A canonical thickening of $\mathbb{Q}$ and the dynamics of continued fractions

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Joint work with Giulio Tiozzo (Harvard PhD student)
Credits

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the images are generated from numerical data computed by Alessandro Profeti for a previous numerical study
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C.C., S.Marmi, A.Profeti, G.Tiozzo: The entropy of alpha-contined fractions: numerical results
arXiv:0912.2379v1 [math.DS]
Regular continued fraction (RCF) expansions and the Gauss map

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where $p_n/q_n$ is the $n$-th convergent of $x$ and $h(T)$ is the entropy of $T$. 
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- $h(T) = \frac{\pi^2}{6 \log 2}$
The maps $T_\alpha$

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$T_\alpha$ for $\alpha = (\sqrt{5} - 1)/2$
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$T_\alpha$ for $\alpha = 1/2$
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$T_\alpha$ for $\alpha = \sqrt{2} - 1$
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$T_\alpha$ for $0 < \alpha << 1$
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$T_\alpha$ for $\alpha = 0$
--continued fractions and the maps $T_\alpha$

The maps $T_\alpha : [\alpha - 1, \alpha] \to [\alpha - 1, \alpha]$ are defined as follows:
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Other features

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Ergodic properties of $T_\alpha$

The maps $T_\alpha$ ($\alpha > 0$) have the following properties

- $T_\alpha$ has an invariant probability measure $\mu_\alpha(x) := \rho(x)dx$
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- The entropy $h(T_\alpha)$ can be computed using Rohlin formula:

$$h(T_\alpha) = \int_{\alpha-1}^{\alpha} \log |T_\alpha'(x)|d\mu_\alpha(x);$$
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- $h(T_\alpha) =$?
An historical account

An historical account

A. Cassa: *Dinamiche caotiche e misure invarianti* (1995) Tesi di Laurea (numerical results)
An historical account

Zooming in
Zooming in
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Question 1.
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Is the entropy really not monotone?
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Is the entropy really not monotone? Yes!
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Matching leads to monotonic behaviour (Thm. 2 in [NN])

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Then $h$ is monotone on $I$; more precisely

- $h$ is strictly increasing on $I$ if $k_1 < k_2$;
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iii. \( h \) is strictly decreasing on \( I \) if \( k_1 > k_2 \).
Some interesting issues
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- each of the cases (i), (ii) and (iii) takes place at least on one infinite family of disjoint matching intervals clustering at the origin ([NN], Thm. 3);
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◮ the matching conditions define a collection of open intervals (called matching intervals);
◮ the entropy is thus a non-monotonic function;
◮ **conjecture:** the union of all matching intervals is a dense, open subset of \([0, 1]\) with **full Lebesgue measure**.
Hierarchy of quadratic intervals
The set $\mathcal{M}$

$$\mathcal{M} = \bigcup_{a \in \mathbb{Q} \cap ]0,1]} I_a.$$ 

► $\mathcal{M}$ is an open neighbourhood of $\mathbb{Q} \cap ]0,1]$;
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- $\mathcal{M}$ is an open neighbourhood of $\mathbb{Q} \cap ]0,1]$;
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- the connected components of $\mathcal{M}$ are quadratic intervals;
- $|\mathcal{M}| = 1$;
- $\dim_H([0,1] \setminus \mathcal{M}) = 1$;
- the entropy function $\alpha \mapsto h(T_\alpha)$ is monotone on each connected component of $\mathcal{M}$. 
Taking off quadratic intervals
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Comparison with the critical case of (a,b)-continued fractions
Comparison with the critical case of (a,b)-continued fractions

![Graph showing comparison with critical case of (a,b)-continued fractions](zoom1.dat)
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Probably not [CMPT].
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See [CT]
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Who knows!?
The end