Hyperordinals and Nonstandard α -Models

MAURO DI NASSO

Abstract

In this paper we extend the traditional set-theoretic notion of standard models and nonstandard models going up to α levels in the cumulative hierarchy, α any given limit ordinal. A proof of the representation theorem is given and the structure of nonstandard models is studied where the "transfer principle" holds for every (not necessarily bounded) formula. These models preserve a stratified structure which is investigated by means of "pseudo-rank" functions taking linearly ordered values (hyperordinals). In particular, such functions show a "rigidity" property of the internal sets, in that each external set has a pseudo-rank which is greater than the pseudo-rank of any internal set.

1 Nonstandard α -Models

We shall work in ZF^-C , i.e. in Zermelo-Fraenkel axiomatic set theory ZF with choice but without the axiom Fnd of foundation (regularity). Thus the axioms of our set theory are precisely:

Ext (Extensionality); *Pair* (Pairing); *Sep* (Separation Schema);
Un (Union); *Pow* (Power Set); *Rep* (Replacement Schema); *Inf*
(Infinity) and *AC* (Choice).

For unexplained set-theoretic and model-theoretic notions and notations we refer to [J] and [CK] respectively.

The traditional set-theoretic approach to nonstandard models is the one streamlined by A. Robinson and E. Zakon in [RZ] and [Z]. It is limited to the first ω levels of the cumulative hierarchy. Here we extend that notion by replacing ω with any limit ordinal α . Namely, a *standard α -model* is an infinite $\in$ -model $\langle V_\alpha(X), \in \rangle$ where $V_\alpha(X)$ is the α -superstructure over the set X .¹

For convenience, X is usually assumed to consist of urelements. However, as pointed out by J. Schmid and J. Schmidt in [SS], in order to apply nonstandard methods to various structures one should then claim that the number of urelements available in the universe exceeds each fixed cardinality. Since we prefer not to postulate the existence of a proper class of urelements (indeed the existence of any) for the sole purpose of nonstandard methods, we only require X to be the set of the atoms relative to the standard model, i.e. $\emptyset \notin X$ and $\forall t \in X \ t \cap V_\alpha(X) = \emptyset$. This definition of standard model seems to be a natural generalization which allows the use of infinite ordinals. Notice that infinite sets are available even in the case of superstructures built up on the sole empty set, provided that $\alpha > \omega$. This will please lovers of reductionism.

Let $\mathcal{M} = \langle M, E \rangle$ be any model of the language of set theory with $E = \in_{\mathcal{M}}$ the $\mathcal{M}$ -interpretation of the membership relation symbol, and let ϑ be a mapping $\vartheta : \langle V_\alpha(X), \in \rangle \longrightarrow \mathcal{M}$. We shall say that ϑ satisfies the *transfer principle* for the formula φ if the equivalence

$$V_\alpha(X) \models \varphi(a_1, \dots, a_n) \iff \mathcal{M} \models \varphi(\vartheta(a_1), \dots, \vartheta(a_n))$$

holds for every assignment $a_1, \dots, a_n \in V_\alpha(X)$. ϑ is a *nonstandard embedding (NSE)* when the *transfer principle* holds for every *bounded* formula.² We shall call $\mathcal{M}$ the *nonstandard α -model (α -NSM)* relative to ϑ . When $\mathcal{M}$ is an $\in$ -model, the additional property $\vartheta(\emptyset) = \emptyset$ is also required. We remark that only $\in$ -models are considered as NSM's in the literature.

In order to simplify the notation, the subset $\subseteq$, the union $\cup$, the intersection $\cap$, the set-minus $\setminus$, the ordered pair $\langle \ , \ \rangle$ and the cartesian product $\times$

¹Remind the cumulative hierarchy defined by transfinite induction: $V_0(X) = X$; $V_{\beta+1}(X) = V_\beta(X) \cup \mathcal{P}(V_\beta(X))$; $V_\alpha(X) = \bigcup_{\beta < \alpha} V_\beta(X)$ if α is a limit ordinal.

²A formula φ in the language of set theory is a *bounded formula* if every quantifier in φ occurs in the form $\forall x (x \in y \rightarrow \dots)$ or $\exists x (x \in y \wedge \dots)$.

symbols will be used. All of these abbreviations are bounded formulae. E.g., we shall write $x = \{a, b\}$ instead of $\forall t \in x (t = a \vee t = b) \wedge a \in x \wedge b \in x$; $x = \langle a, b \rangle$ instead of $x = \{\{a\}, \{a, b\}\}$ and so on. Other basic notions in elementary set theory can be expressed by bounded formulae too. For instance "f is a function", "A is the domain (or the range) of f", "f is one-one (or onto)" etc.

For every $m \in \mathcal{M}$ denote $m_E = \{m' \in \mathcal{M} : m'Em\}$ the set of its E-elements. Every NSE ϑ preserves the basic operations on sets. Namely, if $A, B \in V_\alpha(X)$, then $A \subseteq B$ iff $\vartheta(A)_E \subseteq \vartheta(B)_E$; $\vartheta(A \cup B)_E = \vartheta(A)_E \cup \vartheta(B)_E$; $\vartheta(A \cap B)_E = \vartheta(A)_E \cap \vartheta(B)_E$; $\vartheta(A \setminus B)_E = \vartheta(A)_E \setminus \vartheta(B)_E$ and $\vartheta(A \times B)_E = \vartheta(A)_E \times \vartheta(B)_E$. Also, if R is an n-ary relation on A then $\vartheta(R)_E$ is an n-ary relation on $\vartheta(A)_E$ and if $f : A^n \rightarrow B$ is an n-ary function then $\vartheta(f)_E : \vartheta(A)_E^n \rightarrow \vartheta(B)_E$ is an n-ary function where $\langle \vartheta(a_1), \dots, \vartheta(a_n) \rangle \mapsto \vartheta(f(a_1, \dots, a_n))$ for every $a_1, \dots, a_n \in V_\alpha(X)$. Moreover, $\vartheta(f)_E$ is one-one (resp. onto) iff f is. Membership minimality is also preserved, i.e. A is $\in$ -minimal in $\langle V_\alpha(X), \in \rangle$ iff $\vartheta(A)_E = \emptyset$. All these facts are straightforwardly proved by applying the *transfer* principle to suitable bounded formulae (see [Z] §1.).

Call $b \in \mathcal{M}$ *standard* if $b = \vartheta(a)$ for some $a \in V_\alpha(X)$; *internal* if $bE\vartheta(a)$ for some $a \in V_\alpha(X)$ and *external* if it is not internal. Denote $\mathcal{S}$ the collection of all standard sets and $\mathcal{I}$ the collection of all internal sets. Then $\mathcal{S} \subseteq \mathcal{I} = \bigcup_{\beta < \alpha} \vartheta(V_\beta(X))_E$ and $\mathcal{I}$ is E-transitive, i.e. bEa and $a \in \mathcal{I}$ implies $b \in \mathcal{I}$. Moreover the *transfer* principle also holds between the nonstandard model and the internal model $\langle \mathcal{I}, E \rangle \subseteq \mathcal{M}$. Thus restriction $\vartheta : \langle V_\alpha(X), \in \rangle \rightarrow \langle \mathcal{I}, E \rangle$ is a NSE too.

Keisler's limit ultrapower construction yields an algebraic characterization of the internal models. First, let's fix our notation. Let $V_\alpha(X)_D^I$ be any ultrapower of $V_\alpha(X)$. For every filter F over the set $I \times I$, the corresponding limit ultrapower $V_\alpha(X)_D^I|F \subseteq V_\alpha(X)_D^I$ is the submodel made up by those elements f^D such that $\{\langle i, j \rangle \in I \times I : f(i) = f(j)\} \in F$. Notice that every ultrapower is a limit ultrapower (take $F = I \times I$ the trivial filter). An element $f^D \in V_\alpha(X)_D^I|F$ is *bounded* if $f^D \in V_\beta(X)_D^I|F$ for some $\beta < \alpha$. Call *bounded limit ultrapower* the submodel ${}^bV_\alpha(X)_D^I|F \subseteq V_\alpha(X)_D^I|F$ whose universe is made up by the bounded elements. It is easily seen that the diagonal immersion $d : \langle V_\alpha(X), \in \rangle \rightarrow {}^bV_\alpha(X)_D^I|F$ and the inclusion $\iota : {}^bV_\alpha(X)_D^I|F \hookrightarrow$

$V_\alpha(X)^I_D|F$ are NSE's.

Theorem 1.1 (*REPRESENTATION THEOREM*)

The mapping $\vartheta : \langle V_\alpha(X), \in \rangle \longrightarrow \mathcal{M}$ is a NSE if and only if there exist

- an ultrafilter D over the set I
- a filter F over the set $I \times I$
- an isomorphism $\pi : \langle \mathcal{I}, E \rangle \cong {}^bV_\alpha(X)^I_D|F$

such that the following diagram commutes:

$$\begin{array}{ccc} & {}^bV_\alpha(X)^I_D|F & \\ d \nearrow & & \nwarrow \pi \\ \langle V_\alpha(X), \in \rangle & \xrightarrow{\vartheta} & \langle \mathcal{I}, E \rangle \hookrightarrow \mathcal{M} \end{array}$$

Moreover, the restrictions $\pi_A : \langle \vartheta(A)_E, E \rangle \longrightarrow A^I_D|F$ are isomorphisms for every $A \in V_\alpha(X)$.

The above theorem generalizes a result proved by H.J. Keisler for $\alpha = \omega$ (see [K2] § 1.E* and [CK] §6.4). A detailed proof for the general case is given in the Appendix.

$\vartheta(\{a_1, \dots, a_n\}_E) = \{\vartheta(a_1), \dots, \vartheta(a_n)\}$ for every finite collection of elements in $V_\alpha(X)$. More generally, if $A \in V_\alpha(X)$ then $\vartheta(A)_E$ includes the image set $\vartheta[A] = \{\vartheta(a) : a \in A\}$. With regard to this, the following result holds:

Proposition 1.2 *The following properties are equivalent:*

- (i) For every infinite $A \in V_\alpha(X) \setminus X$, $\vartheta[A] \neq \vartheta(A)_E$
- (ii) For every denumerable $A \in V_\alpha(X) \setminus X$, $\vartheta[A] \neq \vartheta(A)_E$
- (iii) There exists a denumerable $A \in V_\alpha(X) \setminus X$ with $\vartheta[A] \neq \vartheta(A)_E$
- (iv) There exists $A \in V_\alpha(X) \setminus X$ with $|A|$ less than the first uncountable measurable cardinal and $\vartheta[A] \neq \vartheta(A)_E$.

Moreover, if $|X|$ and $|\alpha|$ are less than the first uncountable measurable cardinal, also the following property is equivalent:

- (v) There exist internal elements which are not standard.

PROOF. Implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v)$ are trivial. $(iii) \Rightarrow (i)$ Take A as in the hypothesis. For every infinite $B \in V_\alpha(X)$ take a function $f : B \rightarrow A$ onto A and consider $\vartheta(f)_E : \vartheta(B)_E \rightarrow \vartheta(A)_E$. Suppose by absurd $\vartheta(B)_E = \vartheta[B]$. Then for every $a \in \vartheta(A)$ there exists $b \in B$ such that $\vartheta(f)_E(\vartheta(b)) = \vartheta(f(b)) = a$. We must conclude that $\vartheta[A] = \vartheta(A)_E$, a contradiction. $(iv) \Rightarrow (iii)$ Thanks to the Representation Theorem, we can identify $\vartheta : V_\alpha(X) \rightarrow \mathcal{I}$ with a diagonal immersion $d : V_\alpha(X) \rightarrow {}^bV_\alpha(X)^{I_D}|F$, up to isomorphisms. Now take A as in the hypothesis. $\vartheta[A] \neq \vartheta(A)_E$ iff $d : A \rightarrow A_E^J|F$ is not onto and thus there exists a non-trivial ultrapower A_E^J and an elementary embedding $\tau : A_E^J \rightarrow A_D^I|F$ such that the following diagram commutes:³

$$\begin{array}{ccc} & A_E^J & \\ d \nearrow & & \searrow \tau \\ A & \xrightarrow{d} & A_D^I|F \end{array}$$

Since A_E^J is non-trivial and $|A|$ is less than the first uncountable measurable cardinal, it follows that E is countably incomplete.⁴ Thus for every infinite $B \subseteq A$, $d : B \rightarrow B_E^J \subseteq B_D^I|F$ is not onto, i.e. $\vartheta[B] \neq \vartheta(B)_E$. $(v) \Rightarrow (iv)$ Take $A \in V_\alpha(X)$ such that $\vartheta[A] \neq \vartheta(A)_E$. From the hypotheses follows that $|A|$ is less than the first uncountable measurable cardinal, in that every measurable cardinal is inaccessible.⁵ $\dashv$

From now on, we shall only consider non-trivial NSE's, i.e. NSE's which satisfy the above properties.

Other properties of the internal model are in the following:

Proposition 1.3 *Let Q be the theory whose axioms are $W.Ext$, $Pair$, Δ_o-Sep , Un , Pow , AC , Fnd .⁶ Then*

³This property is a well-known fact about limit ultrapowers: see [K1] Theorem 3.6.

⁴See [CK] Corollary 4.2.8.

⁵See [J] Lemma 27.2.

⁶ $W.Ext$ is the following weak version of the axiom of extensionality:

$$\forall x \forall y [\exists t \in x \wedge \forall s (s \in x \leftrightarrow s \in y)] \rightarrow x = y$$

Contrarily to the axiom of extensionality, $W.Ext$ allows a (possibly empty) collection of atoms. Δ_o-Sep denotes the axiom schema of separation restricted to bounded formulae.

- (i) $\langle \mathcal{I}, E \rangle \models Q$
- (ii) $\langle \mathcal{I}, E \rangle \models Inf \iff \alpha > \omega$
- (iii) Σ_2^Q formulae are preserved under $\vartheta : \langle V_\alpha(X), \in \rangle \longrightarrow \langle \mathcal{I}, E \rangle$, i.e.

$$\langle V_\alpha(X), \in \rangle \models \varphi(a_1, \dots, a_n) \implies \langle \mathcal{I}, E \rangle \models \varphi(\vartheta(a_1), \dots, \vartheta(a_n))$$
for every $\varphi \in \Sigma_2^Q$ and for every assignment $a_1, \dots, a_n \in V_\alpha(X)$.

Remind the *Levi's hierarchy*. The collections of formulae Σ_n, Π_n are defined by induction as follows. $\Delta_o = \Sigma_o = \Pi_o$ is the collection of the bounded formulae; a formula φ is a Σ_{n+1} (resp. Π_{n+1}) formula if $\varphi \equiv \exists x\psi$ (resp. $\varphi \equiv \forall x\psi$) where ψ is a Π_n (resp. Σ_n) formula. If T is any theory, φ is a Σ_n^T (resp. Π_n^T) formula if there exists a Σ_n (resp. Π_n) formula such that $T \vdash \varphi \leftrightarrow \psi$. For a survey of basic results on Levy's hierarchy, see [D] Ch.3, §2.

PROOF OF THE PROPOSITION. (i) is proved by straightforward applications of the *transfer* principle. E.g. the *Pair* axiom is proved by applying the *transfer* principle to the following formulae

$$V_\alpha(X) \models \forall x, y \in V_\beta(X) \exists z \in V_{\beta+1}(X) z = \{x, y\}$$

for every $\beta < \alpha$. (iii) Since $\langle V_\alpha(X), \in \rangle \models Q$ and $\langle \mathcal{I}, E \rangle \models Q$, one can suppose without loss of generality that $\varphi \in \Sigma_2 \subseteq \Sigma_2^Q$, namely $\varphi \equiv \exists x \forall y \psi$ where ψ is bounded. For every given assignment $a_1, \dots, a_n \in V_\alpha(X)$, notice that $V_\alpha(X) \models \exists x \forall y \psi(x, y, a_1, \dots, a_n)$ iff there exists $\hat{a} \in V_\alpha(X)$ such that

$$\langle V_\alpha(X), \in \rangle \models \forall y \in V_\beta(X) \psi(\hat{a}, y, a_1, \dots, a_n)$$

for every $\beta < \alpha$. Then apply the *transfer* principle β times and get

$$\langle \mathcal{I}, E \rangle \models \forall y \psi(\vartheta(\hat{a}), y, \vartheta(a_1), \dots, \vartheta(a_n))$$

which yields the thesis. (ii) *Inf* is a Σ_1 -formula:

$$\exists x [\emptyset \in x \wedge (\forall y \in x \ y \cup \{y\} \in x)]$$

and $\neg Inf$ is a $\Pi_1^Q \subseteq \Sigma_2^Q$ formula. Thus one can apply (iii). $\dashv$

Property (iii) cannot be extended to Π_2^Q -formulae. The formula used in the proof of Prop. 3.1 will give a counter-example.

2 Hyperordinals

In this section we shall extend the notion of Von Neumann ordinal to NSM's. Let $Ord(x, \emptyset)$ be the following bounded formula:

$$\begin{aligned} & \text{"}x \text{ is transitive"} \wedge \text{"}x \text{ is trichotomic"} \wedge \\ & \neg Atom(x, \emptyset) \wedge (\forall x' \in x \neg Atom(x', \emptyset)) \end{aligned}$$

where *"x is trichotomic"* is the formula⁷

$$\text{"}\forall a, b \in x (a = b \text{ aut } a \in b \text{ aut } b \in a)\text{"}$$

and $Atom(y, \emptyset)$ is the formula

$$\text{"}y \neq \emptyset \wedge (\forall t \in y \ t \neq t)\text{"}.$$

Then

$$\langle V_\alpha(X), \in \rangle \models Ord(\beta, \emptyset) \iff \beta < \alpha \text{ is a Von Neumann ordinal}$$

We call *hyperordinal* any element $m \in \mathcal{M}$ satisfying $\mathcal{M} \models Ord(m, \vartheta(\emptyset))$. Denote $Ord_{\mathcal{M}}$ the collection of all the hyperordinals and denote $Ord_{\mathcal{I}}$ the collection of the internal hyperordinals.

The next proposition itemizes some basic properties of $Ord_{\mathcal{I}}$. For the sake of simplicity, if Λ is any collection of elements in $\mathcal{M}$, we shall write $\Lambda \in \mathcal{M}$ ($\Lambda \in \mathcal{I}$ resp.) whenever $\Lambda = m_{\mathbb{E}}$ for some $m \in \mathcal{M}$ ($m \in \mathcal{I}$ resp.).

Proposition 2.1

- (i) $Ord_{\mathcal{I}} = \bigcup_{\beta < \alpha} \vartheta(\beta)_{\mathbb{E}}$
- (ii) $\langle Ord_{\mathcal{I}}, \mathbb{E} \rangle$ is a linearly ordered set
- (iii) $Cofinality \langle Ord_{\mathcal{I}}, \mathbb{E} \rangle = Cofinality \alpha$
- (iv) $\xi' \mathbb{E} \xi \in Ord_{\mathcal{I}} \Rightarrow \xi' \in Ord_{\mathcal{I}}$
- (v) Every $\xi \in Ord_{\mathcal{I}}$ has immediate successor $\xi + 1 \in Ord_{\mathcal{I}}$ where $(\xi + 1)_{\mathbb{E}} = \xi_{\mathbb{E}} \cup \{\xi\}_{\mathbb{E}}$
- (vi) Every non-empty internal collection $\Lambda \in \mathcal{I}$ of hyperordinals admits minimum and supremum in $\langle Ord_{\mathcal{I}}, \mathbb{E} \rangle$
- (vii) $Ord_{\mathcal{I}} \notin \mathcal{I}$

⁷*aut* is the "exclusive or" connective, i.e. *" φ aut ψ "* stands for *" $(\varphi \vee \psi) \wedge (\neg \varphi \vee \neg \psi)$ "*.

PROOF. (i),(ii),(iv),(v) and (vi) are all proved by straightforward applications of the *transfer* principle to suitable formulae. As examples, let's see (i) and (vi) in detail. (i) For every $\beta < \alpha$

$$V_\alpha(X) \models \forall x (x \in V_\beta(X) \wedge \text{Ord}(x, \emptyset)) \leftrightarrow x \in \beta$$

Thus $\text{Ord}_\mathcal{M} \cap \vartheta(V_\beta(X))_\mathbf{E} = \vartheta(\beta)_\mathbf{E}$ and the thesis follows from the equalities $\text{Ord}_\mathcal{I} = \mathcal{I} \cap \text{Ord}_\mathcal{M} = \bigcup_{\beta < \alpha} \vartheta(V_\beta(X))_\mathbf{E} \cap \text{Ord}_\mathcal{M} = \bigcup_{\beta < \alpha} \vartheta(\beta)_\mathbf{E}$. (vi) Suppose $\Lambda \mathbf{E} \vartheta(V_\beta(X))$. Then apply the *transfer* principle to the formula

$$\begin{aligned} V_\alpha(X) \models \forall x \in V_\beta(X) [x \neq \emptyset \wedge \forall t \in x \text{Ord}(t, \emptyset)] \rightarrow \\ \{ [\exists \xi \in x \forall \xi' \in x (\xi = \xi' \vee \xi \in \xi')] \wedge \\ [\exists \eta \in \beta (\forall \xi \in x (\xi = \eta \vee \xi \in \eta)) \wedge \\ \forall \eta' \in \eta (\exists \xi \in x \eta' \in \xi)] \} \end{aligned}$$

(iii) trivially follows from (i). (vii) Suppose $\xi_\mathbf{E} = \text{Ord}_\mathcal{I}$ for some $\xi \in \mathcal{I}$. A glance at the definition of hyperordinal shows that $\xi \in \text{Ord}_\mathcal{M}$ and thus $\xi \mathbf{E} \xi$, which contradicts $\langle \mathcal{I}, \mathbf{E} \rangle \models \text{Fnd}$. $\neg$

We remark that (in general) $\mathbf{E}$ is not a total ordering on $\text{Ord}_\mathcal{M}$. E.g., take $\mathcal{N} = \langle {}^bV_\alpha(X)_D^I, \in_\mathcal{N} \rangle$ any non-trivial bounded ultrapower and consider the structure $\mathcal{M} = \langle M, \in_\mathcal{M} \rangle$ defined as follows:

- The universe M is the union ${}^bV_\alpha(X)_D^I \cup \mathcal{Z}$ where $\langle \mathcal{Z}, < \rangle$ is a copy of the ordered set of the integers (we suppose ${}^bV_\alpha(X)_D^I \upharpoonright F \cap \mathcal{Z} = \emptyset$).
- $m' \in_\mathcal{M} m \stackrel{\text{def}}{\iff} (m, m' \in \mathcal{N} \text{ and } m' \in_\mathcal{N} m) \text{ or } (m, m' \in \mathcal{Z} \text{ and } m' < m)$

The inclusion $\iota : \mathcal{N} \hookrightarrow \mathcal{M}$ preserves the bounded formulae because $\mathcal{M}$ is an $\mathbf{E}$ -transitive extension of $\mathcal{N}$.⁸ Thus the composition

$$\vartheta = \iota \circ d : \langle V_\alpha(X), \in \rangle \longrightarrow \mathcal{M}$$

yields the NSM $\mathcal{M}$. As one can easily verify, $\mathcal{M} \models \text{Ord}(\zeta, \vartheta(\emptyset))$ for every $\zeta \in \mathcal{Z}$ while $\zeta \neq \xi$, $\zeta \notin_\mathcal{M} \xi$ and $\xi \notin_\mathcal{M} \zeta$ for any hyperordinal $\xi \in {}^bV_\alpha(X)_D^I$. We conclude that hyperordinals exist which are not confrontable.

Similarly as the internal model, the internal hyperordinals admit an algebraic characterization.

⁸See [D] Ch.3 Theorem 4.11.

Theorem 2.2 Let $\vartheta : V_\alpha(X) \rightarrow \mathcal{M}$ be a NSE. Take D ultrafilter over I , F filter over $I \times I$ and $\pi : \mathcal{I} \rightarrow {}^bV_\alpha(X)_D^I|F$ isomorphism as in the Representation Theorem. Then the restriction $\tau = \pi \upharpoonright \text{Ord}_{\mathcal{I}} : \text{Ord}_{\mathcal{I}} \rightarrow {}^b\alpha_D^I|F$ is also an isomorphism and the following diagram commutes:

$$\begin{array}{ccc} & {}^b\alpha_D^I|F & \\ d \nearrow & & \nwarrow \tau \\ \langle \alpha, \in \rangle & \xrightarrow{\vartheta \upharpoonright \alpha} & \langle \text{Ord}_{\mathcal{I}}, E \rangle \end{array}$$

PROOF. Notice that for every $b \in \mathcal{I}$ the following properties are equivalent:

$$\begin{aligned} \langle \mathcal{I}, E \rangle &\models \text{Ord}(b, \vartheta(\emptyset)) \\ {}^bV_\alpha(X)_D^I|F &\models \text{Ord}(\pi(b), d(\emptyset)) \\ \{i \in I : V_\alpha(X) &\models \text{Ord}(\pi(b)(i), \emptyset)\} \in D \\ \{i \in I : \pi(b)(i) < \alpha \text{ is a V.N. ordinal}\} &\in D \\ \pi(b) \in \alpha_D^I|F \cap {}^bV_\alpha(X)_D^I|F &= {}^b\alpha_D^I|F \quad \dashv \end{aligned}$$

A subset C of a linearly ordered set $\langle Y, < \rangle$ is a *Dedekind cut* if C is an initial segment (i.e. $y < c \in C \Rightarrow y \in C$) and C lacks maximum. A *gap* is a Dedekind cut C such that $Y \setminus C$ lacks minimum. The whole ordered set Y is a *final gap* if it is a gap of itself (i.e. if Y lacks maximum). An *initial gap* C is an empty gap. A gap C is called *proper* if it is neither final nor initial (i.e. if $C \neq \emptyset$ and $Y \setminus C \neq \emptyset$). It is a well-known fact in Robinsonian Analysis that the standard naturals are a gap of the hypernaturals. Similarly, the following holds:

Proposition 2.3 $\vartheta[\omega]$ is a gap of $\langle \vartheta(\omega)_E, E \rangle$.

In particular, there exist E -descending chains $\xi_1 E \xi$; $\xi_2 E \xi_1$; $\xi_3 E \xi_2 \dots$ for every $\xi E \vartheta(\omega)$ which is not standard. Notice that if $\alpha = \omega$ then every internal hyperordinal is standard: apply the *transfer* principle to

$$V_\omega(X) \models \forall x \in n+1 \rightarrow (x = 0 \vee \dots \vee x = n) \quad n = 0, 1, 2, \dots$$

Finally the internal model is well-founded if $\alpha = \omega$. In fact

$$xEyE\vartheta(V_n(X)) \Rightarrow xE\vartheta(V_{n-1}(X))$$

This completes the proof of the

Proposition 2.4 *The following properties are equivalent:*

- (i) $\alpha > \omega$.
- (ii) $\langle \mathcal{I}, E \rangle$ is non-well-founded.
- (iii) There exist internal hyperordinals which are not standard.

3 Strong NSM's

We now turn to consider the case when the NSM $\mathcal{M}$ is an elementary extension of the standard model, i.e. when the *transfer* principle holds for every (not necessarily bounded) formula. Call such an $\mathcal{M}$ a *strong nonstandard α -model*.

We remark that this notion is not considered in the traditional approach to nonstandard models, because the existence of strong $\in$ -NSM's contradicts the Foundation axiom even for $\alpha = \omega$. In fact the following holds:

Proposition 3.1 *Every strong NSM $\mathcal{M}$ is non-well-founded. In particular, no strong NSE's $\vartheta : \langle V_\alpha(X), \in \rangle \longrightarrow \langle V_\alpha(Y), \in \rangle$ between superstructures can exist.*

PROOF. Consider $\langle N, < \rangle \in V_\alpha(X)$ an order-isomorphic copy of the natural numbers. By taking A_n the Von Neumann ordinal corresponding to the natural number n , one proves that

$$V_\alpha(X) \models \forall n \in N \exists A_n \exists f : [0, n]_N \rightarrow A_n \forall a, b \in A_n (a < b \leftrightarrow f(a) \in f(b))$$

$f : [0, n]_N \rightarrow A_n$ denotes the formula " f is a function $\wedge$ Range(f) = $A_n \wedge$ Domain(f) $\subseteq N \wedge (x \in \text{Domain}(f) \leftrightarrow x \leq n)$ ". The *transfer* principle yields a linearly ordered set $\langle \vartheta(N)_E, <_\vartheta \rangle$. Moreover, for every given $\xi E \vartheta(N)$ one gets an element $B \in \mathcal{M}$ and a function $F : [0, \xi]_{\vartheta(N)} \rightarrow B_E$ with the

property " $x <_{\vartheta} y \Leftrightarrow F(x) \in F(y)$ ". $<_{\vartheta}$ denotes the total ordering on $\vartheta(N)_{\mathbf{E}}$ given by $\vartheta(<_N)_{\mathbf{E}} \subseteq \vartheta(N)_{\mathbf{E}} \times \vartheta(N)_{\mathbf{E}}$ and $[0, \xi]_{\vartheta(N)}$ the initial segment $\{xE\vartheta(N) : x \leq_{\vartheta} \xi\}$. It is a well-known fact in nonstandard analysis that there are infinite descending chains $\xi >_{\vartheta} \xi - 1 >_{\vartheta} \xi - 2 >_{\vartheta} \dots$ for every $\xi E\vartheta(N)$ which is not standard. Hence $\langle \mathcal{M}, \mathbf{E} \rangle$ is non-well-founded. $\dashv$

The above result and Proposition 2.2 in the previous section, show that only non-strong ω -NSE's can yield well-founded NSM's.

Remark that the formula $\varphi(N)$ we used in this proof is a Π_2^Q -formula which is not preserved by the map $\vartheta : \langle V_{\alpha}(X), \in \rangle \rightarrow \langle \mathcal{I}, \mathbf{E} \rangle$ (cfr. Prop. 1.3). In fact, $\langle \mathcal{I}, \mathbf{E} \rangle \models \varphi(\vartheta(N))$ implies that $\langle \mathcal{I}, \mathbf{E} \rangle$ is non-well-founded, while the internal model is well-founded for $\alpha = \omega$.

When $\mathcal{M}$ is a strong NSM, $\langle \text{Ord}_{\mathcal{M}}, \mathbf{E} \rangle$ turns out to be a linearly ordered set satisfying the same properties as the internal hyperordinals in Proposition 2.1.

Proposition 3.2 *Suppose $\mathcal{M}$ is a strong NSM. Then*

- (i) $\langle \text{Ord}_{\mathcal{M}}, \mathbf{E} \rangle$ is a linearly ordered set
- (ii) Every $\xi \in \text{Ord}_{\mathcal{M}}$ has immediate successor $\xi + 1$ where $(\xi + 1)_{\mathbf{E}} = \xi_{\mathbf{E}} \cup \{\xi\}$
- (iii) Every non-empty collection $\Lambda \in \mathcal{M}$ of hyperordinals admits supremum $\text{Sup } \Lambda \in \text{Ord}_{\mathcal{M}}$ and infimum $\text{Inf } \Lambda \in \text{Ord}_{\mathcal{M}}$
- (iv) $\text{Ord}_{\mathcal{M}} \notin \mathcal{M}$

PROOF. The *transfer* principle holds for every formula, thus one can proceed similarly as in the proof of proposition 2.1. $\dashv$

In analogy to Von Neumann ordinals, when dealing with hyperordinals we shall often write $\xi' <_{\mathbf{E}} \xi$ instead of $\xi' E \xi$. Putting together results in propositions 2.1 and 3.2, one gets the

Proposition 3.3 *Suppose $\mathcal{M}$ is a strong NSM. Then $\text{Ord}_{\mathcal{I}}$ is a Dedekind cut of $\langle \text{Ord}_{\mathcal{M}}, <_{\mathbf{E}} \rangle$.*

The question whether the Dedekind cut $\text{Ord}_{\mathcal{I}}$ is a gap of $\text{Ord}_{\mathcal{M}}$ naturally arises. In §5 we show that this problem is closely related to a "rigidity"

property of $\langle \mathcal{I}, E \rangle$ which is in turn related to the existence of non-regular uniform ultrafilters.

4 The Pseudorank Function

Every standard model is provided with a *rank function*, i.e. a function $\rho_X : V_\alpha(X) \rightarrow \alpha$ verifying:

$$\rho_X(A) = \begin{cases} 0 & \text{if } A \in X \text{ or } A = \emptyset ; \\ \text{Sup}\{\rho_X(a) + 1 : a \in A\} & \text{otherwise.} \end{cases}$$

Notice that ρ_X is well defined because X is the set of atoms relative to $\langle V_\alpha(X), \in \rangle$. The following properties straightforwardly follow from the definition of rank.

Proposition 4.1 *Let $\rho_X : V_\alpha(X) \rightarrow \alpha$ be the rank function. Then*

- (i) $\rho_X(\beta) = \beta$ for every ordinal $\beta < \alpha$
- (ii) $\rho_X(A) < \beta \Rightarrow A \in V_\beta(X) \Rightarrow \rho_X(A) < \beta + 1$
- (iii) $a \in A \Rightarrow \rho_X(a) < \rho_X(A)$

We raise the question whether such a function can exist for NSM's. In general, the answer is negative because of non-well-foundedness.

Following an idea by V.M. Tortorelli [T], our aim is to relax the concept of rank so as to provide every strong NSM with a pseudorank function. Precisely, we shall consider functions whose range is the linearly ordered set of the hyperordinals.

Notice that although the whole set ρ_X is not a member of $V_\alpha(X)$, suitable restrictions of ρ_X are elements of the standard model. Call $\rho_a \in V_\alpha(X)$ a *partial rank function (PRF)* for the set $a \in V_\alpha(X) \setminus X$ if the following conditions are verified:

- (i) ρ_a is a function
- (ii) $X \cup \{\emptyset, a\} \subseteq \text{Domain}(\rho_a)$, i.e. the domain of ρ_a contains a and the $\in$ -minimals

- (iii) $Domain(\rho_a)$ is transitive up to elements in X , i.e.
 $t \in x \in Domain(\rho_a) \wedge x \notin X \Rightarrow t \in Domain(\rho_a)$
- (iv) $\rho_a(t) = \begin{cases} 0 & \text{if } t \in X \text{ or } t = \emptyset ; \\ Sup\{\rho_a(s) + 1 : s \in t\} & \text{otherwise.} \end{cases}$
- (v) $Range(\rho_a)$ is a Von Neumann ordinal.

Observe that condition (iv) implies condition (v). The PRF is a definable notion. In fact, notice that

$$V_\alpha(X) \models \psi(x, f, t) \Leftrightarrow x = Sup\{f(s) + 1 : s \in t\}$$

where $\psi(x, f, t)$ denotes the following (bounded) formula

$$\begin{aligned} & (\forall s \in t \ f(s) + 1 = x \vee f(s) + 1 \in x) \wedge \\ & (\forall x' \in x \ \exists s \in t \ x' \in f(s) + 1) \end{aligned}$$

Thus

$$V_\alpha(X) \models Prf(a, f, \emptyset) \Leftrightarrow f \text{ is a PRF for } a$$

where $Prf(a, f, \emptyset)$ is the following Π_1^Q formula:

$$\begin{aligned} & \forall A \ \forall x \ \forall t \ \text{"}f \text{ is a function"} \wedge \\ & \text{"}A \text{ is the range of } f\text{"} \rightarrow Ord(A, \emptyset) \wedge \\ & \text{"}A \text{ is the domain of } f\text{"} \rightarrow \\ & \{ \emptyset \in A \wedge a \in A \wedge \text{"}A \text{ is transitive"} \wedge \\ & [(\forall s \in t \ s \neq s) \rightarrow (t \in A \wedge f(t) = \emptyset)] \wedge \\ & [(\exists s \in t \wedge t \in A \wedge f(t) = x) \rightarrow \psi(x, f, t)] \} \end{aligned}$$

For every $\beta < \alpha$, $\rho_\beta : V_\beta(X) \rightarrow \beta + 1$ will denote the restriction $\rho_x \upharpoonright V_\beta(X)$. If $\rho_x(a) = \beta$ then $\rho_{\beta+1}$ is a PRF for a . Vice versa if f is a PRF for a , then one can proceed by $\in$ -induction and show that $f(t) = \rho_x(t)$ for every $t \in Domain(f)$. Thus the following is proved:

Proposition 4.2

- (i) $V_\alpha(X) \models \forall a \ \exists f \ Prf(a, f, \emptyset)$
- (ii) $V_\alpha(X) \models \forall a, b \ \forall f, g \ \forall t \ [Prf(a, f, \emptyset) \wedge Prf(b, g, \emptyset) \wedge [(t \in Domain(f) \cap Domain(g)) \rightarrow f(t) = g(t)]$

We are ready to extend the notion of rank to strong NSM's.

Theorem 4.3 *There exists a pseudorank function $\psi_{\mathcal{M}} : \mathcal{M} \rightarrow \text{Ord}_{\mathcal{M}}$ such that*

- (i) $\psi_{\mathcal{M}}(m) = \begin{cases} \vartheta(0) & \text{if } m_{\text{E}} = \emptyset ; \\ \text{Sup}\{\psi_{\mathcal{M}}(m') + 1 : m' \text{E} m\} & \text{otherwise.} \end{cases}$
where $\{\psi_{\mathcal{M}}(m') + 1 : m' \text{E} m\} \in \mathcal{M}$ for every $m \in \mathcal{M}$.
- (ii) $\psi_{\mathcal{M}}(\xi) = \xi$ for every hyperordinal ξ .
- (iii) $\psi_{\mathcal{M}}(m) <_{\text{E}} \vartheta(\beta) \Rightarrow m \text{E} \vartheta(V_{\beta}(X)) \Rightarrow \psi_{\mathcal{M}}(m) <_{\text{E}} \vartheta(\beta + 1)$.
- (iv) $m' \text{E} m \Rightarrow \psi_{\mathcal{M}}(m') <_{\text{E}} \psi_{\mathcal{M}}(m)$.
- (v) $m \in \mathcal{I} \Leftrightarrow \psi_{\mathcal{M}}(m) \in \text{Ord}_{\mathcal{I}}$.

PROOF. Take the formulae in the previous proposition and apply the *transfer* principle. Then

- For every $m \in \mathcal{M}$, there exists a hyperordinal valued function ψ_m whose domain contains m_{E} and the E-minimals, and satisfying

$$\psi_m(t) = \begin{cases} \vartheta(0) & \text{if } t_{\text{E}} = \emptyset ; \\ \text{Sup}\{\psi_m(s) + 1 : s \text{E} t\} & \text{otherwise.} \end{cases}$$

It is easily proved by the *transfer* principle that $\{\psi_m(s) + 1 : s \in t\} \in \mathcal{M}$.

- Any two such functions coincide over the intersection set of the domains.

Thus, putting together the various ψ_m 's, one gets a function $\psi_{\mathcal{M}} : \mathcal{M} \rightarrow \text{Ord}_{\mathcal{M}}$ satisfying (i). Besides, (ii), (iii) and (iv) are proved by *transfer* from the properties (i), (ii) and (iii) of Prop. 4.1 respectively. Finally, (v) follows from (iii). $\dashv$

For $\alpha = \omega$, the submodel of the internal sets is well-founded and thus it can be isomorphically collapsed into a transitive submodel of an ω -superstructure.⁹ A similar result can be obtained also in our more general context $\alpha > \omega$. Precisely, the internal model is provided with a pseudorank function

⁹This fact follows from a modification of arguments in [CK] §4.4. A detailed proof can be found in [DN1]. Here, transitivity has to be intended up to atoms.

even for non-strong NSE's. Moreover, assuming a suitable Anti-Foundation Axiom, every internal model can be isomorphically collapsed into a transitive submodel of a *pseudosuperstructure*, which is a full and transitive $\in$ -structure.¹⁰ provided with a pseudorank function (see [DN2]) See also the recent paper [BH] by D. Ballard and K. Hrbacek, where a different but related anti-foundational approach is presented.

5 Regular Nonstandard Models

In this last section, we investigate the structure of strong NSM's by means of hyperordinals and show how the internal sets make up a "core" which is in some sense isolated from the rest of the model. A first result is the following

Proposition 5.1 *Suppose $\mathcal{M}$ is a strong nonstandard model. If $\Lambda \in \mathcal{M}$ is a bounded collection of internal sets, i.e. $\Lambda_E \subseteq \vartheta(V_\beta(X))_E$ for some $\beta < \alpha$, then Λ is internal too.*

PROOF. Apply *transfer* to $V_\alpha(X) \models \forall x (x \subseteq V_\beta(X) \rightarrow x \in V_{\beta+1}(X))$. $\dashv$

Thus every bounded collection of internal sets which cannot be in itself internal, is not a member of the nonstandard model. This is the case of many collections which play a central role in the practice of Robinsonian analysis, e.g. the standard reals, the infinite reals, the infinitesimals, the monads, the galaxies, etc.

We now raise the question whether the above property holds for every (not necessarily bounded) collection of internal sets. The next result shows that this "stability" property for the internal sets is closely related to the order structure of the hyperordinals.

Theorem 5.2 *Suppose $\mathcal{M}$ is a strong nonstandard model. Then the followings are equivalent:*

¹⁰An $\in$ -structure S is *full* if " $\forall s \in S \mathcal{P}(s) \subseteq S$ ".

- (i) For every $m \in \mathcal{M}$, $m_E \subseteq \mathcal{I} \Rightarrow m \in \mathcal{I}$
- (ii) Every external element contains external E -elements.
- (iii) $Ord_{\mathcal{I}}$ is a gap of the ordered set $\langle Ord_{\mathcal{M}}, <_E \rangle$
- (iv) $Ord_{\mathcal{I}} \notin \mathcal{M}$
- (v) $\mathcal{I} \notin \mathcal{M}$

PROOF. Equivalence (i) $\Leftrightarrow$ (ii) is trivial. (i) $\Rightarrow$ (v) Otherwise, let $m \in \mathcal{M}$ be such that $m_E = \mathcal{I}$. Now $m_E \subseteq \mathcal{I}$ implies $m \in \mathcal{I}$, hence mEm which contradicts $\langle \mathcal{I}, E \rangle \models Fnd$. (v) $\Rightarrow$ (iv) Suppose $Ord_{\mathcal{I}} \in \mathcal{M}$ and apply *transfer* to

$$\begin{aligned} V_\alpha(X) \models \forall \Lambda [\forall \lambda \in \Lambda \text{ } Ord(\lambda, \emptyset)] \rightarrow \\ \exists V \forall t [t \in V \leftrightarrow \exists \lambda \in \Lambda \forall f (Prf(t, f, \emptyset) \rightarrow f(t) \in \lambda)] \end{aligned}$$

One gets an element $\Upsilon \in \mathcal{M}$ such that

$$\Upsilon_E = \{m \in \mathcal{M} : \exists \xi \in Ord_{\mathcal{I}} \psi_{\mathcal{M}}(m) <_E \xi\} = \mathcal{I}$$

(iv) $\Rightarrow$ (iii) Otherwise let ξ be $\xi = \text{Min}(Ord_{\mathcal{M}} \setminus Ord_{\mathcal{I}})$. Then $\xi \in \mathcal{M}$ and $\xi_E = Ord_{\mathcal{I}}$, which contradicts (iv). (iii) $\Rightarrow$ (i) Suppose $m \in \mathcal{M}$ be such that $m_E \subseteq \mathcal{I}$ while $m \notin \mathcal{I}$. Take $\xi = \psi_{\mathcal{M}}(m) = \text{Sup}\{\psi_{\mathcal{M}}(m') + 1 : m'Em\}$. ξ is an external supremum of internal hyperordinals, hence $\xi = \text{Sup } Ord_{\mathcal{I}} = \text{Min}(Ord_{\mathcal{M}} \setminus Ord_{\mathcal{I}})$ and $Ord_{\mathcal{I}}$ is not a gap. $\dashv$

We shall call *regular* any strong nonstandard model which satisfies the above properties. The use of the adjective "regular" originates from theorem 5.6. For $\alpha = \omega$, the regularity question is settled by the

Theorem 5.3 *Every strong ω -NSM is regular.*

PROOF. Since every ordinal $0 < n < \omega$ is a successor, $\rho_x(A) = \text{Sup}\{\rho_x(a) + 1 : a \in A\}$ is a maximum for every $A \in V_\omega(X)$ which is not $\in$ -minimal. Thus

$$V_\omega(X) \models \forall A \forall f [\exists x \in A \wedge Prf(A, f, \emptyset)] \rightarrow (\exists a \in A \ f(A) = f(a) + 1)$$

By application of *transfer* it follows that for every given $m \in \mathcal{M}$ $\psi_{\mathcal{M}}(m) = \psi_{\mathcal{M}}(m') + 1$ for some $m'Em$. In particular, if $m_E \subseteq \mathcal{I}$ then $m \in \mathcal{I}$. $\dashv$

The above theorem yields a characterization for the internal sets of a strong ω -NSM. This result was first proved in [T] by other means.

Theorem 5.4 *Suppose $\mathcal{M}$ is a strong ω -NSM. Then the collection $\mathcal{I} \subseteq \mathcal{M}$ of the internal sets is characterized by the following properties:*

- (i) $\mathcal{I}$ contains the standard elements.
- (ii) $\mathcal{I}$ is E -transitive.
- (iii) $\langle \mathcal{I}, E \rangle$ is well-founded.

PROOF. It is known that $\mathcal{I}$ satisfies the above properties. Since $\mathcal{I}$ is the E -transitive closure of the standards, any other collection $\Lambda \subseteq \mathcal{M}$ satisfying (i) and (ii) contains the internal sets. Moreover no external set can be a member of Λ . Otherwise let $m \in \Lambda$ be external. The regularity of $\mathcal{M}$ yields the existence of an external element m_1Em . By iteration, one gets an E -descending chain $m_1Em, m_2Em_1, m_3Em_2, \dots$ which contradicts (iii). $\dashv$

Let's now concentrate on the regularity problem when $\alpha > \omega$. This problem happens to be in some way connected to the existence of non-regular uniform ultrafilters.

Remind that a filter D over the set I is *regular*¹¹ if it satisfies the following equivalent conditions: (i) There exists a family $\{D_i : i \in I\} \subseteq D$ of members of D such that $\bigcap \{D_i : i \in I_o\} = \emptyset$ for every infinite $I_o \subseteq I$; (ii) there exists a family $\{\mathcal{J}_i : i \in I\}$ of finite subsets of I such that $\{i \in I : j \in \mathcal{J}_i\} \in D$ for every $j \in I$. The filter D is called *uniform* if $|X| = |I|$ for every $X \in D$. Regular uniform ultrafilters exist over any given infinite set.

The notion of regularity of ultrafilters was isolated as a consequence of the following result:

- Suppose $\mathcal{M}$ is any infinite structure of cardinality μ and D is a uniform regular ultrafilter on an index set I of cardinality κ . Then $\mathcal{M}_D^I$ has the largest cardinality possible, i.e. μ^κ .

The next proposition is a generalization to limit ordinals of a known result.¹²

Proposition 5.5 *Suppose α is a limit ordinal and D a regular ultrafilter over a set I with $|I| \geq |\alpha|$. If $f^D \in \alpha_D^I$ is "unbounded", i.e. $f^D > d(\gamma)$ for every $\gamma < \alpha$, then $f^D > g^D$ for some unbounded g^D .*

¹¹For a collection of basic results on regular ultrafilters, see [CK] §4.3.

¹²See [J] §38.

PROOF. Let $\{\mathcal{J}_i : i \in I\}$ be a family of finite subsets of I such that $\{i \in I : j \in \mathcal{J}_i\} \in D$ for every $j \in I$, and $\sigma : \kappa \rightarrow \alpha$ a surjective map. Then for every $\gamma < \alpha$, $\{i \in I : \gamma \in \sigma[\mathcal{J}_i]\} \in D$.¹³ Notice that $f^D > d(0)$ implies

$$\Xi = \{i \in \kappa : \exists \gamma \in \sigma[\mathcal{J}_i] \text{ with } \gamma < f(i)\} \in D$$

because $\Xi \supseteq \{i \in I : 0 \in \sigma[\mathcal{J}_i] \text{ and } 0 < f(i)\} \in D$. This fact and the finiteness of the $\mathcal{J}_i$'s gives sense to the following definition:

$$g(i) = \begin{cases} \text{Max}\{\gamma \in \sigma[\mathcal{J}_i] : \gamma < f(i)\} & \text{if } i \in \Xi \\ 0 & \text{otherwise.} \end{cases}$$

Trivially $f^D > g^D$. Moreover, if $f^D > d(\gamma + 1)$ then $g^D > d(\gamma)$ because $\{i \in I : g(i) > \gamma\} \supseteq \{i \in I : \gamma \in \sigma[\mathcal{J}_i] \text{ and } f(i) > \gamma + 1\} \in D$. We conclude that also g^D is unbounded. $\dashv$

Regular ultrafilters yield regular NSM's (and this originates the name).

Theorem 5.6 *Take D any regular ultrafilter over a set I with $|I| \geq |\alpha|$. Then the diagonal immersion $d : V_\alpha(X) \rightarrow V_\alpha(X)^I_D$ yields a regular α -NSM.*

PROOF. Take $d : V_\alpha(X) \rightarrow V_\alpha(X)^I_D$ the NSE given by the diagonal immersion. Notice that $\xi^D \in \text{Ord}_M \Leftrightarrow \{i \in I : \xi(i) \text{ is a V.N. ordinal}\} \in D \Leftrightarrow \xi^D \in \alpha^I_D$. Moreover $\xi^D \notin \text{Ord}_I \Leftrightarrow \xi^D$ is unbounded. Thus one can apply the previous proposition and get the thesis. $\dashv$

Using the core model, H.-D. Donder [DO] proved that if ZFC is consistent, then so is $ZFC +$ "every uniform ultrafilter over an infinite set is regular". Thus the following holds:

Corollary 5.7 *There is a model of ZFC where every diagonal immersion $d : V_\alpha(X) \rightarrow V_\alpha(X)^I_D$ of the standard model into any of its ultrapowers with $|I| \geq |\alpha|$ yields a regular NSM.*

An extensive literature indicates that the existence problem for uniform ultrafilters which are not regular is closely connected with the existence of large cardinals.¹⁴ We shall use the following:¹⁵

¹³ $\sigma[\mathcal{J}_i]$ denotes the image set $\{\sigma(j) : j \in \mathcal{J}_i\}$.

¹⁴E.g. see [FMS].

¹⁵See [J] §38.

Proposition 5.8 *Suppose D is a uniform non-regular ultrafilter over ω_1 . Then the D -equivalence class of the diagonal function $\Delta : \gamma \mapsto \gamma \ \gamma < \alpha$, is the least unbounded element in $\omega_1^{\omega_1}_D$.*

A recent and deep result by M. Foreman, M. Magidor and S. Shelah [FMS], states that if a huge cardinal exists then there is a model provided with an ultrafilter D such that $|\omega_D^{\omega_1}| = \omega_1$. In particular such a D yields a non-regular uniform ultrafilter over ω_1 . If one proceeds as in the proof of Theorem 5.6, this fact together with the previous proposition yields the

Theorem 5.9 *The existence of non-regular ω_1 -NSM's is consistent relative to $ZFC + \text{"there exists a huge cardinal"}$.*

We conclude by raising two interesting problems:

- Is there a model of ZFC where every strong NSM is regular ?
- Which relative consistency relations hold between the existence of non-regular strong NSM's and various large cardinal axioms ?

Appendix.

PROOF OF THE REPRESENTATION THEOREM. Denote $\mathcal{L}_*$ the language of set theory whose set of variables is $\{x_b : b \in \mathcal{I}\}$ instead of $\{x_n : n < \omega\}$. For every model $\mathcal{N} = \langle N, \in_{\mathcal{N}} \rangle$, for every formula φ of $\mathcal{L}_*$ and for every assignment $\chi : \mathcal{I} \rightarrow N$, the satisfaction relation $\models_*$ is defined as follows:

$$\mathcal{N} \models_* \varphi(x_{b_1}, \dots, x_{b_n}) [\chi] \stackrel{\text{def}}{\iff} \mathcal{N} \models \varphi(\chi(b_1), \dots, \chi(b_n))$$

Now take $\mathcal{B}_* = \{\psi_i : i \in I\}$ the set of all the bounded formulae of $\mathcal{L}_*$ such that $\langle \mathcal{I}, E \rangle \models_* \psi_i [1]$ ($1 : \mathcal{I} \rightarrow \mathcal{I}$ denotes the identity map). Fix $\psi_i \in \mathcal{B}_*$ a formula all of whose free variables are among $x_{b_1}, \dots, x_{b_n}$. Without loss of generality, suppose that $b_s = \vartheta(a_s)$ is standard if and only if $s = 1, \dots, k$ and denote $\beta_s = \text{Min}\{\beta : b_s E \vartheta(V_\beta(X))\}$ $s = k+1, \dots, n$. Then apply *transfer* to $\langle \mathcal{I}, E \rangle \models_* \psi_i(x_{b_1}, \dots, x_{b_n}) [1]$ and get that $\langle V_\alpha(X), \in \rangle \models \psi_i(a_1, \dots, a_k, a_{k+1}, \dots, a_n)$ for some $a_s \in V_{\beta_s+1}(X)$ $s = k+1, \dots, n$. This fact yields the following property:

$\forall i \in I$ there exists an assignment $\chi_i : \mathcal{I} \rightarrow V_\alpha(X)$ such that

- (i) $\langle V_\alpha(X), \in \rangle \models_* \psi_i [\chi_i]$
- (ii) $\forall \beta < \alpha \ bE\vartheta(V_\beta(X)) \Rightarrow \chi_i(b) \in V_\beta(X)$
- (iii) $\forall a \in V_\alpha(X) \ \chi_i(\vartheta(a)) = a$

The family $\mathcal{D} = \{D_i : i \in I\}$ where $D_i = \{j \in I : \langle V_\alpha(X), \in \rangle \models_* \psi_i [\chi_j]\}$ has the *finite intersection property (FIP)* (notice that the formula $\psi_{i_1} \wedge \dots \wedge \psi_{i_n}$ belongs to $D_{i_1} \cap \dots \cap D_{i_n}$). Thus we can take an ultrafilter $D \supseteq \mathcal{D}$ on I and define $\pi : \mathcal{I} \rightarrow V_\alpha(X)^I_D$ as follows. For every $b \in \mathcal{I}$, let $\pi(b) = f_b^D$ be the D -equivalence class of the function $f_b : i \mapsto \chi_i(b)$. From (ii) it follows that $\text{Range}(\pi) \subseteq {}^bV_\alpha(X)^I_D$. Besides, (iii) yields that $\pi \circ \vartheta = d : V_\alpha(X) \hookrightarrow {}^bV_\alpha(X)^I_D|F$ is the diagonal immersion. Now take φ any bounded formula, $b_1, \dots, b_n \in \mathcal{I}$ and suppose $\langle \mathcal{I}, E \rangle \models \varphi(b_1, \dots, b_n)$. Then

$$\begin{aligned} {}^bV_\alpha(X)^I_D \models \varphi(\pi(b_1), \dots, \pi(b_n)) &\iff \\ \{j \in I : V_\alpha(X) \models \varphi(\pi(b_1)(j), \dots, \pi(b_n)(j))\} &= \\ \{j \in I : V_\alpha(X) \models \varphi(\chi_j(b_1), \dots, \chi_j(b_n))\} &= \\ \{j \in I : V_\alpha(X) \models_* \varphi(x_{b_1}, \dots, x_{b_n}) [\chi_j]\} &= D_i \in D \end{aligned}$$

and this last property is true by the definition of D . Our next step is to single out a suitable submodel of ${}^bV_\alpha(X)^I_D$ in such a way that the *transfer* principle still holds and π is onto (hence an isomorphism). The family $\mathcal{F} = \{\text{eq}(f) : f^D \in \text{Range}(\pi)\}$ where $\text{eq}(f) = \{\langle i, j \rangle \in I \times I : f(i) = f(j)\}$ has the FIP. Take F the filter over $I \times I$ generated by $\mathcal{F}$. We claim that $\pi : \mathcal{I} \rightarrow {}^bV_\alpha(X)^I_D|F$ is onto. Fix $f^D \in {}^bV_\alpha(X)^I_D$ with $\text{eq}(f) \in F$. By the definition, there exist $g_s^D = \pi(b_s)$ $s = 1, \dots, n$ such that

$$\text{eq}(g_1) \cap \dots \cap \text{eq}(g_n) \subseteq \text{eq}(f).$$

Take $\beta < \alpha$ such that $b_1, \dots, b_n \in \vartheta(V_\beta(X))$ (hence $g_1^D, \dots, g_n^D \in V_\beta(X)^I_D$) and also $f^D \in V_\beta(X)^I_D$. The above inclusion assures the existence of an n -ary function $\sigma : V_\beta(X)^n \rightarrow V_\beta(X)$, with $\sigma : \langle g_1(i), \dots, g_n(i) \rangle \mapsto f(i) \ \forall i \in I$. Notice that $\sigma \in V_{\beta+2n+2} \subseteq V_\alpha(X)$. Thus

$${}^bV_\alpha(X)^I_D|F \models d(\sigma) : \langle g_1^D, \dots, g_n^D \rangle \mapsto f^D$$

Take $\widehat{b}E\vartheta(V_\beta(X))$ with $\vartheta(\sigma)_E : \langle b_1, \dots, b_n \rangle \mapsto \widehat{b}$ and apply the *transfer* to the mapping π so to get

$${}^bV_\alpha(X)^I_D|F \models \pi(\vartheta(\sigma)) : \langle \pi(b_1), \dots, \pi(b_n) \rangle \mapsto \pi(\widehat{b})$$

Recall that $\pi(\vartheta(\sigma)) = d(\sigma)$, $\pi(b_s) = g_s^D$ $s = 1, \dots, n$ and compare the two above formulae which hold in ${}^bV_\alpha(X)^I_D|F$. Then it necessarily follows that $f^D = \pi(\widehat{b}) \in \text{Range}(\pi)$.

The last statement of the theorem is proved by noticing that for every $a \in \mathcal{I}$, $aE\vartheta(A) \Leftrightarrow \pi(a)E\pi(\vartheta(A)) = d(A) \Leftrightarrow \pi(a) \in A^I_D|F$. $\dashv$

Acknowledgments. The author would like to thank Marco Forti for many helpful suggestions on preliminary drafts of this paper and Piero Ghilardi for his encouragement.

References

- [BH] D. BALLARD and K. HRBACEK, *Standard Foundations for Nonstandard Analysis*, **J. Symb. Logic**, vol. **57** (1992), pp. 741-748.
- [CK] C.C. CHANG and H.J. KEISLER, **Model Theory**, 3rd edition, North-Holland, Amsterdam, 1990 (1st edition 1973).
- [D] F.R. DRAKE, **Set Theory**, North-Holland, Amsterdam, 1974.
- [DN1] M. DI NASSO, *Ph.D. Thesis*, in preparation.
- [DN2] M. DI NASSO, *Pseudo-superstructures as Nonstandard Universes*, in preparation.
- [Do] H.-D. DONDER, *Regularity of Ultrafilters and the Core model*, **Israel J. Math.**, vol. **63** (1988), pp. 289-322.
- [FMS] M. FOREMAN, M. MAGIDOR and S. SHELAH, *Martin's Maximum, Saturated Ideals and non-Regular Ultrafilters. Part II*, **Ann. Math.**, vol. **127** (1988), pp. 521-545.

- [J] T. JECH, **Set Theory**, Academic Press, New York, 1978.
- [K1] H.J. KEISLER, *Limit Ultrapowers*, **Trans. Amer. Math. Soc.**, vol. **107** (1963), pp. 382-408.
- [K2] H.J. KEISLER, **Foundations of Infinitesimal Calculus**, Prindle, Weber & Schmidt, Boston, 1976.
- [RZ] A. ROBINSON and E. ZAKON, *A Set-Theoretical Characterization of Enlargements*, in **Applications of Model Theory to Algebra, Analysis and Probability** (W.A.J. Luxemburg, ed.), Holt, Rinehart and Winston, New York, 1969, pp. 109-122.
- [SS] J. SCHMID and J. SCHMIDT, *Enlargements without Urelements*, **Colloq. Math.**, vol. **52** (1987), pp. 1-22.
- [T] V.M. TORTORELLI, *A Characterization of Internal Sets*, **Rend. Sem. Mat. Univ. Padova**, vol. **81** (1989), pp. 193-200.
- [Z] E. ZAKON, *A New Variant of Nonstandard Analysis*, in **Victoria Symposium on Nonstandard Analysis** (A.E. Hurd and P.A. Loeb, eds.), *Lecture Notes in Mathematics*, Vol. **369**, Springer-Verlag, Berlin, 1974, pp. 313-339.

DIPARTIMENTO DI MATEMATICA
 UNIVERSITÀ DI SIENA
 VIA DEL CAPITANO, 15
 53100 SIENA, ITALY.
 E-MAIL: dinasso@unisi.it