Esercizio 1. (A)

a) f è di classe C^1. Il suo gradiente è

$$\nabla f = \left(\frac{1+3y^2-3x^2-6xy}{(1+3x^2+3y^2)^2}, \frac{1+3x^2-3y^2-6xy}{(1+3x^2+3y^2)^2} \right)$$

Si ha $\quad \nabla f = (0,0) \iff$

$$\begin{cases} 1+3y^2-3x^2-6xy = 0 \\ 1+3x^2-3y^2-6xy = 0 \end{cases} \iff$$

$$\begin{cases} 6x^2-6y^2 = 0 \\ 1+3x^2-3y^2-6xy = 0 \end{cases} \iff$$

$$\begin{cases} y = \pm x \\ 1 \mp 6x^2 = 0 \end{cases} \iff$$

$$\begin{cases} y = x \\ x^2 = 1/6 \end{cases} \iff \qquad x = y = \pm 1/\sqrt{6}$$

da cui $\quad P = (1/\sqrt{6}, 1/\sqrt{6}) \in \mathring{B}(1)$ è l'unico punto stazionario interno. Si ha $\quad f(P) = 1/\sqrt{6}$.

Inoltre, con riferimento alla figura 1, $\partial B(1) = OA \cup \widehat{AB} \cup BO$. Da cui

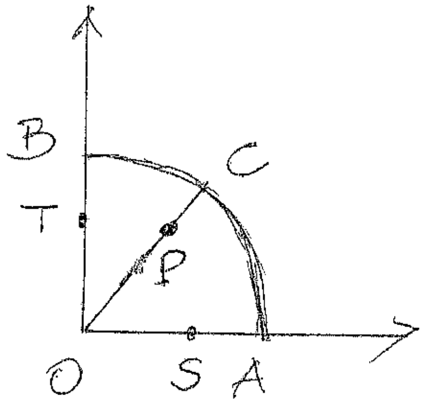


figura 1

$$f|_{OA} = \frac{x}{1+3x^2}$$

$$\frac{d}{dx}\left(\frac{x}{1+3x^2}\right) = \frac{1-3x^2}{(1+3x^2)^2} =$$

$$= 0 \iff x = 1/\sqrt{3} \in]0,1[$$

con $f(1/\sqrt{3}, 0) = f(S) = 1/2\sqrt{3}$

e, analogamente

$$f|_{BO} = \frac{y}{1+3y^2}, \quad \frac{d}{dy}\left(\frac{y}{1+3y^2}\right) = 0 \iff$$

$y = 1/\sqrt{3} \in]0,1[$, con

$$f(0, 1/\sqrt{3}) = f(T) = 1/2\sqrt{3}$$

Si ha inoltre $f(O) = f(0,0) = 0$,

$f(A) = f(1,0) = f(B) = f(0,1) = 1/4$.

Infine

$$f|_{\overset{\frown}{AB}} = \frac{\cos\theta + \sin\theta}{4} \quad (\theta \in]0, \pi/2[)$$

$$\frac{d}{d\theta}(\cos\theta + \sin\theta) = \cos\theta - \sin\theta = 0 \iff$$

$\theta = \pi/4$ e $f|_{\overset{\frown}{AB}}(\pi/4) = f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) =$

$= 1/2\sqrt{2} = f(C)$. Confrontando

$1/\sqrt{6}$, $1/2\sqrt{3}$, $1/4$, $1/2\sqrt{2}$, 0, si vede che, per il teo di Weierstrass,

$$1/\sqrt{6} = \max_{B(1)} f \qquad e \qquad 0 = \min_{B(1)} f$$

(infatti: $\frac{1}{\sqrt{6}} > 1/2\sqrt{2} \Leftrightarrow 1/6 > 1/8$,

$\frac{1}{2\sqrt{2}} > \frac{1}{2\sqrt{3}} \Leftrightarrow 1/2 > 1/3$, $\frac{1}{2\sqrt{3}} > 1/4$

$\Leftrightarrow 2 > \sqrt{3} \Leftrightarrow 4 > 3$).

(Valutazione 6/30).

b) Passando a coordinate polari

$$\iint_{B(R)} f(x,y)\,dx\,dy = \int_0^{\pi/2} (\cos\theta + \sin\theta)\,d\theta \cdot \int_0^R \frac{\rho^2}{1+3\rho^2}\,d\rho = 2\int_0^R \frac{\rho^2}{1+3\rho^2}\,d\rho$$

poiché $\int_0^{\pi/2} (\cos\theta + \sin\theta)\,d\theta = \left[\sin\theta - \cos\theta\right]_0^{\pi/2} = 2$

Si ha poi

$$\int_0^R \frac{\rho^2}{1+3\rho^2}\,d\rho = \frac{1}{3}\int_0^R \frac{3\rho^2+1-1}{1+3\rho^2}\,d\rho = \frac{1}{3}\int_0^R d\rho +$$

$$-\frac{1}{3}\int_0^R \frac{d\rho}{1+3\rho^2} = \frac{R}{3} - \frac{1}{3\sqrt{3}}\operatorname{arctg}(\sqrt{3}R)$$

da cui

$$\iint_{B(R)} f(x,y)\,dx\,dy = \frac{2}{3}\left(R - \frac{1}{\sqrt{3}}\operatorname{arctg}(\sqrt{3}R)\right).$$

(<u>Valutazione</u> 4/30).

c) Per definizione e usando il punto b)

$$\iint_B f(x,y)\,dx\,dy = \lim_{R\to+\infty} \iint_{B(R)} f(x,y)\,dx\,dy =$$

$$= \frac{2}{3}\lim_{R\to+\infty}\left(R - \frac{1}{\sqrt{3}}\operatorname{arctg}(\sqrt{3}R)\right) =$$

$$= +\infty$$

poiché $1/\sqrt{3}\,\operatorname{arctg}(\sqrt{3}R) \xrightarrow[R\to+\infty]{} \frac{\pi}{2\sqrt{3}}$.

Pertanto l'integrale improprio

$$\iint_B f(x,y)\,dx\,dy$$

non converge.

(<u>Valutazione</u> 3/30)

Esercizio 3

A

(a) $\mathrm{rot}(\vec{F}) = (x-x)\vec{i} + (y-y)\vec{j} + (z-z)\vec{k} = \underline{0}$

(b) $\vec{F}$ è un campo irrotazionale definito su un dominio semplicemente connesso (tutto $\mathbb{R}^2$) $\Rightarrow$ $\vec{F}$ è conservativo

cioè $\exists\, U:\mathbb{R}^3 \to \mathbb{R}$ tale che

$$\frac{\partial U}{\partial x}(x,y,z) = yz - 2 \qquad (1)$$

$$\frac{\partial U}{\partial y}(x,y,z) = xz + 1 \qquad (2)$$

$$\frac{\partial U}{\partial z}(x,y,z) = xy \qquad (3)$$

Integrando l'equazione (3) rispetto a z otteniamo

$$U(x,y,z) = xyz + u(x,y) \qquad (4)$$

Derivando (4) rispetto ad x ed y e confrontando il risultato con le equazioni (1) e (2) otteniamo:

$$\frac{\partial u}{\partial x}(x,y) = -2, \qquad \frac{\partial u}{\partial y}(x,y) = 1;$$

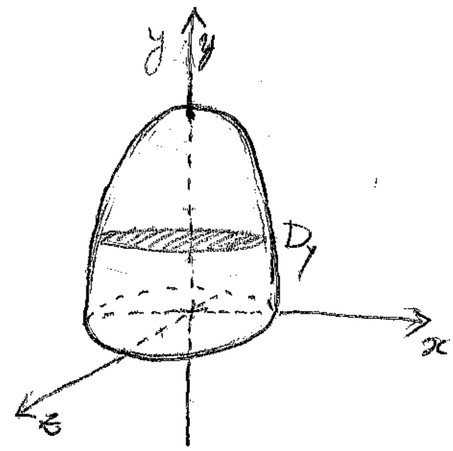
da cui $u(x,y) = -2x + y + c$ e $\boxed{U(x,y,z) = xyz - 2x + y + c}$

Pertanto $\boxed{\int_\gamma \vec{F}\cdot d\vec{\gamma} = U(\gamma(\pi)) - U(\gamma(0)) = 4}$

Esercizio 2

(a)

D è l'interno di un tronco di paraboloide di asse y



$\partial D = \Sigma_0 \cup \Sigma_1$

coperchio

calotta

con

$\Sigma_0 = \{(x,y,z): \ y=0, \ \ 0 \le 4 - x^2 - z^2\}$

$\Sigma_1 = \{(x,y,z): \ y = 4 - x^2 - z^2, \ \ y \ge 0\}$

$$\text{Area}(\partial D) = \text{Area}(\Sigma_0) + \text{Area}(\Sigma_0)$$

$\text{Area}(\Sigma_0) = 4\pi$ $\qquad$ $\text{Area}(\Sigma_1) = \iint_{\{x^2+z^2 \le 2\}} \sqrt{1 + |\nabla f|^2}\, dx\, dz$ $\quad$ con $f(x,z) := 4 - x^2 - z^2$

$$= \iint_{\{x^2+z^2 \le 2\}} \sqrt{1 + 4x^2 + 4z^2}\, dx\, dz = \int_0^{2\pi}\int_0^2 \sqrt{1+4\rho^2}\,\rho\, d\rho\, d\theta = 2\pi \left[\frac{1}{12}(1+4\rho^2)^{3/2}\right]_0^2 = \frac{\pi}{6}\left[(17)^{3/2} - 1\right]$$

Dunque $\boxed{\text{Area}(\partial D) = 4\pi + \frac{\pi}{6}\left[(17)^{3/2} - 1\right]}$

(b) $\boxed{\iint_{\partial D} \vec{F}\cdot \hat{n}_e \, dS} = \iiint_D \text{div}\, \vec{F}\, dx\,dy\,dz$. $\qquad$ $\text{div}\,\vec{F} = 2$

$$= 2\, \text{Vol}(D) = 2\int_0^4 \left(\iint_{D_y} dx\,dz\right) dy = 2\int_0^4 \pi(4-y)\,dy$$

integrazione per strati; $D_y := \{(x,z): \ x^2 + z^2 \le 4 - y\}$

$$\boxed{= 16\pi.}$$

Esercizio 1. (B)

a) $$\nabla f = \left(\frac{1+2y^2-2x^2-4xy}{(1+2x^2+2y^2)^2} \,,\ \frac{1+2x^2-2y^2-4xy}{(1+2x^2+2y^2)^2} \right)$$

$$\nabla f = (0,0) \Longleftrightarrow \begin{cases} 4x^2-4y^2=0 \\ 1+2x^2-2y^2-4xy=0 \end{cases} \Longleftrightarrow$$

$$\begin{cases} y=x \\ x^2=1/4 \end{cases} \Longleftrightarrow \quad x=y=\pm 1/2 \Rightarrow P=(1/2,1/2)$$

$f(P)=1/2$. Con riferimento alla figura 2,

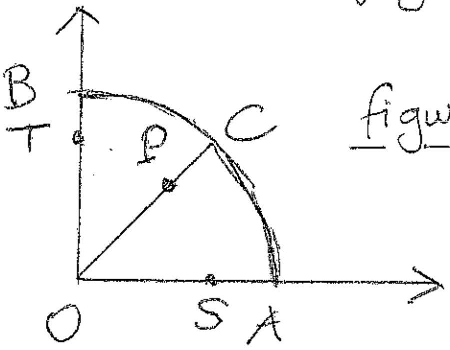


figura 2

$$f_{|OA} = \frac{x}{1+2x^2}$$

$d\left(\frac{x}{1+2x^2}\right)=0 \Longleftrightarrow x=1/\sqrt{2} \in$
$\in \,]0,1[$ e $f_{|BO}=\dfrac{y}{1+2y^2}$ con $\dfrac{d}{dy}\left(\dfrac{y}{1+2y^2}\right)=0$
$\Longleftrightarrow y=1/\sqrt{2} \in \,]0,1[$. Si ha

$$f(1/\sqrt{2},0)=f(S)=\frac{1}{2\sqrt{2}}=f(0,1/\sqrt{2})=f(T)$$

$$f(O)=f(0,0)=0 \quad f(A)=f(1,0)=f(B)=$$
$=f(0,1)=1/3$. Infine

$$f_{|AB} = \frac{\cos\theta+\sin\theta}{3} \qquad \left(\theta \in \left]0,\pi/2\right[\right)$$

e

$$\frac{d}{d\theta}(\cos\theta + \sin\theta) = 0 \iff \theta = \pi/4 \Rightarrow$$

$$C = \left(1/\sqrt{2},\ 1/\sqrt{2}\right) \quad e \quad f(C) = 2/3\sqrt{2}.$$

Poiché

$$\frac{1}{3} < \frac{1}{2\sqrt{2}} \iff 9 > 8$$

$$\frac{1}{2\sqrt{2}} < \frac{2}{3\sqrt{2}} \iff \frac{1}{2} < \frac{2}{3}$$

$$\frac{2}{3\sqrt{2}} < \frac{1}{2} \iff 16 < 18$$

se ne conclude

$$\max_{B(1)} f = \frac{1}{2} \qquad \min_{B(1)} f = 0$$

(Valutazione 6/30).

b) Passando a coordinate polari

$$\iint_{B(R)} f(x,y)\,dx\,dy = \int_0^{\pi/2} (\cos\theta + \sin\theta)\,d\theta \cdot \int_0^R \frac{\rho^2}{1+2\rho^2}\,d\rho = 2\int_0^R \frac{\rho^2}{1+2\rho^2}\,d\rho =$$

$$= \int_0^R \left(\frac{2\rho^2+1}{2\rho^2+1} - \frac{1}{2\rho^2+1}\right) d\rho = R - \int_0^R \frac{d\rho}{1+2\rho^2} =$$

$$= R - \frac{1}{\sqrt{2}} \operatorname{arctg}(\sqrt{2}R).$$

(Valutazione 4/30)

c) Per definizione e per il punto b) si ha

$$\iint_B f(x,y)\,dx\,dy = \lim_{R\to+\infty} \iint_{B(R)} f(x,y)\,dx\,dy =$$

$$= \lim_{R\to+\infty}\left(R - \frac{1}{\sqrt{2}}\operatorname{arctg}(\sqrt{2}R)\right) =$$

$$= +\infty .$$

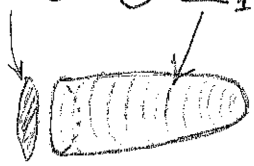
L'integrale improprio non converge

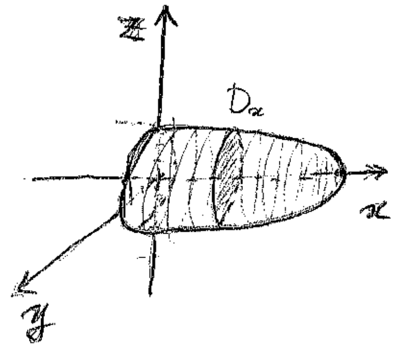
(Valutazione: 3/30).

Esercizio 2 $\boxed{B}$

(a) L'insieme D è ~~un~~ l'interno di un tronco di paraboloide di asse x

$$\partial D = \overset{\text{tappo}}{\Sigma_0} \cup \overset{\text{tronco di paraboloide}}{\Sigma_1}$$





con

$$\Sigma_0 := \{(x,y,z) : \ x = 0, \ \ y^2+z^2 \leq 9\}$$

$$\Sigma_1 := \{(x,y,z) : \ 0 \leq x, \ \ x = 9 - y^2 - z^2\}$$

$$\text{Area}(\partial D) = \text{Area}(\Sigma_0) + \text{Area}(\Sigma_1)$$

$$\boxed{\text{Area}(\Sigma_1)} = \iint_{\{y^2+z^2\leq 9\}} \sqrt{1+|\nabla f|^2}\, dy\,dz \qquad \boxed{\text{Area}(\Sigma_0) = 9\pi}$$

con $f(y,z) = 9 - y^2 - z^2$

$$= \iint_{\{y^2+z^2\leq 9\}} \sqrt{1+4y^2+4z^2}\, dy\,dz = \int_0^{2\pi}\int_0^3 \sqrt{1+4\rho^2}\,\rho\, d\rho\, d\theta = 2\pi\left[\frac{1}{12}(1+4\rho^2)^{3/2}\right]_0^3 = \boxed{\frac{\pi}{6}\left[37^{3/2}-1\right]}$$

Pertanto $\boxed{\text{Area}(\partial D) = 9\pi + \frac{\pi}{6}\left[37^{3/2}-1\right]}$

(b) $$\iint_{\partial D} \vec{F}\cdot\vec{n}_e\, dS = \iiint_D \text{div}\,\vec{F}\, dx\,dy\,dz \qquad \text{div}\,F = 2$$

$$= 2\,\text{Vol}(D) = 2\int_0^9 \left(\iint_{D_x} dy\,dz\right) dx = 2\int_0^9 \pi(9-x)\,dx =$$

integrazione per strati

$$D_x := \{(y,z) : \ y^2+z^2 \leq 9-x\}$$

$$= 81\pi$$

B

Esercizio 3

(a) $rot(\vec{F}) = (x-x)\vec{i} + (y-y)\vec{j} + (z-z)\vec{k} = \underline{0}$

(b) $\vec{F}$ è un campo irrotazionale definito su un dominio semplicemente connesso, quindi è conservativo. Pertanto $\exists\, U: \mathbb{R}^3 \to \mathbb{R}$ tale che

$$\frac{\partial U}{\partial x}(x,y,z) = yz \qquad (1)$$

$$\frac{\partial U}{\partial y}(x,y,z) = xz + 1 \qquad (2)$$

$$\frac{\partial U}{\partial z}(x,y,z) = xy - 1 \qquad (3)$$

Integrando l'equazione (3) rispetto alla variabile z si ha

$$U(x,y,z) = xyz - z + u(x,y) \qquad (4)$$

Derivando (4) rispetto ad x ed y e confrontando il risultato con le equazioni (1) e (2) otteniamo:

$$\frac{\partial u}{\partial x}(x,y) = 0 \;, \qquad \frac{\partial u}{\partial y}(x,y) = 2 \;;$$

da cui $u(x,y) = 2y + \text{cte}$ e $U(x,y,z) = xyz - z + 2y + c$

Pertanto

$$\int_\gamma \vec{F}\cdot d\vec{\gamma} = U(\gamma(\pi)) - U(\gamma(0)) = +2$$