INFINITE PATHS AND CLIQUES IN RANDOM GRAPHS

ALESSANDRO BERARDUCCI, PIETRO MAJER AND MATTEO NOVAGA

Abstract. We study the thresholds for the emergence of various properties in random subgraphs of $(\mathbb{N}, <)$. In particular, we give sharp sufficient conditions for the existence of (finite or infinite) cliques and paths in a random subgraph. No specific assumption on the probability is made. The main tools are a topological version of Ramsey theory, exchangeability theory and elementary ergodic theory.

Contents

1. Introduction

In this paper we introduce a new method in order to deal with some combinatorial problems in random graphs, originally proposed in [EH:64]. Some of this questions have been successfully addressed in [FT:85], using different techniques. We obtain new and self-contained proofs of some of the results in [FT:85]; moreover with this method we expect to be able to treat similar problems in more general random graphs.

Let $G = (\mathbb{N}, \mathbb{N}^{(2)})$ be the directed graph over $\mathbb{N}$ with set of edges $\mathbb{N}^{(2)} := \{(i, j) \in \mathbb{N}^2 : i < j\}$. Let us randomly choose some of the edges of G, that is, we associate to the edge $(i, j) \in \mathbb{N}^{(2)}$ a measurable set $\mathbb{X}_{i,j} \subseteq \Omega$, where $(\Omega, \mathcal{A}, \mu)$ is a base probability space. Assuming $\mu(\mathbb{X}_{i,j}) \geq \lambda$ for each (i, j), we then ask whether the resulting random subgraph $\mathbb{X}$ of $(\mathbb{N}, \mathbb{N}^{(2)})$ contains an infinite path:

Problem 1. Let $(\Omega, \mathcal{A}, \mu)$ be a probability space. Let $\lambda > 0$ and for all $(i, j) \in \mathbb{N}^{(2)}$, let $\mathbb{X}_{i,j}$ be a measurable subset of Ω with $\mu(\mathbb{X}_{i,j}) \geq \lambda$. Is there an infinite increasing sequence $\{n_i\}_{i\in\mathbb{N}}$ such that $\bigcap_{i\in\mathbb{N}} \mathbb{X}_{n_i,n_{i+1}}$ is non-empty?

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More formally, a random subgraph $\mathbb{X}$ of a directed graph $G = (V_G, E_G)$ (with set of edges $E_G \subset V_G \times V_G$), is a measurable function $\mathbb{X} : \Omega \to 2^{E_G}$ where $\Omega = (\Omega, \mathcal{A}, \mu)$ is a probability space, and 2^{E_G} is the power set of E_G, identified with the set of all functions from E_G to $\{0, 1\}$ (with the product topology and the σ-algebra of its Borel sets). For each $x \in \Omega$, we identify $\mathbb{X}(x)$ with the subgraph of G with vertices V_G and edges $\mathbb{X}(x)$. Given $e \in E_G$, the set $\mathbb{X}_e := \{x \in \Omega : e \in \mathbb{X}(x)\}$ represents the event that the random graph $\mathbb{X}$ contains the edge $e \in E_G$. The family $(\mathbb{X}_e)_{e\in E_G}$ determines $\mathbb{X}$ putting: $\mathbb{X}(x) = \{e \in E_G : x \in \mathbb{X}_e\}$. So a random subgraph of G can be equivalently defined as a function from E_G to 2^Ω assigning to each $e \in E_G$ a measurable subset $\mathbb{X}_e$ of Ω.

As in classic percolation theory, we wish to estimate the probability that $\mathbb{X}$ contains an infinite path, in terms of a parameter λ that bounds from below the probability $\mu(\mathbb{X}_e)$ that an edge e belongs to $\mathbb{X}$. Note that it is not a priori obvious that the existence of an infinite path has a well-defined probability, since it corresponds to the uncountable union of the sets $\bigcap_{k\in\mathbb{N}} \mathbb{X}_{i_k,i_{k+1}}$ over all strictly increasing sequences $i : \mathbb{N} \to \mathbb{N}$. However, it turns out that it belongs to the μ-completion of the σ-algebra generated by the $\mathbb{X}_{i,j}$. It has to be noticed that the analogy with classic bond percolation is only formal, the main difference being that in the usual percolation models (see for instance [G:99]) the events $\mathbb{X}_{i,j}$ are supposed *independent*, whereas in the present case the probability distribution is completely general, i.e. we do not impose any restriction on the events $\mathbb{X}_{i,j}$, and on the probability space Ω.

Problem 1 has been originally proposed by P. Erdős and A. Hajnal in [EH:64], and an answer was given by D. H. Fremlin and M. Talagrand in [FT:85], where other related and more general problems are also considered. In particular they show that the threshold for the existence of infinite paths is $\lambda = 1/2$, under the assumption that the probability space $(\Omega, \mathcal{A}, \mu)$ is $[0, 1]$ equipped with the Lebesgue measure. We point out that our result holds for any probability space $(\Omega, \mathcal{A}, \mu)$. One of the main goals of this paper is to present a general method, different from the one in [FT:85], which in particular allows us to recover the same result as in [FT:85] (see Theorem 4.5). Our approach relies on the reduction to the following dual problem:

Problem 2. Given a directed graph F, determine the minimal λ_c such that, whenever $\inf_{e\in\mathbb{N}^{(2)}} \mu(\mathbb{X}_e) > \lambda_c$, there is a graph morphism $f \colon \mathbb{X}(x) \to F$ for some $x \in \Omega$.

Problem 1 can be reformulated in this setting by letting F be the graph $(\omega_1, >)$ where ω_1 is the first uncountable ordinal. This depends on the fact that a subgraph H of $(\mathbb{N}, \mathbb{N}^{(2)})$ does not contain an infinite path if and only if it admits a rank function with values in ω_1. Therefore, if a random subgraph $\mathbb{X}$ of $(\mathbb{N}, \mathbb{N}^{(2)})$ has no infinite paths, it is defined a μ-measurable map $\varphi \colon \Omega \to \omega_1^{\mathbb{N}}$ where $\varphi(x)(i)$ is the rank of the vertex $i \in \mathbb{N}$ in the graph $\mathbb{X}(x)$. It turns out that $\varphi_\#(\mu)$ is a compactly supported Borel measure on $\omega_1^{\mathbb{N}}$, and that $\varphi(\mathbb{X}_{i,j}) \subseteq A_{i,j} := \left\{x \in \omega_1^{\mathbb{N}} : x_i > x_j\right\}$. As a consequence, in

the determination of the threshold for existence of infinite paths
(1.1)

$$\lambda_c := \sup \left\{ \inf_{(i,j)\in\mathbb{N}^{(2)}} \mu(\mathbb{X}_{i,j}) : \ \mathbb{X} \text{ random graph without infinite paths} \right\}$$

we can set $\Omega = \omega_1^{\mathbb{N}}$, $\mathbb{X}_{i,j} = A_{i,j}$, and reduce to the variational problem on the convex set $\mathcal{M}_c^1(\omega_1^{\mathbb{N}})$ of compactly supported probability measures on $\omega_1^{\mathbb{N}}$:

$$\lambda_c = \sup_{m\in\mathcal{M}_c^1(\omega_1^{\mathbb{N}})} \inf_{(i,j)\in\mathbb{N}^{(2)}} m\left(A_{i,j}\right). \tag{1.2}$$

As a next step, we show that in (1.2) we can equivalently take the supremum in the smaller class of all the compactly supported *exchangeable measures* on $\omega_1^{\mathbb{N}}$ (see Appendix B and references therein for a precise definition). Thanks to this reduction, we can explicitly compute $\lambda_c = 1/2$ (Theorem 4.5). We note that the supremum in (1.2) is not attained, which implies that for $\mu(\mathbb{X}_{i,j}) \geq 1/2$ infinite paths occur with positive probability.

In Section 5, we consider again Problem 2 and we give a complete solution when F is a finite graph, showing in particular that

$$\lambda_c = \sup_{\lambda\in\Sigma_F} \sum_{(a,b)\in E_F} \lambda_a\lambda_b$$

where Σ_F is the set of all sequences $\{\lambda_a\}_{a\in V_F}$ with values in $[0,1]$ and such that $\sum_{a\in V_F} \lambda_a = 1$. By the appropriate choice of F we can determine the threholds for the existence of paths of a given finite length (Section 3 and Remark 5.2), or for the property of having chromatic number $\geq n$ (Section 6).

We can consider Problems 1 and 2 for a random subgraph $\mathbb{X}$ of an arbitrary directed graph G, not necessarily equal to $(\mathbb{N}, \mathbb{N}^{(2)})$. However, it can be shown that, if we replace $(\mathbb{N}, \mathbb{N}^{(2)})$ with a finitely branching graph G (such as a finite dimensional network), the probability that $\mathbb{X}$ has an infinite path may be zero even if $\inf_{e\in E_G} \mu(\mathbb{X}_e)$ is arbitrarily close to 1 (Proposition 4.8). Another variant is to consider subgraphs of $\mathbb{R}^{(2)}$ rathen than $\mathbb{N}^{(2)}$ but it turns out that this makes no difference in terms of the threshold for having infinite paths in random subgraphs (Remark 4.9).

In Section 6 we fix again $G = (\mathbb{N}, \mathbb{N}^{(2)})$ and we ask if a random subgraph $\mathbb{X}$ of G contains an infinite clique, i.e. a copy of G itself. More generally we consider the following problem.

Problem 3. Let $(\Omega, \mathcal{A}, \mu)$ be a probability space. Let $\lambda > 0$ and for all $(i_1, \dots, i_k) \in \mathbb{N}^{(k)}$, let $\mathbb{X}_{i_1,\dots,i_k}$ be a measurable subset of X with $\mu(\mathbb{X}_{i_1,\dots,i_k}) \geq \lambda$. Is there an infinite set $J \subset \mathbb{N}$ such that $\bigcap_{(i_1,\dots,i_k)\in J^{(k)}} \mathbb{X}_{i_1,\dots,i_k}$ is nonempty?

This problem is a random version of the classical Ramsey theorem [R:28] (we refer to [GP:73, PR:05], and references therein, for various generalization of Ramsey theorem). Clearly Ramsey theorem implies that the answer to Problem 3 is positive when Ω is finite. Moreover it can be shown that the answer remains positive when Ω is countable (Example 6.3). However when $\Omega = [0,1]$ (with the Lebesgue measure) the probability that $\mathbb{X}$ contains an infinite clique may be zero even when $\inf_{e\in E_G} \mu(\mathbb{X}_e)$ is arbitrarily close to 1 (see Example 6.2). We will show that Problem 3 has a positive answer if the indicator functions of the sets $\mathbb{X}_{i_1,\dots,i_k}$ all belong to a compact subset of $L^1(\Omega,\mu)$ (see Theorem 6.5).

Our original motivation for the above problems came from the following situation. Suppose we are given a space E and a certain family Ω of sequences on E (e.g., minimizing sequences of a functional, or orbits of a discrete dynamical system, etc). A typical general problem asks for existence of a sequence in the family Ω, that admits a subsequence with a prescribed property. One approach to it is by means of measure theory. The archetypal situation here comes from recurrence theorems: one may ask if there exists a subsequence which belongs frequently to a given subset C of the "phase" space Ω (we refer to such sequences as "C-recurrent orbits"). If we consider the set $\mathbb{X}_i := \{x \in \Omega : x_i \in C\}$, then a standard sufficient condition for existence of C-recurrent orbits is $\mu(\mathbb{X}_i) \geq \lambda > 0$, for some probability measure μ on Ω. In fact is easy to check that the set of C-recurrent orbits has measure at least λ by an elementary version of a Borel-Cantelli lemma (see Proposition 6.1). This is indeed the existence argument in the Poincaré Recurrence Theorem for measure preserving transformations. A more subtle question arises when one looks for a subsequence satisfying a given relation between two successive (or possibly more) terms: given a subset R of $E \times E$ we look for a subsequence x_{i_k} such that $(x_{i_k}, x_{i_{k+1}}) \in R$ for all $k \in \mathbb{N}$. As before, we may consider the subset of Ω, with double indices $i < j$, $\mathbb{X}_{i,j} := \{x \in \Omega : (x_i, x_j) \in R\}$ and we are then led to Problem 1.

2. Notations

We follow the set-theoretical convention of identifying a natural number p with the set $\{0, 1, \dots, p-1\}$ of its predecessors. More generally an ordinal number α coincides with the set of its predecessors. With these conventions the set of natural numbers $\mathbb{N}$ coincides with the least infinite ordinal ω. As usual ω_1 denotes the first uncountable ordinal, namely the set of all countable ordinals.

Given two sets X, Y we denote by X^Y the set of all functions from Y to X. If X, Y are linearly ordered we denote by $X^{(Y)}$ the set of all increasing functions from Y to X. In particular $\mathbb{N}^{(p)}$ (with $p \in \mathbb{N}$) is the set of all increasing p-tuples from $\mathbb{N}$, where a p-tuple $\boldsymbol{i} = (i_0, \dots, i_{p-1})$ is a function $\boldsymbol{i}\colon p \to \mathbb{N}$. The case $p = 2$, with the obvious identifications, takes the form $\mathbb{N}^{(2)} = \{(i,j) \in \mathbb{N}^2 : i < j\}$.

Any function $f\colon X \to X$ induces a function $f_*\colon X^Y \to X^Y$ by $f(u) = f \circ u$. On the other hand a function $f\colon Y \to Z$ induces a function $f^*\colon X^Z \to X^Y$ by $f^*(u) = u \circ f$. In particular if $\mathsf{S}\colon \mathbb{N} \to \mathbb{N}$ is the successor function, $\mathsf{S}^*\colon X^{\mathbb{N}} \to X^{\mathbb{N}}$ is the *shift map.*

We let $\mathfrak{S}_c(\mathbb{N}), \mathrm{Inj}(\mathbb{N}), \mathrm{Incr}(\mathbb{N}) \subset \mathbb{N}^{\mathbb{N}}$ be the families of maps $\sigma : \mathbb{N} \to \mathbb{N}$ which are compactly supported permutations (namely they fix all but finitely many points), injective functions and strictly increasing functions, respectively. Note that with the above conventions $\mathrm{Incr}(\mathbb{N}) = \mathbb{N}^{(\omega)}$.

Given a measurable function $\psi\colon X \to Y$ between two measurable spaces and given a measure m on X, we denote as usual by $\psi_\#(m)$ the induced measure on Y.

Given a compact metric space Λ, the space $\mathcal{M}(\Lambda^{\mathbb{N}})$ of signed Borel measures on $\Lambda^{\mathbb{N}}$ can be identified with $C(\Lambda^{\mathbb{N}})^*$, i.e. the dual of the Banach space of all continuous functions on $\Lambda^{\mathbb{N}}$. By the Banach-Alaoglu theorem the subset $\mathcal{M}^1(\Lambda^{\mathbb{N}}) \subset \mathcal{M}(\Lambda^{\mathbb{N}})$ of probability measures is a compact (metrizable) subspace of $C(\Lambda^{\mathbb{N}})^*$ endowed with the weak* topology.

Given $\sigma\colon \mathbb{N} \to \mathbb{N}$ we have $\sigma^*\colon \Lambda^{\mathbb{N}} \to \Lambda^{\mathbb{N}}$ and $\sigma^*_{\#}\colon \mathcal{M}^1(\Lambda^{\mathbb{N}}) \to \mathcal{M}^1(\Lambda^{\mathbb{N}})$. To simplify notations we also write $\sigma \cdot m$ for $\sigma^*_{\#}m$. Note the contravariance of this action:

$$\theta \cdot \sigma \cdot m = (\sigma \circ \theta) \cdot m\,. \tag{2.1}$$

Similarly given $r \in \mathbb{N}$ and $\iota \in \mathbb{N}^{(r)}$, we have $\iota^*_{\#}\colon \mathcal{M}^1(\Lambda^{\mathbb{N}}) \to \mathcal{M}^1(\Lambda^r)$ and we define $\iota \cdot m = \iota^*_{\#}(m)$.

Given a family $\mathcal{F} \subset \mathbb{N}^{\mathbb{N}}$, we say that m is $\mathcal{F}$-invariant if $\sigma \cdot m = m$ for all $\sigma \in \mathcal{F}$.

3. Finite paths in random subgraphs

As a preparation for the study of infinite paths (Problem 1) we first consider the case of finite paths. The following example shows that there are random subgraphs $\mathbb{X}$ of $(\mathbb{N}, \mathbb{N}^{(2)})$ such that $\inf_{e \in \mathbb{N}^{(2)}} \mathbb{X}_e$ is arbitrarily close to $1/2$, and yet $\mathbb{X}$ has probability zero of having infinite paths.

Example 3.1. Let $p \in \mathbb{N}$ and let $\Omega = p^{\mathbb{N}}$ with the Bernoulli probability measure $\mu = B_{(1/p,\dots,1/p)}$. For $i < j$ in $\mathbb{N}$ let $\mathbb{X}_{i,j} = \{x \in p^{\mathbb{N}} : x_i > x_j\}$. Then $\mu(\mathbb{X}_{i,j}) = \frac{1}{2}(1 - \frac{1}{p})$ for all $(i,j) \in \mathbb{N}^{(2)}$ and yet for each $x \in \Omega$ the graph $\mathbb{X}(x) = \{(i,j) \in \mathbb{N}^{(2)} : x_i > x_j\}$ has no paths of length $\geq p$ (where the length of a path is the number of its edges).

We will next show that the bounds in Example 3.1 are optimal. We need:

Lemma 3.2. *Let $p \in \mathbb{N}$ and let $m \in \mathcal{M}^1(p^{\mathbb{N}})$. Let*

$$A_{i,j} := \{x \in p^{\mathbb{N}} : x_i > x_j\}\,. \tag{3.1}$$

Then

$$\inf_{(i,j) \in \mathbb{N}^{(2)}} m(A_{i,j}) \leq \frac{1}{2}\left(1 - \frac{1}{p}\right) \tag{3.2}$$

Proof. The proof is a reduction to the case of *exchangeable measures* (see Appendix B). Note that if $\sigma \in \mathrm{Incr}(\mathbb{N})$, then $(\sigma \cdot m)(A_{i,j}) = m(A_{\sigma(i),\sigma(j)})$. Hence, replacing m with $\sigma \cdot m$ in (3.2) can only increase the infimum, as it is equivalent to the infimum of $m(A_{i,j})$ over a subset of $\mathbb{N}^{(2)}$. By Theorem B.8 we can then assume that m is asymptotically exchangeable, so that in particular the sequence $m_k = \mathsf{S}^k \cdot m$ converges, in the weak* topology, to an exchangeable measure $m' \in \mathcal{M}^1(p^{\mathbb{N}})$. Since p is finite, the sets $A_{i,j}$ are clopen, and therefore $\lim_{k\to\infty} m_k(A_{i,j}) = m'(A_{i,j}) = m'(A_{0,1})$. Noting that

$m_k(A_{i,j}) = m(A_{i+k,j+k})$, it follows that

$$\begin{aligned} \inf_{(i,j)\in\mathbb{N}^{(2)}} m(A_{i,j}) &\leq \lim_{k\to\infty} m_k(A_{0,1}) \\ &= m'(A_{0,1}) \\ &= \frac{1}{2}\left(1 - m'\{x : x_0 = x_1\}\right) \\ &\leq \frac{1}{2}\left(1 - \frac{1}{p}\right) \end{aligned} \tag{3.3}$$

where the latter inequality follows from Corollary B.11. □

Theorem 3.3. *Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and let $\mathbb{X} : \Omega \to 2^{E_G}$ be a random subgraph of $G := (\mathbb{N}, \mathbb{N}^{(2)})$. Consider the set*

$$P := \{x \in \Omega : \mathbb{X}(x) \text{ has a path of length } \geq p\}.$$

Assume $\inf_{e\in\mathbb{N}^{(2)}} \mu(\mathbb{X}_e) > \frac{1}{2}(1 - \frac{1}{p})$. Then $\mu(P) > 0$.

A different proof of this result has been given in [FT:85, 3F] (when the probability space Ω is $[0, 1]$ equipped with the Lebesgue measure).

Proof. Suppose for a contradiction that $\mu(P) = 0$. We can then assume $P = \emptyset$ (otherwise replace Ω with $\Omega - P$). For $x \in \Omega$ let $\varphi(x) \colon \mathbb{N} \to p$ assign to each $i \in \mathbb{N}$ the length of the longest path starting from i in $\mathbb{X}(x)$. We thus obtain a function $\varphi \colon \Omega \to p^{\mathbb{N}}$ which is easily seen to be measurable (this is a special case of Lemma 4.3). Let $m = \varphi_{\#}(\mu) \in \mathcal{M}^1(p^{\mathbb{N}})$. Since $\varphi(\mathbb{X}_{i,j}) \subset A_{i,j}$, we have $m(A_{i,j}) \geq \mu(\mathbb{X}_{i,j}) > 1/2(1 - \frac{1}{p})$ for all i, j, contradicting Lemma 3.2. □

Having determined the critical threshold $\lambda_p = \frac{1}{2}(1 - \frac{1}{p})$, it follows that if $\inf_{e\in\mathbb{N}^{(2)}} \mu(\mathbb{X}_e) \geq \lambda \geq \lambda_p$, the lower bound for $\mu(P)$ grows linearly with λ. More precisely we have:

Corollary 3.4. *In the setting of Theorem 3.3, let $\lambda \in [0, 1]$ and suppose that $\inf_{e\in\mathbb{N}^{(2)}} \mu(\mathbb{X}_e) \geq \lambda$. Then $\mu(P) \geq \frac{\lambda-\lambda_p}{1-\lambda_p}$ where $\lambda_p = \frac{1}{2}(1 - \frac{1}{p})$.*

Proof. Suppose $\inf_{e\in\mathbb{N}^{(2)}} \mu(\mathbb{X}_e) \geq \lambda$. Consider the conditional probability $\mu(\cdot \mid \Omega - P) \in \mathcal{M}^1(\Omega)$. We have

$$\begin{aligned} \mu(\mathbb{X}_e \mid \Omega - P) &\geq \frac{\mu(\mathbb{X}_e) - \mu(P)}{1 - \mu(P)} \\ &\geq \frac{\lambda - \mu(P)}{1 - \mu(P)}. \end{aligned} \tag{3.4}$$

Clearly $\mu(P \mid \Omega - P) = 0$. Applying Theorem 3.3 to $\mu(\cdot \mid \Omega - P)$ it then follows that $\frac{\lambda-\mu(P)}{1-\mu(P)} \leq \lambda_p$, or equivalently $\mu(P) \geq \frac{\lambda-\lambda_p}{1-\lambda_p}$. □

4. Infinite paths

By Theorem 3.3, if $\inf_{e\in\mathbb{N}^{(2)}} \mu(\mathbb{X}_{i,j}) \geq 1/2$, then the random subgraph $\mathbb{X}$ of $(\mathbb{N}, \mathbb{N}^{(2)})$ has arbitrarily long finite paths, namely for each p there is $x \in \Omega$ (depending on p) such that $\mathbb{X}(x)$ has a path of length $\geq p$. We want to show that for some $x \in \Omega$, $\mathbb{X}(x)$ has an infinite path. To this aim it is not enough to find a single x that works for all p. Indeed, $\mathbb{X}(x)$ could have arbitrarily long finite paths without having an infinite path. The existence of infinite paths can be neatly expressed in terms of the following definition.

Definition 4.1. Let G be a countable directed graph and let ω_1 be the first uncountable ordinal. We recall that the *rank function* $\varphi_G \colon V_G \to \omega_1 \cup \{\infty\}$ of G is defined as follows. For $i \in V_G$,

$$\varphi_G(i) = \sup_{j:(i,j)\in E_G} \big(\varphi_G(j) + 1\big).$$

This is a well defined countable ordinal if G has no infinite paths starting at i. In the opposite case we set

$$\varphi_G(i) = \infty$$

where ∞ is a conventional value bigger than all the countable ordinals. For notational convenience we will take $\infty = \omega_1$ so that $\omega_1 \cup \{\infty\} = \omega_1 \cup \{\omega_1\} = \omega_1 + 1$. Note that if i is a leaf, $\varphi_G(i) = 0$. Also note that G has an infinite path if and only if φ_G assumes the value ∞.

Given a random subgraph $\mathbb{X} \colon \Omega \to 2^{E_G}$ of G, we let $\varphi_{\mathbb{X}}(x) = \varphi_{\mathbb{X}(x)}$, namely $\varphi_{\mathbb{X}}(x)(i)$ is the rank of the vertex i in the graph $\mathbb{X}(x)$. So $\varphi_{\mathbb{X}}$ is a map from Ω to $(\omega_1+1)^{V_G}$. It can also be considered as a map from $\Omega \times V_G$ to $\omega_1 + 1$ by writing $\varphi_{\mathbb{X}}(x,i)$ instead of $\varphi_{\mathbb{X}}(x)(i)$.

Remark 4.2. We have $\varphi_{\mathbb{X}}(x,i) = \varphi_{\omega_1}(x,i)$ where $\varphi_\alpha \colon \Omega \to (\omega_1+1)^{V_G}$ is the truncation $\varphi_\alpha := \min(\varphi, \alpha)$, that we can equivalently define by induction on $\alpha \leq \omega_1$ as follows.

$$\begin{array}{rcl} \varphi_0(x,i) &=& 0 \\ \varphi_\alpha(x,i) &=& \sup\{\varphi_\beta(x,j) + 1 \,:\, \beta < \alpha,\ (i,j) \in \mathbb{X}(x)\} \end{array}$$

The above representation will be of use in the following lemma in connection to measurability properties of the map $\varphi_{\mathbb{X}}$.

Lemma 4.3. *Let G be a countable directed graph, let $(\Omega, \mathcal{A}, \mu)$ be a probability space and let $\mathbb{X} : \Omega \to 2^{E_G}$ be a random subgraph of G.*

(1) *For all $\alpha < \omega_1$ and $i \in V_G$, the set $\{x \in \Omega \,:\, \varphi_{\mathbb{X}}(x,i) = \alpha\}$ belongs to the σ-algebra $\mathcal{A}$.*
(2) *The set $P := \{x \in \Omega \,:\, \mathbb{X}(x)$ has an infinite path$\}$ is μ-measurable, that is, it is measurable in the μ-completion of the σ-algebra $\mathcal{A}$.*
(3) *$\varphi_{\mathbb{X}} \colon \Omega \to (\omega_1+1)^{V_G}$ is μ-measurable and its restriction to $\Omega - P$ is essentially bounded, namely it takes values in $\alpha_0^{V_G}$ for some $\alpha_0 < \omega_1$, out of a μ-null set.*

Proof. Since taking the supremum over a countable set preserves measurability, from Remark 4.2 it follows that for all $i \in V_G$ and $\alpha < \omega_1$ the sets $\{x : \varphi_{\mathbb{X}}(x,i) = \alpha\}$ are measurable. We will show that $\{x : \varphi_{\mathbb{X}}(x,i) = \omega_1\}$ is μ-measurable. Fix $i \in V_G$. The sequence of values $\mu\left(\{x : \varphi_{\mathbb{X}}(x,i) \leq \beta\}\right)$ is increasing with respect to the countable ordinal β and uniformly bounded by $1 = \mu(\Omega)$, therefore it is stationary at some finite value. So there is $\alpha_0 < \omega_1$ such that

$$\mu\left(\{x \in \Omega :\ \varphi_{\mathbb{X}}(x,i) = \beta\}\right) = 0 \qquad \text{for } \alpha_0 \leq \beta < \omega_1\,. \tag{4.1}$$

Notice that

$$P = \{x : \varphi_{\mathbb{X}}(x) = \omega_1\} = \big(\Omega - \{x : \varphi_{\mathbb{X}}(x) < \alpha_0\}\big) - \{x : \alpha_0 \le \varphi_{\mathbb{X}}(x) < \omega_1\}.$$

Since

$$\{x : \alpha_0 \le \varphi_{\mathbb{X}}(x) < \omega_1\} \subseteq \bigcup_{i\in V_G} \{x \in \Omega : \ \varphi_{\mathbb{X}}(x,i) = \alpha_0\}$$

and, by (4.1),

$$\mu\left(\bigcup_{i\in V_G} \{x \in \Omega : \ \varphi_{\mathbb{X}}(x,i) = \alpha_0\}\right) = 0,$$

it follows that P is μ-measurable and $\varphi_{\mathbb{X}}$ is μ-measurable, too. □

Notice that the set P is universally measurable with respect to $\mathcal{A}$, that is, it is measurable in the completion of any measure μ defined on the σ-algebra $\mathcal{A}$.

Given an ordinal α, we put on α the topology generated by the open intervals. Note that a non-zero ordinal is compact if and only if it is a successor ordinal, and it is metrizable if and only if it is countable. Let $\mathcal{M}_c(\omega_1^{\mathbb{N}})$ be the set of compactly supported Borel measures on $\omega_1^{\mathbb{N}}$, namely the measures with support in ${\alpha_0}^{\mathbb{N}}$ for some $\alpha_0 < \omega_1$. The following Lemma reduces to Lemma 3.2 if α_0 is finite.

Lemma 4.4. *Let $m \in \mathcal{M}_c(\omega_1^{\mathbb{N}})$ be a non-zero measure with compact support. Let*

$$A_{i,j} := \{x \in \omega_1^{\mathbb{N}} \ : \ x_i > x_j\}\,. \tag{4.2}$$

Then

$$\inf_{(i,j)\in\mathbb{N}^{(2)}} m(A_{i,j}) < \frac{m\left(\omega_1^{\mathbb{N}}\right)}{2}\,. \tag{4.3}$$

Proof. With no loss of generality we can assume that $m \in \mathcal{M}^1(\omega_1^{\mathbb{N}})$, i.e. $m\left(\omega_1^{\mathbb{N}}\right) = 1$. We divide the proof into four steps.

Step 1. Letting $\partial\omega_1$ be the derived set of ω_1, that is the subset of all countable limit ordinals, we can assume that

$$m\left(\{x : \ x_i \in \partial\omega_1\}\right) = 0 \qquad \forall i \in \mathbb{N}\,.$$

Indeed, it is enough to observe that the left-hand side of equation (4.3) can only increase if we replace m with $s_\#(m)$, where $s : \omega_1 \to \omega_1 \setminus \partial\omega_1$ is the successor map sending $\alpha < \omega_1$ to $\alpha + 1$, and $s_\#(m) = (s_*)_\#$, namely $s_\#(m)(X) := m(\{x \in \omega_1^{\mathbb{N}} \ : \ s \circ x \in X\})$.

Step 2. Since the support of m is contained in $\alpha_0^{\mathbb{N}}$, for some ordinal $\alpha_0 < \omega_1$, thanks to Theorem B.8 we can assume that m is asymptotically exchangeable, i.e. the sequence $m_k = \mathsf{S}^k \cdot \theta \cdot m$ converges, in the weak* topology, to an exchangeable measure $m' \in \mathcal{M}^1(\omega_1^{\mathbb{N}})$, with support in $\alpha_0^{\mathbb{N}}$, for all $\theta \in \omega^{(\omega)}$. Note however that, unless α_0 is finite, we cannot conclude that $\lim_{k\to\infty} m_k(A_{i,j}) = m'(A_{i,j})$ since the sets $A_{i,j} = \{x \in \omega_1^{\mathbb{N}} \ : \ x_i > x_j\}$ are not clopen.

Step 3. We shall prove by induction on $\alpha < \omega_1$ that

$$\liminf_{(i,j)\to+\infty} m\left(\{x : \ x_j < x_i \le \alpha\}\right) \le m'\left(\{x : \ x_1 < x_0 \le \alpha\}\right). \tag{4.4}$$

Indeed, for $\alpha = 0$ we have $\{x: \ x_j < x_i \leq 0\} = \emptyset$, and (4.4) holds.

As inductive step, let us assume that (4.4) holds for all $\alpha < \beta < \omega_1$, and we distinguish whether β is a successor or a limit ordinal.

In the former case let $\beta = \alpha + 1$. For $(i, j) \to +\infty$ (with $i < j$) we have:

$$\begin{aligned} m\left(\{x_j < x_i \leq \beta\}\right) &= m\left(\{x_j < x_i \leq \alpha\}\right) + m\left(\{x_j \leq \alpha, \ x_i = \beta\}\right) \\ &\leq m'\left(\{x_1 < x_0 \leq \alpha\}\right) + m'\left(\{x_1 \leq \alpha, \ x_0 = \beta\}\right) + o(1) \\ &= m'\left(\{x_1 < x_0 \leq \beta\}\right) + o(1), \end{aligned}$$

where we used the induction hypothesis, and the fact that $\{x_j \leq \alpha, \ x_i = \beta\}$ is clopen.

Let us now assume that β is a limit ordinal. For all $i \in \mathbb{N}$ we have

$$\bigcap_{\alpha < \beta} \{x: \ \alpha < x_i < \beta\} = \emptyset. \tag{4.5}$$

In particular, for all $\varepsilon > 0$ there exists $\alpha < \beta$ such that

$$m'\left(\{\alpha < x_0 < \beta\}\right) < \varepsilon.$$

Since m' is exchangeable, we also have

$$m'\left(\{\alpha < x_i < \beta\}\right) < \varepsilon$$

for all $i \in \mathbb{N}$. Moreover by assumption $m(\{x_i = \beta\}) = 0$ for every $i \in \mathbb{N}$. Hence, again by (4.5), for all $i \in \mathbb{N}$ there exists $\alpha \leq \alpha_i < \beta$ such that

$$m(\{\alpha_i \leq x_i \leq \beta\}) < \varepsilon\,.$$

Given $i < j$, distinguishing the relative positions of x_i, x_j with respect to α and α_i, we have

$$\begin{aligned} \{x_j < x_i \leq \beta\} \subseteq \ & \{x_j < x_i \leq \alpha\} \\ & \cup \{x_j \leq \alpha < x_i \leq \beta\} \\ & \cup \{\alpha < x_j \leq \alpha_i\} \\ & \cup \{\alpha_i < x_i \leq \beta\}\,. \end{aligned}$$

which gives

$$\begin{aligned} m\left(\{x_j < x_i \leq \beta\}\right) \leq \ & m\left(\{x_j < x_i \leq \alpha\}\right) \\ & + m\left(\{x_j \leq \alpha < x_i \leq \beta\}\right) \\ & + m\left(\{\alpha < x_j \leq \alpha_i\}\right) \\ & + m\left(\{\alpha_i < x_i \leq \beta\}\right). \end{aligned} \tag{4.6}$$

Since $\{x_j \leq \alpha < x_i \leq \beta\}$ and $\{\alpha < x_j \leq \alpha_i\}$ are both clopen, we can approximate their m-measure by their m'-measure. So we have:

$$m\,\{x_j \leq \alpha < x_i \leq \beta\} = m'\left(\{x_1 \leq \alpha < x_0 \leq \beta\}\right) + o(1) \quad \text{for } (i, j) \to \infty$$

and

$$m\left(\{\alpha < x_j \leq \alpha_i\}\right) = m'\left(\{\alpha < x_1 \leq \alpha_i\}\right) + o(1) \quad \text{for } j \to \infty,$$

where we used Remark B.7 to allow $j \to \infty$ keeping i fixed.

Note that, by the choice of α, we have $m'(\{\alpha < x_1 \leq \alpha_i\}) < \varepsilon$, and by induction hypothesis $\liminf_{(i,j)\to+\infty} m(\{x_j < x_i \leq \alpha\}) < m'(\{x_1 < x_0 \leq \beta\})$. Hence, from (4.6) we obtain:

$$\begin{aligned}\liminf_{(i,j)\to+\infty} m(\{x_j < x_i \leq \beta\}) \quad &\leq \quad m'(\{x_1 < x_0 \leq \alpha\}) \\ &\quad +m'(\{x_1 \leq \alpha < x_0 \leq \beta\}) \\ &\quad +\varepsilon + \varepsilon\,.\end{aligned}$$

Therefore,

$$\liminf_{(i,j)\to+\infty} m(\{x_j < x_i \leq \beta\}) \quad \leq \quad m'(\{x_1 < x_0 \leq \beta\}) + 2\varepsilon$$

Inequality (4.4) is then proved for all $\alpha < \omega_1$.
Step 4. We now conclude the proof of the theorem. From (4.4) it follows (4.7)

$$\inf_{(i,j)\in\mathbb{N}^{(2)}} m(A_{i,j}) \leq m'(\{x:\ x_1 < x_0\}) = \frac{1}{2}\big(1 - m'(\{x:\ x_1 = x_0\})\big) < \frac{1}{2}\,.$$

where we used the fact the m' is exchangeable and Corollary B.10. □

Theorem 4.5. *Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and let $\mathbb{X} : \Omega \to 2^{E_G}$ be a random subgraph of $G := (\mathbb{N}, \mathbb{N}^{(2)})$. Consider the set*

$$P := \{x \in \Omega \,:\, \mathbb{X}(x) \text{ has an infinite path}\}.$$

Assume $\inf_{e\in\mathbb{N}^{(2)}} \mu(\mathbb{X}_e) \geq \frac{1}{2}$. *Then* $\mu(P) > 0$.

As observed in the Introduction, we recall that this result follows from [FT:85, 4D], when $\Omega = [0, 1]$ with the Lebesgue measure.

Proof. Suppose for a contradiction $\mu(P) = 0$. We can then assume $P = \emptyset$ (replacing Ω with $\Omega - P$). Hence the rank function $\varphi := \varphi_{\mathbb{X}} \colon \Omega \to (\omega_1 + 1)^{\mathbb{N}}$ takes values in $\omega_1^{\mathbb{N}}$. Let $m = \varphi_{\#}(\mu) \in \mathcal{M}^1(\omega_1^{\mathbb{N}})$. Note that $\varphi(\mathbb{X}_{i,j}) \subset A_{i.j} := \{x \in p^{\mathbb{N}} \,:\, x_i > x_j\}$. Hence $m(A_{i,j}) \geq \mu(\mathbb{X}_{i,j}) \geq 1/2$ for all $(i, j) \in \mathbb{N}^{(2)}$. This contradicts Lemma 4.4. □

Remark 4.6. Note that the bound 1/2 is optimal by Example 3.1.

Reasoning as in Corollary 3.4 we obtain:

Corollary 4.7. *Let $0 \leq \lambda < 1$. If $\inf_{e\in\mathbb{N}^{(2)}} \mu(\mathbb{X}_e) \geq \lambda$, then $\mu(P) > \frac{\lambda-1/2}{1-1/2}$.*

Note that if we replace $(\mathbb{N}, \mathbb{N}^{(2)})$ with a finitely branching countable graph G, then the threshold for the existence of infinite paths becomes 1, namely we cannot ensure the existence of infinite paths even if each edge of G belongs to the random subgraph $\mathbb{X}$ with probability very close to 1. In fact, the following more general result holds:

Proposition 4.8. *Let $G = (V_G, E_G)$ be graph admitting a coloring function $c \colon E_G \to \mathbb{N}$ such that each infinite path in G meets all but finitely many colours (it is easy to see, considering the distance from a fixed vertex in each connected component, that a finitely branching countable graph G has this property). Then for every $\varepsilon > 0$ there is a probability space $(\Omega, \mathcal{A}, \mu)$ and a random subgraph $\mathbb{X} \colon \Omega \to 2^{E_G}$ of G such that for all $x \in \Omega$, $\mathbb{X}(x)$ has no infinite paths, and yet $\mu(\mathbb{X}_e) > 1 - \varepsilon$ for all $e \in E_G$.*

Proof. Let μ be a probability measure on $\Omega := \mathbb{N}$ with $\mu(\{n\}) < \varepsilon$ for every n. Given $n \in \Omega$ let $\mathbb{X}(n)$ be the subgraph of G (with vertices V_G) containing all edges $e \in E_G$ of colour $c(e) \neq n$. Given $e \in E_G$ there is at most one n such that $c(e) \in Z_n$. Hence clearly $\mu(\mathbb{X}_e) \geq 1 - \varepsilon$, and yet $\mathbb{X}(n)$ has no infinite paths for any $n \in \Omega$. □

Remark 4.9. It is natural to ask whether the answer to Problem 1 changes if we substitute $\mathbb{N}$ with the set of the real numbers. Since $\mathbb{N} \subset \mathbb{R}$, the probability threshold for the existence of infinite paths can only decrease, but the following example shows that it still equals 1/2. Let $\Omega = [0,1]^{\mathbb{R}}$ equipped with the product Lebesgue measure $\mathcal{L}$, let $\varepsilon > 0$, and let

$$\mathbb{X}_{i,j} := \left\{ x \in \Omega : \, x_i > x_j + \varepsilon \right\},$$

for all $i < j \in \mathbb{R}$. The assertion follows observing that $\mathcal{L}(\mathbb{X}_{i,j}) = (1-\varepsilon)^2/2$ for all $i < j \in \mathbb{R}$, and

$$\bigcap_{i \in \{1,\dots,N\}} \mathbb{X}_{n_i, n_{i+1}} = \emptyset$$

whenever n_i is a strictly increasing sequence of real numbers, and $N > 1/\varepsilon$.

5. Threshold functions for graph morphisms

Definition 5.1. Let F and G be directed graphs. A graph morphism $\varphi \colon G \to F$ is a map $\varphi : V_G \to V_F$ such that $(\varphi(a), \varphi(b)) \in E_F$ for all $(a,b) \in E_G$. We write $G \to F$ if there is a graph morphism from G to F.

The results of the previous sections were implicitly based on following observation:

Remark 5.2. Let G be a directed graph.

(1) G has a path of length $\geq p$ if and only if $G \not\to (p, p^{(2)})$.
(2) G has an infinite path if and only if $G \not\to (\omega_1, \omega_1^{(2)})$.

This suggests to generalize the above results considering other properties of graphs that can be expressed in terms of non-existence of graph morphisms. Let us give the relevant definitions.

Definition 5.3. Given two directed graphs F, G and given $i, j \in V_G$ let

$$A_{i,j}(F,G) := \{u \in V_F^{V_G} : \, (u(i), u(j)) \in E_F\} \tag{5.1}$$

and define the *relative capacity* of F with respect to G as

$$c(F,G) := \sup_{m \in \mathcal{M}^1(V_F^{V_G})} \inf_{(i,j) \in E_G} m\left(A_{i,j}\left(F,G\right)\right) \in [0,1]. \tag{5.2}$$

Theorems 3.3 and 4.5 have the following counterpart.

Theorem 5.4. *Let F and G be directed countable graphs, let $(\Omega, \mathcal{A}, \mu)$ be a probability space and let $\mathbb{X} \colon \Omega \to 2^{E_G}$ be a random subgraph of G. Let $P := \{x \in \Omega \, : \, \mathbb{X}(x) \not\to F\}$. Assume $\inf_{e \in E_G} \mu(\mathbb{X}_e) > c(F,G)$. Then $\mu(P) > 0$. Moreover there are examples in which P is empty and $\inf_{e \in E_G} \mu(\mathbb{X}_e)$ is as close to $c(F,G)$ as required. So $c(F,G)$ is the threshold for non-existence of graph morphisms $f \colon \mathbb{X}(x) \to F$. To prove the second part it suffices to take $\Omega = V_F^{V G}$ and $\mathbb{X}_{i,j} = A_{i,j}(F,G)$.*

Proof. Suppose for a contradiction $\mu(P) = 0$. We can then assume $P = \emptyset$ (replacing Ω with $\Omega - P$). Hence for each $x \in \Omega$ there is a graph morphism $\varphi(x)\colon \mathbb{X}(x) \to F$, which can be seen as an element of $V_F^{V_G}$. We thus obtain a map $\varphi\colon \Omega \to V_F^{V_G}$. By Lemma 5.7 below, φ can be chosen to be μ-measurable. Since $x \in \mathbb{X}_{i,j}$ implies $(\varphi(x)(i), \varphi(x)(j)) \in E_F$, we have $\varphi(\mathbb{X}_{i,j}) \subset A_{i,j}(F,G)$ for all $(i,j) \in E_G$. Let $m := \varphi_\#(\mu) \in \mathcal{M}^1(V_F^{V_G})$. Then $m(A_{i,j}(F,G)) \geq \mu(\mathbb{X}_{i,j}) > c(F,G)$. This is absurd by definition of $c(F,G)$. □

Reasoning as in Corollary 3.4 we obtain:

Corollary 5.5. *Suppose $c(F,G) < 1$. If $\inf_{e\in\mathbb{N}^{(2)}} \mu(\mathbb{X}_e) \geq \lambda$, then $\mu(P) \geq \frac{\lambda - c(F,G)}{1-c(F,G)}$.*

Remark 5.6. If the sup in the definition of $c(F,G)$ is not reached, it suffices to have the weak inequality $\inf_{e\in E_G} \mu(\mathbb{X}_e) \geq c(F,G)$ in order to have $\mu(P) > 0$ (this is indeed the case of Theorem 4.5).

It remains to show that the map $\varphi\colon \Omega \to V_F^{V_G}$ in the proof of Theorem 5.4 can be taken to be μ-measurable.

Lemma 5.7. *Let F, G be countable directed graphs, let $(\Omega, \mathcal{A}, \mu)$ be a probability space, and let $\mathbb{X}\colon \Omega \to 2^{E_G}$ be a random subgraph of G.*

1. *The set $\Omega_0 := \{x \in \Omega : \mathbb{X}(x) \to F\}$ is μ-measurable (i.e. measurable with respect to the μ-completion of $\mathcal{A}$).*
2. *There is an μ-measurable function $\varphi\colon \Omega_0 \to V_F^{V_G}$ that selects, for each $x \in \Omega_0$, a graph morphism $\varphi(x)\colon \mathbb{X}(x) \to F$.*
3. *If F is finite, then Ω_0 is measurable and φ can be chosen measurable.*

Proof. Given a function $f\colon V_G \to V_F$, we have $f\colon \mathbb{X}(x) \to F$ (i.e., f is a graph morphism from $\mathbb{X}(x)$ to F) if and only if $x \in \bigcap_{(i,j)\in V_G} \bigcup_{(a,b)\in V_F} B_{i,j,a,b}$, where $x \in B_{i,j,a,b}$ says that $f(i) = a, f(j) = b$ and $x \in \mathbb{X}_{i,j}$. This shows that $B := \{(x,f) : f\colon \mathbb{X}(x) \to F\}$ is a measurable subset of $\Omega \times V_G^{V_F}$. We are looking for a (μ-)measurable function $\varphi\colon \pi_X(B) \to V_F^{V_G}$ whose graph is contained in B.

Special case: Let us first assume that Ω is a Polish space (i.e., a complete separable metric space) with its algebra $\mathcal{A}$ of Borel sets. By Jankov - von Neumann uniformization theorem (see [K:95, Thm. 29.9]), if X, Y are Polish spaces and $Q \subset X \times Y$ is a Borel set, then the projection $\pi_X(Q) \subset X$ is universally measurable (i.e. it is m-measurable for every σ-finite Borel measure m on X), and there is a universally measurable function $f\colon \pi_X(Q) \to Y$ whose graph is contained in Q. We can apply this to $X = \Omega, Y = V_F^{V_G}$ and $Q = B$ to obtain (1) and (2). It remains to show that if F is finite $\pi_X(Q)$ and f can be chosen to be Borel measurable. To this aim it suffices to use the following uniformization theorem of Arsenin - Kunugui (see [K:95, Thm. 35.46]): if X, Y, Q are as above and each section $Q_x = \{y \in Y : (x,y) \in Q\}$ is a countable unions of compact sets, then $p_X(Q)$ is Borel and there is a Borel measurable function $f\colon \pi_X(Q) \to Y$ whose graph is contained in Q.

General case: We reduce to the special case as follows. Let $X = 2^{V_G}, Y = V_F^{V_G}$ and consider the set $B' \subset X \times Y$ consisting of those pairs (H, f) such that H is a subgraph of G (with the same vertices) and $f\colon H \to F$ is a graph morphism. Consider the pushforward measure $m = \mathbb{X}_{\#}(\mu)$ defined on the Borel algebra of 2^{V_G}. By the special case there is a (m-)measurable function $\psi\colon \pi_X(B') \to V_F^{V_G}$ whose graph is contained in B'. To conclude it suffices to take $\varphi := \psi \circ \mathbb{X}$. □

We now show how to compute the relative capacity $c(F, (\mathbb{N}, \mathbb{N}^{(2)}))$ (see Definition 5.3) for any finite graph F. The following invariant of directed graphs has been studied in [R:82] and [FT:85, Section 3].

Definition 5.8. Given a directed graph F, we define the *capacity* of F as

$$c_0(F) := \sup_{\lambda \in \Sigma_F} \sum_{(a,b)\in E_F} \lambda_a \lambda_b \quad \in [0,1]\,, \tag{5.3}$$

where Σ_F is the symplex of all sequences $\{\lambda_a\}_{a\in V_F}$ of real numbers such that $\lambda_a \geq 0$ and $\sum_{a\in V_F} \lambda_a = 1$.

Proposition 5.9. *If F is a finite directed graph, then*

$$c\left(F, (\mathbb{N}, \mathbb{N}^{(2)})\right) = c_0(F)\,. \tag{5.4}$$

Proof. Let $G = (\mathbb{N}, \mathbb{N}^{(2)})$. The proof is a series of reductions.
Step 1. Note that if $\sigma \in \mathrm{Incr}(\mathbb{N})$, then $\sigma \cdot m(A_{i,j}(F,G)) = m(A_{\sigma(i),\sigma(j)})$. Hence the infimum in (5.2) can only increase replacing m with $\sigma^*_{\#}(m)$. By Theorem B.8 there is $\sigma \in \mathrm{Incr}(\mathbb{N})$ such that $\sigma \cdot m$ is asymptotically exchangeable. It then follows that we can equivalently take the supremum in (5.2) among the measures $m \in \mathcal{M}^1(V_F^{\mathbb{N}})$ which are asymptotically exchangeable.
Step 2. By definition if m is asymptotically exchangeable there is an exchangeable measure m' such that $\lim_{k\to\infty} m_k = m'$, where $m_k = \mathsf{S}^k \cdot m$. Clearly

$$\inf_{(i,j)\in E_G} m(A_{i,j}(F,G)) \leq \lim_{k\to\infty} m_k(A_{0,1}(F,G)) = m'(A_{0,1}(F,G)).$$

So the supremum in (5.2) coincides with $\sup_m m\left(A_{0,1}\left(F,G\right)\right)$, for m ranging over the exchangeable measures.
Step 3. Recalling (B.13), every exchangeable measure is a convex integral combination of Bernoulli measures B_λ, with $\lambda \in \Sigma_F$. It follows that it is sufficient to compute the supremum on the Bernoulli measures B_λ. We have:

$$\begin{aligned} B_\lambda\left(\left\{x \in V_F^{\mathbb{N}} \,:\, (x_0,x_1) \in E_F\right\}\right) &= \sum_{(a,b)\in E_F} B_\lambda\left(\{x \,:\, x_0 = a\,,\ x_1 = b\}\right) \\ &= \sum_{(a,b)\in E_F} \lambda_a \lambda_b \end{aligned}$$

so that (5.2) reduces to (5.3). □

Notice that if there is a morphism of graphs from G to F, then $c_0(G) \leq c_0(F)$. Also note that $c_0(F) = 1$ if there is some $a \in V_F$ with $(a,a) \in E_F$. Recall that F is said to be: *irreflexive* if $(a,a) \notin E_F$ for all $a \in V_F$; *symmetric* if $(a,b) \in E_F \iff (b,a) \in E_F$ for all $a,b \in V_F$; *anti-symmetric* if $(a,b) \in E_F \implies (b,a) \notin E_F$ for all $a,b \in V_F$.

The *clique number* $\mathrm{cl}(F)$ of F is defined as the largest integer n such that there is a subset $S \subset V_F$ of size n which forms a clique, namely $(a,b) \in E_F$ or $(b,a) \in E_F$ for all $a,b \in S$.

Proposition 5.10. (see also [FT:85, Section 3]) *Let F be a finite irreflexive directed graph. If F is anti-symmetric, then*

$$c_0(F) = \frac{1}{2}\left(1 - \frac{1}{\mathrm{cl}(F)}\right). \tag{5.5}$$

If F is symmetric, then

$$c_0(F) = 1 - \frac{1}{\mathrm{cl}(F)}. \tag{5.6}$$

In particular $c_0(K_p) = 1 - \frac{1}{p}$.

Proof. The anti-symmetric case follows from the symmetric one taking the symmetric closure. So we can assume that F is symmetric. Let $\lambda \in \Sigma_F$ be a maximizing distribution, meaning that $c_0(F) = \sum_{(a,b)\in E_F} \lambda_a \lambda_b$, and let S_λ be the subgraph of F spanned by the support of λ, that is $V_{S_\lambda} = \{a \in V_F : \lambda_a > 0\}$. Given $a \in S_\lambda$ note that $\frac{\partial}{\partial \lambda_a} \sum_{(u,v)\in E_F} \lambda_u \lambda_v = 2\sum_{b \in V_F:\,(a,b)\in E_F} \lambda_b$. From Lagrange's multiplier Theorem it then follows that $\sum_{b\in V_F:\,(a,b)\in E_F} \lambda_b$ is constant, namely it does not depend on the choice of $a \in S_\lambda$. Since $\sum_{a \in S_\lambda} (\sum_{b:\,(a,b)\in E_F} \lambda_a) = c_0(F)$, it follows that for each $a \in S_\lambda$ we have:

$$\sum_{b\in V_F:\,(a,b)\in E_F} \lambda_b = c_0(F). \tag{5.7}$$

If $c, c' \in V_{S_\lambda}$, we can consider the distribution $\lambda' \in \Sigma_F$ such that $\lambda'_c = 0$, $\lambda'_{c'} = \lambda_c + \lambda_{c'}$, and $\lambda'_b = \lambda_b$ for all $b \in V_F \setminus \{c, c'\}$. From (5.7) it then follows that λ' is also a maximizing distribution whenever $(c, c') \notin E_F$. (In fact $\sum_{(a,b)\in E_F} \lambda'_a \lambda'_b = \sum_{(a,b)\in E_F} \lambda_a \lambda_b - \lambda_c \sum_{b:\,(c,b)\in E_F} \lambda_b + \lambda_c \sum_{b:\,(c',b)\in E_F} \lambda b = c_0(F) - \lambda_c c_0(F) + \lambda_c c_0(F)$.)

As a first consequence, S_λ is a clique whenever λ is a maximizing distribution with minimal support. Indeed, let K be a maximal clique contained in S_λ, and assume by contradiction that there exists $a \in V_{S_\lambda} \setminus V_K$. Letting $a' \in V_K$ be a vertex of F independent of a (such an element exists since K is a maximal clique), and letting $\lambda' \in \Sigma_F$ as above, we have $c_0(F) = \sum_{(a,b)\in E_F} \lambda'_a \lambda'_b$, contradicting the minimality of V_{S_λ}.

Once we know that S_λ is a clique, again from (5.7) we get that λ is a uniform ditribution, that is $\lambda_a = \lambda_b$, for all $a, b \in V_{S_\lambda}$. It follows

$$c_0(F) \;=\; 1 - \frac{1}{|S_\lambda|} \;\le\; 1 - \frac{1}{\mathrm{cl}(F)},$$

which in turn implies (5.5), the opposite inequality being realized by a uniform distribution on a maximal clique. □

Notice that the proof of Proposition 5.10 shows that there exists a maximizing $\lambda \in \Sigma_F$ whose support is a clique (not necessarily of maximal order).

5.1. **Chromatic number.** We will apply the results of the previous section to study the chromatic number of a random subgraph of $(\mathbb{N}, \mathbb{N}^{(2)})$. We point out that an alternative proof of this result follows from [EH:64, Theorem 1].

We recall that the chromatic number $\chi(G)$ of a directed graph G is the smallest n such that there is a colouring of the vertices of G with n colours in such a way that $a, b \in V_G$ have different colours whenever $(a, b) \in E_G$ (see [B:79]).

For $p \in \mathbb{N}$, let K_p be the complete graph on p vertices, namely K_p has set of vertices $p = \{0, 1, \ldots, p-1\}$ and set of edges $\{(x, y) \in p^2 : x \neq y\}$. Clearly $\chi(K_p) = p$. Note also that:

$$G \to K_p \iff \chi(G) \leq p\,. \tag{5.8}$$

Now let $(\Omega, \mathcal{A}, m)$ be a probability space, and let $\mathbb{X}\colon \Omega \to 2^{E_G}$ be a random subgraph of $G = (\mathbb{N}, \mathbb{N}^{(2)})$. Let $P = \{x \in \Omega : \chi(\mathbb{X}(x)) \geq p\}$. By Equation (5.8) and the results of the previous section, if $\inf_{e \in \mu(\mathbb{X}_e)} > c(K_p, (\mathbb{N}, \mathbb{N}^{(2)})$, then $\mu(P) > 0$. This however does not say much unless we manage to determine $c(K_p, (\mathbb{N}, \mathbb{N}^{(2)}))$. We will show that $c(K_p, (\mathbb{N}, \mathbb{N}^{(2)})) = (1 - \frac{1}{p})$, so we have:

Theorem 5.11. *Let $(\Omega, \mathcal{A}, m)$ be a probability space, and let $\mathbb{X}\colon \Omega \to 2^{E_G}$ be a random subgraph of $(\mathbb{N}, \mathbb{N}^{(2)})$. If $\inf_{e \in \mu(\mathbb{X}_e)} > 1 - \frac{1}{p}$, then*

$$\mu(\{x \in \Omega : \chi(\mathbb{X}(x)) \geq p + 1\}) > 0.$$

6. Infinite cliques

We recall the following standard Borel-Cantelli type result, which shows that Problem 3 has a positive answer for $k = 1$.

Proposition 6.1. *Let $(\Omega, \mathcal{A}, \mu)$ be a probability space. Let $\lambda > 0$ and for each $i \in \mathbb{N}$ let $X_i \subseteq \Omega$ be a measurable set such that $\mu(X_i) \geq \lambda$. Then there is an infinite set $J \subset \mathbb{N}$ such that*

$$\bigcap_{i \in J} X_i \neq \emptyset.$$

Proof. The set $Y := \bigcap_n \bigcup_{i > n} X_i$ is a decreasing intersection of sets of (finite) measure greater than $\lambda > 0$, hence $\mu(Y) \geq \lambda$ and, in particular, Y is non-empty. Now it suffices to note that any element x of Y belongs to infinitely many X_i's. □

Proposition 6.1 has the following interpretation: if we choose each element of $\mathbb{N}$ with probability greater or equal to λ, we obtain an infinite subset with probability greater or equal to λ.

The following example shows that Problem 3 has in general a negative answer for $k > 1$.

Example 6.2. Let $p \in \mathbb{N}$ and consider the Cantor space $\Omega = p^{\mathbb{N}}$, equipped with the Bernoulli measure $B_{(1/p, \ldots, 1/p)}$, and let $\mathbb{X}_{i,j} := \{x \in \Omega : x_i \neq x_j\}$. Then each $\mathbb{X}_{i,j}$ has measure $\lambda = 1 - 1/p$, and for all $x \in X$ the graph $\mathbb{X}(x) := \{(i, j) \in \mathbb{N}^{(2)} : x \in \mathbb{X}_{i,j}\}$ does not contains cliques (i.e. complete subgraphs) of cardinality $(p + 1)$.

In view of Example 6.2, we need further assumptions in order to get a positive answer to Problem 3.

Example 6.3. By Ramsey theorem, Problem 3 has a positive answer if there is a finite set $S \subset \Omega$ such that each $X_{i_1,\dots,i_k}$ has a non-empty intersection with S. In particular, this is the case if Ω is countable.

Proposition 6.4. *Let $r > 0$. Assume that Ω is a compact metric space and each set $\mathbb{X}_{i_1,\dots,i_k}$ contains a ball $B_{i_1,\dots,i_k}$ of radius $r > 0$. Then Problem 3 has a positive answer.*

Proof. Applying Lemma A.1 to the centers of the balls $B_{i_1,\dots,i_k}$ it follows that for all $0 < r' < r$ there exists an infinite set J and a ball B of radius r' such that

$$B \subset \bigcap_{(j_1,\dots,j_k)\in J^{(k)}} X_{j_1,\dots,j_k}.$$

□

We now give a sufficient condition for a positive answer to Problem 3.

Theorem 6.5. *Let $(\Omega, \mathcal{A}, \mu)$ be a probability space. Let $\lambda > 0$ and assume that we have the sets $\mu(\mathbb{X}_{i_1\dots i_k}) \geq \lambda$ for each $(i_1, \dots, i_k) \in \mathbb{N}^{(k)}$. Assume further that the indicator functions of $\mathbb{X}_{i_1,\dots,i_k}$ belong to a compact subset $\mathcal{K}$ of $L^1(\Omega, \mu)$. Then, for any $\varepsilon > 0$ there exists an infinite set $J \subset \mathbb{N}$ such that*

$$\mu\left(\bigcap_{(i_1,\dots,i_k)\in J^{(k)}} X_{i_1\dots i_k}\right) \geq \lambda - \varepsilon.$$

Proof. Consider first the case $k = 1$. By compactness of $\mathcal{K}$, for all $\varepsilon > 0$ there exist an increasing sequence $\{i_n\}$ and a set $X_\infty \subset X$, with $\mu(X_\infty) \geq \lambda$, such that

$$\mu\left(X_\infty \Delta X_{i_n}\right) \leq \frac{\varepsilon}{2^n} \qquad \forall n \in \mathbb{N}.$$

As a consequence, letting $J := \{i_n : n \in \mathbb{N}\}$ we have

$$\mu\left(\bigcap_{n\in\mathbb{N}} X_{i_n}\right) \geq \mu\left(X_\infty \cap \bigcap_{n\in\mathbb{N}} X_{i_n}\right) \geq \mu\left(X_\infty\right) - \sum_{n\in\mathbb{N}} \mu\left(X_\infty \Delta X_{i_n}\right) \geq \lambda - \varepsilon.$$

For $k > 1$, we apply Lemma A.1 with

$$\begin{aligned} M &= \mathcal{K} \subset L^1(\Omega, \mu) \\ f(i_1, \dots, i_k) &= \chi_{X_{i_1\dots i_k}} \in L^1(\Omega, \mu). \end{aligned}$$

In particular, recalling Remark A.4, for all $\varepsilon > 0$ there exist $J = \sigma(\mathbb{N})$, $X_\infty \subset \Omega$, and $X_{i_1\dots i_m} \subset X$, for all $(i_1, \dots, i_m) \in J^{(m)}$ with $1 \leq m < k$, such that $\mu(X_\infty) \geq \lambda$ and for all $(i_1, \dots, i_k) \in J^{(k)}$ it holds

$$\begin{aligned} \mu\left(X_\infty \Delta X_{i_1}\right) &\leq \frac{\varepsilon}{2^{\sigma^{-1}(i_1)}} \\ \mu\left(X_{i_1\dots i_m} \Delta X_{i_1\dots i_{m+1}}\right) &\leq \frac{\varepsilon}{2^{\sigma^{-1}(i_{m+1})}}. \end{aligned}$$

Reasoning as above, it then follows

$$\mu\left(X_\infty \Delta \bigcap_{(i_1,\dots,i_k)\in J^{(k)}} X_{i_1\dots i_k}\right) \leq$$
$$\sum_{i_1\in\mathbb{N}} \mu\left(X_\infty \Delta X_{i_1}\right) + \sum_{i_1<i_2} \mu\left(X_{i_1}\Delta X_{i_1 i_2}\right) +$$
$$\cdots + \sum_{i_1<\cdots<i_k} \mu\left(X_{i_1\dots i_{k-1}}\Delta X_{i_1\dots i_k}\right) \leq C(k)\varepsilon\,,$$

where $C(k)>0$ is a constant depending only on k. Therefore

$$\begin{aligned}\mu\left(\bigcap_{(i_1,\dots,i_k)\in J^{(k)}} X_{i_1\dots i_k}\right) &\geq \mu\left(X_\infty \cap \bigcap_{(i_1,\dots,i_k)\in J^{(k)}} X_{i_1\dots i_k}\right)\\ &\geq \mu\left(X_\infty\right) - \mu\left(X_\infty \Delta \bigcap_{(i_1,\dots,i_k)\in J^{(k)}} X_{i_1\dots i_k}\right)\\ &\geq \lambda - C(k)\varepsilon.\end{aligned}$$

□

Notice that from Theorem 6.5 it follows that Problem 3 has a positive answer if there exist an infinite $J \subseteq \mathbb{N}$ and sets $\widetilde{\mathbb{X}}_{i_1,\dots,i_k} \subseteq X_{i_1\dots i_k}$ with $(i_1,\dots,i_k) \in J^{(k)}$, such that $\mu\left(\widetilde{\mathbb{X}}_{i_1,\dots,i_k}\right) \geq \lambda$ for some $\lambda > 0$, and the indicator functions of $\widetilde{\mathbb{X}}_{i_1,\dots,i_k}$ belong to a compact subset of $L^1(\Omega,\mu)$.

Remark 6.6. We recall that, when Ω is a compact subset of $\mathbb{R}^n$ and the perimeters of the sets $\mathbb{X}_{i_1,\dots,i_k}$ are uniformly bounded, then the family $\chi_{\mathbb{X}_{i_1,\dots,i_k}}$ has compact closure in $L^1(\Omega,\mu)$ (see for instance [AFP:00, Thm. 3.23]). In particular, if the sets $\mathbb{X}_{i_1,\dots,i_k}$ have equibounded Cheeger constant, i.e. if there exists $C>0$ such that

$$\min_{E\subset \mathbb{X}_{i_1,\dots,i_k}} \frac{\mathrm{Per}(E)}{|E|} \leq C \qquad \forall (i_1,\dots,i_k)\in \mathbb{N}^{(k)},$$

then Problem 3 has a positive answer.

Appendix A. A topological Ramsey theorem

The following metric version of Ramsey theorem reduces to the classical Ramsey theorem when M is finite.

Lemma A.1. *Let M be a compact metric space, let $k\in\mathbb{N}$, and let $f : \mathbb{N}^{(k)} \to M$. Then there exists an infinite set $J \subset \mathbb{N}$ such that the limit*

$$\lim_{\substack{(i_1,\dots,i_k)\to+\infty \\ (i_1,\dots,i_k)\in J^{(k)}}} f(i_1,\dots,i_k)$$

exists.

Proof. Notice first that the thesis is trivial for $k = 1$, since the space M is compact. Assuming that the thesis holds for some $k \in \mathbb{N}$, we want to prove it for $k+1$. So let $f\colon \mathbb{N}^{(k+1)} \to M$. By inductive assumption, for all $j \in \mathbb{N}$ there exist a infinite set $J_j \subset \mathbb{N}$ and a point $x_j \in M$ such that $x_j = \lim_{i_1,\dots,i_k\to\infty} f(j, i_1, \dots, i_k)$, with $(i_1, \dots, i_k) \in J_j^{(k)}$. Possibly extracting further subsequences we can also assume that

$$d(x_j, f(j, i_1, \dots, i_k)) \le 1/2^j \tag{A.1}$$

for all $(i_1, \dots, i_k) \in J_j^{(k)}$. Moreover, by a recursive construction, we can assume that $J_{j+1} \subseteq J_j$. Now define $\tau \in \mathrm{Incr}(\mathbb{N})$ by choosing $\tau(0) \in \mathbb{N}$ and inductively $\tau(n+1) \in J_{\tau(n)}$. Since $J_{j+1} \subset J_j$ for all j, this implies $\tau(m) \in J_{\tau(n)}$ for all $m > n$. By compactness of M, there exists $\lambda \in \mathrm{Incr}(\mathbb{N})$ and a point $x \in M$ such that $x_{\tau(\lambda(n))} \to x$ for $n \to \infty$. Take $J = \mathrm{Im}(\tau \circ \lambda)$. The thesis follows the triangle inequality $d(x, f(j, i_1, \dots, i_k)) \le d(x, x_j) + d(x_j, f(j, i_1, \dots, i_k))$, noting that if $j < i_1 < \ldots < i_k$ are in J, then $i_1, \dots, i_k \in J_j$, and Equation (A.1) applies. □

Note that in Lemma A.1, the condition $(i_1, \dots, i_k) \to +\infty$ is equivalent to $i_1 \to \infty$ (since $i_1 < i_2 < \ldots < i_k$). We would like to strengthen Lemma A.1 by requiring the existence of all the partial limits

$$x = \lim_{i_{j(1)}\to\infty} \lim_{i_{j(2)}\to\infty} \cdots \lim_{i_{j(r)}\to\infty} x_{i_1\ldots i_k}$$

where $1 \le r \le k$ and $(i_{j(1)}, \dots, i_{j(r)}) \in J^{(r)}$ is a subsequence of the (finite) sequence $(i_1, \dots, i_k) \in J^{(k)}$. Note that the existence of all these 2^{k-1} partial limits does not follow directly from Lemma A.1.

To prove the desired strengthening it is convenient to introduce some terminology. Let $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ be the one-point compactification of $\mathbb{N}$. Given a distance δ on $\mathbb{N}$, we consider on $\mathbb{N}^{(k)}$ the induced metric

$$\delta_k((n_1, \dots, n_k), (m_1, \dots, m_k)) := \max_i \delta(n_i, m_i)\,.$$

Given $\sigma \in \mathrm{Incr}(\mathbb{N})$, let $\sigma_*\colon \mathbb{N}^{(k)} \to \mathbb{N}^{(k)}$ be the induced map defined by $\sigma_*(n_1, \dots, n_k) := (\sigma(n_1), \dots, \sigma(n_k))$. Given $f\colon \mathbb{N}^{(k)}$, by the following theorem there is an infinite $J \subset \mathbb{N}$ such that all the partial limits of $f\restriction_{J^{(k)}}$ exist. Moreover the arbitrarity of δ shows that we can impose an arbitrary modulus of convergence on all the partial limits of $f \circ \sigma_*$, where $\sigma \in \mathrm{Incr}(\mathbb{N})$ is an increasing enumeration of J.

Theorem A.2. *Let M be a compact metric space, let $k \in \mathbb{N}$, and let $f : \mathbb{N}^{(k)} \to M$. Then, for any distance δ on $\overline{\mathbb{N}}$ there exists $\sigma \in \mathrm{Incr}(\mathbb{N})$ such that $f \circ \sigma_* : \mathbb{N}^{(k)} \to M$ is 1-Lipschitz. As a consequence, it can be extended to a 1-Lipschitz function on the closure of $\mathbb{N}^{(k)}$ in $\overline{\mathbb{N}}^k$.*

Lemma A.3. *Let δ be a metric on $\overline{\mathbb{N}}$. Then there is another metric δ^* on $\overline{\mathbb{N}}$ such that*

1. *$\delta^*(x, y) \le \delta(x, y)$ for all x, y.*
2. *δ^* is* monotone *in the following sense: $\delta^*(x', y') \le \delta^*(x, y)$ for all x, x, y, y', provided $x < \min(y, x', y')$.*

(3) $\varepsilon^*(x) \geq \varepsilon^*(y)$ *for all* $x \leq y$*, where*

$$\varepsilon^*(x) := \min_{y \geq x+1} \delta^*(x,y). \tag{A.2}$$

Proof. We shall define a distance of the form $\delta^*(x,y) = \delta(\psi(x),\psi(y))$ for a suitable strictly increasing function

$$\psi\colon \overline{\mathbb{N}} \to \overline{\mathbb{N}} \,.$$

To this aim, let us consider, for any $x \in \overline{\mathbb{N}}$, the diameter of the interval $[x,\infty] \cap \overline{\mathbb{N}}$

$$\eta(x) := \max_{x \leq y \leq z} \delta(y,z), \tag{A.3}$$

and the point-set distance from x to the interval $[x+1,\infty] \cap \overline{\mathbb{N}}$

$$\varepsilon(x) := \min_{y \geq x+1} \delta(x,y). \tag{A.4}$$

Since $\varepsilon(x) > 0$ for all $x < \infty$ and $\eta(x) = o(1)$ as $x \to \infty$, there exists a recursively defined, strictly increasing function $\psi\colon \overline{\mathbb{N}} \to \overline{\mathbb{N}}$ such that for any $x \in \mathbb{N}$

$$\begin{aligned} \eta(\psi(x)) &\leq \varepsilon(x) \\ \eta(\psi(x+1)) &\leq \varepsilon(\psi(x))\,. \end{aligned} \tag{A.5}$$

As a consequence, the distance

$$\delta^*(x,y) := \delta(\psi(x),\psi(y))$$

verifies, for all $x < y \leq \infty$

$$\delta^*(x,y) = \delta(\psi(x),\psi(y)) \leq \eta(\psi(x)) \leq \varepsilon(x) \leq \delta(x,y),$$

and, assuming also $x < x' \leq \infty$ and $x < y' \leq \infty$,

$$\begin{aligned} \delta^*(x',y') &= \delta(\psi(x'),\psi(y')) \leq \eta(\psi(x')) \leq \eta(\psi(x+1)) \\ &\leq \varepsilon(\psi(x)) \leq \delta(\psi(x),\psi(y)) = \delta^*(x,y)\,. \end{aligned}$$

To prove the last statement we observe that

$$\varepsilon^*(x) \geq \varepsilon(\psi(x)) \geq \eta(\psi(x+1)) \geq \varepsilon^*(x+1).$$

□

Proof of Theorem A.2. By Lemma A.3 we can assume that δ is monotone in the sense of Lemma A.3 (2).

We proceed by induction on k. When $k = 1$, consider the function $\varepsilon(n) := \min_{m \geq n+1} \delta(n,m)$ as in (A.2). By compactness of M there exist $x \in M$ and a subsequence $f \circ \sigma$ of f converging to x with the property

$$d_M\left(f\left(\sigma n\right), x\right) \leq \frac{\varepsilon(n)}{2}\,. \tag{A.6}$$

Recalling Lemma A.3 (3), for $n \neq m$ we have

$$d_M\left(f\left(\sigma n\right), f\left(\sigma m\right)\right) \leq \frac{\varepsilon(n) + \varepsilon(m)}{2} \leq \delta(n,m)\,. \tag{A.7}$$

So $f \circ \sigma$ is 1-Lipschitz.

Now assume inductively that the thesis holds for some $k \in \mathbb{N}$, and let us prove it for $k+1$. So let $f\colon \mathbb{N}^{(k+1)} \to M$. We need to prove the existence of $\sigma \in \mathrm{Incr}(\mathbb{N})$ such that

$$d_M\left(f\left(\sigma_*(n,\boldsymbol{m})\right), f\left(\sigma_*(n',\boldsymbol{m}')\right)\right) \leq \delta_{k+1}((n,\boldsymbol{m}),(n',\boldsymbol{m}')) \tag{A.8}$$

for all $(n,\boldsymbol{m}) \in \mathbb{N}^{(k+1)}$ and $(n',\boldsymbol{m}') \in \mathbb{N}^{(k+1)}$, where $\boldsymbol{m} = (m_1,\dots,m_k)$ and $\boldsymbol{m}' = (m'_1,\dots,m'_k)$.

Given $n \in \mathbb{N}$ define $f_n\colon \mathbb{N}^{(k)} \to M$ by

$$f_n(\boldsymbol{m}) := \begin{cases} f(n,\boldsymbol{m}) & \text{if } n < m_1, \\ \perp & \text{if } n \geq m_1 \end{cases} \tag{A.9}$$

where $\perp$ is an arbitrary element of M. Note that the condition $n < m_1$ is equivalent to $(n,\boldsymbol{m}) \in \mathbb{N}^{(k+1)}$.

By inductive assumption, for all $n \in \mathbb{N}$ there exists $\theta_n \in \mathrm{Incr}(\mathbb{N})$ such that $f_n \circ \theta_{n*}\colon \mathbb{N}^{(k)} \to M$ is 1-Lipschitz. By a recursive construction, we can also assume that θ_{n+1} is a subsequence of θ_n, namely $\theta_{n+1} = \theta_n \circ \gamma_n$ for some $\gamma_n \in \mathrm{Incr}(\mathbb{N})$. Indeed to obtain θ_{n+1} as desired it suffices to apply the induction hypothesis to $f_{n+1} \circ \theta_{n*}\colon \mathbb{N}^{(k)} \to M$ rather than directly to f_{n+1}.

Since $f_n \circ \theta_{n*}$ is 1-Lipschitz, there exist the limit

$$g(n) := \lim_{\min(\boldsymbol{m})\to\infty} f(n, \theta_{n*}(\boldsymbol{m}))$$

Passing to a subsequence we can further assume that all the values of $f_n \circ \theta_n$ are within distance $\varepsilon(n)/4$ from its limit, namely:

$$d_M\left(g\left(n\right), f\left(n, \theta_n((\boldsymbol{m}))\right)\right) < \frac{\varepsilon(n)}{4}\,. \tag{A.10}$$

Let $J_n := \theta_n(\mathbb{N}) \subset \mathbb{N}$ and let $\tau \in \mathrm{Incr}(\mathbb{N})$ be such that:

$$\tau(n+1) \in J_{\tau(n)} \tag{A.11}$$

It then follows that

$$\forall n, m \in \tau(\mathbb{N}) \quad m > n \Longrightarrow m \in J_n\,. \tag{A.12}$$

For later purposes we need to define $\tau(n+1)$ as an element of $J_{\tau(n)}$ bigger than its $n+1$-th element, namely $\tau(n+1) > \theta_{\tau(n)}(n+1)$. So, for the sake of concreteness, we define inductively $\tau(0) := 0$ and $\tau(n+1) := \theta_{\tau(n)}(n+2)$. It then follows that:

$$\forall i, j \in \tau(\mathbb{N})\ \forall k \in \mathbb{N} \quad j > i, j \geq k \Longrightarrow \tau(j) > \theta_{\tau(i)}(k)\,. \tag{A.13}$$

Reasoning as in the case $k = 1$, there is $\lambda \in \mathrm{Incr}(\mathbb{N})$ and $x_\infty \in M$ such that

$$d_M\left(g\left(\tau\left(\lambda\left(n\right)\right)\right), x_\infty\right) < \frac{\varepsilon(n)}{4} \tag{A.14}$$

Now define $\sigma := \tau \circ \lambda \in \mathrm{Incr}(\mathbb{N})$. Note that $\sigma(\mathbb{N}) \subset \tau(\mathbb{N})$ so (A.12) and (A.13) continue to hold with σ instead of τ. We claim that $f \circ \sigma_*\colon \mathbb{N}^{(k+1)} \to M$ is 1-Lipschitz.

As a first step we show that

$$\exists \boldsymbol{k} > \boldsymbol{m}\ :\ (f \circ \sigma_*)(n,\boldsymbol{m}) = (f_{\sigma(n)} \circ \theta_{\sigma(n)})(n,\boldsymbol{k}) \tag{A.15}$$

where $\boldsymbol{k} > \boldsymbol{m}$ means that $k_i > m_i$ for all respective components. To prove (A.15) recall that $(f \circ \sigma_*)(n,\boldsymbol{m}) = f(\sigma(n), \sigma(m_1), \dots, \sigma(m_k))$. Since $n <$

$\min(\boldsymbol{m})$, by (A.12) the elements $\sigma(m_1), \ldots, \sigma(m_k)$ are in the image of $\theta_{\sigma(n)}$, namely for each i we have $\sigma(m_i) = \theta_{\sigma(n)}(k_i)$ for some $k_i \in \mathbb{N}$. Moreover applying (A.13) we must have $k_i > m_i$. The proof of (A.15) is thus complete.

It follows from (A.15) and (A.10) that $(f \circ \sigma_*)(n, \boldsymbol{m})$ is within distance $\varepsilon(\sigma(n))/4$ from its limit $g(\sigma(n))$, which in turn is within distance $\varepsilon(n)/4$ from its limit x_∞ by (A.14). We thus proved:

$$d_M\left(f\left(\sigma_*\left(n, \boldsymbol{m}\right)\right), x_\infty\right) < \frac{1}{4}\varepsilon(\sigma(n)) + \frac{1}{4}\varepsilon(n)\,. \tag{A.16}$$

Recalling that for $x \neq y$ we have $\varepsilon(x) + \varepsilon(y) \leq 2\delta(x, y)$, it follows that for $n \neq n'$ the left-hand side of (A.8) is bounded by $[\delta(\sigma(n), \sigma(n')) + \delta(n, n')]/2$, which in turn is $\leq \delta(n, n')$ by monotonicity of δ.

If remains to prove (A.8) in the case $n = n'$. Given $\boldsymbol{m}, \boldsymbol{m'}$ as in (A.8), we apply (A.15) to get $\boldsymbol{k} > \boldsymbol{m}, \boldsymbol{k'} > \boldsymbol{m'}$ with $(f \circ \sigma_*)(n, \boldsymbol{m}) = (f_{\sigma(n)} \circ \theta_{\sigma(n)})(n, \boldsymbol{k})$ and $(f \circ \sigma_*)(n, \boldsymbol{m'}) = (f_{\sigma(n)} \circ \theta_{\sigma(n)})(n, \boldsymbol{k'})$.

Using the monotonicity of δ and the fact that $f_{\sigma(n)} \circ \theta_{\sigma(n)}$ is 1-Lipschitz, it follows that:

$$d_M\left(f\left(\sigma_*(n, \boldsymbol{m})\right), f\left(\sigma_*(n, \boldsymbol{m'})\right)\right) \leq \delta_k(\boldsymbol{k}, \boldsymbol{k'}) \leq \delta_k(\boldsymbol{m}, \boldsymbol{m'})\,. \tag{A.17}$$

□

Remark A.4. Theorem A.2 implies that there exists an infinite set $J = \sigma(\mathbb{N}) \subset \mathbb{N}$ such that, for all $0 \leq m < k$ and $(i_1, \ldots, i_m) \in J^{(m)}$, there are limit points $x_{i_1 \ldots i_m} \in M$ with the property

$$x_{i_1 \ldots i_m} = \lim_{\substack{(i_{m+1}, \ldots, i_k) \to \infty \\ (i_1 \ldots i_k) \in J^{(k)}}} x_{i_1 \ldots i_k}\,,$$

where we set $x_{i_1 \ldots i_k} := f\,(i_1, \ldots, i_k)$. Moreover, by choosing the distance $\delta(n, m) = \varepsilon|2^{-n} - 2^{-m}|$, we may also require

$$d_M\left(x_{i_1 \ldots i_m}, x_{i_1 \ldots i_k}\right) \leq \frac{\varepsilon}{2^{\sigma^{-1}(i_{m+1})}} \qquad \forall (i_1, \ldots, i_k) \in J^{(k)}.$$

Appendix B. Exchangeable measures

Let Λ be a compact metric space. We recall a classical notion of *exchangeable measure* due to De Finetti [DF:74], showing some equivalent conditions.

Proposition B.1. *Given $m \in \mathcal{M}^1(\Lambda^{\mathbb{N}})$, the following conditions are equivalent:*

a) *m is $\mathfrak{S}_c(\mathbb{N})$-invariant;*
b) *m is $\mathrm{Inj}(\mathbb{N})$-invariant;*
c) *m is $\mathrm{Incr}(\mathbb{N})$-invariant.*

Definition B.2. If m satisfies one of these equivalent conditions we say that m is *exchangeable.*

Notice that an exchangeable measure is always shift-invariant, while there are shift-invariant measures which are not exchangeable. To prove Proposition B.1 we need some preliminary results concerning measures satisfying condition (c).

Definition B.3. Given $m \in \mathcal{M}(\Lambda^{\mathbb{N}})$ and $f \in L^p(\Lambda^{\mathbb{N}})$, with $p \in [1, +\infty]$, we let

$$\tilde{f} = E\left(f|\mathcal{A}_s\right) \in L^p(\Lambda^{\mathbb{N}})$$

be the conditional probability of f with respect to the σ-algebra $\mathcal{A}_s$ of the shift-invariant Borel subsets of $\Lambda^{\mathbb{N}}$. In particular, $\tilde{f}$ is shift-invariant, and by Birkhoff's theorem (see for instance [P:82]) we have

$$\tilde{f} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ \mathsf{S}^{*k},$$

where the limit holds almost everywhere and in the strong topology of $L^1(\Lambda^{\mathbb{N}})$.

Lemma B.4. *Assume that $m \in \mathcal{M}^1(\Lambda^{\mathbb{N}})$ is $\mathrm{Incr}(\mathbb{N})$-invariant. Then for all $f \in L^\infty(\Lambda^{\mathbb{N}}, m)$ we have*

$$\tilde{f} = \lim_{n\to\infty} f \circ \mathsf{S}^{*n}, \tag{B.1}$$

where the limit is taken in the weak topology of $L^\infty(\Lambda^{\mathbb{N}})$, namely for every $g \in L^1(\Lambda^{\mathbb{N}}, m)$ we have*

$$\lim_{n\to\infty} \int_{\Lambda^{\mathbb{N}}} g\left(f \circ \mathsf{S}^{*n}\right) dm = \int_{\Lambda^{\mathbb{N}}} g\tilde{f}\, dm \tag{B.2}$$

Proof. It suffices to prove that $\lim_{n\to\infty} f \circ \mathsf{S}^{*n}$ exists, since in that case it is necessarily equal to the (weak*) limit of the arithmetic means $\frac{1}{n}\sum_{k=0}^{n-1} f \circ \mathsf{S}^{*k}$, and therefore to $\tilde{f}$ (since $\tilde{f} = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f \circ \mathsf{S}^{*k}$ in an even stronger topology). Since the sequence $f \circ \mathsf{S}^{*n}$ is equibounded in $L^\infty(\Lambda^{\mathbb{N}}, m)$, it is enough to prove (B.2) for all g in a dense subset D of $L^1(\Lambda^{\mathbb{N}})$. We can take D to be the set of those functions $g \in L^1(\Lambda^{\mathbb{N}}, m)$ depending on finitely many coordinates (namely $g(x) = h(x_1, \dots, x_r)$ for some $r \in \mathbb{N}$ and some $h \in L^1(\Lambda^r, m)$). The convergence of (B.2) for $g(x) = h(x_1, \dots, x_r)$ follows at once from the fact that $\sigma \cdot m = m$ for all $\sigma \in \mathrm{Incr}(\mathbb{N})$, which implies that the quantity in (B.2) is constant for all $n > r$. Indeed to prove that $\int_{\Lambda^{\mathbb{N}}} g\left(f \circ \mathsf{S}^{*n}\right) dm = \int_{\Lambda^{\mathbb{N}}} g\left(f \circ \mathsf{S}^{*n+l}\right) dm$ it suffices to consider the function $\sigma \in \mathrm{Incr}(\mathbb{N})$ which fixes $0, \dots, r-1$ and sends i to $i + l$ for $i \geq r$. □

We are now ready to prove the equivalence of the conditions in the definition of exchangeable measure.

Proof of Proposition B.1. Since $\mathfrak{S}_c(\mathbb{N}) \subset \mathrm{Inj}(\mathbb{N})$ and $\mathrm{Incr}(\mathbb{N}) \subset \mathrm{Inj}(\mathbb{N})$, the implications b) ⇒ a) and b) ⇒ c) are obvious. The implication a) ⇒ b) is also obvious since it is true on the Borel subsets of $\Lambda^{\mathbb{N}}$ of the form $\{x \in \Lambda^{\mathbb{N}} : x_{i_1} \in A_1, \dots, x_{i_r} \in A_r\}$, which generate the whole Borel σ-algebra of $\Lambda^{\mathbb{N}}$.

Let $m \in \mathcal{M}^1(\Lambda^{\mathbb{N}})$ be $\mathrm{Incr}(\mathbb{N})$-invariant, and let us prove that m is $\mathrm{Inj}(\mathbb{N})$-invariant. So let $\sigma \in \mathrm{Inj}(\mathbb{N})$. We must show that

$$\int_{\Lambda^{\mathbb{N}}} g\, dm = \int_{\Lambda^{\mathbb{N}}} g \circ \sigma^*\, dm, \tag{B.3}$$

for all $g \in C(\Lambda^{\mathbb{N}})$. It suffices to prove (B.3) for g in a dense subset D of $C(\Lambda^{\mathbb{N}})$. So we can assume that $g(x)$ has the form $g_0(x_0) \cdot \ldots \cdot g_r(x_r)$ for some $r \in \mathbb{N}$ and $g_1, \dots, g_r \in C(\Lambda)$. Note that $g_i(x_i) = (g_i \circ P_i)(x)$ where

$P_i\colon \Lambda^{\mathbb{N}} \to \Lambda$ is the projection on the i-th coordinate. Since $P_i = P_0 \circ \mathsf{S}^*$ where S^* is the shift, we can apply Lemma B.4 to obtain

$$\int_{\Lambda^{\mathbb{N}}} g\, dm = \int_{\Lambda^{\mathbb{N}}} \widetilde{g_1 \circ P_1} \cdots \widetilde{g_r \circ P_1}\, dm\,.$$

Reasoning in the same way for the function $g \circ \sigma^*$, we finally get

$$\int_{\Lambda^{\mathbb{N}}} g \circ \sigma^*\, dm = \int_{\Lambda^{\mathbb{N}}} \widetilde{g_1 \circ P_1} \cdots \widetilde{g_r \circ P_1}\, dm = \int_{\Lambda^{\mathbb{N}}} g\, dm\,.$$

□

Definition B.5. We say that $m \in \mathcal{M}^1(\Lambda^{\mathbb{N}})$ is *asymptotically exchangeable* if the limit $m' = \lim_{\substack{\min\theta\to\infty \\ \theta\in \mathrm{Incr}(\mathbb{N})}} \theta \cdot m$ exists in $\mathcal{M}^1(\Lambda^{\mathbb{N}})$ and is an exchangeable measure.

Remark B.6. Note that if m is asymptotically exchangeable, then:

$$m' := \lim_{\substack{\min\theta\to\infty \\ \theta\in \mathrm{Incr}(\mathbb{N})}} \theta \cdot m \tag{B.4}$$

$$= \lim_{k\to\infty} \mathsf{S}^k \cdot m\,. \tag{B.5}$$

However it is possible that $\lim_{k\to\infty} \mathsf{S}^k \cdot m$ exists and is exchangeable, and yet m is not asymptotically exchangeable. As an example one may start with the Bernoulli probability measure μ on $2^{\mathbb{N}}$ with $\mu(\{x_i = 0\}) = 1/2$ and then consider the conditional probability $m(\cdot) = \mu(\cdot|A)$ where $A \subset 2^{\mathbb{N}}$ is the set of those sequences $x \in 2^{\mathbb{N}}$ satisfying $x_{(n+1)^2} = 1 - x_{n^2}$ for all n.

Remark B.7. If m is asymptotically exchangeable and $m' = \lim_{k\to\infty} \mathsf{S}^k \cdot m$, then for all $r \in \mathbb{N}$ and $g_1, \ldots, g_r \in C(\Lambda)$ we have

$$\lim_{\substack{i_1\to+\infty \\ (i_1,\ldots,i_r)\in\mathbb{N}^{(r)}}} \int_{\Lambda^{\mathbb{N}}} g_1(x_{i_1}) \cdots g_r(x_{i_r})\, dm = \int_{\Lambda^{\mathbb{N}}} g_1(x_1) \cdots g_r(x_r)\, dm'. \tag{B.6}$$

Theorem B.8. *Given $m \in \mathcal{M}^1(\Lambda^{\mathbb{N}})$ there is $\sigma \in \omega^{(\omega)}$ such that $\sigma \cdot m$ is asymptotically exchangeable.*

Proof. Fix $m \in \mathcal{M}^1(\Lambda^{\mathbb{N}})$. Given $r \in \omega$ consider the function $f\colon \omega^{(r)} \to \mathcal{M}^1(\Lambda^r)$ sending ι to $\iota \cdot m \in \mathcal{M}^1(\Lambda^r)$. By Lemma A.1 there is an infinite set $J_r \subset \omega$ such that

$$\lim_{\substack{\min(\iota)\to\infty \\ \iota\in J_r^{(r)}}} \iota \cdot m \tag{B.7}$$

exists in $\mathcal{M}^1(\Lambda^r)$. By a diagonal argument we choose the same set $J = J_r$ for all r. Let $\sigma \in \mathrm{Incr}(\mathbb{N})$ be such that $\sigma(\mathbb{N}) = J$. We claim that $\sigma \cdot m$ is asymptotically exchangeable. To this aim consider $m_k := \mathsf{S}^k \cdot \sigma \cdot m \in \mathcal{M}^1(\Lambda^{\mathbb{N}})$. By compactness there is an accumulation point $m' \in \mathcal{M}^1(\Lambda^{\mathbb{N}})$ of $\{m_k\}_{k\in\mathbb{N}}$. We claim that

$$\lim_{\substack{\min(\theta)\to\infty \\ \theta\in J^{(\omega)}}} \theta \cdot \sigma \cdot m = m'\,, \tag{B.8}$$

hence in particular $m_k \to m'$ (taking $\theta = \mathsf{S}^k$). Note that the claim also implies that m' is exchangeable. Indeed, given an increasing function $\gamma\colon \mathbb{N} \to$

$\mathbb{N}$, to show $\gamma \cdot m' = m'$ it suffices to replace θ with $\theta \circ \gamma$ in equation (B.8). Since the subset of $C(\Lambda^{\mathbb{N}})$ consising of the functions depending on finitely many coordinates is dense, it suffices to prove that for all $r \in \mathbb{N}$ and $\iota \in \mathbb{N}^{(r)}$ the limit

$$\lim_{\substack{\min(\theta)\to\infty \\ \theta\in J^{(\omega)}}} \iota \cdot \theta \cdot \sigma \cdot m \tag{B.9}$$

exists in $\mathcal{M}^1(\Lambda^r)$ (the limit being necessarily $\iota \cdot m'$). This is however just a special case of equation B.7. □

We give below some representation results for exchangeable measures. First note that if Λ is countable, a measure $m \in \mathcal{M}^1(\Lambda^{\mathbb{N}})$ is determined by the values it takes on the sets of the form $\{x : x_{i_1} = a_1, \dots, x_{i_r} = a_r\}$.

Lemma B.9. *If Λ is countable, a measure $m \in \mathcal{M}(\Lambda^{\mathbb{N}})$ is exchangeable if and only if it admits a representation of the following form. There is a probability space (Ω, μ) (which in fact can be taken to be $(\Lambda^{\mathbb{N}}, m)$) and a family $\{\psi_a\}_{a\in\Lambda}$ in $L^\infty(\Omega, \mu)$ such that for all $i_1 < \dots < i_r$ in $\mathbb{N}$ we have*

$$m\left(\{x : x_{i_1} = a_1, \dots, x_{i_r} = a_r\}\right) = \int_\Omega \psi_{a_1} \cdot \dots \cdot \psi_{a_n} d\mu\,. \tag{B.10}$$

Proof. Since the right-hand side of the equation does not depend on $i_1, \dots, i_r$ a measure $m \in \mathcal{M}^1(\Lambda^{\mathbb{N}})$ admitting the above representation is clearly exchangeable. Conversely if m is exchangeable it suffices to take $\psi_a = \widetilde{\chi_a}$ where χ_a is the characteristic function of the set $\{x : x_0 = a\}$. We can in fact obtain the desired result by a repeated application of Equation (B.2) after observing that the characteristic function $\chi_{\{x\,:\,x_{i_1}=a_1,\dots,x_{i_r}=a_r\}}$ is the product $\chi_{\{x\,:\,x_{i_1}=a_1\}} \cdot \dots \cdot \chi_{\{x_{i_r}=a_r\}}$ and $\chi_{\{x\,:\,x_i=a\}} = \chi_a \circ (\mathsf{S}^*)^i$. □

Corollary B.10. *If Λ is countable and $m \in \mathcal{M}^1(\Lambda^{\mathbb{N}})$ is exchangeable, then $m(\{x \in \Lambda^{\mathbb{N}} : x_0 = x_1\}) \neq 0$.*

Proof. By (B.10) $m(\{x \in \Lambda^{\mathbb{N}} : x_0 = x_1\}) = \sum_{a\in\Lambda} \int {\psi_a}^2 d\mu \neq 0$. □

Corollary B.11. *If $p \in \mathbb{N}$ and $m \in \mathcal{M}^1(p^{\mathbb{N}})$ is exchangeable, then $m(\{x \in \Lambda^{\mathbb{N}} : x_0 = x_1\}) \geq \frac{1}{p}$.*

Proof. Write $m(\{x \in \Lambda^{\mathbb{N}} : x_0 = x_1\}) = \sum_{a\in\Lambda} \int_\Omega {\psi_a}^2$ and apply the Cauchy-Schwarz inequality to the linear operator $\sum \int$ on $p \times \Omega$ to obtain

$$\left(\sum_{a<p} \int_\Omega {\psi_a}^2 d\mu\right) \cdot \left(\sum_{a<p} \int_\Omega 1 d\mu\right) \geq \left(\sum_{a<p} \int_\Omega \psi_a d\mu\right)^2 \tag{B.11}$$

which gives the desired result. □

Thanks to a theorem of De Finetti, suitably extended in [HS:55] there is an integral representation *à la Choquet* for the exchangeable measures on $\Lambda^{\mathbb{N}}$, where Λ is a compact metric space. More precisely, in [HS:55] it is shown that the extremal points of the (compact) convex set of all exchangeable measures are given by the product measures $\sigma^{\mathbb{N}}$, with $\sigma \in \mathcal{M}^1(\Lambda)$. As a consequence, Choquet theorem [C:69] provides an integral representation for any exchangeable measure m on $\Lambda^{\mathbb{N}}$, i.e. there is a probability measure $\mu \in \mathcal{M}^1(\Lambda)$ such that

$$m = \int_{\mathcal{M}^1(\Lambda)} \sigma^{\mathbb{N}} \, d\mu(\sigma) \,. \tag{B.12}$$

When Λ is finite, i.e. $\Lambda = p = \{0, \dots, p-1\}$ for some $p \in \mathbb{N}$, we can identify $\mathcal{M}^1(\Lambda)$ with the symplex Σ_p of all $\lambda \in [0,1]^p$ such that $\sum_{i=0}^{p-1} \lambda_i = 1$. Given $\lambda \in \Sigma_p$, we denote by B_λ the product measure on $p^{\mathbb{N}}$, namely the unique measure making all the events $\{x \,:\, x_i = a\}$ independent with measure $B_\lambda(\{x \,:\, x_i = a\}) = \lambda_a$. In this case, (B.12) becomes

$$m = \int_{\Sigma_p} B_\lambda \, d\mu(\lambda) \,, \tag{B.13}$$

where μ is a probability measure on Σ_p.

We finish this excursus on exchangeable measures with the following result:

Proposition B.12. *Let $m \in \mathcal{M}^1(\Lambda^{\mathbb{N}})$ be exchangeable, then for all $f \in L^1(\Lambda^{\mathbb{N}})$ the following conditions are equivalent:*

a) *f is $\mathfrak{S}_c(\mathbb{N})$-invariant;*
b) *f is $\mathrm{Inj}(\mathbb{N})$-invariant;*
c) *f is shift-invariant.*

Proof. Since $\mathfrak{S}_c(\mathbb{N}) \subset \mathrm{Inj}(\mathbb{N})$ and $s \in \mathrm{Inj}(\mathbb{N})$, the implications b) $\Rightarrow$ a) and b) $\Rightarrow$ c) are obvious.

In order to prove that a) $\Rightarrow$ b), we let $\mathcal{F} = \{\sigma \in \mathrm{Inj}(\mathbb{N}) \,:\, f = f \circ \sigma^*\}$, which is a closed subset of $\mathrm{Inj}(\mathbb{N})$ containing $\mathfrak{S}_c(\mathbb{N})$. Then, it is enough to observe that $\mathfrak{S}_c(\mathbb{N})$ is a dense subset of $\mathrm{Inj}(\mathbb{N}) \subset \mathbb{N}^{\mathbb{N}}$, with respect to the product topology of $\mathbb{N}^{\mathbb{N}}$, so that $\mathcal{F} = \overline{\mathfrak{S}_c(\mathbb{N})} = \mathrm{Inj}(\mathbb{N})$.

Let us prove that c) $\Rightarrow$ a). Let $\sigma \in \mathfrak{S}_c(\mathbb{N})$ and let n be such that $\sigma(i) = i$ for all $i \geq n$. It follows that ${\mathsf{S}^*}^k \circ \sigma^* = \mathsf{S}^k$, for all $k \geq n$. As a consequence, for m-almost every $x \in \Lambda^{\mathbb{N}}$ it holds

$$f \circ \sigma^*(x) = f \circ {\mathsf{S}^*}^n \circ \sigma^*(x) = f \circ {\mathsf{S}^*}^n(x) = f(x),$$

where the first equality holds since the measure m is $\mathfrak{S}_c(\mathbb{N})$-invariant. □

Notice that from Proposition B.12 it follows that $\tilde{f}$ is $\mathrm{Inj}(\mathbb{N})$-invariant for all $f \in L^1(\Lambda^{\mathbb{N}})$. In particular, for an exchangeable measure, the σ-algebra of the shift-invariant sets coincides with the (a priori smaller) σ-algebra of the $\mathrm{Inj}(\mathbb{N})$-invariant sets.

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Alessandro Berarducci, Dipartimento di Matematica, Università di Pisa, Largo B. Pontecorvo 5, 56127 Pisa, Italy, email: *berardu@dm.unipi.it*

Pietro Majer, Dipartimento di Matematica, Università di Pisa, Largo B. Pontecorvo 5, 56127 Pisa, Italy, email: *majer@dm.unipi.it*

Matteo Novaga, Dipartimento di Matematica, Università di Padova, Via Trieste 63, 35121 Padova, Italy, email: *novaga@math.unipd.it*