Orthogonal similarity reduction of any symmetric matrix into a diagonal-plus-semiseparable one with free choice of the diagonal

Ellen Van Camp, Raf Vandebril, Marc Van Barel
and Nicola Mastronardi

I. Algorithms

Orthogonal similarity transformation of any symmetric matrix into

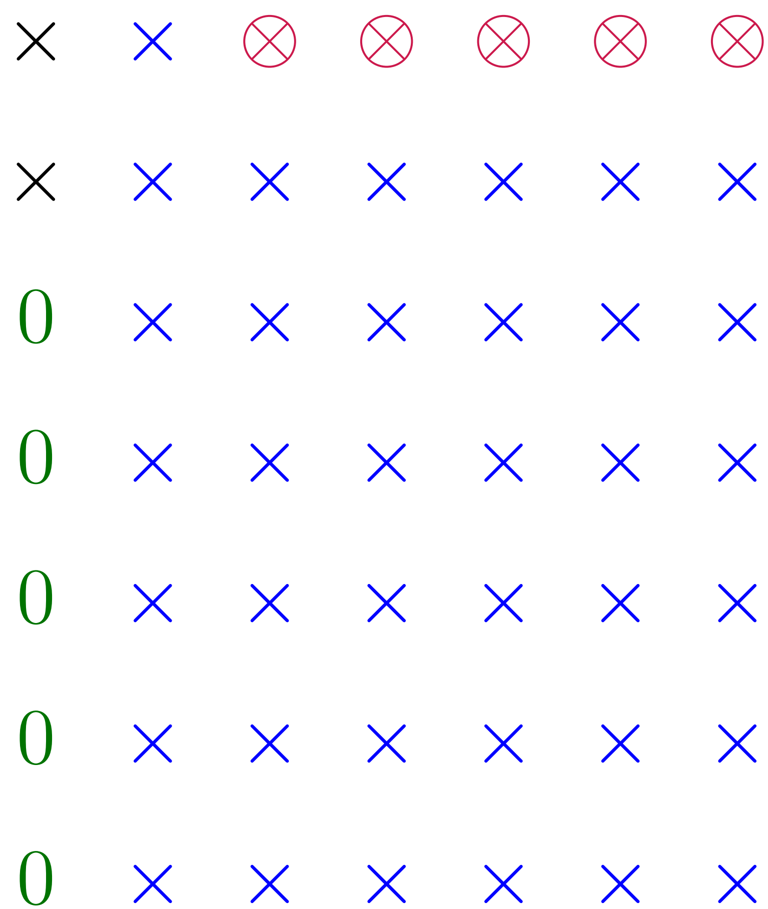
1. **tridiagonal form (Golub, Van Loan)**
2. **semiseparable form (Vandebril, Van Barel, Mastronardi)**
3. **diagonal-plus-semiseparable form with free choice of the diagonal (2 algorithms)**

1. Reduction to tridiagonal form

Any symmetric matrix can be transformed into a tridiagonal one by means of orthogonal similarity transformations in order $\mathbf{O}(\frac{4}{3}\mathbf{n}^3)$.

Definition

$$\begin{matrix} \times & \times & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 \\ 0 & \times & \times & \times & 0 \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{matrix}$$


0
0
0
0
0

$$
\begin{matrix}
\times & \times & 0 & 0 & 0 & 0 & 0 \\
\times & \times & \times & \times & \times & \times & \times \\
0 & \times & \times & \times & \times & \times & \times \\
0 & \times & \times & \times & \times & \times & \times \\
0 & \times & \times & \times & \times & \times & \times \\
0 & \times & \times & \times & \times & \times & \times \\
0 & \times & \times & \times & \times & \times & \times
\end{matrix}
$$

$$
\begin{matrix}
\times & \times & 0 & 0 & 0 & 0 & 0 \\
\times & \times & \times & \times & \times & \times & \times \\
0 & \times & \times & \times & \times & \times & \times \\
0 & \otimes & \times & \times & \times & \times & \times \\
0 & \otimes & \times & \times & \times & \times & \times \\
0 & \otimes & \times & \times & \times & \times & \times \\
0 & \otimes & \times & \times & \times & \times & \times
\end{matrix}
$$

$$
\begin{matrix}
\times & \times & 0 & 0 & 0 & 0 & 0 \\
\times & \times & \times & \otimes & \otimes & \otimes & \otimes \\
0 & \times & \times & \times & \times & \times & \times \\
0 & 0 & \times & \times & \times & \times & \times \\
0 & 0 & \times & \times & \times & \times & \times \\
0 & 0 & \times & \times & \times & \times & \times \\
0 & 0 & \times & \times & \times & \times & \times
\end{matrix}
$$

$$
\begin{matrix}
\times & \times & 0 & 0 & 0 & 0 & 0 \\
\times & \times & \times & 0 & 0 & 0 & 0 \\
0 & \times & \times & \times & \times & \times & \times \\
0 & 0 & \times & \times & \times & \times & \times \\
0 & 0 & \times & \times & \times & \times & \times \\
0 & 0 & \times & \times & \times & \times & \times \\
0 & 0 & \times & \times & \times & \times & \times
\end{matrix}
$$

$$
\begin{matrix}
\times & \times & 0 & 0 & 0 & 0 & 0 \\
\times & \times & \times & 0 & 0 & 0 & 0 \\
0 & \times & \times & \times & \times & \times & \times \\
0 & 0 & \times & \times & \times & \times & \times \\
0 & 0 & \otimes & \times & \times & \times & \times \\
0 & 0 & \otimes & \times & \times & \times & \times \\
0 & 0 & \otimes & \times & \times & \times & \times
\end{matrix}
$$

$$
\begin{matrix}
\times & \times & 0 & 0 & 0 & 0 & 0 \\
\times & \times & \times & 0 & 0 & 0 & 0 \\
0 & \times & \times & \times & \otimes & \otimes & \otimes \\
0 & 0 & \times & \times & \times & \times & \times \\
0 & 0 & 0 & \times & \times & \times & \times \\
0 & 0 & 0 & \times & \times & \times & \times \\
0 & 0 & 0 & \times & \times & \times & \times
\end{matrix}
$$

$$
\begin{matrix}
\times & \times & 0 & 0 & 0 & 0 & 0 \\
\times & \times & \times & 0 & 0 & 0 & 0 \\
0 & \times & \times & \times & 0 & 0 & 0 \\
0 & 0 & \times & \times & \times & \times & \times \\
0 & 0 & 0 & \times & \times & \times & \times \\
0 & 0 & 0 & \times & \times & \times & \times \\
0 & 0 & 0 & \times & \times & \times & \times
\end{matrix}
$$

$$
\begin{matrix}
\times & \times & 0 & 0 & 0 & 0 & 0 \\
\times & \times & \times & 0 & 0 & 0 & 0 \\
0 & \times & \times & \times & 0 & 0 & 0 \\
0 & 0 & \times & \times & \times & \times & \times \\
0 & 0 & 0 & \times & \times & \times & \times \\
0 & 0 & 0 & \otimes & \times & \times & \times \\
0 & 0 & 0 & \otimes & \times & \times & \times
\end{matrix}
$$

$$
\begin{array}{ccccccc}
\times & \times & 0 & 0 & 0 & 0 & 0 \\
\times & \times & \times & 0 & 0 & 0 & 0 \\
0 & \times & \times & \times & 0 & 0 & 0 \\
0 & 0 & \times & \times & \times & \otimes & \otimes \\
0 & 0 & 0 & \times & \times & \times & \times \\
0 & 0 & 0 & 0 & \times & \times & \times \\
0 & 0 & 0 & 0 & \times & \times & \times
\end{array}
$$

$$
\begin{matrix}
\times & \times & 0 & 0 & 0 & 0 & 0 \\
\times & \times & \times & 0 & 0 & 0 & 0 \\
0 & \times & \times & \times & 0 & 0 & 0 \\
0 & 0 & \times & \times & \times & 0 & 0 \\
0 & 0 & 0 & \times & \times & \times & \times \\
0 & 0 & 0 & 0 & \times & \times & \times \\
0 & 0 & 0 & 0 & \times & \times & \times
\end{matrix}
$$

$$
\begin{array}{ccccccc}
\times & \times & 0 & 0 & 0 & 0 & 0 \\
\times & \times & \times & 0 & 0 & 0 & 0 \\
0 & \times & \times & \times & 0 & 0 & 0 \\
0 & 0 & \times & \times & \times & 0 & 0 \\
0 & 0 & 0 & \times & \times & \times & \times \\
0 & 0 & 0 & 0 & \times & \times & \times \\
0 & 0 & 0 & 0 & \otimes & \times & \times
\end{array}
$$

$$
\begin{matrix}
\times & \times & 0 & 0 & 0 & 0 & 0 \\
\times & \times & \times & 0 & 0 & 0 & 0 \\
0 & \times & \times & \times & 0 & 0 & 0 \\
0 & 0 & \times & \times & \times & 0 & 0 \\
0 & 0 & 0 & \times & \times & \times & \otimes \\
0 & 0 & 0 & 0 & \times & \times & \times \\
0 & 0 & 0 & 0 & 0 & \times & \times
\end{matrix}
$$

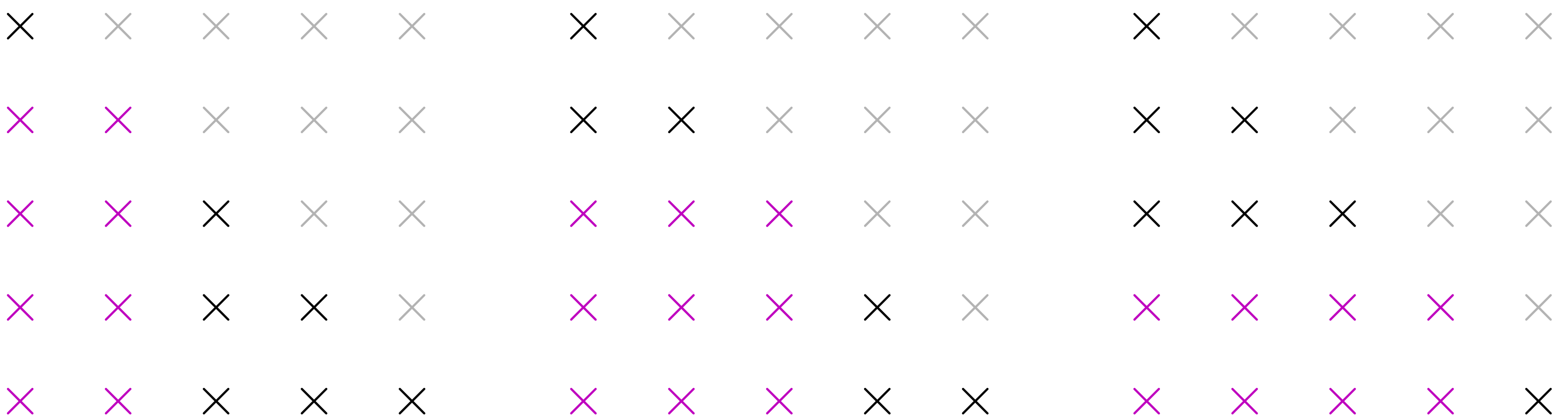
$$
\begin{array}{ccccccc}
\times & \times & 0 & 0 & 0 & 0 & 0 \\
\times & \times & \times & 0 & 0 & 0 & 0 \\
0 & \times & \times & \times & 0 & 0 & 0 \\
0 & 0 & \times & \times & \times & 0 & 0 \\
0 & 0 & 0 & \times & \times & \times & 0 \\
0 & 0 & 0 & 0 & \times & \times & \times \\
0 & 0 & 0 & 0 & 0 & \times & \times
\end{array}
$$

2. Reduction to semiseparable form

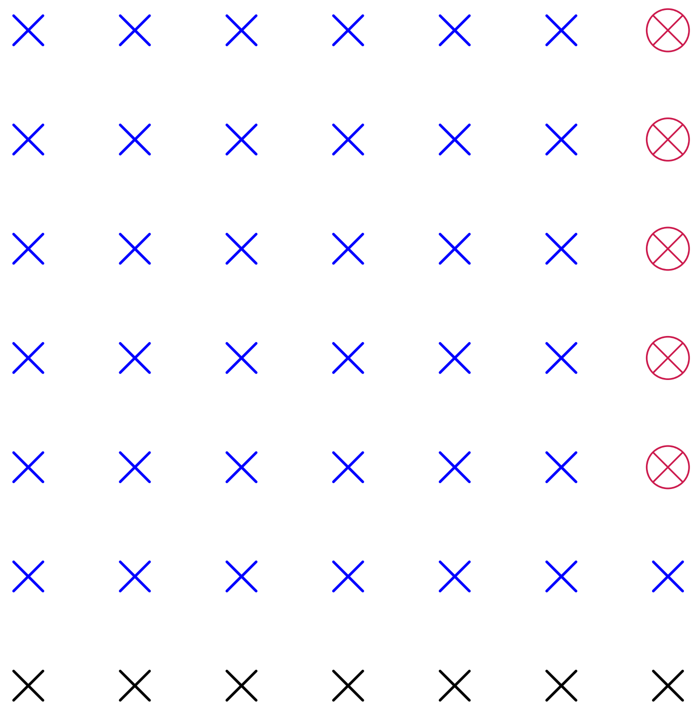
Any symmetric matrix can be transformed into a semiseparable one by means of orthogonal similarity transformations in order $O(\frac{4}{3}n^3)$.

Definition

When every submatrix that can be taken out of the lower-, resp. upper-, triangular part of a symmetric matrix has rank at most 1, this matrix is called a semiseparable matrix.



Step 1




0
0
0
0
0

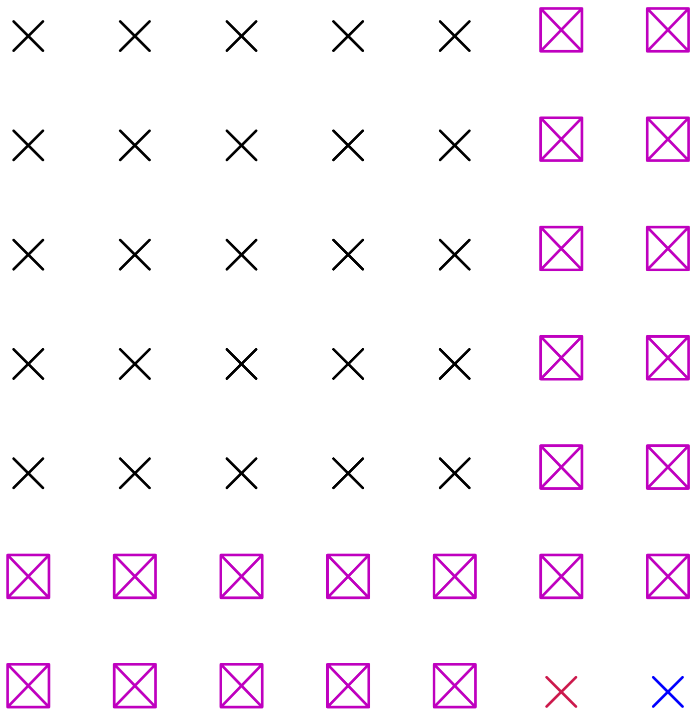
$$
\begin{matrix}
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & \times & \times \\
0 & 0 & 0 & 0 & 0 & \times & \times
\end{matrix}
$$

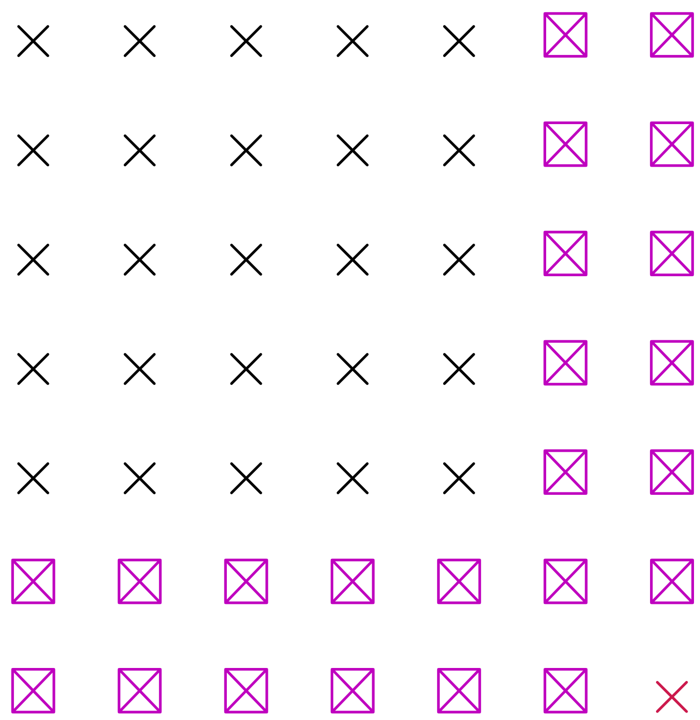
$$
\begin{matrix}
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & \times & \otimes \\
0 & 0 & 0 & 0 & 0 & \times & \times
\end{matrix}
$$

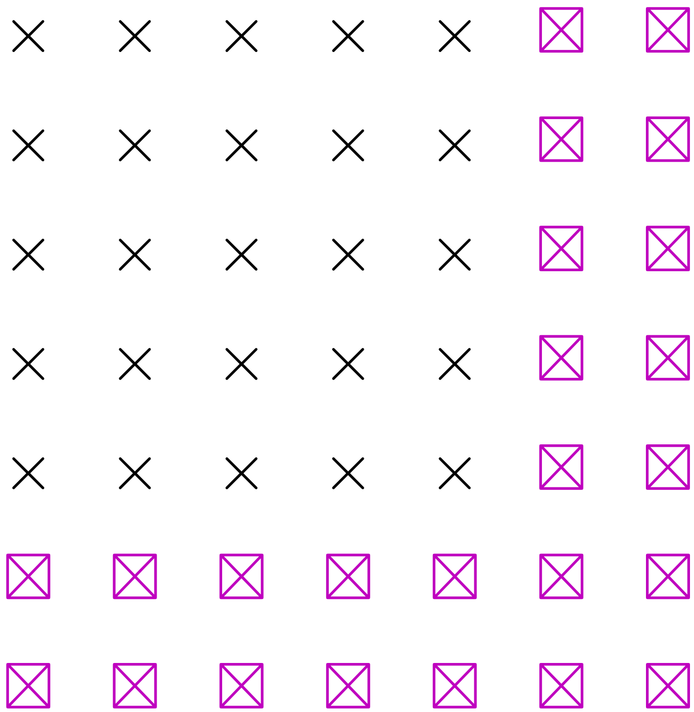
$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{pmatrix} \times \\ 0 \end{pmatrix} = \begin{pmatrix} c\times \\ -s\times \end{pmatrix}$$

$$= \begin{pmatrix} \boxtimes \\ \boxtimes \end{pmatrix}$$

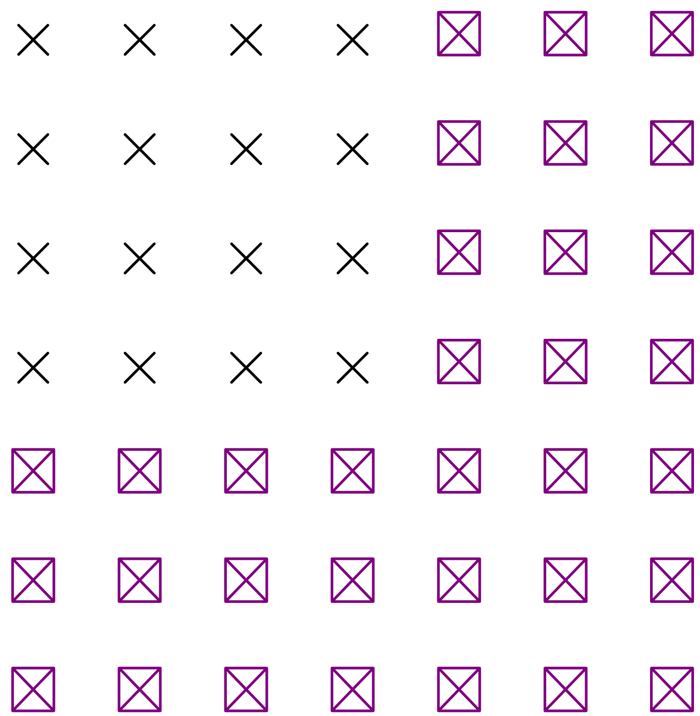
$$
\begin{array}{ccccccc}
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & \times & 0 \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \times & 0 \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \times & \times
\end{array}
$$

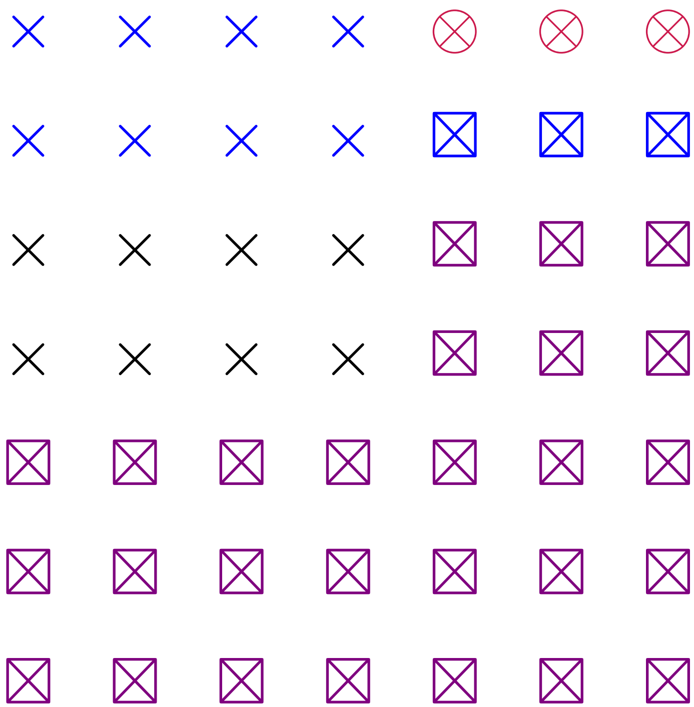


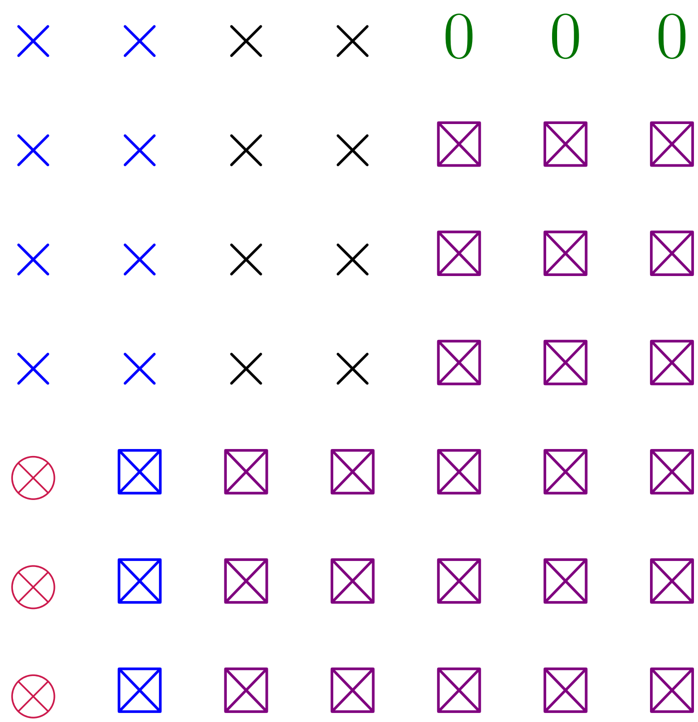




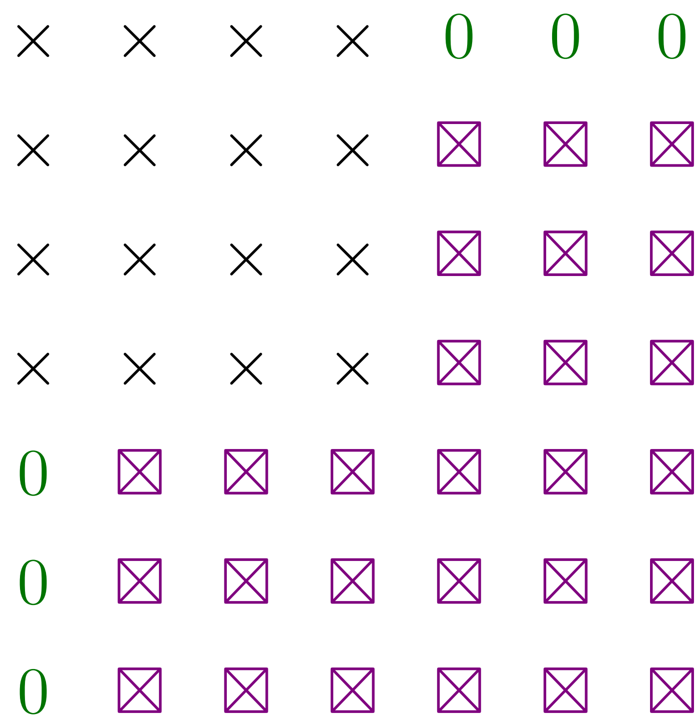
Step 3

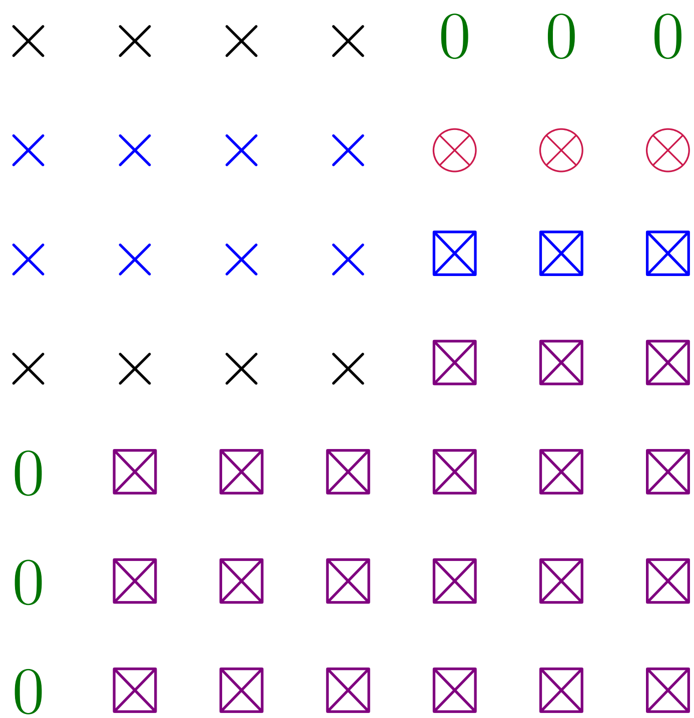


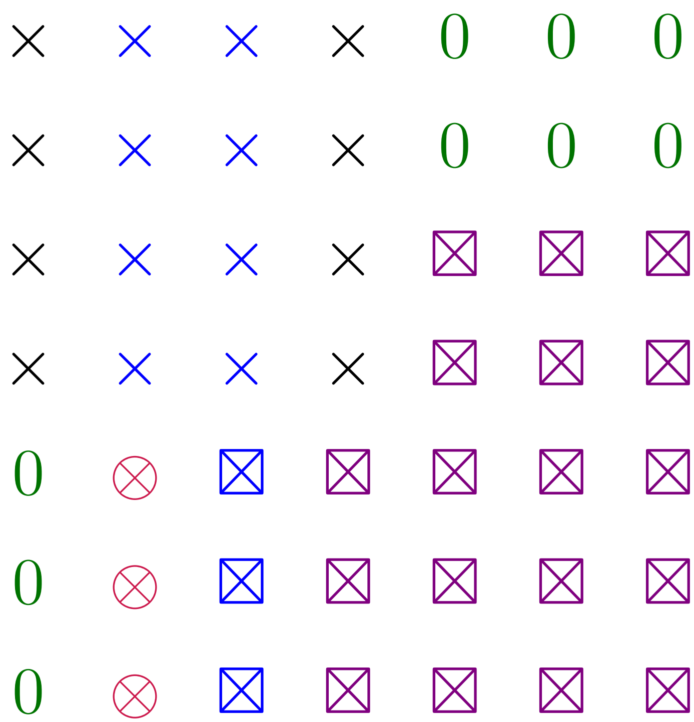




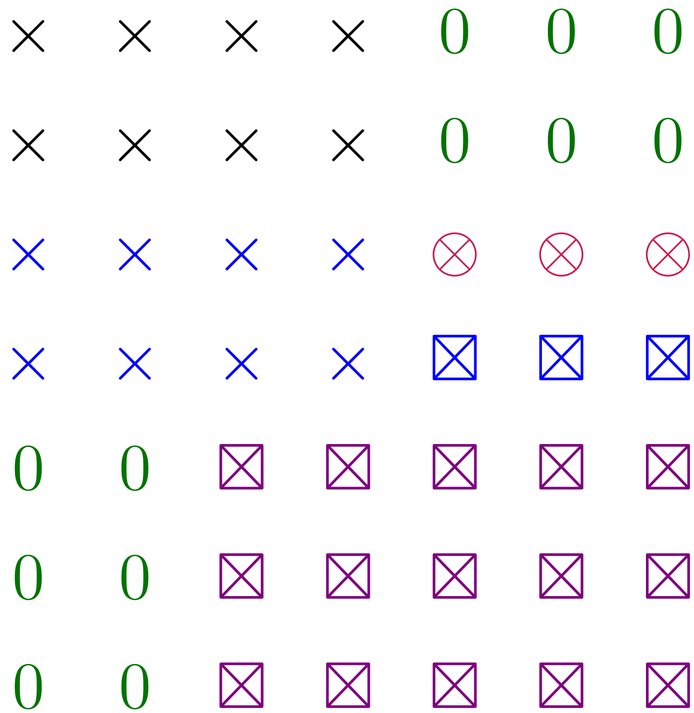
0 0 0

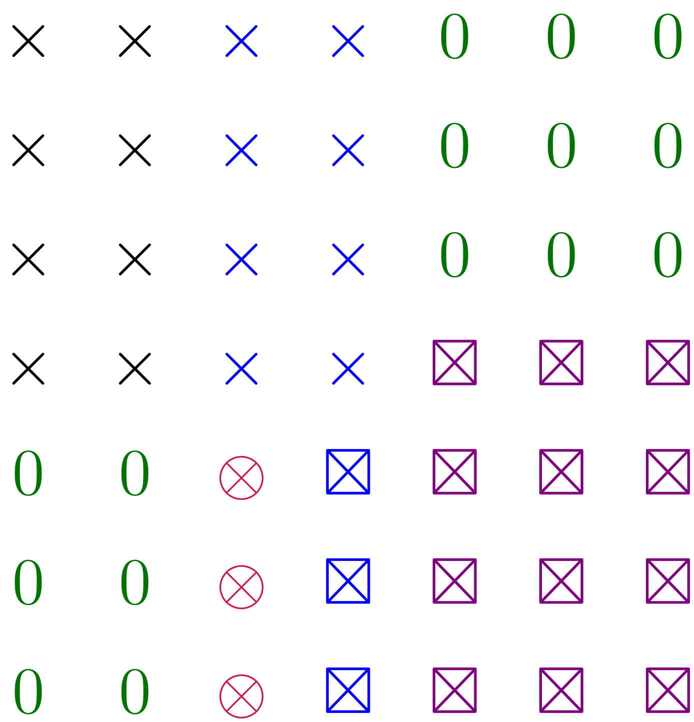






$$
\begin{array}{ccccccc}
\times & \times & \times & \times & 0 & 0 & 0 \\
\times & \times & \times & \times & 0 & 0 & 0 \\
\times & \times & \times & \times & \boxtimes & \boxtimes & \boxtimes \\
\times & \times & \times & \times & \boxtimes & \boxtimes & \boxtimes \\
0 & 0 & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
0 & 0 & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
0 & 0 & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes
\end{array}
$$

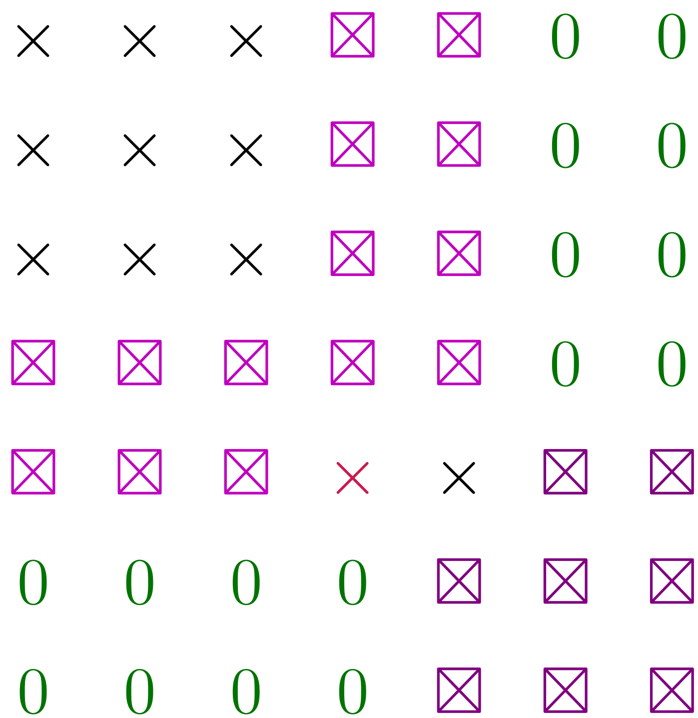




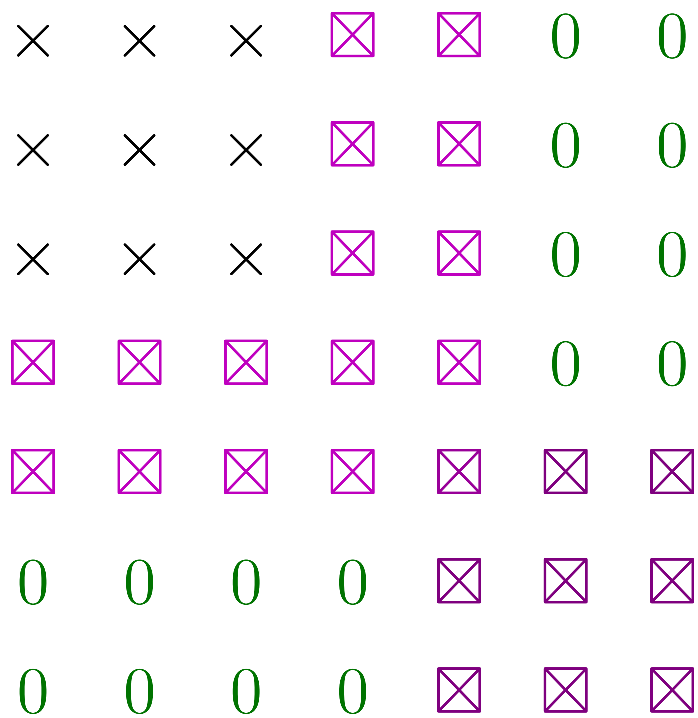
$$
\begin{array}{ccccccc}
\times & \times & \times & \times & 0 & 0 & 0 \\
\times & \times & \times & \times & 0 & 0 & 0 \\
\times & \times & \times & \times & 0 & 0 & 0 \\
\times & \times & \times & \times & \boxtimes & \boxtimes & \boxtimes \\
0 & 0 & 0 & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
0 & 0 & 0 & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
0 & 0 & 0 & \boxtimes & \boxtimes & \boxtimes & \boxtimes
\end{array}
$$

$$
\begin{array}{ccccccc}
\times & \times & \times & \times & 0 & 0 & 0 \\
\times & \times & \times & \times & 0 & 0 & 0 \\
\times & \times & \times & \times & 0 & 0 & 0 \\
\times & \times & \times & \times & \otimes & \otimes & \otimes \\
0 & 0 & 0 & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
0 & 0 & 0 & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
0 & 0 & 0 & \boxtimes & \boxtimes & \boxtimes & \boxtimes
\end{array}
$$

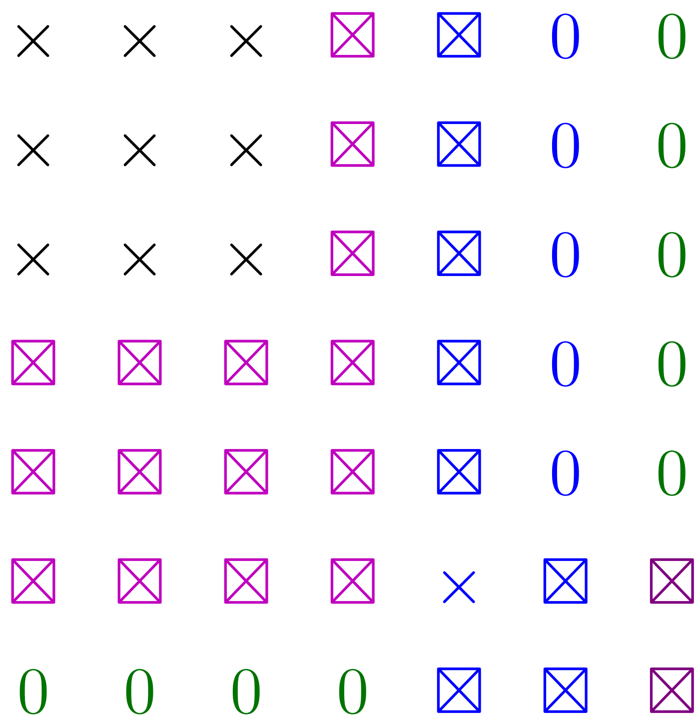
$$
\begin{matrix}
\times & \times & \times & \times & 0 & 0 & 0 \\
\times & \times & \times & \times & 0 & 0 & 0 \\
\times & \times & \times & \times & 0 & 0 & 0 \\
\boxtimes & \boxtimes & \boxtimes & \times & 0 & 0 & 0 \\
\boxtimes & \boxtimes & \boxtimes & \times & \boxtimes & \boxtimes & \boxtimes \\
0 & 0 & 0 & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
0 & 0 & 0 & \boxtimes & \boxtimes & \boxtimes & \boxtimes
\end{matrix}
$$

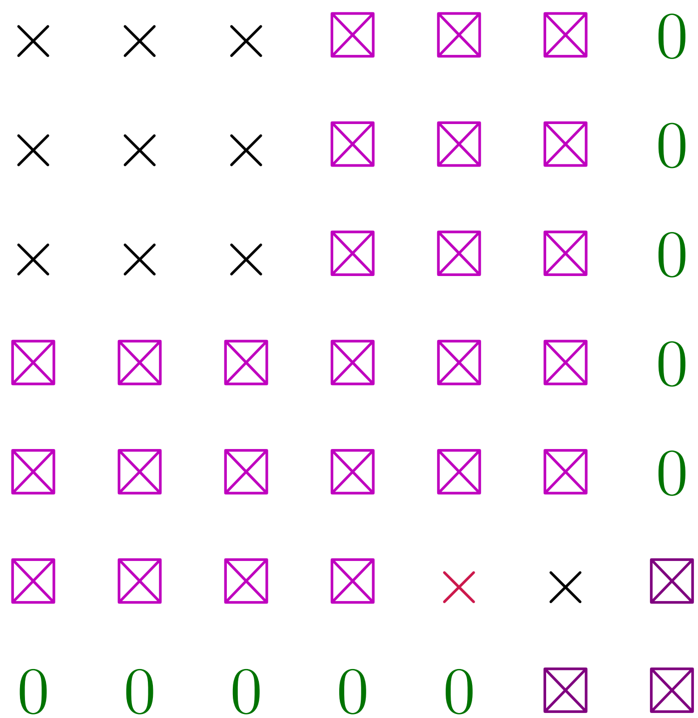


$$
\begin{matrix}
\times & \times & \times & \boxtimes & \boxtimes & 0 & 0 \\
\times & \times & \times & \boxtimes & \boxtimes & 0 & 0 \\
\times & \times & \times & \boxtimes & \boxtimes & 0 & 0 \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & 0 & 0 \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \times & \boxtimes & \boxtimes \\
0 & 0 & 0 & 0 & \boxtimes & \boxtimes & \boxtimes \\
0 & 0 & 0 & 0 & \boxtimes & \boxtimes & \boxtimes
\end{matrix}
$$



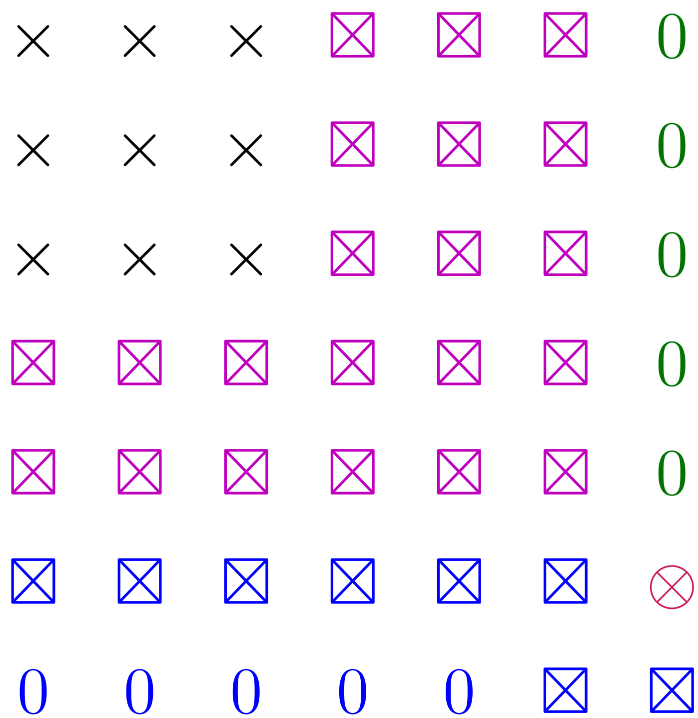
$$
\begin{matrix}
\times & \times & \times & \boxtimes & \boxtimes & 0 & 0 \\
\times & \times & \times & \boxtimes & \boxtimes & 0 & 0 \\
\times & \times & \times & \boxtimes & \boxtimes & 0 & 0 \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & 0 & 0 \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \otimes & \otimes \\
0 & 0 & 0 & 0 & \boxtimes & \boxtimes & \boxtimes \\
0 & 0 & 0 & 0 & \boxtimes & \boxtimes & \boxtimes
\end{matrix}
$$

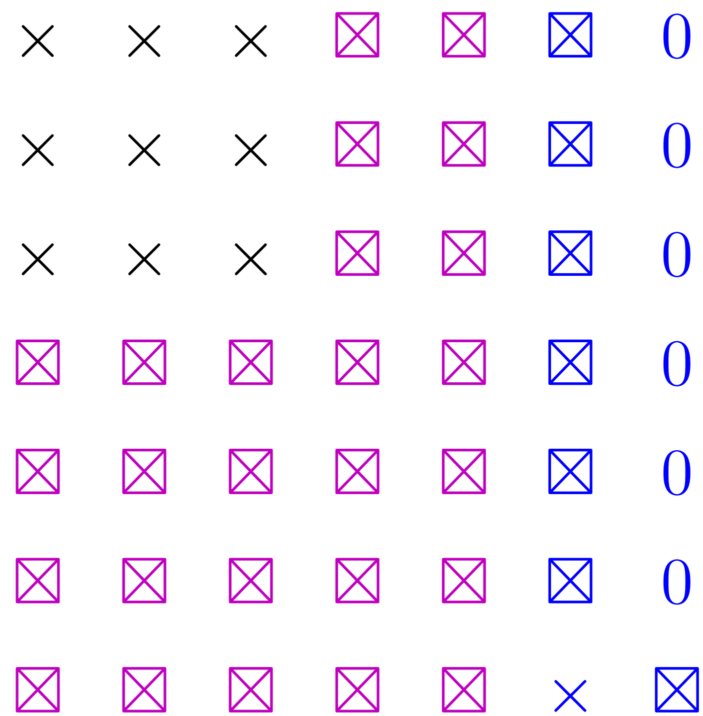


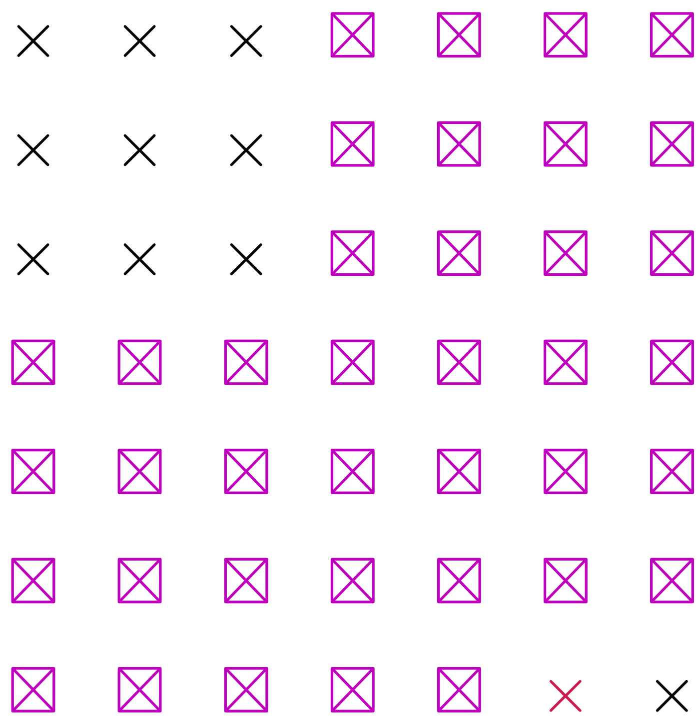


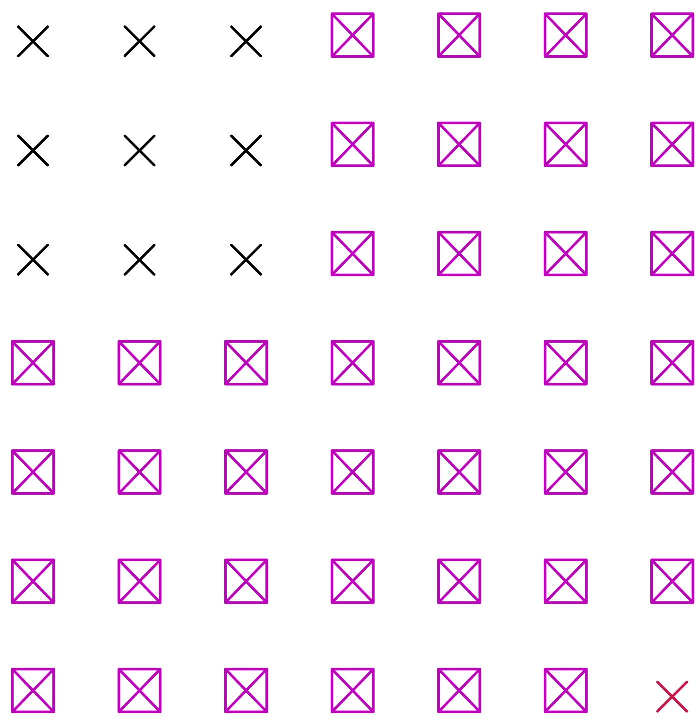
×	×	×	⊠	⊠	⊠	0
×	×	×	⊠	⊠	⊠	0
×	×	×	⊠	⊠	⊠	0
⊠	⊠	⊠	⊠	⊠	⊠	0
⊠	⊠	⊠	⊠	⊠	⊠	0
⊠	⊠	⊠	⊠	⊠	×	⊠
0	0	0	0	0	⊠	⊠

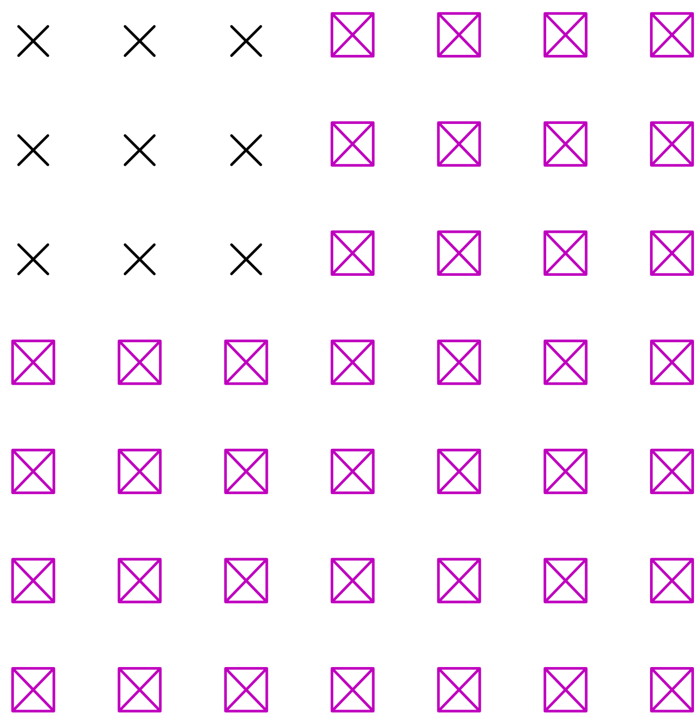
$$
\begin{matrix}
\times & \times & \times & \boxtimes & \boxtimes & \boxtimes & 0 \\
\times & \times & \times & \boxtimes & \boxtimes & \boxtimes & 0 \\
\times & \times & \times & \boxtimes & \boxtimes & \boxtimes & 0 \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & 0 \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & 0 \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
0 & 0 & 0 & 0 & 0 & \boxtimes & \boxtimes
\end{matrix}
$$











3. Reduction to diagonal-plus-semiseparable form with free choice of the diagonal

Any symmetric matrix can be transformed into a diagonal-plus-semiseparable one where the diagonal can be chosen in advance, by means of orthogonal similarity transformations in order $O(\frac{4}{3}n^3)$.

Definition

The sum of a symmetric semiseparable matrix and a diagonal matrix is called a diagonal-plus-semiseparable matrix.

So choose a diagonal $d = [d_1, d_2, \ldots, d_n]$.

A first algorithm

$$\begin{array}{ccccccc} \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times \end{array} \longrightarrow D + S$$

Step 1

$$
\begin{matrix}
\times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \color{red}{\times}
\end{matrix}
+
\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \color{red}{d_1}
\end{matrix}
$$

$$
\begin{matrix}
\times & \times & \times & \times & \times & \times & \otimes \\
\times & \times & \times & \times & \times & \times & \otimes \\
\times & \times & \times & \times & \times & \times & \otimes \\
\times & \times & \times & \times & \times & \times & \otimes \\
\times & \times & \times & \times & \times & \times & \otimes \\
\times & \times & \times & \times & \times & \times & \times \\
\otimes & \otimes & \otimes & \otimes & \otimes & \times & \times
\end{matrix}
+
\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_1
\end{matrix}
$$

$$
\begin{matrix}
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & \times & \times \\
0 & 0 & 0 & 0 & 0 & \times & \times
\end{matrix}
+
\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_1
\end{matrix}
$$

$$
\begin{matrix}
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & \times & \otimes \\
0 & 0 & 0 & 0 & 0 & \otimes & \times
\end{matrix}
+
\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_1
\end{matrix}
$$

Problem

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{pmatrix} 0 & 0 \\ 0 & d_1 \end{pmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix} = \begin{pmatrix} s^2 d_1 & csd_1 \\ csd_1 & c^2 d_1 \end{pmatrix}$$

$$\Rightarrow ???$$

Solution

$$\begin{aligned} & \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{pmatrix} d_1 & 0 \\ 0 & d_1 \end{pmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \\ = & \begin{bmatrix} c & s \\ -s & c \end{bmatrix} d_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \\ = & \begin{pmatrix} d_1 & 0 \\ 0 & d_1 \end{pmatrix} \end{aligned}$$

$$
\begin{matrix}
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & \times & \otimes \\
0 & 0 & 0 & 0 & 0 & \otimes & \times
\end{matrix}
+
\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_1
\end{matrix}
$$

$$
\begin{matrix}
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times & + & \otimes \\
0 & 0 & 0 & 0 & 0 & \otimes & \times
\end{matrix}
+
\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_1
\end{matrix}
$$

$$
\begin{matrix}
\times & \times & \times & \times & \times & \boxtimes & \boxtimes \\
\times & \times & \times & \times & \times & \boxtimes & \boxtimes \\
\times & \times & \times & \times & \times & \boxtimes & \boxtimes \\
\times & \times & \times & \times & \times & \boxtimes & \boxtimes \\
\times & \times & \times & \times & \times & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes
\end{matrix}
+
\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_1
\end{matrix}
$$

$$
\begin{matrix}
\times & \times & \times & \times & \times & \boxtimes & \boxtimes \\
\times & \times & \times & \times & \times & \boxtimes & \boxtimes \\
\times & \times & \times & \times & \times & \boxtimes & \boxtimes \\
\times & \times & \times & \times & \times & \boxtimes & \boxtimes \\
\times & \times & \times & \times & \times & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxplus
\end{matrix}
+
\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_2
\end{matrix}
$$

$$
\begin{array}{ccccccc}
\times & \times & \times & \times & \times & \boxtimes & \boxtimes \\
\times & \times & \times & \times & \times & \boxtimes & \boxtimes \\
\times & \times & \times & \times & \times & \boxtimes & \boxtimes \\
\times & \times & \times & \times & \times & \boxtimes & \boxtimes \\
\times & \times & \times & \times & \times & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes
\end{array}
+
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_2
\end{array}
$$

Step 3

$$\begin{matrix} \times & \times & \times & \times & \boxtimes & \boxtimes & \boxtimes \\ \times & \times & \times & \times & \boxtimes & \boxtimes & \boxtimes \\ \times & \times & \times & \times & \boxtimes & \boxtimes & \boxtimes \\ \times & \times & \times & \times & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \end{matrix} + \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d_3 \end{matrix}$$

$$\begin{array}{ccccccc} \times & \times & \times & \times & \otimes & \otimes & \otimes \\ \times & \times & \times & \times & \boxtimes & \boxtimes & \boxtimes \\ \times & \times & \times & \times & \boxtimes & \boxtimes & \boxtimes \\ \times & \times & \times & \times & \boxtimes & \boxtimes & \boxtimes \\ \otimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \otimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \otimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \end{array} + \begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d_3 \end{array}$$

$$
\begin{array}{ccccccc}
\times & \times & \times & \times & 0 & 0 & 0 \\
\times & \times & \times & \times & \otimes & \otimes & \otimes \\
\times & \times & \times & \times & \boxtimes & \boxtimes & \boxtimes \\
\times & \times & \times & \times & \boxtimes & \boxtimes & \boxtimes \\
0 & \otimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
0 & \otimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
0 & \otimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes
\end{array}
+
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_3
\end{array}
$$

$$
\begin{array}{ccccccc}
\times & \times & \times & \times & 0 & 0 & 0 \\
\times & \times & \times & \times & 0 & 0 & 0 \\
\times & \times & \times & \times & \otimes & \otimes & \otimes \\
\times & \times & \times & \times & \boxtimes & \boxtimes & \boxtimes \\
0 & 0 & \otimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
0 & 0 & \otimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
0 & 0 & \otimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes
\end{array}
+
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_3
\end{array}
$$

$$
\begin{array}{ccccccc}
\times & \times & \times & \times & 0 & 0 & 0 \\
\times & \times & \times & \times & 0 & 0 & 0 \\
\times & \times & \times & \times & 0 & 0 & 0 \\
\times & \times & \times & \times & \otimes & \otimes & \otimes \\
0 & 0 & 0 & \otimes & \boxtimes & \boxtimes & \boxtimes \\
0 & 0 & 0 & \otimes & \boxtimes & \boxtimes & \boxtimes \\
0 & 0 & 0 & \otimes & \boxtimes & \boxtimes & \boxtimes
\end{array}
+
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_3
\end{array}
$$

$$
\begin{matrix}
\times & \times & \times & \times & 0 & 0 & 0 \\
\times & \times & \times & \times & 0 & 0 & 0 \\
\times & \times & \times & \times & 0 & 0 & 0 \\
\times & \times & \times & + & \otimes & \otimes & \otimes \\
0 & 0 & 0 & \otimes & \boxtimes & \boxtimes & \boxtimes \\
0 & 0 & 0 & \otimes & \boxtimes & \boxtimes & \boxtimes \\
0 & 0 & 0 & \otimes & \boxtimes & \boxtimes & \boxtimes
\end{matrix}
\;+\;
\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & d_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_3
\end{matrix}
$$

$$
\begin{array}{ccccccc}
\times & \times & \times & \boxtimes & \boxtimes & 0 & 0 \\
\times & \times & \times & \boxtimes & \boxtimes & 0 & 0 \\
\times & \times & \times & \boxtimes & \boxtimes & 0 & 0 \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & 0 & 0 \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
0 & 0 & 0 & 0 & \boxtimes & \boxtimes & \boxtimes \\
0 & 0 & 0 & 0 & \boxtimes & \boxtimes & \boxtimes
\end{array}
+
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & d_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_3
\end{array}
$$

$$
\begin{matrix}
\times & \times & \times & \boxtimes & \boxtimes & 0 & 0 \\
\times & \times & \times & \boxtimes & \boxtimes & 0 & 0 \\
\times & \times & \times & \boxtimes & \boxtimes & 0 & 0 \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & 0 & 0 \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxplus & \boxtimes & \boxtimes \\
0 & 0 & 0 & 0 & \boxtimes & \boxtimes & \boxtimes \\
0 & 0 & 0 & 0 & \boxtimes & \boxtimes & \boxtimes
\end{matrix}
+
\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & d_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_3
\end{matrix}
$$

$$
\begin{matrix}
\times & \times & \times & \boxtimes & \boxtimes & 0 & 0 \\
\times & \times & \times & \boxtimes & \boxtimes & 0 & 0 \\
\times & \times & \times & \boxtimes & \boxtimes & 0 & 0 \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & 0 & 0 \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \otimes & \otimes \\
0 & 0 & 0 & 0 & \otimes & \boxtimes & \boxtimes \\
0 & 0 & 0 & 0 & \otimes & \boxtimes & \boxtimes
\end{matrix}
\quad + \quad
\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & d_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_3
\end{matrix}
$$

$$
\begin{array}{ccccccc}
\times & \times & \times & \boxtimes & \boxtimes & \boxtimes & 0 \\
\times & \times & \times & \boxtimes & \boxtimes & \boxtimes & 0 \\
\times & \times & \times & \boxtimes & \boxtimes & \boxtimes & 0 \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & 0 \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & 0 \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
0 & 0 & 0 & 0 & 0 & \boxtimes & \boxtimes
\end{array}
\quad + \quad
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & d_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_3
\end{array}
$$

$$
\begin{array}{ccccccc}
\times & \times & \times & \boxtimes & \boxtimes & \boxtimes & 0 \\
\times & \times & \times & \boxtimes & \boxtimes & \boxtimes & 0 \\
\times & \times & \times & \boxtimes & \boxtimes & \boxtimes & 0 \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & 0 \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & 0 \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxplus & \boxtimes \\
0 & 0 & 0 & 0 & 0 & \boxtimes & \boxtimes
\end{array}
\quad + \quad
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & d_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_3
\end{array}
$$

$$
\begin{array}{ccccccc}
\times & \times & \times & \boxtimes & \boxtimes & \boxtimes & 0 \\
\times & \times & \times & \boxtimes & \boxtimes & \boxtimes & 0 \\
\times & \times & \times & \boxtimes & \boxtimes & \boxtimes & 0 \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & 0 \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & 0 \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \otimes \\
0 & 0 & 0 & 0 & 0 & \otimes & \boxtimes
\end{array}
\;+\;
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & d_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_3
\end{array}
$$

$$
\begin{matrix}
\times & \times & \times & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\times & \times & \times & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\times & \times & \times & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes
\end{matrix}
+
\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & d_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_3
\end{matrix}
$$

$$
\begin{matrix}
\times & \times & \times & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\times & \times & \times & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\times & \times & \times & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxplus
\end{matrix}
+
\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & d_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_4
\end{matrix}
$$

$$
\begin{array}{ccccccc}
\times & \times & \times & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\times & \times & \times & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\times & \times & \times & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes
\end{array}
+
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & d_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_4
\end{array}
$$

A second algorithm

Before the last step of the first algorithm:

$$
\begin{matrix}
\times & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes
\end{matrix}
+
\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & d_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & d_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & d_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d_5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_6
\end{matrix}
$$

When applying the first algorithm starting with $\mathbf{D} = [\mathbf{d_2}, \mathbf{d_3}, \ldots, \mathbf{d_n}, \star]$ with $\star$ an arbitrary element, instead of $[\mathbf{d_1}, \mathbf{d_2}, \ldots, \mathbf{d_n}]$, we get the following situation before the last step:

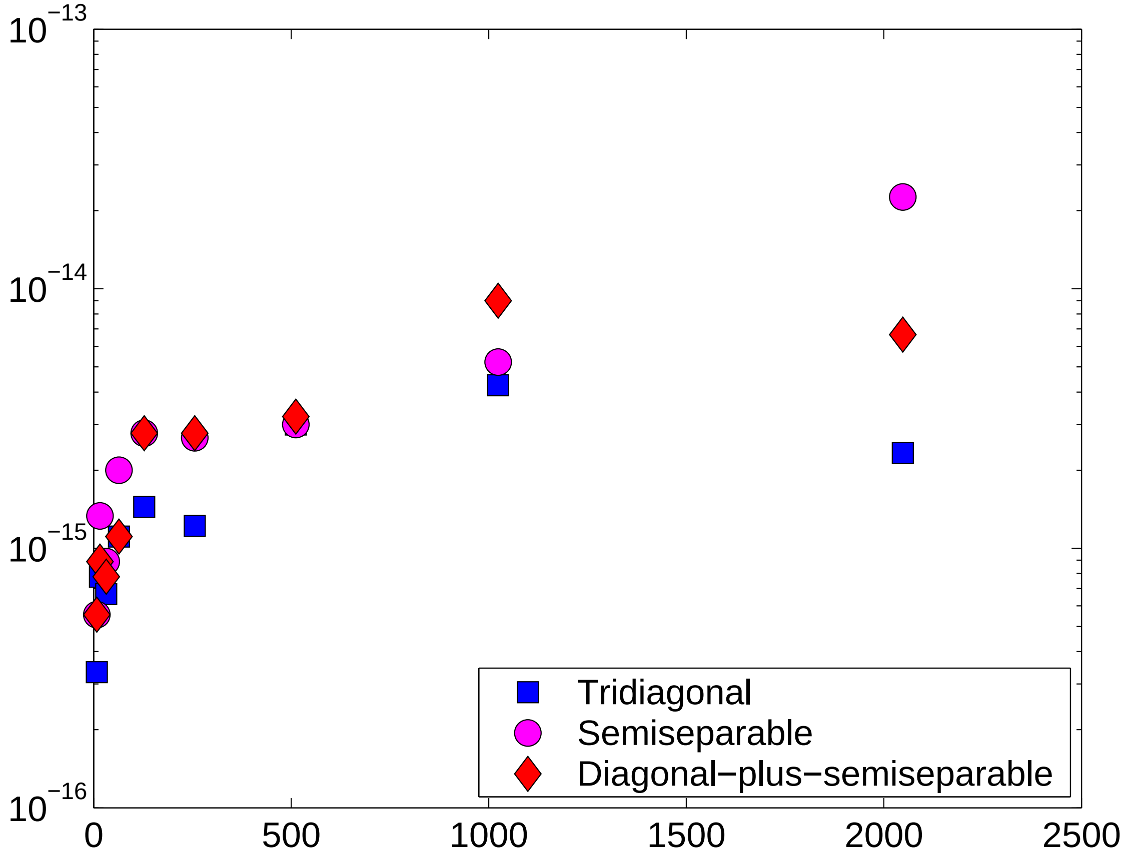
$$
\begin{matrix}
\times & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes
\end{matrix}
+
\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & d_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & d_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & d_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d_6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_7
\end{matrix}
$$

No last step necessary:

$$
\begin{matrix}
\boxplus & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes
\end{matrix}
+
\begin{matrix}
d_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & d_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & d_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & d_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d_6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_7
\end{matrix}
$$

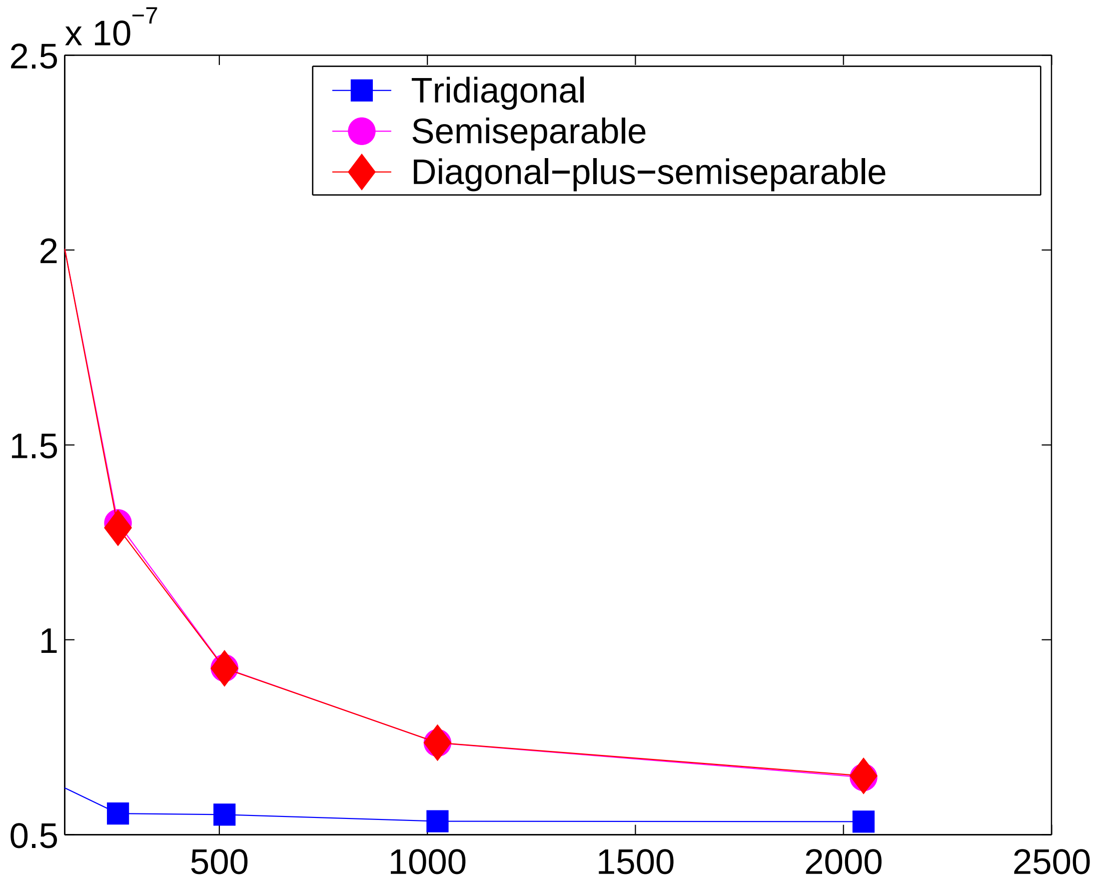
Any arbitrary symmetric matrix can be transformed into a symmetric diagonal-plus-semiseparable one with free choice of the diagonal by means of an orthogonal similarity transformation Q such that $\mathbf{Q}\mathbf{e}_1 = \mathbf{e}_1$.

II. Accuracy



10^{-13}
10^{-14}
10^{-15}
10^{-16}
0
500
1000
1500
2000
2500
Tridiagonal
Semiseparable
Diagonal−plus−semiseparable

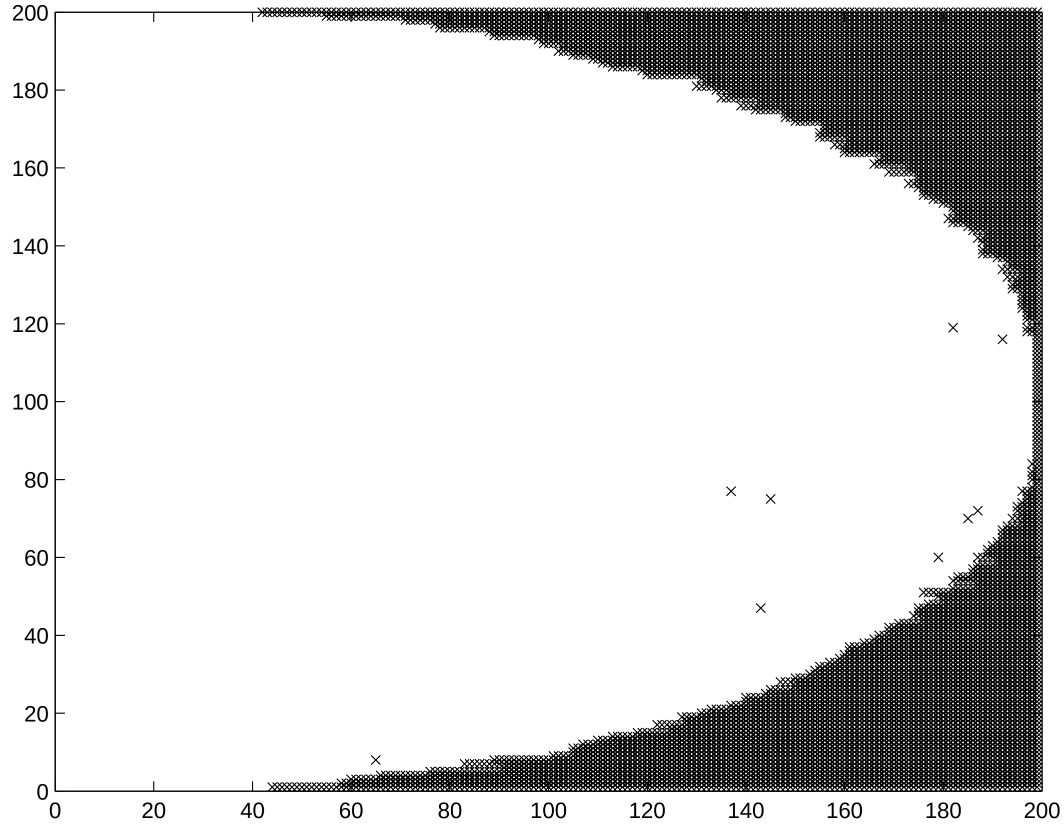
III. Computational complexity



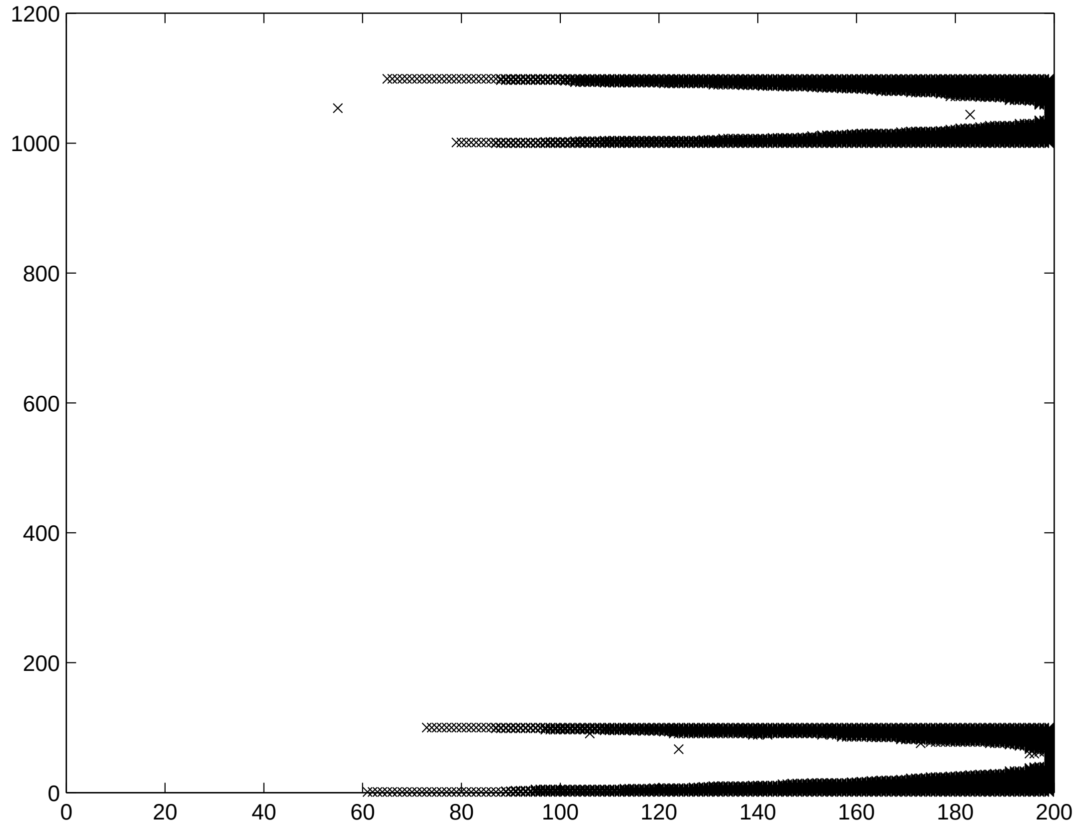
IV. Convergence behavior of reduction algorithm into diagonal-plus-semiseparable form

For the reduction to semiseparable form

Eigenvalues are equidistant $1 : 200$.



Eigenvalues $1 : 100$ and $1000 : 1100$.



For the reduction to diagonal-plus-semiseparable form

Some notation

$$
\begin{aligned}
A^{(0)} &= A \\
A^{(m)} &= Q_m^T A^{(m-1)} Q_m \\
&= \left(\begin{array}{c|c} A_m & R_1^T \\ \hline R_1 & (D+S)_m \end{array} \right) \\
&= Q_{1:m}^T A Q_{1:m}
\end{aligned}
$$

where $(\mathbf{D}+\mathbf{S})_\mathbf{m}$ is a square diagonal-plus-semiseparable matrix of dimensions $(\mathbf{m}+1) \times (\mathbf{m}+1)$.

Lemma

$$\mathbf{Q_{1:m} < e_n >= (A - d_m I)(A - d_{m-1} I) \ldots (A - d_1 I) < e_n >,}$$

for $\mathbf{m = 1, 2, \ldots}$ **and** $\mathbf{Q_{1:0} = I.}$

Proof.

. For $\mathbf{m = 0}$: $\mathbf{Q_{1:0} < e_n > = < e_n >}$.

. Suppose the theorem is true for $\mathbf{m - 1,}$ i.e.,

$$\mathbf{Q_{1:m-1} < e_n > = (A - d_{m-1}I) \dots (A - d_2I)(A - d_1I) < e_n >} .$$

The structure of $\mathbf{Q_m^T A^{(m-1)}}$ is of the form:

$$\left(\begin{array}{ccc|ccc} \times & \dots & \times & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \times & \dots & \times & 0 & \dots & 0 \\ \hline \times & \dots & \times & \times & \dots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ \times & \dots & \times & \times & \dots & \times \end{array}\right) + Q_m^T \left(\begin{array}{ccc|ccc} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ \hline & & & d_1 & & \\ & & & & \ddots & \\ & & & & & d_m \end{array}\right)$$

$$= H + Q_m^T D$$

Hence,

$$\begin{aligned} Q_m^T(A^{(m-1)}) &= H + Q_m^T D \\ Q_m^T(Q_{1:m-1}^T A Q_{1:m-1}) &= H + Q_m^T D \\ \Rightarrow AQ_{1:m-1} - Q_{1:m-1}D &= Q_{1:m}H \end{aligned}$$

Applying the former equality on $< e_n >$ **and using the induction hypothesis, we derive that:**

$$\begin{aligned} & (AQ_{1:m-1} - Q_{1:m-1}D) < e_n >= Q_{1:m}H < e_n > \\ & (AQ_{1:m-1} - Q_{1:m-1}d_m I) < e_n >= Q_{1:m} < e_n > \\ \Rightarrow \quad & (A - d_m I)(A - d_{m-1} I) \ldots (A - d_1 I) < e_n >= Q_{1:m} < e_n > \end{aligned}$$

Lanczos-Ritz convergence behavior

a) Lanczos-Ritz values

Because $AQ_{1:m} = Q_{1:m}A^{(m)}$ **equals:**

$$A\left[\overleftarrow{Q}_{1:m}|\overrightarrow{Q}_{1:m}\right] = \left[\overleftarrow{Q}_{1:m}|\overrightarrow{Q}_{1:m}\right]\left(\begin{array}{c|c} A_m & R_1^T \\ \hline R_1 & (D+S)_m \end{array}\right).$$

Hence, the eigenvalues of $(\mathbf{D}+\mathbf{S})_\mathbf{m}$ **are the Ritz-values of** $\mathbf{A}$ **with respect to the subspace spanned by the columns of** $\overrightarrow{Q}_{\mathbf{1:m}}$**.**

b) Connection with the Krylov subspace

Some notation

$$\mathcal{K}_{\mathbf{m}} = < \mathbf{e_n}, \mathbf{A}\mathbf{e_n}, \mathbf{A^2}\mathbf{e_n}, \ldots, \mathbf{A^m}\mathbf{e_n} >$$

$$\mathbf{Q_{m+1}} = \left(\begin{array}{c|c|ccccc} \tilde{H} & h & & & hq^T & & \\ \hline 0 & \times & \times & \cdots & & & \times \\ & 0 & \times & & & & \\ \vdots & \vdots & & \ddots & & & \vdots \\ 0 & 0 & 0 & & & \times & \times \end{array}\right) \left\{ \begin{array}{l} \tilde{H} \in \mathbb{R}^{(n-m-1)\times(n-m-2)} \\ h \in \mathbb{R}^{(n-m-1)\times 1} \\ q \in \mathbb{R}^{(m+1)\times 1} \end{array} \right. .$$

We want to prove by induction that:

$$\mathbf{span}\left(\mathbf{col}(\overrightarrow{Q}_{\mathbf{1:m+1}})\right) = \mathcal{K}_{\mathbf{m+1}}.$$

We have:

$$\begin{aligned}
\overrightarrow{Q}_{1:m+1} &= [\overleftarrow{Q}_{1:m}|\overrightarrow{Q}_{1:m}] \left(\begin{array}{c|cccc}
h & & hq^T & & \\
\hline
\times & \times & \cdots & & \times \\
0 & \times & & & \\
\vdots & & \ddots & & \vdots \\
0 & 0 & & \times & \times
\end{array}\right). \\
&= \overleftarrow{Q}_{1:m}h[1, q^T] + \overrightarrow{Q}_{1:m} \left(\begin{array}{c|cccc}
\times & \times & \cdots & & \times \\
0 & \times & & & \\
\vdots & & \ddots & & \vdots \\
0 & 0 & & \times & \times
\end{array}\right).
\end{aligned}$$

Because:

- $span\left(col(\overrightarrow{Q}_{1:m})\right) = \mathcal{K}_m =< e_n, Ae_n, \ldots, A^m e_n >$
- $\overrightarrow{Q}_{1:m+1} < e_n >= (A - d_{m+1}I)\ldots(A - d_1 I) < e_n >$

$$\Rightarrow \quad \overleftarrow{Q}_{1:m} h \in \mathcal{K}_{m+1} \backslash \mathcal{K}_m$$

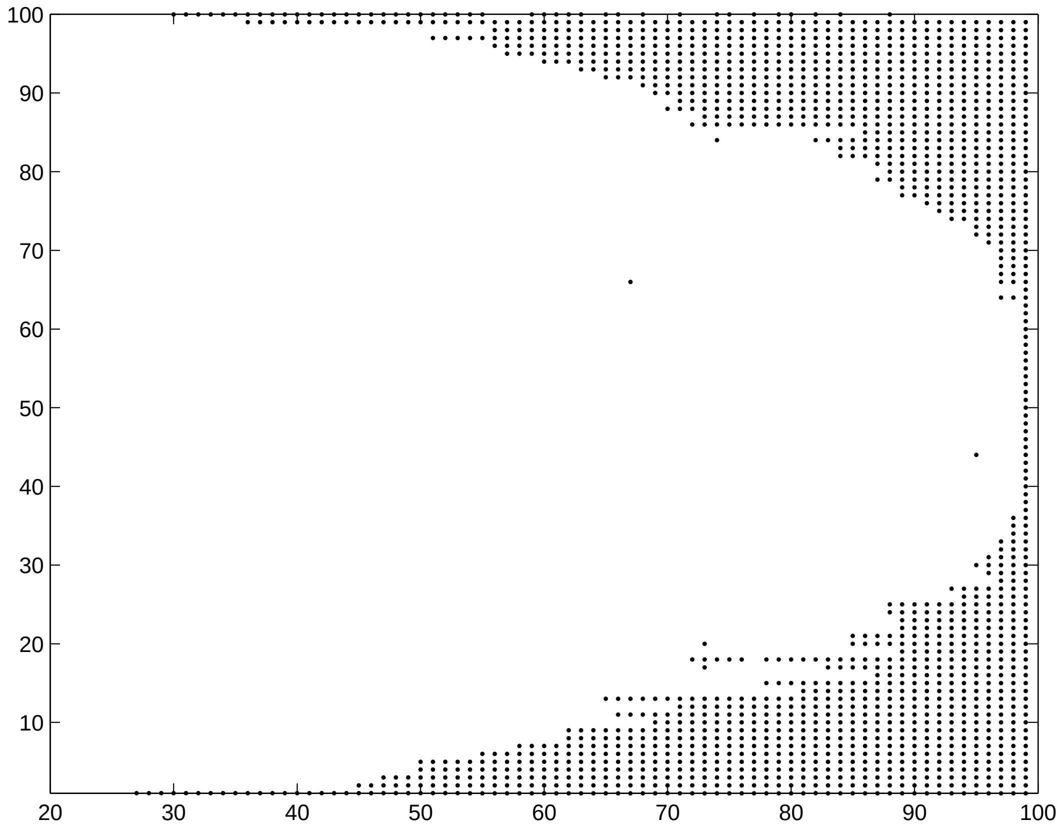
We get:

$$\mathbf{span}\left(\mathbf{col}(\overrightarrow{Q}_{\mathbf{1:m+1}})\right) = \mathcal{K}_{\mathbf{m+1}} =< \mathbf{e_n}, \mathbf{Ae_n}, \ldots, \mathbf{A^{m+1}e_n} >$$

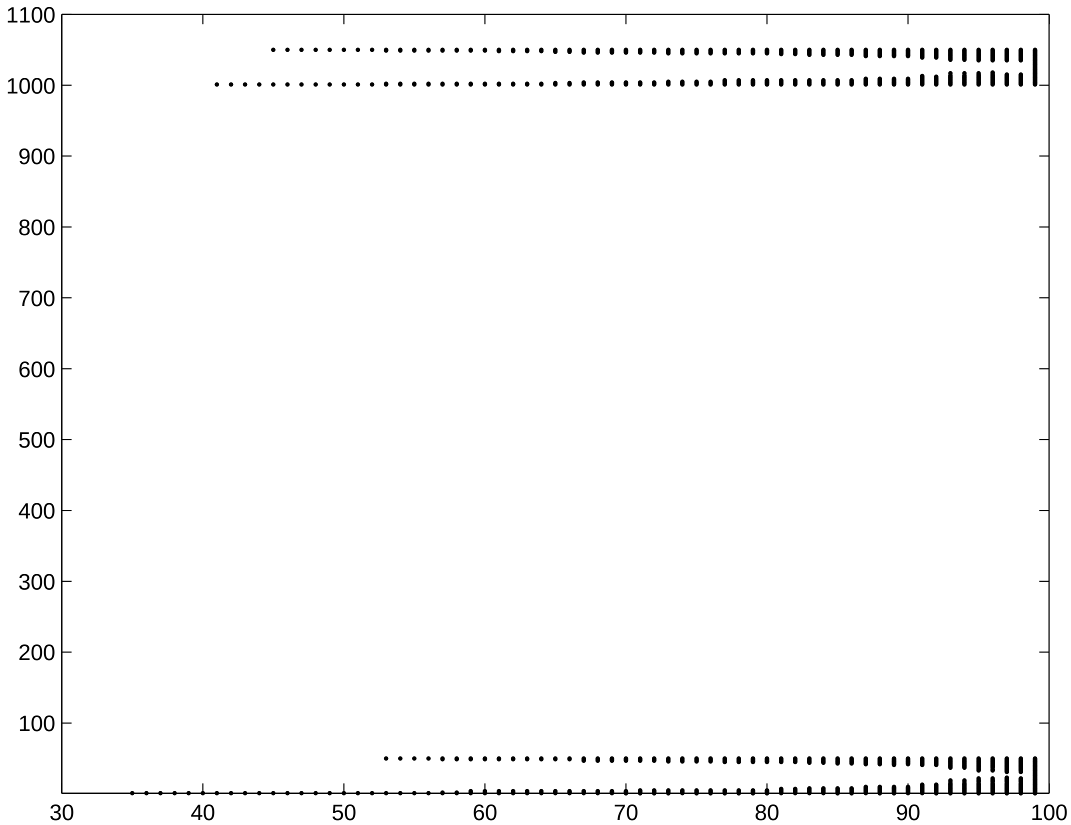
Theorem

The eigenvalues of $(\mathbf{D}+\mathbf{S})_{\mathbf{m}}$, the lower diagonal blocks that appear during the reduction algorithm, are the Lanczos-Ritz values of A with respect to the Krylov subspace $\mathcal{K}_{\mathbf{m}}$.

Eigenvalues are equidistant $1:100$ and d=random.



Eigenvalues $1:50$ and $10001:1050$ and d=random.



Subspace iteration

The semiseparable case

Demo with two clusters of eigenvalues.

The diagonal-plus-semiseparable case

first step

$$
\begin{aligned}
A &= A^{(0)} = Q_1(Q_1^T A^{(0)}) \\
&= Q_1 \left(\left(\begin{array}{ccc|c} \times & \dots & \times & 0 \\ \vdots & & \vdots & \vdots \\ \times & \dots & \times & 0 \\ \hline \times & \dots & \times & \times \end{array} \right) + Q_1^T \left(\begin{array}{ccc|c} 0 & & & \\ & \ddots & & \\ & & 0 & \\ \hline & & & d_1 \end{array} \right) \right)
\end{aligned}
$$

Hence,

$$
(A - d_1 I) < e_n >= Q_1 < e_n >= q_n^{(1)}.
$$

Transformation of basis:

$$\mathbf{A^{(1)} = Q_1^T A Q_1}$$

A vector y in the old basis, becomes $\mathbf{Q_1^T y}$ in the new basis. This means that $\mathbf{q_n^{(1)}}$ becomes $\mathbf{Q_1^T q_n^{(1)} = e_n}$ and hence, $\mathbf{(A - d_1 I) < e_n >}$ becomes $\mathbf{< e_n >}$.

mth step

$$\mathbf{(A - d_m I) \dots (A - d_1 I) < e_{n-j}, \dots, e_n > = < q_{n-m+1}^{(m)}, \dots, q_n^{(m)} >}$$

for $\mathbf{j = 0, \dots, m+1}$.

Conclusion

The reduction algorithm proposed in this talk in order to transform any symmetric matrix into a diagonal-plus-semiseparable one with free choice of the diagonal has

- **A Lanczos-Ritz behavior - Krylov subspace**
- **Subspace iteration.**

$\Rightarrow$ As soon as the Lanczos-Ritz values approximate some eigenvalues good enough, the subspace iteration starts converging.