^{00})$. Since the covering spaces are classified by the subgroups of the fundamental group ([Ha:02, Proposition 1.36 and 1.37]), the two coverings $f: \mathbf{H} \rightarrow G/G^{00}$ and $F(p): H/H^{00} \rightarrow G/G^{00}$ are isomorphic. \square

9. DEFINABLY COMPACT SEMISIMPLE GROUPS

Let G be a definably compact definably connected semisimple group. In this section we show that the (Lie-)isomorphism type of G/G^{00} determines the definable isomorphism type of G .

Lemma 9.1. *Work in an o -minimal expansion of \mathbb{R} . Let $f: X \rightarrow B$ be a definable continuous map. Let $p: E \rightarrow B$ be a definable covering map. Let $\tilde{f}: X \rightarrow E$ be a lifting of f (i.e. a continuous function, not necessarily definable, such that $p \circ \tilde{f} = f$). Then \tilde{f} is definable.*

Proof. By definition of definable covering, we have a definable finite cover \mathcal{U} of B by definably connected definable open sets such that, for any $U \in \mathcal{U}$, the preimage $p^{-1}(U)$ is a finite disjoint union of definably connected open sets on each of which p is an homeomorphism onto U . Fix $U \in \mathcal{U}$ and let E_1, \dots, E_m be the definably connected components of $p^{-1}(U)$ and X_1, \dots, X_n be the definably connected components of $f^{-1}(U)$. Note that all these sets are definable. Fix an $i \in \{1, \dots, n\}$. Since we are working over \mathbb{R} , a definably connected set is connected. So, by continuity of f , there is a $j \in \{1, \dots, m\}$ such that $\tilde{f}(X_i) \subset E_j$. Hence $(\tilde{f}|_{X_i})(x) = y$ if and only if $x \in X_i \wedge y \in E_j \wedge f(x) = p(y)$. This proves that $\tilde{f}|_{X_i}$ is definable, and the definability of \tilde{f} follows observing that the same hold for any $U \in \mathcal{U}$ and any i . \square

Fact 9.2 ([EdJoPe:07, Theorem 3.1] or [HrPePi:08b, Theorem 4.4(ii)]). *Let G_1 be a semisimple definable group. Then there is a group G_2 , semialgebraic without parameters, definably isomorphic to G_1 .*

Lemma 9.3. *Let G_1 and G_2 be definably connected semialgebraic semisimple groups defined over \mathbb{R} . By [Pi:88], $G_1(\mathbb{R})$ and $G_2(\mathbb{R})$ have a natural Lie group structure. Suppose that $f: G_1(\mathbb{R}) \rightarrow G_2(\mathbb{R})$ is a Lie isomorphism. Then f is semialgebraic over \mathbb{R} .*

Proof. We first prove the result under the additional assumption that G_1 and G_2 are centerless. The isomorphism $f: G_1(\mathbb{R}) \rightarrow G_2(\mathbb{R})$ induces an isomorphism $\phi: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ of the corresponding Lie algebras. Since we are in the centerless case, the adjoint representation $\text{Ad}_{G_1}: G_1(\mathbb{R}) \rightarrow \text{Aut}(\mathfrak{g}_1)$ is an isomorphism onto $\text{Aut}^0(\mathfrak{g}_1)$ and similarly for $G_2(\mathbb{R})$. Fixing a basis of the vector spaces \mathfrak{g}_1 and \mathfrak{g}_2 , we can consider Ad_{G_1} and Ad_{G_2} as semialgebraic maps. Let $\tilde{\phi}: \text{Aut}^0(\mathfrak{g}_1) \rightarrow \text{Aut}^0(\mathfrak{g}_2)$ be the isomorphism induced by ϕ . Then $f = \text{Ad}_{G_2} \circ \tilde{\phi} \circ \text{Ad}_{G_1}^{-1}$ and therefore f is semialgebraic over \mathbb{R} .

To reduce the general case to the centerless case we use the fact that $G_1/Z(G_1)$ and $G_2/Z(G_2)$ are centerless. Clearly f induces an isomorphism $g: G_1/Z(G_1) \rightarrow G_2/Z(G_2)$. By the centerless case g is semialgebraic. By Lemma 9.1, we have that f is semialgebraic. \square

Remark 9.4. In the above lemma we cannot ensure that f is semialgebraic over \mathbb{R}^{alg} even assuming that G_1 and G_2 are semialgebraic over \mathbb{R}^{alg} . In fact let $G_1 = G_2 = SO_3$. The group of inner automorphisms of SO_3 is non-trivial and connected, so it has the cardinality of the continuum. Therefore there is some inner automorphism $f: SO_3 \rightarrow SO_3$ which is not definable over \mathbb{R}^{alg} .

Theorem 9.5. *Let G_1 and G_2 be definably compact definably connected semisimple definable groups in an \mathcal{o} -minimal structure M . Suppose that there is a Lie isomorphism $\psi: G_1/G_1^{00} \rightarrow G_2/G_2^{00}$, where by convention $G/G^{00} := G(M')/G^{00}(M')$ for some $M' \succ M$ sufficiently saturated². Then, in M , there is a definable isomorphism $f: G_1 \rightarrow G_2$. If $M = M'$ (namely M was already sufficiently saturated), we can consider the projections $\mathfrak{p}^{G_1}: G_1 \rightarrow G_1/G_1^{00}$ and $\mathfrak{p}^{G_2}: G_2 \rightarrow G_2/G_2^{00}$ and we can choose f so that $\mathfrak{p}^{G_2} \circ f = \psi \circ \mathfrak{p}^{G_1}$.*

Proof. By fact 9.2 we may assume G_1 and G_2 to be semialgebraic without parameters. So it makes sense to consider the groups $G_1(\mathbb{R})$ and $G_2(\mathbb{R})$. If M is sufficiently saturated there is an elementary embedding of \mathbb{R} into M (in the language of fields) and there is a surjective homomorphism $G_1(M) \rightarrow G_1(\mathbb{R})$ (given by the *standard part map*) whose kernel is $G_1^{00} = G_1^{00}(M)$: see [Pi:04]. Similarly for G_2 . So $G_1/G_1^{00} \cong G_1(\mathbb{R})$ and $G_2/G_2^{00} \cong G_2(\mathbb{R})$ (with the logic topology). Hence we have a Lie isomorphism $\psi': G_1(\mathbb{R}) \rightarrow G_2(\mathbb{R})$ induced by ψ . By Lemma 9.3 ψ' is semialgebraic over \mathbb{R} . The same formula defines an isomorphism $f: G_1(M) \rightarrow G_2(M)$ with $\mathfrak{p}^{G_2} \circ f = \psi \circ \mathfrak{p}^{G_1}$. If M is not sufficiently saturated, then we can go to a saturated extension M' to get an M' -definable isomorphism $f: G_1(M') \rightarrow G_2(M')$ as above, and therefore (quantifying over the parameters) also an M -definable isomorphism from $G_1(M)$ to $G_2(M)$. \square

10. DEFINABLY COMPACT ABELIAN GROUPS

In this section we try to understand, in the abelian case, up to which extent G/G^{00} determines G . It is known that there are definably compact definably connected abelian groups G_1 and G_2 of the same dimension (hence with $G_1/G_1^{00} \cong G_2/G_2^{00}$) which are not definably isomorphic ([St:94, PeSt:99]). However by [BeMaOt:09] any two definably compact definably connected abelian groups of the same dimension are definably homotopy equivalent. The same proof yields the following:

Lemma 10.1. *Let G_1 and G_2 be definably compact definably connected abelian groups of the same dimension n . (So $\pi_1(G_1) \cong \pi_1(G_2) \cong \mathbb{Z}^n$ by [EdOt:04].) Let $\theta: \pi_1(G_1) \rightarrow \pi_1(G_2)$ be an isomorphism. Then there is a definable continuous map $f: G_1 \rightarrow G_2$ with $\pi_1(f) = \theta$ and $f(e_{G_1}) = e_{G_2}$. Moreover, any such map f is a definable homotopy equivalence.*

Proof.

Special case. Suppose that G_1 is a direct product of 1-dimensional definable subgroups. Choose free generators $[a_1], \dots, [a_n]$ of $\pi_1^{\text{def}}(G_1)$ such that each $x \in G_1$ can be written uniquely in the form $x = a_1(t_1) + \dots + a_n(t_n)$ with $0 \leq t_i < 1$. Choose definable loops b_1, \dots, b_n in G_2 such that $[b_1], \dots, [b_n] \in \pi_1^{\text{def}}(G_2)$ are the images of $[a_1], \dots, [a_n]$ under θ . Define $f(x) = b_1(t_1) + \dots + b_n(t_n)$. Then clearly $\pi_1^{\text{def}}(f) = \theta$. Since the higher definable homotopy groups of G_1 and G_2 are zero by [BeMaOt:09], f is a definable homotopy equivalence by the \mathcal{o} -minimal version of Whitehead theorem in [BaOt:09].

General case. We reduce to the special case as follows. Let T be a definably compact definably connected one-dimensional abelian group, and T^n be the direct product of n copies of T , where $n = \dim G_1$. By [EdOt:04], we have $\pi_1^{\text{def}}(G_1) \cong \pi_1(T^n) \cong \mathbb{Z}^n$.

²If the groups are definable with parameters from A , then $2^{|T|+|A|}$ -saturation suffices

Choose an isomorphism $\lambda: \pi_1^{\text{def}}(T^n) \rightarrow \pi_1^{\text{def}}(G_1)$. Then $\theta \circ \lambda: \pi_1^{\text{def}}(T^n) \rightarrow \pi_1^{\text{def}}(G_2)$ is an isomorphism. By the special case we get definable homotopy equivalences g and h with $\pi_1(g) = \lambda$ and $\pi_1(h) = \theta \circ \lambda$. Let g' be a definable homotopy inverse of g . So $f := h \circ g'$ satisfies $\pi_1(f) = \theta$. \square

To improve on the above result we need a definition.

Definition 10.2. Let G_1 and G_2 be definable groups. Given a subgroup $\Gamma_1 < G_1$ we say that a definable map $f: G_1 \rightarrow G_2$ is a Γ_1 -equivariant definable homotopy equivalence if $f|_{\Gamma_1}$ is an isomorphism onto its image Γ_2 , and f admits a definable homotopy inverse f' such that the following holds:

- $f(e_{G_1}) = e_{G_2}$ and $f(cx) = f(c)f(x)$ for any $c \in \Gamma_1$ and $x \in G_1$;
- $f'(e_{G_2}) = e_{G_1}$ and $f'(c'x') = f'(c')f'(x')$ for any $c' \in \Gamma_2$ and $x' \in G_2$;
- there is a definable homotopy $h: I \times G_1 \rightarrow G_1$ relative to Γ_1 between $f' \circ f$ and the identity map on G_1 such that $h_t(cx) = ch_t(x)$ for any $c \in \Gamma_1$, $x \in G_1$, and $t \in I$;
- there is a definable homotopy $h': I \times G_2 \rightarrow G_2$ relative to Γ_2 between $f \circ f'$ and the identity map on G_2 such that $h'_t(c'x') = c'h'_t(x')$ for any $c' \in \Gamma_2$, $x' \in G_2$, and $t \in I$.

Note that $f'|_{\Gamma_2}$ is the inverse of $f|_{\Gamma_1}$.

To prove the existence of Γ -equivariant homotopy equivalences we need some preliminary results. The following lemma says that given a definable covering map $p: E \rightarrow B$ we can always lift a definable map $f: X \rightarrow B$ to a map $\tilde{f}: X \rightarrow E$ provided there are no obstructions coming from the fundamental group.

Lemma 10.3. *Let $p: E \rightarrow B$ be a definable covering map, with B definably connected. And let $f: X \rightarrow B$ be a definable map from a definable definably connected set X to B . Fix base points $e_0 \in E$, $b_0 \in B$ and $x_0 \in X$ with $f(x_0) = p(e_0) = b_0$. Consider the homomorphisms $\pi_1(p)$ and $\pi_1(f)$ induced by p and f on the definable fundamental groups. If $\text{Im } \pi_1^{\text{def}}(f) \subset \text{Im } \pi_1^{\text{def}}(p)$ then there is a unique definable map $\tilde{f}: X \rightarrow E$ lifting f (i.e. such that $p \circ \tilde{f} = f$) with $\tilde{f}(x_0) = e_0$.*

Proof. The proof of the corresponding classical result (see [Sp:66, Theorem 2.4.5]) can be adapted to the o-minimal category thanks to the definable homotopy lifting property in [BaOt:09]. More precisely, for each $x \in X$ choose, uniformly in x , a definable path a_x from x_0 to x in X . Then $b_x := f \circ a_x$ is a definable path in B . Let \tilde{b}_x be its (unique) lifting to a definable path in E with starting point e_0 . Define $\tilde{f}(x)$ as the final point of \tilde{b}_x . This is independent on the choice of the paths and works. \square

Theorem 10.4. *Let G_1 and G_2 be definably compact definably connected abelian groups. Let*

$$\psi: G_1/G_1^{00} \rightarrow G_2/G_2^{00}$$

be an isomorphism of Lie groups. Let Γ_1 be a finite subgroup of G_1 . Then there is a Γ_1 -equivariant definable homotopy equivalence $f^{G_1}: G_1 \rightarrow G_2$ which agrees with ψ on Γ_1 (more precisely for each $c \in \Gamma_1$ we have $f^{G_1}(c)G_2^{00} = \psi(cG_1^{00})$).

Proof. To simplify notations let $\mathbf{G}_1 = G_1/G_1^{00}$ and $\mathbf{G}_2 = G_2/G_2^{00}$. Since Γ_1 is finite, the projection $G_1 \rightarrow \mathbf{G}_1$ maps Γ_1 isomorphically onto its image

$$\mathbf{\Gamma}_1 := \Gamma_1 G_1^{00}/G_1^{00} < \mathbf{G}_1$$

Let $\mathbf{\Gamma}_2 = \psi(\mathbf{\Gamma}_1) < \mathbf{G}_2$ and let Γ_2 be the unique finite subgroup of G_2 which is mapped to $\mathbf{\Gamma}_2$ under the projection $G_2 \rightarrow \mathbf{G}_2$.

Passing to the quotient, the isomorphism $\psi: \mathbf{G}_1 \rightarrow \mathbf{G}_2$ induces an isomorphism $\phi: \mathbf{G}_1/\mathbf{\Gamma}_1 \rightarrow \mathbf{G}_2/\mathbf{\Gamma}_2$ making the following diagram commute (where the vertical arrows are the projections):

$$\begin{array}{ccc} \mathbf{G}_1 & \xrightarrow{\psi} & \mathbf{G}_2 \\ \downarrow & & \downarrow \\ \mathbf{G}_1/\mathbf{\Gamma}_1 & \xrightarrow{\phi} & \mathbf{G}_2/\mathbf{\Gamma}_2 \end{array}$$

Since $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$ are mapped to the identity of $\mathbf{G}_1/\mathbf{\Gamma}_1$ and $\mathbf{G}_2/\mathbf{\Gamma}_2$ respectively, we obtain an induced commutative diagram in the category of groupoids:

$$(*) \quad \begin{array}{ccc} \pi(\mathbf{G}_1, \mathbf{\Gamma}_1) & \xrightarrow{\pi(\psi)} & \pi(\mathbf{G}_2, \mathbf{\Gamma}_2) \\ \downarrow & & \downarrow \\ \pi(\mathbf{G}_1/\mathbf{\Gamma}_1, e) & \xrightarrow{\pi(\phi)} & \pi(\mathbf{G}_2/\mathbf{\Gamma}_2, e) \end{array}$$

where $\mathbf{G}_1/\mathbf{\Gamma}_1$ can be naturally identified with $(G_1/\Gamma_1)/(G_1/\Gamma_1)^{00}$ and similarly on the G_2 side. By Theorem 6.1 and Remark 4.6 we have commutative diagrams

$$(**) \quad \begin{array}{ccccccc} \pi^{\text{def}}(G_1, \Gamma_1) & \longrightarrow & \pi^{\text{def}}(G_1/\Gamma_1, e) & & \pi^{\text{def}}(G_2, \Gamma_2) & \longrightarrow & \pi^{\text{def}}(G_2/\Gamma_2, e) \\ \downarrow \mathfrak{p}_*^{G_1} & & \downarrow \mathfrak{p}_*^{G_1/\Gamma_1} & & \downarrow \mathfrak{p}_*^{G_2} & & \downarrow \mathfrak{p}_*^{G_2/\Gamma_2} \\ \pi(\mathbf{G}_1, \mathbf{\Gamma}_1) & \longrightarrow & \pi(\mathbf{G}_1/\mathbf{\Gamma}_1, e) & & \pi(\mathbf{G}_2, \mathbf{\Gamma}_2) & \longrightarrow & \pi(\mathbf{G}_2/\mathbf{\Gamma}_2, e) \end{array}$$

where the vertical arrows are isomorphisms and the horizontal arrows are induced by the quotient maps. By the composition of diagrams (*) and (**) we obtain a commutative diagram in the category of groupoids

$$\begin{array}{ccc} \pi^{\text{def}}(G_1, \Gamma_1) & \xrightarrow{\theta} & \pi^{\text{def}}(G_2, \Gamma_2) \\ \downarrow & & \downarrow \\ \pi^{\text{def}}(G_1/\Gamma_1, e) & \xrightarrow{\lambda} & \pi^{\text{def}}(G_2/\Gamma_2, e) \end{array}$$

where θ is uniquely determined by the equation

$$\pi(\psi) \circ \mathfrak{p}_*^{G_1} = \mathfrak{p}_*^{G_2} \circ \theta$$

For each $c \in \Gamma_1$ and $[a] \in \pi(G_1, \Gamma_1)$ we have $\theta(c \cdot [a]) = \overline{\psi}(c) \cdot \theta([a])$, where $\overline{\psi}: \Gamma_1 \rightarrow \Gamma_2$ is defined by $\overline{\psi}(c)G_1^{00} = \psi(cG_1^{00})$. In fact, using Theorem 5.2 we can verify that $\overline{\psi}(c) \cdot \theta([a])$ meets the definition of $\theta(c \cdot [a])$:

$$\begin{aligned} \pi(\psi) \circ \mathfrak{p}_*^{G_1}(c \cdot [a]) &= \pi(\psi)(\mathfrak{p}^{G_1}(c) \cdot \mathfrak{p}_*^{G_1}([a])) \\ &= (\psi \circ \mathfrak{p}^{G_1})(c) \cdot (\pi(\psi) \circ \mathfrak{p}_*^{G_1})([a]) \\ &= (\psi \circ \mathfrak{p}^{G_1})(c) \cdot (\mathfrak{p}_*^{G_2} \circ \theta)([a]) \\ &= (\mathfrak{p}^{G_2} \circ \overline{\psi})(c) \cdot (\mathfrak{p}_*^{G_2} \circ \theta)([a]) = \mathfrak{p}_*^{G_2}(\overline{\psi}(c) \cdot \theta([a])) \end{aligned}$$

Using Lemma 10.1 we can obtain a definable homotopy equivalence $f: G_1/\Gamma_1 \rightarrow G_2/\Gamma_2$ with $\pi_1(f) = \lambda$ and $f(e_{G_1/\Gamma_1}) = e_{G_2/\Gamma_2}$. So, by Lemma 10.3 there is a definable continuous map $f^{G_1}: G_1 \rightarrow G_2$ with $f^{G_1}(e_{G_1}) = e_{G_2}$ making the following diagram commute.

$$\begin{array}{ccc} G_1 & \xrightarrow{f^{G_1}} & G_2 \\ \downarrow p & & \downarrow p' \\ G_1/\Gamma_1 & \xrightarrow{f} & G_2/\Gamma_2 \end{array}$$

It remains to be shown that f^{G_1} is a Γ_1 -equivariant definable homotopy equivalence. The equation $f^{G_1}(c \cdot -) = f^{G_1}(c)f^{G_1}(-)$ for $c \in \Gamma_1$ holds because both maps coincide with the unique lifting of $f \circ p$ mapping e_{G_1} to $f^{G_1}(c) \in G_2$. Now let $f': G_2/\Gamma_2 \rightarrow G_1/\Gamma_1$ be a homotopy inverse of f . Define $f^{G_1'}$ to be the unique lifting of $f' \circ p'$ at e_{G_1} . Then as above $f^{G_1'}(c'y) = f^{G_1'}(c')f^{G_1'}(y)$ for any $c' \in \Gamma_2$ and $y \in G_2$. Let $h: I \times G_1/\Gamma_1 \rightarrow G_1/\Gamma_1$ be a definable homotopy between the identity h_0 on G_1/Γ_1 and $f' \circ f$, then let $h': I \times G_2/\Gamma_2 \rightarrow G_2/\Gamma_2$ be a definable homotopy between the identity h'_0 on G_2/Γ_2 and $f \circ f' = h'_1$. We may assume $h_t(1) = e_{G_1}$ for all t (otherwise use $(t, x) \rightarrow (h_t(1))^{-1}h_t(x)$ instead of h) and the same for h' . Finally, define $\tilde{h}: I \times G_1 \rightarrow G_1$ as the unique lifting of $h \circ (\text{Id} \times p): I \times G_1 \rightarrow G_1/\Gamma_1$ to G_1 and $\tilde{h}': I \times G_2 \rightarrow G_2$ as the unique lifting of $h' \circ (\text{Id} \times p'): I \times G_2 \rightarrow G_2/\Gamma_2$ to G_2 . By uniqueness of liftings, \tilde{h} is a definable homotopy between the identity and $f^{G_1} \circ f^{G_2}$. Similarly \tilde{h}' is a definable homotopy between the identity and $f^{G_2} \circ f^{G_1}$. Moreover \tilde{h} and \tilde{h}' are constant on $I \times \Gamma_1$ and $I \times \Gamma_2$ since h and h' are constant on $I \times \{e\}$. The equations $\tilde{h}_t(cx) = c\tilde{h}_t(x)$ and $\tilde{h}'_t(c'x') = c'\tilde{h}'_t(x')$, where $c \in \Gamma_1$ and $c' \in \Gamma_2$, follow by uniqueness of liftings. \square

11. ALMOST DIRECT PRODUCTS

Given a group G and two subgroups A and B of G , we recall that G is the *almost direct product* of A and B if $G = AB$ and the function $m: A \times B \rightarrow G$ sending (x, y) to xy is a surjective group homomorphism with a finite kernel. This implies that $ab = ba$ for all $a \in A$ and $b \in B$, and that $\Gamma := A \cap B$ is a finite central subgroup of G . In this situation we write $G = A \times_{\Gamma} B$. Note that the kernel of $m: A \times B \rightarrow A \times_{\Gamma} B$ is $\Gamma^{\Delta} := \{(c, c^{-1})\}_{c \in \Gamma}$.

Every definably compact definably connected group is an almost direct product of a definably connected abelian subgroup and a semisimple definable subgroup. More precisely we have:

Fact 11.1. *Let G be a definably compact definably connected group. Let $Z^0(G)$ be the definable identity component of the center $Z(G)$ of G . By [HrPePi:08b] the commutator subgroup $[G, G]$ is definable and semisimple, and G is an almost direct product of $Z^0(G)$ and $[G, G]$. The corresponding statement holds in the category of compact connected Lie groups.*

Lemma 11.2. *Consider almost direct products of definable groups $G_1 = A_1 \times_{\Gamma_1} B_1$ and $G_2 = A_2 \times_{\Gamma_2} B_2$. Suppose that there are:*

- an isomorphism $f^{\Gamma_1}: \Gamma_1 \rightarrow \Gamma_2$,
- a Γ_1 -equivariant definable homotopy equivalence $f^{A_1}: A_1 \rightarrow A_2$,
- a Γ_1 -equivariant definable homotopy equivalence $f^{B_1}: B_1 \rightarrow B_2$,

satisfying $f^{A_1} \upharpoonright_{\Gamma_1} = f^{B_1} \upharpoonright_{\Gamma_1} = f^{\Gamma_1}$. Then there is a Γ_1 -equivariant definable homotopy equivalence $f^{G_1}: G_1 \rightarrow G_2$ such that $f^{G_1}(ab) = f^{A_1}(a)f^{B_1}(b)$ for all $a \in A_1$ and $b \in B_1$. In particular $f^{G_1} \upharpoonright_{A_1} = f^{A_1}$ and $f^{G_1} \upharpoonright_{B_1} = f^{B_1}$.

Proof. By definition of almost direct product there is a (unique) well defined function $f^{G_1}: G_1 \rightarrow G_2$ satisfying $f^{G_1}(ab) = f^{A_1}(a)f^{B_1}(b)$ for $a \in A_1$ and $b \in B_1$. Moreover f^{G_1} is continuous since $m: A_1 \times B_1 \rightarrow G_1$ is a definable covering map (hence locally $f^{G_1} = (f^{A_1} \otimes f^{B_1}) \circ m^{-1}$). Let f'^{A_1} and f'^{B_1} be homotopy inverses for f^{A_1} and f^{B_1} satisfying the conditions of Definition 10.2 and let $f'^{G_1}: G_2 \rightarrow G_1$ be defined symmetrically. We claim that f^{G_1} is a definable homotopy equivalence with homotopy inverse f'^{G_1} . In fact, let $h^{A_1}: I \times A_1 \rightarrow A_1$ be a definable homotopy between $f'^{A_1} \circ f^{A_1}$ and the identity satisfying the conditions of Definition 10.2, and let $h^{B_1}: I \times B_1 \rightarrow B_1$ be the same for $f'^{B_1} \circ f^{B_1}$. Define

$$(1) \quad h_t^{G_1}(ab) = h_t^{A_1}(a)h_t^{B_1}(b)$$

for $a \in A_1$ and $b \in B_1$. The fact that $h_t^{G_1}$ is well defined follows by the conditions in Definition 10.2 and the definition of almost direct product. A definable homotopy between $f^{G_1} \circ f'^{G_1}$ and the identity can be defined symmetrically. The lemma is thus proved. \square

Fact 11.3 ([Co:09]). *Let A and B be type-definable subgroups of a definable group G , with A normal in G . Then AB is a type-definable subgroup of G and $(AB)^{00} = A^{00}B^{00}$.*

Lemma 11.4. *Let $G = A \times_{\Gamma} B$ be an almost direct product of definable groups. Let $\mathfrak{p}: G \rightarrow G/G^{00}$ be the projection map. Then $G/G^{00} = \mathfrak{p}(A) \times_{\mathfrak{p}(\Gamma)} \mathfrak{p}(B)$.*

Proof. Consider the homomorphism

$$\begin{aligned} m: \mathfrak{p}(A) \times \mathfrak{p}(B) &\rightarrow G/G^{00} \\ (aG^{00}, bG^{00}) &\mapsto abG^{00} \end{aligned}$$

Since $G^{00} = A^{00}B^{00}$ (fact 11.3), if abG^{00} is the identity of G/G^{00} we have $aa' = b^{-1}b'$ for some $a' \in A^{00}$ and $b' \in B^{00}$. But $A \cap B = \Gamma$, so there is $c \in \Gamma$ such that $aa' = b^{-1}b' = c$. It follows that $aG^{00} = cG^{00}$ and $bG^{00} = c^{-1}G^{00}$. We have thus proved that the kernel of m is

$$\ker m = \mathfrak{p}(\Gamma)^{\Delta} := \{(cG^{00}, c^{-1}G^{00})\}_{c \in \Gamma}$$

a finite subgroup. \square

Remark 11.5. Let G be a definably compact group and let A be a definable subgroup of G . Let $\mathfrak{p}: G \rightarrow G/G^{00}$ be the projection map. By [HrPePi:08] we have $A \cap G^{00} = A^{00}$ (see [Be:07] for the non-abelian case). Therefore

$$\mathfrak{p}(A) = AG^{00}/G^{00} \cong A/A^{00}$$

via the natural homomorphism sending $aA^{00} \in A/A^{00}$ to $aG^{00} \in AG^{00}/G^{00}$.

Lemma 11.6. *Let G be a definably compact definably connected group. Write $G = Z^0(G) \times_{\Gamma} [G, G]$. Let $\mathfrak{p}: G \rightarrow G/G^{00}$ be the projection map. Then*

$$\begin{aligned} \mathfrak{p}(Z^0(G)) &= Z^0(G/G^{00}), \\ \mathfrak{p}([G, G]) &= [G/G^{00}, G/G^{00}], \\ G/G^{00} &= \mathfrak{p}(Z^0(G)) \times_{\mathfrak{p}(\Gamma)} \mathfrak{p}([G, G]) \\ &= Z^0(G/G^{00}) \times_{\mathfrak{p}(\Gamma)} [G/G^{00}, G/G^{00}]. \end{aligned}$$

Proof. By fact 11.1 we have $\dim(G) = \dim(Z(G)) + \dim([G, G])$ and similarly for G/G^{00} . By [HrPePi:08] the dimension of A as a definable group equals the dimension of A/A^{00} as a Lie group. So \mathfrak{p} preserves dimensions. The equality $\mathfrak{p}([G, G]) = [G/G^{00}, G/G^{00}]$ is clear. The inclusion $\mathfrak{p}(Z^0(G)) \subset Z^0(G/G^{00})$ is also clear (using the fact the image under \mathfrak{p} of a definably connected set is connected). The result follows by counting dimensions. \square

Theorem 11.7. *Let G_1 and G_2 be definably compact definably connected groups. Suppose that there is a Lie isomorphism $\psi: G_1/G_1^{00} \rightarrow G_2/G_2^{00}$. Then:*

- (1) *There is a definable homotopy equivalence $f: G_1 \rightarrow G_2$.*
- (2) *Given a finite central subgroup Γ of G_1 we can choose f to be Γ -equivariant, or even Γ' -equivariant where $\Gamma' = \Gamma[G_1, G_1]$ (so in particular $f \upharpoonright_{[G_1, G_1]}$ is an isomorphism onto $[G_2, G_2]$).*
- (3) *Moreover, assuming saturation, we can ensure that $\mathfrak{p}^{G_2} \circ f \upharpoonright_{\Gamma'} = \psi \circ \mathfrak{p}^{G_1} \upharpoonright_{\Gamma'}$ where $\mathfrak{p}_{G_1}: G_1 \rightarrow G_1/G_1^{00}$ and $\mathfrak{p}_{G_2}: G_2 \rightarrow G_2/G_2^{00}$ are the projections.*

This should be compared with the following result of E. Baro, obtained independently by different methods.

Theorem 11.8 ([Ba:09]). *Let G_1 and G_2 be definably compact definably connected groups. Then G_1 and G_2 are definably homotopy equivalent if and only if G_1/G_1^{00} and G_2/G_2^{00} are homotopy equivalent.*

Proof of Theorem 11.7. We can write $G_1 = Z^0(G_1) \times_{\Gamma_0} [G_1, G_1]$ and we can assume $\Gamma \supset \Gamma_0$. Let $\Gamma_1 = \Gamma[G_1, G_1] \cap Z^0(G_1)$ and note that $\Gamma_1[G_1, G_1] = \Gamma[G_1, G_1]$. Moreover Γ_1 is finite: take any $c \in \Gamma$, fix an $x \in c[G_1, G_1] \cap Z^0(G_1)$, then for any $y \in c[G_1, G_1] \cap Z^0(G_1)$ we have $xy^{-1} \in [G_1, G_1] \cap Z^0(G_1)$, which is finite. By Theorem 10.4 (and Lemma 11.6) there is a definable Γ_1 -equivariant homotopy equivalence $f^Z: Z^0(G_1) \rightarrow Z^0(G_2)$ such that $\mathfrak{p} \circ f^Z \upharpoonright_{\Gamma} = \psi \circ \mathfrak{p} \upharpoonright_{\Gamma}$. By Theorem 9.5 (and Lemma 11.6) there is a definable isomorphism

$$f^{[G_1, G_1]}: [G_1, G_1] \rightarrow [G_2, G_2]$$

In particular both f^Z and $f^{[G_1, G_1]}$ are Γ_0 -equivariant definable homotopy equivalences. So by Lemma 11.2 there is a Γ_0 -equivariant definable homotopy equivalence $f^{G_1}: G_1 \rightarrow G_2$ such that $f^{G_1}(ab) = f^Z(a)f^{[G_1, G_1]}(b)$ for all $a \in Z^0(G_1)$ and $b \in [G_1, G_1]$. By construction, using Equation (1) in Lemma 11.2, we have that f^{G_1} is in fact a $\Gamma_1[G_1, G_1]$ -equivariant definable homotopy equivalence. If the o-minimal structure is sufficiently saturated (and we choose $f^{[G_1, G_1]}$ as in Theorem 9.5) we obtain $\mathfrak{p}^{G_2} \circ f \upharpoonright_{\Gamma'} = \psi \circ \mathfrak{p}^{G_1} \upharpoonright_{\Gamma'}$. \square

Another application of Fact 11.1 and Lemma 11.2 is the following.

Corollary 11.9. *For each definably compact definably connected group G , there is a semialgebraic group over \mathbb{R}^{alg} , which is definably homotopy equivalent to G and has the same associated Lie group.*

Proof. We can write $G = Z^0(G) \times_{\Gamma} [G, G]$. Let $d = \dim Z^0(G) = \dim Z(G)$. Let $T = SO(2, M)$. Then T^d is semialgebraic over \mathbb{R}^{alg} and has the same associated Lie group as $Z^0(G)$. So there is a definable homotopy equivalence $f: Z^0(G) \rightarrow T^d$ which is Γ -equivariant. Now let S be a semialgebraic group over \mathbb{R}^{alg} definably isomorphic to $[G, G]$. So in particular there is a Γ -equivariant definable homotopy equivalence $g: [G, G] \rightarrow S$. We can assume that f and g are the identity on Γ . By

Lemma 11.2 G is definably homotopy equivalent to $H := T^d \times_{\Gamma} S$. Note that H is semialgebraic over \mathbb{R}^{alg} and has the same associated Lie group as G by Lemma 11.4. \square

The following corollary was proved in the abelian case in [BeMaOt:09]. Granted the above Corollary the same proof extends to the general case.

Corollary 11.10. *For each definably compact definably connected group G ,*

$$H_*^{\text{def}}(G; \mathbb{Z}) \cong H_*(G/G^{00}; \mathbb{Z})$$

where H^{def} is the o-minimal homology functor.

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