Structures preserved by matrix inversion

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   - Definition
   - Examples

3 Rank structures
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   - Generally positioned rank structures
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Rank structures

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Inversion of rank structures: some extensions

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Introduction

Suppose given a matrix $A$.

Which structures of $A$ carry over to the inverse matrix $A^{-1}$?

- Displacement structures
  - Toeplitz-like and Hankel-like
  - Cauchy-like and Vandermonde-like.
  - circulant
- Rank structures
  - upper triangular
  - Hessenberg and (lower) semiseparable
  - (lower) semiseparable plus diagonal.
- Some other structures
  - Hermitian
  - unitary
  - normal.

$\Rightarrow$ We will handle these structures from a general point of view.
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Displacement structure

- Stein type displacement

\[ FAG + A = \operatorname{Rk} r \]

implies

\[ A^{-1} + GA^{-1}F = \operatorname{Rk} r. \]

- Sylvester type displacement

\[ FA + AG = \operatorname{Rk} r \]

implies

\[ A^{-1}F + GA^{-1} = \operatorname{Rk} r. \]

**Proof.** Multiply by \( A^{-1} \) on the left and by \( A^{-1} \) on the right.

- Straightforward generalizations:
  - adding a constant and a quadratic term.
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  \[ AEA + FA + AG + H = \text{Rk } r \]

  implies

  \[ E + A^{-1}F + GA^{-1} + A^{-1}HA^{-1} = \text{Rk } r. \]

**Proof.** Multiply by \( A^{-1} \) on the left and by \( A^{-1} \) on the right.

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  - ‘decoupling’ the variable \( A \) into \( A \) and \( B \).
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\[ FAG + B = \text{Rk } r \]

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\[ BEA + FA + BG + H = \text{Rk } r \]

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**Proof.** Multiply by \( B^{-1} \) on the left and by \( A^{-1} \) on the right.

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Why should we ‘decouple’?

Illustration:

**Corollary**

Let $A$ and $B$ be nonsingular matrices satisfying

$$A - B = \text{Rk} \; r,$$

then

$$A^{-1} - B^{-1} = \text{Rk} \; r.$$

Examples:

- $B = \text{Herm}$: Hermitian plus low rank.
- $B = \text{Uni}$: unitary plus low rank.
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Example: Hermitian plus low rank

We supposed Herm to be nonsingular.

What if Herm is singular?

**Theorem**

The following are equivalent:

(i) \( A = \text{Herm} + \text{Rk} \ r; \)

(ii) \( A - A^H = \text{Rk} \ 2r, \) where \( \text{Inertia}(\sqrt{-1}\text{Rk} \ 2r) = (\pi, \nu, \zeta) \) with \( \max\{\pi, \nu\} \leq r. \)

**Corollary**

The structure \( A = \text{Herm} + \text{Rk} \ r \) is always preserved under matrix inversion.
Example: Hermitian plus low rank

We supposed \texttt{Herm} to be nonsingular. What if \texttt{Herm} is singular?

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**Definition**

We define a rank structure on $\mathbb{C}^{n \times n}$ as a collection $\{B_k\}_k$ where each $B_k$ is a ‘structure block’.

$$B_k = (i_k, j_k, r_k, \Lambda_k) :$$

- $i_k$: row index,
- $j_k$: column index,
- $r_k$: rank upper bound,
- $\Lambda_k \in \mathbb{C}^{(j_k-i_k+1) \times (j_k-i_k+1)}$: shift matrix.

We say a matrix $A \in \mathbb{C}^{n \times n}$ to satisfy $B_k$ if, after subtracting $\Lambda_k$ as illustrated, we have $\text{Rank } A(i_k : n, 1 : j_k) \leq r_k$. 
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A structure block $B_k = (i_k, j_k, r_k, \Lambda_k)$ is called pure if either

- $\Lambda_k = 0$,
- $i_k - j_k + 1 \leq 0$, i.e. the structure block does not intersect the diagonal.

Notation: $B_{\text{pure}, k}$.

Here are two examples of pure structure blocks $B_{\text{pure}, k}$:

- Left: $i_k, j_k, R_k = 6$
- Right: $i_k, j_k, R_k = 3$
Definition

(Continuation)

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![Example 1](image1)

![Example 2](image2)
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![Diagram](image-url)
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Here is an example of pure structure $\{B_{\text{pure, } k}\}_{k=1}^{n}$, yielding the class of lower semiseparable matrices:

Allowing shift matrices $\lambda_k \in \mathbb{C}^{1 \times 1}$, we get the structure $\{B_{k}\}_{k=1}^{n}$ of lower semiseparable...
Here is an example of pure structure $\{B_{\text{pure},k}\}_{k=1}^{n}$, yielding the class of lower semiseparable matrices:

Allowing shift matrices $\lambda_k \in \mathbb{C}^{1 \times 1}$, we get the structure $\{B_k\}_{k=1}^{n}$ of lower semiseparable plus diagonal matrices.
Here is an example of pure structure $\{B_{\text{pure},k}\}_{k=1}^n$, yielding the class of lower semiseparable matrices:

Allowing shift matrices $\lambda_k \in \mathbb{C}^{1 \times 1}$, we get the structure $\{B_k\}_{k=1}^n$ of lower semiseparable \textit{plus diagonal} matrices.

The diagonal correction $\text{diag}(\lambda_k)_{k=1}^n$ is part of the structure.
Example

Here is an example of pure structure $\{B_{\text{pure},k}\}_{k=1}^{n}$, yielding the class of lower semiseparable matrices:

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Examples of rank structures

- upper triangular matrices (including diagonal elements)
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- Hessenberg matrices
- lower semiseparable matrices (+symmetry: semiseparable)
- lower semiseparable plus diagonal (+symmetry: semiseparable plus diagonal)
- Higher semiseparability ranks
- Also ‘poorly ordered’ structures are possible
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Theorem

Let $A \in \mathbb{C}^{n \times n}$ be a nonsingular matrix satisfying the structure block $B = (i, j, r, \Lambda)$, with $\Lambda$ nonsingular. Then the inverse matrix satisfies the structure block $B^{-1} := (i, j, r, \Lambda^{-1})$.

**Proof.** We have $A - B = \text{Rk } r$ with $B = \begin{bmatrix} X & X & X \\ 0 & \Lambda & X \\ 0 & 0 & X \end{bmatrix}$. Therefore, $A^{-1} - B^{-1} = \text{Rk } r$. □
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Structures preserved by matrix inversion
Inversion of rank structures

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\[
A = \begin{bmatrix}
    & & & Rk 6 \\
    & & \Lambda & \\
    & \Lambda & & \\
    Rk 6 & & & \\
\end{bmatrix} \quad \longrightarrow \quad A^{-1} = \begin{bmatrix}
    & & & Rk 6 \\
    & & \Lambda^{-1} & \\
    & \Lambda^{-1} & & \\
    Rk 6 & & & \\
\end{bmatrix}
\]

**PROOF.** We have \( A - B = Rk \ r \) with \( B = \begin{bmatrix} X & X & X \\ 0 & \Lambda & X \\ 0 & 0 & X \end{bmatrix} \).

\[ A^{-1} - B^{-1} = Rk \ r. \]
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Until now we assumed that $\Lambda^{-1}$ exists. What happens if $\Lambda$ is singular?

Singular value decomposition

$\Lambda = U \Sigma V^H$.

Behaviour of $U, V$ under inversion is easy to handle.

$\Rightarrow$ sufficient to define the structure block $B = (i, j, r, \Sigma^{-1})$.

We suppose from now on that $\Lambda = \Sigma$ is diagonal.
Until now we assumed that $\Lambda^{-1}$ exists. What happens if $\Lambda$ is singular?

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- We suppose from now on that $\Lambda = \Sigma$ is diagonal.
Shift elements $\infty$

Let $\Lambda$ be diagonal, with some diagonal entries $\lambda_i = 0$. Corresponding diagonal entries of $\Lambda^{-1}$ are $\frac{1}{\lambda_i} = \infty$.

**Definition**

We define $B = (i, j, r, \Lambda)$, having $\Lambda = \text{diag}(\lambda_i)$ with $\lambda_i \in \mathbb{C} \cup \{\infty\}$, as follows. We identify $B$ with the ‘structure block’ obtained by dropping all rows and columns involving $\infty$, and with $r$ decreased by the number of these dropped rows.

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**Definition**

We define $B = (i, j, r, \Lambda)$, having $\Lambda = \text{diag}(\lambda_l)$ with $\lambda_l \in \mathbb{C} \cup \{\infty\}$, as follows. We identify $B$ with the ‘structure block’ obtained by dropping all rows and columns involving $\infty$, and with $r$ decreased by the number of these dropped rows.
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Of course, we must motivate our definition for \( \lambda_l = \infty \).

**Theorem**

1. Our definition for \( \lambda_l = \infty \) was made in a ‘continuous’ way;
2. For \( B = (i, j, r, \Lambda) \) having \( \Lambda = \text{diag}(\lambda_l)_l \) with \( \lambda_l \in \mathbb{C} \cup \{\infty\} \), \( B \) can be written as the ‘limit’ of a family \( B_\epsilon \) with each \( \Lambda_\epsilon \) having entries in \( \mathbb{C} \).

⇒ Problems involving \( \lambda_l = \infty \) reduce to finite problems.

Example: Hessenberg matrices are limits of lower ss+d matrices:
Shift elements $\infty$

Of course, we must motivate our definition for ‘$\lambda_l = \infty$’.

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$\implies$ Problems involving $\lambda_l = \infty$ reduce to *finite* problems.

**Example:** Hessenberg matrices are limits of lower ss+1 matrices:
Of course, we must motivate our definition for ‘$\lambda_l = \infty$’.

**Theorem**

1. **Our definition for $\lambda_l = \infty$ was made in a ‘continuous’ way;**
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Example: Hessenberg matrices are limits of lower ss+d matrices:
Theorem (General inversion theorem)

Let $B = (i, j, r, \Lambda)$ where $\Lambda$ is diagonal with diagonal entries $\lambda_i \in \mathbb{C} \cup \{\infty\}$. Then $B^{-1}$ is precisely the structure block $(i, j, r, \Lambda^{-1})$, using the rules $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$.

Example: Similarly: Hessenberg and lower semiseparable structure.
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Example:

\[
\begin{array}{c}
\begin{array}{c}
A = \\
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
\text{Rk 6}
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
\rightarrow
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
A^{-1} =
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
\text{Rk 6}
\end{array}
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**Theorem (General inversion theorem)**

Let \( B = (i, j, r, \Lambda) \) where \( \Lambda \) is diagonal with diagonal entries \( \lambda_l \in \mathbb{C} \cup \{\infty\} \). Then \( B^{-1} \) is precisely the structure block \((i, j, r, \Lambda^{-1})\), using the rules \( \frac{1}{0} = \infty \) and \( \frac{1}{\infty} = 0 \).

Example:

\[
A = \begin{bmatrix}
& & & & & Rk 6 \\
& & & & \\
& & & \\
& & \\
& \\
\end{bmatrix} \quad \rightarrow \quad A^{-1} = \begin{bmatrix}
& & & & & Rk 2 \\
& & & & \\
& & & \\
& & \\
& \\
\end{bmatrix}
\]

Similarly: Hessenberg and lower semiseparable structure.
Example: arrowhead matrix

Let $A$ be diagonal plus rank 1, with diagonal correction $\Lambda$.

- $\Lambda$ nonsingular: then $A^{-1}$ is again diagonal plus rank 1, with diagonal correction $\Lambda^{-1}$.
- $\Lambda$ singular, say $\lambda_n = 0$. Then $\frac{1}{\lambda_n} = \infty$:

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$$A = \begin{bmatrix} x & x & x & x & 0 \\ x & x & x & x \\ x & x \\ x \\ 0 \end{bmatrix} \quad \rightarrow \quad A^{-1} = \begin{bmatrix} x & x & x & x & \infty \\ x & x & x \\ x \\ x \\ \infty \end{bmatrix}$$

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$A = \begin{bmatrix} * & * & \cdots & * \\ * & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & * & * \end{bmatrix}$  $\Rightarrow A^{-1}$ is arrowhead.
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\[
A = \begin{bmatrix}
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
0 & 0 & 0 & 0 \\
\end{bmatrix} \quad \Rightarrow \quad A^{-1} = \begin{bmatrix}
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\end{bmatrix}
\]

$A^{-1}$ is arrowhead.
Example

Given a matrix \( A = \begin{bmatrix}
A_{1,1} & A_{1,2} \\
A_{2,1} & \Lambda
\end{bmatrix} \) with \( \Lambda \in \mathbb{C}^{k \times k} \).

Then

\[
A^{-1} - \begin{bmatrix}
0 & 0 \\
0 & \Lambda^{-1}
\end{bmatrix} = \text{Rk } n - k.
\]

Idea: Connection with the Schur complement formula.

\[
(A^{-1})_{2,2} = S := A_{2,2} - A_{2,1}A_{1,1}^{-1}A_{1,2}.
\]
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Connection with the Schur complement formula

\[ (A^{-1})_{2,2} = S^{-1} \quad \text{with} \quad S := A_{2,2} - A_{2,1}A_{1,1}^{-1}A_{1,2}. \]
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Example: unitary plus low rank

General inversion theorem relates $A$ and $A^{-1}$.
What if $A = \text{Uni} + \text{Rk } r$?
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Example: unitary plus low rank

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What if $A = \text{Uni} + \text{Rk } r$?
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$\Rightarrow$ General inversion theorem relates $A$ with itself.

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Example:

$$
\text{Uni} = \\
\text{Rk } 2
$$
Example: unitary plus low rank

General inversion theorem relates $A$ and $A^{-1}$. What if $A = \text{Uni} + \text{Rk } r$? Then $A^{-1} - A^H = \text{Rk } 2r$. 

$\Rightarrow$ General inversion theorem relates $A$ with itself.

Example:

\[
\text{Uni } = \begin{bmatrix}
\infty & & & & \\
& \infty & & & \\
& & \infty & & \\
& & & \infty & \\
& & & & \infty
\end{bmatrix}
\]

\[
\begin{bmatrix}
\text{Rk } 6 \\
& \infty & & \\
& & \infty & & \\
& & & \infty & \\
& & & & \infty
\end{bmatrix}
\]

\[
\begin{bmatrix}
& & & & \\
& & & & \\
& & \infty & & \\
& & & \infty & \\
& & & & \infty
\end{bmatrix}
\]

\[
\begin{bmatrix}
& & & & \\
& & & & \\
& & \infty & & \\
& & & \infty & \\
& & & & \infty
\end{bmatrix}
\]

\[
\begin{bmatrix}
& & & & \\
& & & & \\
& & \infty & & \\
& & & \infty & \\
& & & & \infty
\end{bmatrix}
\]

\[
\begin{bmatrix}
& & & & \\
& & & & \\
& & \infty & & \\
& & & \infty & \\
& & & & \infty
\end{bmatrix}
\]
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Then $A^{-1} - A^H = \text{Rk } 2r$.

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Example:

$$
\text{Uni} = \\
\begin{array}{c|c}
\text{Rk 6} & \\
\hline
\text{Rk 2} & \\
\end{array}
$$
Introduction

Displacement structures

Rank structures

Inversion of rank structures: some extensions

Singular shift matrices

Generally positioned rank structures

Outline

1. Introduction
   - Structures preserved by matrix inversion

2. Displacement structures
   - Definition
   - Examples

3. Rank structures
   - Definition
   - Examples

4. Inversion of rank structures: some extensions
   - Singular shift matrices
   - Generally positioned rank structures
Generally positioned rank structures

For all $P_1, P_2$,

$$B := P_1^{-1}AP_2 \quad \Rightarrow \quad B^{-1} = P_2^{-1}A^{-1}P_1.$$  

Choosing $P_1$ and $P_2$ permutation matrices: structure blocks can be moved to a general matrix position.

Here are some illustrations:
Generally positioned rank structures

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Choosing $P_1$ and $P_2$ permutation matrices: structure blocks can be moved to a **general matrix position**.

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Here are some illustrations:
For all $P_1, P_2$,

$$B := P_1^{-1}AP_2 \implies B^{-1} = P_2^{-1}A^{-1}P_1.$$ 

Choosing $P_1$ and $P_2$ permutation matrices: structure blocks can be moved to a general matrix position.

Here are some illustrations:

$$A = \begin{bmatrix}
\Lambda & Rk 4 \\
\end{bmatrix} \quad \begin{bmatrix}
\Lambda \\
Rk 4 \\
\end{bmatrix}
$$

$$A^{-1} = \begin{bmatrix}
\Lambda & Rk 4 \\
\end{bmatrix} \quad \begin{bmatrix}
\Lambda \\
Rk 4 \\
\end{bmatrix}$$
Generally positioned rank structures

For all $P_1, P_2$,

$$B := P_1^{-1} A P_2 \implies B^{-1} = P_2^{-1} A^{-1} P_1.$$  

Choosing $P_1$ and $P_2$ permutation matrices: structure blocks can be moved to a general matrix position.

Here are some illustrations:

\[ A = \begin{bmatrix}
\Lambda & 0 \\
0 & \Lambda
\end{bmatrix} \]

\[ A^{-1} = \begin{bmatrix}
\Lambda^{-1} & 0 \\
0 & \Lambda^{-1}
\end{bmatrix} \]
Nullity theorem

What happens in the limiting case $\Lambda \to 0$ or $\Lambda \to \infty$?

Theorem (Fiedler)

Let $A \in \mathbb{C}^{n \times n}$ be nonsingular. Then for all index sets $I$ and $J$,

$$\text{Null } A^{-1}(I, J) = \text{Null } A(N \setminus J, N \setminus I).$$

(1)

where $N := \{1, 2, \ldots, n\}$. 
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\[ A = \begin{bmatrix} Rk & 4 \\ \hline I & \end{bmatrix} \]

\[ A^{-1} = \begin{bmatrix} \text{Rk} & 4 \\ \hline N \setminus I & \end{bmatrix} \]
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\[ A = \begin{bmatrix} \text{Rk 4} \\
\end{bmatrix}
\]

\[ A^{-1} = \begin{bmatrix} \text{Rk 2} \\
\end{bmatrix}
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(1)

where $N := \{1, 2, \ldots, n\}$. 

\[ A = \begin{bmatrix} Rk 4 & & & \hline & Rk 4 & & \\ & & & \\ & & & \\
\end{bmatrix}, \quad A^{-1} = \begin{bmatrix} Rk 2 & & & \hline & & & \\ & & & \\ & & & \\
\end{bmatrix} \]
Nullity theorem

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(1)

where $N := \{1, 2, \ldots, n\}$. 

\[ A = \begin{pmatrix} 1 & \text{Rk 4} \\ \text{Rk 4} \\ \text{Rk 2} \end{pmatrix} \quad A^{-1} = \begin{pmatrix} \text{Rk 2} \\ \text{Rk 2} \end{pmatrix} \]
Nullity theorem

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(1)

where $N := \{1, 2, \ldots, n\}$. 

\[ A = \begin{bmatrix}
1 & & & & \\
 & Rk 4 & & & \\
 & & Rk 4 & & \\
 & & & Rk 2 & \\
& N \setminus I & & N \setminus J & \\
N \setminus J & & & & \\
& & & Rk 2 & \\
& & & & Rk 4 \\
\end{bmatrix} \] 

\[ A^{-1} = \begin{bmatrix}
 & & & & \\
 & & & & \\
 & & & & \\
 & & & & \\
& & & Rk 2 & \\
& & N \setminus I & & \\
& & & & N \setminus J \\
& & N \setminus I & & \\
& & & & \\
\end{bmatrix} \]