DATA 22/03/16

$A\bar{x} = \bar{b}$ $\quad \forall \bar{b} = 0$ FORMANO UN SOTTOSPAZIO

ES: $f: \mathbb{R}[x]^{\leq 3} \to \mathbb{R}[x]^{\leq 3}$

$$f(ax^3 + bx^2 + cx + d) = (a+b)x^3 + (b+c)x^2 + (c+d)x + (a-d) \quad *$$

$f: V \to V$ $\quad$ f È LINEARE? $\quad v_1, v_2 \in \mathbb{R}[x]^{\leq 3}$

VERIFICHIAMO $f(\bar{0}) = \bar{0}$, $f(c_1 v_1 + c_2 v_2) = c_1 f(v_1) + c_2 f(v_2)$

$$v_1 = ax^3 + bx^2 + cx + d$$

$$v_2 = a'x^3 + b'x^2 + c'x + d'$$

$$f(v_1 + v_2) = f[(a+a')x^3 + (b+b')x^2 + (c+c')x + (d+d')] =$$

RIFERITO $*$ $(a+a'+b+b')x^3 + (b+b'+c+c')x^2 + (c+c'+d+d')x + (a+a'-d-d')$

DIAMO COME INPUT $V_1 =$ $(a+b)x^3 + (b+c)x^2 + (c+d) + (a-d) +$

$(a'+b')x^3 + (b'+c')x^2 + (c'+d') + (a'-d') =$

$f(ax^3 + bx^2 + cx + d) + f(a'x^3 + b'x^2 + c'x + d') =$

$f(v_1) + f(v_2)$

OSS: RIMANE DA VERIFICARE CHE $f(cv) = c f(v)$

Scelgo COME BASE $B = (x^3, x^2, x, 1)$ di $\mathbb{R}[x]^{\leq 3}$

$\varepsilon_1 \quad \varepsilon_2 \quad \varepsilon_3 \quad \varepsilon_4$

SCRIVERE LA MATRICE DI f IN QUESTA BASE (IN PARTENZA E ARRIVO)

$\mathbb{R}[x]^{\leq 3}$ $\quad f: V \to V$

COORDINATE $\quad ax^3+bx^2+cx+d \longrightarrow \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4$

$= a\mathcal{E}_1 + b\mathcal{E}_2 + c\mathcal{E}_3 + d\mathcal{E}_4$

OSS: La MATRICE ASSOCIATA "A" ASSOCIATA a f DEVE MANDARE LE COORDINATE DELL'INPUT NELLE COORDINATE DELL'OUTPUT

QUINDI:

$$A(a,b,c,d) = \begin{pmatrix} a+b \\ b+c \\ c+d \\ a-d \end{pmatrix} \Rightarrow A\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} /$$

$$A\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \ / \ A\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \ / \ A\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$$A\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

Se APPLICASSIMO $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ RAVREMO LA 1° COLONNA

$$\Rightarrow A\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a+b \\ b+c \\ c+d \\ a-d \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + d\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$\rightsquigarrow$ STUDIARE IL Ker di f e l'Im (f)

$f: \mathbb{R}[x]^{\leq 3} \to \mathbb{R}[x]^{\leq 3} \qquad f(ax^3+bx^2+cx+d) = (a+b)x^3 + (b+c)x^2 + (c+d)x + (a-b)$

STUDIAMO: $\operatorname{Ker}(f) = \{ v \in V \mid f(v) = \bar{0} \} \Rightarrow$ STUDIARE $\operatorname{Ker}(f) := \operatorname{Ker}(A) = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} : A\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

$\operatorname{Im}(f) = \{ f(v) \mid v \in V \}$

$$\operatorname{Ker}(A) = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} : \begin{matrix} a+b=0 \\ b+c=0 \\ c+d=0 \\ a-d=0 \end{matrix} \right\} \Rightarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \xrightarrow{R_4 - R_1} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 \end{pmatrix}$$

LE MOSSE DI RIGA NON CAMBIA IL Ker

$R_4 + R_2$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -2 \end{pmatrix} \xleftarrow{R_4 - R_3} \begin{pmatrix} \cdots \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ \cdots \end{pmatrix}$$

OSS: IL Ker(A) È SOLO IL VETTORE $\bar{0}$

TEO: $f: V \to W$ lineare $\dim \operatorname{Ker}(f) + \dim \operatorname{Im}(f) = \dim(V)$

QUINDI: $\dim V = 4$ BASE $(x^3, x^2, x, 1)$ $\dim W = 4$ $\dim \operatorname{Ker}(f) = 0$

$f(ax^3 + bx^2 + cx + d) = 5x^3 + 4x^2 + 3x + 17$ $\exists a, b, c, d$? Sì perché $\dim \operatorname{Im} f =$
$\Rightarrow \dim f = \mathbb{R}[x]^{\leq 3}$

ESERCIZIO CON $\mathbb{Z}_2$

$f: \mathbb{Z}_2[x]^{\leq 3} \to \mathbb{Z}_2[x]^{\leq 3} = V$ ALLORA $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Im $\mathbb{Z}_2$ NON CAMBIA NULLA ECCETTO $\operatorname{Ker}(f) = \left\{ ax^3 + bx^2 + cx + d \;\middle|\; \begin{matrix} a + d = 0 \\ b + c = 0 \\ c + d = 0 \\ 0 = 0 \end{matrix} \right\}$

SOLUZIONI PARTICOLARI $\to \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \leftarrow$ BASE $\operatorname{Ker}(A)$

$\Downarrow$

$c = -d$

$b = -c$

$a = +d$

$\operatorname{SPAN}\left[x_0 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + y_0 \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right] \Rightarrow \operatorname{Ker}(A) = \begin{pmatrix} -y_0 \\ y_0 \\ -y_0 \\ y_0 \end{pmatrix} \Rightarrow \operatorname{Ker}(f) = \{ -dx^3 + dx^2 - dx + d \mid d \in \mathbb{Z}_2 \}$

$= \{ d(-x^3 + x^2 - x + 1) \mid d \in \mathbb{Z}_2 \}$

$= \operatorname{Span}_{\mathbb{Z}_2} \{ -x^3 + x^2 - x + 1 \}$

$\Uparrow$

N_1

STUDIAMO L'IMMAGINE $= x^3 - x^2 + 2x - 3 \in \operatorname{Im}(f)$?

$\Downarrow$

$(x^3 + x^2 + 1) \in \operatorname{Im}(f)$?

OSS: LE COORDINATE È COME CHIEDERSI SE ESISTONO $a, b, c, d \in \mathbb{Z}_2$

$A \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{\text{LA MATRICE}} \left(\begin{matrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{matrix} \middle| \begin{matrix} 1 \\ 1 \\ 0 \end{matrix} \right)$ RIDUCIAMOLA

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{R_4 - R_1} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_4 + R_2} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{R_4 + R_3} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \Rightarrow \begin{cases} a+b=1 \\ b+c=1 \\ c+d=1 \\ \underline{0=1} \end{cases}$$

NON CI SONO SOLUZIONI

x^3+x^2+1 NON È IN $\mathrm{Im}(f)$

INVECE $x^3+x^2+x+1 \in \mathrm{Im}(f)$

TROVARE TUTTI I POLINOMI IN $\mathrm{Im}(f)$

$ex^3+fx^2+gx+h \in \mathrm{Im}(f)$

ES: $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \Leftrightarrow \begin{pmatrix} a+b \\ b+c \\ c+d \\ a-d \end{pmatrix}$ OPPURE

$a\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + d\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \mathrm{Im}(A) = \mathrm{span}(\text{colonne di } A)$

$\Rightarrow \cancel{\begin{pmatrix} y_0+x_0 \\ x_0+t_0 \\ t_0+k_0 \\ y_0-k_0 \end{pmatrix}}$

$\begin{pmatrix} ? \end{pmatrix}$ LA SOLUZIONE È 3 PAGINE AVANTI.

TROVARE LA MATRICE INVERSA

$A\bar{x} = \bar{y} \iff A^{-1}\bar{y} = \bar{x}$

$$A\begin{pmatrix} \cancel{4} & 3 \\ 4 & 1 \end{pmatrix} \xrightarrow{\mathbb{Z}_3} \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \leftarrow A$$

FACCIAMO $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ $\quad A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x = b_1 \\ x+y = b_2 \end{pmatrix}$

$\Downarrow$

in $\mathbb{Z}_3$

$$\begin{cases} x = 2b_1 \\ 2b_1 + y = b_2 \end{cases} \Rightarrow \begin{cases} x = 2b_1 \\ y = b_2 - 2b_1 \end{cases}$$

$\Downarrow$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

$A^{-1}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

OSS: A CORRISPONDE PER PURO CASO AD A^{-1}

ESERCIZIO

$$\left(\begin{array}{cc|cc} 2 & 0 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{array}\right) \xrightarrow{R_2 + R_1} \left(\begin{array}{cc|cc} 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right) \xrightarrow{2R_1} \left(\begin{array}{cc|cc} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right)$$

ESERCIZIO

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 6 & 0 & 1 \end{array}\right) \xrightarrow{R_2 - 3R_1} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{array}\right)$$

NON ESISTE L'INVERSA

OSS: A MATRICE $n \cdot n$

$L_A : \mathbb{R}^n \to \mathbb{R}^n$ $\qquad$ A HA UN'INVERSA sse:

L_A È SURGETTIVA $\qquad$ sse

L_A È INIETTIVA $\qquad$ sse

$\ker(A) = 0$

OSS: Se RIDUCENDOLA A SCALINI, ABBIAMO UNO SCALINO + LUNGO $\Rightarrow$ LA MATRICE "A" NON HA UN'INVERSA

ESERCIZIO: $A\vec{x} = \vec{b} : A\vec{b} = \cancel{\vec{x}}$

$$\Downarrow$$

$$A A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} c_{il} \end{pmatrix} \quad i \leq m, \ l \leq j \qquad A = (a_{ij}) \ i \leq m \ \ j \leq k \qquad B(b_{jl}) \ j \leq k, \ l \leq s$$

$$c_{il} = \sum_j a_{ij} \cdot b_{jl}$$

ESERCIZIO:

BASE DI $\mathbb{R}^3$ $\qquad \text{Span}\left(\begin{array}{c|c|c} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 2 \end{array}\right) = \text{Span}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 2 \end{pmatrix} =$

$$\text{Span}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix} = \text{Span}\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & -1 \end{pmatrix} \neq \mathbb{R}^3$$

Riproviamo

$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix} \xrightarrow[\text{colonne}]{} \text{span}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 3 \\ 1 & -1 & 2 \end{pmatrix} = \mathbb{R}^3 ?$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \end{array}$$

TROVARE LE COORDINATE DI $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ PER VERIFICARE SE SI OTTIENE $\mathbb{R}^3$

$$\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} \cancel{0} \\ \cancel{0} \\ - \end{pmatrix} + c_3 \begin{pmatrix} \cancel{0} \\ 3 \\ 2 \end{pmatrix} \Longleftrightarrow$$

$1 = c_1 + 2c_2 +$ $\qquad$ $4 = c_1 + c_2 + 2c_3$

$2 = c_1 + 2c_2 + 3c_3$

RISCRIVIAMO IN MATRICE $\to \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 1 & 2 & 3 & 2 \\ 1 & 1 & 2 & 4 \end{array}\right]$ CON LE RIDUZIONI $\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 3 & 1 \end{array}\right]$

$\Downarrow$ GAUSS JORDAN

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & -\frac{7}{3} \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right]$$

$\Downarrow$ GAUSS JORDAN

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{17}{3} \\ 0 & 1 & 0 & -\frac{7}{3} \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right]$$

RISOLVIAMO IL SISTEMA:

$$\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \frac{17}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{7}{3}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 \\ 3 \\ 8 \end{pmatrix}$$

LE COORDINATE DI $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ RISPETTO ALLA BASE $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$

SONO $\begin{pmatrix} \frac{17}{3} \\ -\frac{7}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ STANDARD

SOLUZIONE 3 PAGINE PRECEDENTI

$$f(ax^3+bx^2+cx+d) = (a+b)x^3+(b+c)x^2+(c+d)x+(a-d)$$

BASE $x^3, x^2, x, 1$ MATRICE $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}$

$ex^3+fx^2+gx+h \in Im(f)$? $\exists\, abcd$ t.c. $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \Big($

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & e \\ 0 & 1 & 1 & 0 & f \\ 0 & 0 & 1 & 1 & g \\ 1 & 0 & 0 & -1 & h \end{array}\right] \xRightarrow{\text{RIDOTTA}} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & e \\ 0 & 1 & 1 & 0 & f \\ 0 & 0 & 1 & 1 & g \\ 0 & 0 & 0 & 0 & +h-e+f+g \end{array}\right]$$

SOLUZIONE: d libera

$c = g - d$

$b = f - c = f + g + d$

$a = e - b = e - f + g + d$

Quindi l'immagine di $f: \mathbb{Z}_2[x]^{\leq 3} \to \mathbb{Z}_2[x]^{\leq 3}$ *

* è data dai polinomi $ex^3 + fx^2 + gx + h$ tali che

$e + f + g + h = 0$

Es: $x^3 + x^2 + x + h \in \mathrm{Im}(f) \Rightarrow 1+1+1+1=0$ in $\mathbb{Z}_2$

Base di $\mathrm{Im}(f) = \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} \in \mathrm{Im}(A) \Leftrightarrow e+f+g+h=0 \Leftrightarrow \underbrace{[1,1,1,1]}_{1\cdot 4}\underbrace{\begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix}}_{4\cdot 1} = \underbrace{[0]}_{1\cdot 1}$

Quindi: $\begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} \in \mathrm{Ker}([1,1,1,1])$

f, g, h LIBERE

$e = -f - g - h$

I vettori in $\mathrm{Im}(A)$ $\begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix}$ sono della forma $\begin{pmatrix} -f+g-h \\ f \\ g \\ h \end{pmatrix} =$

$= f\underset{v_1}{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}} + g\underset{v_2}{\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}} + h\underset{v_3}{\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}}$ = ~~sono~~ ~~una~~ BASE DI $\mathrm{Im}(A)$

$\mathrm{Im}(f) = \mathrm{span}(-x^3+x^2, -x^3+x, -x^3+1) \Rightarrow \mathrm{Im}(f) = f(x^3+x^2) + g(x^3+x)$
$+ h(x^3+1)$