On the dimension of bounded hyperdefinable sets

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Outline

A new look at:

1. Torsion elements in definably compact groups [EO04]
2. Pillay's conjectures [BOPP05, HPP08]
3. Compact domination [HP11]

Achille-Ber., A Vietoris-Smale mapping theorem for the homotopy of hyperdefinable sets, ArXiv 1706.02094 (2017), 1--24.

Smooth measures

- Given a structure M and a definable set X, a **Keisler measure** $\mu \in \mathfrak{M}_X(M)$ is a finitely additive real valued probability measure μ on M-definable subsets of X.
- μ is called **smooth** if it has a unique extension $\nu \in \mathfrak{M}_X(N)$ for any $N \succ M$.
- This is equivalent to say that every N-definable subset Y of X can be approximated by M-definable sets, namely for every $\varepsilon > 0$ there are M-definable Y_1, Y_2 with $Y_1 \subseteq Y \subseteq Y_2$ and $\mu(Y_1 \setminus Y_2) \leq \varepsilon$ [Sim15, Lemma 7.8]

Boundaries

- $\mu \in \mathfrak{M}_X(M)$ extends to a σ-additive Borel measure on the space of types $S_X(M)$.
- Given $Y \subseteq X$ definable with parameters in the moster model $N \succ M$, the boundary of Y is the set of types $p \in S_X(M)$ which have realizations a, b with $a \in Y$ and $b \notin Y$.
- $\mu \in \mathfrak{M}_X(M)$ is smooth iff the boundary ∂Y of every N-definable set $Y \subseteq X$ has measure zero [Sim15].

Jordan measure

- Given an o-minimal expansion M of a field, there is a (unique) translation-invariant finitely additive measure μ on $\mathbb{Q}$-bounded definable sets $X \subseteq M^n$ normalizing the unite cube [BO04].
- Over the reals it coincide with the Jordan measure (the restriction of the Lebesgue measure to sets whose boundary has Lebesgue measure zero), so we call μ Jordan measure (even in the non-standard context).

The torus

- The Jordan measure μ induces a smooth left invariant Keisler measure on the torus $\mathbb{T}^n(M)$, defined as $[0,1)^n$ with addition modulo 1.
- If M is sufficiently saturated, the standard part map $\mathrm{st} : \mathbb{T}^n(M) \to \mathbb{T}^n(\mathbb{R})$ is surjective and for every definable set $X \subseteq \mathbb{T}^n$ the intersection $\mathrm{st}(X) \cap \mathrm{st}(X^{\complement}) \subseteq \mathbb{T}^n(\mathbb{R})$ has Haar measure zero [BO04, Cor 4.4].
- The μ measure of $X \subseteq \mathbb{T}^n(M)$ coincides with the Haar-measure of $\mathrm{st}(X) \subseteq \mathbb{T}^n(\mathbb{R})$.

Compact domination

- For every definable group G, the infinitesimal subgroup G^{00} exists and G/G^{00} with the logic topology is a real Lie group [BOPP05].
- One says that G is **compactly dominated** if for every definable set $X \subseteq G$, $\mathrm{p}(X) \cap \mathrm{p}(X^{\complement})$ has Haar measure zero, where $\mathrm{p} : G \to G/G^{00}$ is the projection.
- With this terminology the results in [BO04] say that $\mathbb{T}^n$ has a smooth measure and it is compactly dominated ($\mathrm{p} : \mathbb{T}^n(M) \to \mathbb{T}^n(M)/\mathbb{T}^n(M)^{00}$ can be identified with $\mathrm{st} : \mathbb{T}^n(M) \to \mathbb{T}^n(\mathbb{R})$)
- In NIP context, G admits a smooth left-invariant measure $\iff$ G is compactly dominated [Sim15, 8.38, 8.41].

o-minimal case

- The following are equivalent:
  - G has a smooth left-invariant measure
  - G is compactly dominated
  - G is definably compact
  - G is fsg.

  [HPP08, Thm. 8.1] (definably compact $\to$ fsg), [Sim14] (fsg$\to$smooth), [Sim15] (smooth$\leftrightarrow$compact domination).

- **fsg** does not require a topology: it says that if a definable set $X \subseteq G$ is syndetic (i.e. finitely many tranlates of X cover G), then X has points with coordinates in any small model; we also require that there is a type containing only syndetic sets.

- fsg fails for the additive group $(M, +)$ of a sufficiently saturated real closed field $(M, +, \cdot, <)$.

Amenable groups

- The mere existence of a left-invariant Keisler measure (not necessarily smooth), does not imply the the group is compactly dominated, for instance all the definable abelian groups have such measures (being amenable).

From compact domination to Pillay's conjectures

- Let X be a definable set in o-minimal M and let $E \subseteq X \times X$ be a type-definable equivalence relation of bounded index. Put on X/E the logic topology.
- Under suitable assumptions, including a form of compact domination, we show that $\dim(X) = \dim_{\mathbb{R}}(X/E)$.
- In particular $\dim(G) = \dim_{\mathbb{R}}(G/G^{00})$.
- The form of compact domination that we need is the following: for every definable $Y \subseteq X$, the set $\mathrm{p}(Y) \cap \mathrm{p}(Y^{\complement}) \subseteq X/E$ has empty interior.
- We also assume that X/E is a triangulable topological space and every E-equivalence class is a decreasing intersection of definable proper balls. These are technical conditions which in the case of definable groups are implied by compact domination.

Homotopy

- First we prove that there is a natural isomorphism $\pi_n^{\text{def}}(X) \cong \pi_n(X/E)$.
- Given a definable continuous map $f : S^n(M) \to X$ (between pointed spaces), its homotopy class $[f]$ is an element of $\pi_n^{\text{def}}(X)$, and we need to map it to $[f^*] \in \pi_n(X/E)$ for some continuous $f^* : S^n(\mathbb{R}) \to X/E$.
- How do we choose f^*?

Approximations

- Let $\mathcal{U}$ be good open cover of X/E. We say that f^* is a $\mathcal{U}$-approximation of f, if the following diagram commutes "up to $\mathcal{U}$",

$$\begin{tikzcd} S^n(M) \arrow[r, "\mathrm{st}"] \arrow[d, "f"'] & S^n(\mathbb{R}) \arrow[d, "f^*"] \\ X \arrow[r, "\mathrm{p}"'] & X/E \end{tikzcd}$$

  in the sense that for every $x \in S^n(M)$ there is $U \in \mathcal{U}$ such that $(f^* \circ \mathrm{st})(x) \in U$ and $(\mathrm{p} \circ f)(x) \in U$.
- Main idea: we cannot ensure that every f has a $\mathcal{U}$-approximation f^*, but we can prove that every f is definably homotopic to a map g which has a $\mathcal{U}$-approximation g^*. This suffices to define a natural map $\pi_n^{\mathrm{def}}(X) \to \pi_n(X/E)$ and we can prove that it is an isomorphism.
- The same argument yields $\pi_n^{\mathrm{def}}(\mathrm{p}^{-1}(U)) \cong \pi_n(U)$ for any open $U \subseteq X/E$ (homotopy transfer).

Dimension

- To obtain $\dim(X) = \dim_{\mathbb{R}}(X/E)$ we use $\pi_n^{\mathrm{def}}(\mathrm{p}^{-1}(U)) \cong \pi_n(U)$ for a suitable $U \subseteq X/E$ (an open ball B with a hole $B_1 \subseteq B$ in it).
- The idea is to use the following link between homotopy and dimension: given a ball B and a concentric ball B_1, the dimension of B is the least $i \in \mathbb{N}$ such that $\pi_{i-1}(B \setminus B_1) \neq 0$.
- we need to be careful since homotopy alone cannot detect the dimension of a topological space: $Y \times \mathbb{R}$ and Y are homotopy equivalent.
- We use compact domination to ensure that if a definable set $Y \subseteq X$ of a certain dimension fills a "hole" in $\mathrm{p}^{-1}(U)$, its image in X/E cannot fill an open hole, unless $\dim(Y) = \dim(X)$.

Torsion

- The proof of $\dim(X) = \dim_{\mathbb{R}}(X/E)$ is thus completed, modulo some cheating, including the fact that the preimage in X of an open subset of X/E is not definable, so we need to approximate it with definable sets.
- As a special case of $\dim(X) = \dim_{\mathbb{R}}(X/E)$ we get $\dim(G) = \dim_{\mathbb{R}}(G/G^{00})$ for any definably compact group.
- If G is abelian G and G/G^{00} have the same torsion and we deduce that the k-torsion subgroup of G is isomorphic to $(\mathbb{Z}/k\mathbb{Z})^d$ where $d = \dim(G)$.

O-minimal vs classical homotopy

- Given a closed and bounded $\emptyset$-semialgebraic set X, consider the standard part $\mathrm{st} : X(M) \to X(\mathbb{R})$.
- We can view $X(\mathbb{R})$ as X/E where $E = \ker(\mathrm{st})$, so by the main result we obtain an isomorphism

  $$\pi_n^{\mathrm{def}}(X(M)) \cong \pi_n(X(\mathbb{R})),$$

  yielding a new proof of results Delfs-Knebush, Ber.-Otero, Baro-Otero.
- Conclusion: we have a general result about $X \to X/E$ specializing to $G \to G/G^{00}$ and $X \to X(\mathbb{R})$.

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