NONSTANDARD ANALYSIS BY MEANS OF IDEAL VALUES OF SEQUENCES

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Abstract

A notion of *ideal* value of N -sequences is axiomatized through elementary properties. The resulting theory $\text{ZFC}[\alpha]$ provides nonstandard analysis with a general foundational framework.

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In this paper, the axiomatic system $\text{ZFC}[\alpha]$ is presented. It is a generalization in a set theoretic context of an approach to nonstandard analysis recently given by V. Benci in [BE]. In that paper, nonstandard analysis is presented by postulating, besides a nonstandard embedding $*$: $V(X) \rightarrow V(X)$ from a superstructure into itself, the existence of a *pre-extension* for each function defined on the set N of natural numbers. Axioms are then given that correlate pre-extensions with the enlarging map $*$. Here, we formalize the idea of pre-extension by means of a notion of *ideal value*. Precisely, we enrich the language of set theory with a constant symbol α (to be intended as a new *ideal* natural number) and axiomatize the *ideal value* $\varphi(\alpha)$ of N -sequences φ .

The idea of adding a new logical constant to behave as an “infinite” natural number is not new. Precisely, it goes back to the pioneering work by C. Schmieden and D. Laugwitz [SL], appeared before Robinson’s presentation of nonstandard analysis. A general way of formally giving a “nonstandard extension” of a given first order theory by means of a new constant symbol, can be found in Laugwitz’ work [LA].

In this paper, an approach is given in the full generality of set theory. As a consequence of our axioms, a nonstandard embedding $*$: $\mathbf{V} \rightarrow \mathbf{V}$ is determined for the universal class of all sets, and all basic principles of nonstandard analysis, including *transfer* and countable *saturation*, are proved. All principles of mathematics as formalized by Zermelo-Fraenkel set theory with choice are assumed, with the only exception of regularity. Thus, working in $\text{ZFC}[\alpha]$, one can manipulate (external) sets as in ordinary mathematics with no restrictions. Also, nonstandard arguments can be iterated, because the enlarging map $*$ is defined for all sets. For instance, one can consider the *hyper*-hyperreal numbers

$^*(^*R)$ or the nonstandard extension $^*\mathcal{L}$ of a *Loeb measure space* $\mathcal{L}$ etc. Another feature of our axiomatic system $\text{ZFC}[\alpha]$ is that it consists of few simple properties expressed in elementary terms, thus making it possible to present them even at the freshman level.¹ For basic notions from set theory and model theory, we refer to [JE] and [CK], respectively.

Let $L = \{\in, \alpha\}$ be the language of set theory together with a new constant symbol α . $\text{ZFC}[\alpha]$ is the theory of L given by the following seven axioms.²

1. ZFC^-

All principles of mathematics, as axiomatized by Zermelo-Fraenkel set theory with choice but without regularity, are assumed. The replacement and separation schemata are also considered for formulas where the symbol α occurs.³

2. $\alpha \notin N$.

3. Every sequence $\varphi : N \rightarrow A$ has a unique ideal value $\varphi(\alpha)$.⁴

For simplicity, we will use the same symbol to denote a sequence $\varphi : N \rightarrow A$ and its *pre-extension* $\varphi : N \cup \{\alpha\} \rightarrow A \cup \{\varphi(\alpha)\}$. The remaining axioms give some natural properties of the ideal values.

4. The ideal extension of the identity map $1_N : n \mapsto n$ on N is the identity map on $N \cup \{\alpha\}$, i.e. $1_N(\alpha) = \alpha$.

5. For every sequence φ :

$$\varphi(\alpha) = \{\psi(\alpha) : \psi \text{ is a sequence with } \psi(n) \in \varphi(n) \text{ for all } n \in N\}$$

In other words, the elements of the ideal value $\varphi(\alpha)$ are precisely the ideal values of sequences that are “point-wise” elements of φ . In particular, if $c_\emptyset$ is the sequence constantly equal to the emptyset, then $c_\emptyset(\alpha) = \emptyset$. The next axiom states that pairs are preserved at the ideal point α .

6. If $\vartheta(n) = \{\varphi(n), \psi(n)\}$ for all $n \in N$, then $\vartheta(\alpha) = \{\varphi(\alpha), \psi(\alpha)\}$.

The last axiom gives a natural *coherence* property of ideal values with respect to compositions.

7. If φ, ψ are sequences taking values in a set E , and if f is any function defined on E , then

$$\varphi(\alpha) = \psi(\alpha) \Rightarrow (f \circ \varphi)(\alpha) = (f \circ \psi)(\alpha)$$

¹A more general nonstandard set theory where unlimited levels of saturation are available, is presented in [BD].

²In the following, by sequence we will always mean a function whose domain is the set N of natural numbers.

³Actually, this is not an axiom but an axiom scheme.

⁴We remark that in general $\varphi(\alpha) \notin A$.

Quite surprisingly, we will see that the above axioms are all that is needed in order to get a countably saturated nonstandard embedding $*$ defined for all sets. We remark that in the above presentation, not a single technical notion from mathematical logic is used. In fact, axiom 1 can be skipped by implicitly assuming all “usual” principles of mathematics; and the remaining axioms are all expressed in truly basic terms, such as pairs or compositions of functions. Let us start developing our theory. Axiom 6 says that pairs are preserved at ideal values. As a first immediate result, we see next that ideal values also preserve inclusions, ordered pairs, and other basic set operations.

Proposition 1. *Let φ, ψ, ϑ be any N -sequences. Then*

- (i) $\varphi(n) \subseteq \psi(n)$ for all $n \in N \Rightarrow \varphi(\alpha) \subseteq \psi(\alpha)$.
- (ii) $\varphi(n) = \bigcup \psi(n)$ for all $n \in N \Rightarrow \varphi(\alpha) = \bigcup \psi(\alpha)$.⁵
- (iii) $\vartheta(n) = \varphi(n) \cup \psi(n)$ for all $n \in N \Rightarrow \vartheta(\alpha) = \varphi(\alpha) \cup \psi(\alpha)$.
- (iv) $\vartheta(n) = \varphi(n) \cap \psi(n)$ for all $n \in N \Rightarrow \vartheta(\alpha) = \varphi(\alpha) \cap \psi(\alpha)$.
- (v) $\vartheta(n) = \varphi(n) \setminus \psi(n)$ for all $n \in N \Rightarrow \vartheta(\alpha) = \varphi(\alpha) \setminus \psi(\alpha)$.
- (vi) $\vartheta(n) = \langle \varphi(n), \psi(n) \rangle$ for all $n \in N \Rightarrow \vartheta(\alpha) = \langle \varphi(\alpha), \psi(\alpha) \rangle$.
- (vii) $\vartheta(n) = \varphi(n) \times \psi(n)$ for all $n \in N \Rightarrow \vartheta(\alpha) = \varphi(\alpha) \times \psi(\alpha)$.

Proof. All proofs are straightforward applications of the axioms, thus here we only give two of them as examples. (iii) Notice that $\vartheta(n) = \varphi(n) \cup \psi(n) \Leftrightarrow \vartheta(n) = \bigcup \xi(n)$ where $\xi(n) = \{\varphi(n), \psi(n)\}$. By (ii) $\vartheta(\alpha) = \bigcup \xi(\alpha)$, and by axiom 6 $\xi(\alpha) = \{\varphi(\alpha), \psi(\alpha)\}$. Thus $\vartheta(\alpha) = \bigcup \{\varphi(\alpha), \psi(\alpha)\}$, the thesis. (v) Notice that $\vartheta(n) = \varphi(n) \setminus \psi(n) \Leftrightarrow \vartheta(n) \cap \psi(n) = c_\emptyset(n)$ and $\vartheta(n) \cup (\varphi(n) \cap \psi(n)) = \varphi(n)$. By considering the ideal values and properties (iii) and (iv), one gets $\vartheta(\alpha) \cap \psi(\alpha) = \emptyset$ and $\vartheta(\alpha) \cup (\varphi(\alpha) \cap \psi(\alpha)) = \varphi(\alpha)$. We conclude that $\vartheta(\alpha) = \varphi(\alpha) \setminus \psi(\alpha)$. $\dashv$

It is straightforwardly seen by induction that the above properties also hold for finite unions, finite intersections, finite (ordered) tuples and finite cartesian products. Axiom 5 implies that the sequence constantly equal to the emptyset is stable at the ideal value. The next result extends that property to every (von Neumann) natural number.

Proposition 2. *Let n be any given natural number and let c_n be the sequence constantly equal to n . Then $c_n(\alpha) = n$.*

Proof. Recall that in Zermelo-Fraenkel set theory, natural numbers are coded as von Neumann naturals, i.e. 0 is the emptyset and, by induction, $n+1$ is the set $n \cup \{n\}$. Thus for $n=0$, the thesis follows from axiom 5. Now proceed by induction and consider c_{n+1} . For all $m \in N$, $c_{n+1}(m) = n+1 = n \cup \{n\} = c_n(m) \cup \{c_n(m)\}$. By inductive hypothesis, $c_{n+1}(\alpha) = c_n(\alpha) \cup \{c_n(\alpha)\} = n \cup \{n\} = n+1$. $\dashv$

The next proposition shows that sequences that are eventually equal have

⁵Recall that, by definition, $\bigcup X = \{y : y \in x \text{ for some } x \in X\}$.

the same ideal values, while sequences that are eventually different have different ideal values.

Proposition 3. *Let φ and ψ be two given sequences.*

- (i) *If $\varphi(m) = \psi(m)$ for all but a finite number of m , then $\varphi(\alpha) = \psi(\alpha)$.*
- (ii) *If $\varphi(m) \neq \psi(m)$ for all but a finite number of m , then $\varphi(\alpha) \neq \psi(\alpha)$.*

Proof. (i). Let $\{m \in N : \varphi(m) \neq \psi(m)\} = \{n_1, \dots, n_k\}$. Fix $n \in N$ with $\varphi(n) = \psi(n)$, and define the sequence ζ as follows. $\zeta(m) = n$ if $\varphi(m) = \psi(m)$, and $\zeta(m) = m$ otherwise. We claim that $\zeta(\alpha) = n$. Notice that for all $m \in N$, $\{\zeta(m)\} \setminus \{m\} \subseteq \{n\}$, hence $\{\zeta(\alpha)\} \setminus \{\alpha\} \subseteq \{n\}$. Moreover, $\zeta(\alpha) \neq \alpha$ because $\zeta(m) \in \{n_1, \dots, n_k, n\}$ for all $m \in N \Rightarrow \zeta(\alpha) \in \{n_1, \dots, n_k, n\}$, while $\alpha \notin N$. Now, take $A = \text{range}(\varphi) \cup \text{range}(\psi)$ and define the sequence ξ by setting $\xi(n) = \emptyset$ and $\xi(m) = A$ for $m \neq n$. Then $\{\varphi(m)\} \setminus \{\psi(m)\} \subseteq \xi(\zeta(m))$ for all $m \in N$ implies $\{\varphi(\alpha)\} \setminus \{\psi(\alpha)\} \subseteq (\xi \circ \zeta)(\alpha) = (\text{by axiom 7}) (\xi \circ c_n)(\alpha) = c_\emptyset(\alpha) = \emptyset$, and $\varphi(\alpha) = \psi(\alpha)$. (ii). Since $\{\varphi(m)\} \setminus \{\psi(m)\} = \{\varphi(m)\}$ for all but a finite number of m , by (i) it follows that $\{\varphi(\alpha)\} \setminus \{\psi(\alpha)\} = \{\varphi(\alpha)\}$, that is $\varphi(\alpha) \neq \psi(\alpha)$. $\dashv$

We are now ready to define the nonstandard mapping $*$.

Definition. For any set A , let $*A$ be the ideal value of the sequence c_A constantly equal to A , that is $*A = c_A(\alpha)$.

The map $*$ gives a 1-1 $\in$ -homomorphism from the universal class of all sets into itself. Moreover, $*$ preserves all basic operations of sets.⁶

Proposition 4. *For every sets A, B , the following hold.*

- (i) $*\{A, B\} = \{*A, *B\}$.
- (ii) $A \subseteq B \Leftrightarrow *A \subseteq *B$.
- (iii) $A \in B \Leftrightarrow *A \in *B$.
- (iv) $A = B \Leftrightarrow *A = *B$.
- (v) $*(\bigcup A) = \bigcup(*A)$.
- (vi) $*(A \cup B) = *A \cup *B$.
- (vii) $*(A \cap B) = *A \cap *B$.
- (viii) $*(A \setminus B) = *A \setminus *B$.
- (ix) $*\langle A, B \rangle = \langle *A, *B \rangle$.
- (x) $*(A \times B) = *A \times *B$.
- (xi) $*\{\langle x, x \rangle : x \in A\} = \{\langle \xi, \xi \rangle : \xi \in *A\}$.
- (xii) $*\{\langle x, y \rangle : x \in y \in A\} = \{\langle \xi, \eta \rangle : \xi \in \eta \in *A\}$.
- (xiii) $*\{\langle x, y \rangle : \langle y, x \rangle \in A\} = \{\langle \xi, \eta \rangle : \langle \eta, \xi \rangle \in *A\}$.
- (xiv) $*\{\langle x, y, z \rangle : \langle x, z, y \rangle \in A\} = \{\langle \xi, \eta, \zeta \rangle : \langle \xi, \zeta, \eta \rangle \in *A\}$.
- (xv) $*\{x : \exists y \langle x, y \rangle \in A\} = \{\xi : \exists \eta \langle \xi, \eta \rangle \in *A\}$, i.e. $*\text{dom}(A) = \text{dom}(*A)$.
- (xvi) $*\{y : \exists x \langle x, y \rangle \in A\} = \{\eta : \exists \xi \langle \xi, \eta \rangle \in *A\}$, i.e. $*\text{range}(A) = \text{range}(*A)$.

⁶Notice that the list given in proposition 4 includes all Gödel operations (see [JE] §11). As a consequence, the *transfer* principle could be directly proved (see [D2]). In this paper, the transfer principle will be proved by different means, through the representation theorem 8.

Proof. (i) Trivial by axiom 6. (ii), (v), (vi), (vii), (viii), (ix) and (x) directly follow from proposition 1. (iii) $A \in B \Leftrightarrow \{A\} \subseteq B \Leftrightarrow * \{A\} = \{ * A \} \subseteq * B \Leftrightarrow * A \in * B$. Also (iv) is a corollary of (ii). Property (xi) is straightforward. The remaining properties are all proved similarly one to the other. For instance, let us consider (xv). If ξ is a sequence with $\xi(n) \in \text{dom}(A)$ for all $n \in N$, then there is a sequence η with $\langle \xi(n), \eta(n) \rangle \in A$ for all $n \in N$. Hence $\langle \xi(\alpha), \eta(\alpha) \rangle \in * A$, and $\xi(\alpha) \in \text{domain}(*A)$. Vice versa, let $p \in \text{domain}(A)$, i.e. $\langle p, q \rangle \in A$ for some q . Then $\langle p, q \rangle = \zeta(\alpha)$ for some sequence $\zeta : N \rightarrow A$. Since $A \subseteq D \times R \Rightarrow *A \subseteq *D \times *R$ where $D = \text{dom}(A)$ and $R = \text{range}(A)$, there are sequences $\xi : N \rightarrow D$ and $\eta : N \rightarrow R$ with $p = \xi(\alpha)$ and $q = \eta(\alpha)$. Notice that $\langle p, q \rangle = \nu(\alpha)$ where ν is the sequence $n \mapsto \langle \xi(n), \eta(n) \rangle$. Now, let $\pi : D \times R \rightarrow D$ be the projection on the first coordinate. $\xi(n) = (\pi \circ \nu)(n)$ for all $n \in N$ implies $p = \xi(\alpha) = (\pi \circ \nu)(\alpha)$. But $\nu(\alpha) = \langle p, q \rangle = \zeta(\alpha)$, thus by axiom 7, $p = (\pi \circ \nu)(\alpha) = (\pi \circ \zeta)(\alpha)$ where $\pi \circ \zeta : N \rightarrow D$. We conclude that $p \in *D = *\text{dom}(A)$. $\dashv$

For every A , trivially $\{ *a : a \in A \} \subseteq *A$. The following result gives a characterization of the fixed points of $*$.

Proposition 5. $\{ *a : a \in A \} = *A$ if and only if A is finite.

Proof. If $A = \{a_1, \dots, a_k\}$ is finite, we already saw that $*A = \{ *a_1, \dots, *a_k \}$. Vice versa, if A is infinite there is a 1-1 mapping $f : N \rightarrow A$. Take a left inverse $g : A \rightarrow N$, i.e. a mapping with $g(f(n)) = n$ for all $n \in N$. We claim that $f(\alpha) \in *A$ and $f(\alpha) \neq *a$ for every $a \in A$. Assume by contradiction that $f(\alpha) = *a = c_a(\alpha)$ for some $a \in A$. Then, $\alpha = 1_N(\alpha) = (g \circ f)(\alpha) = (g \circ c_a)(\alpha) = c_{g(a)}(\alpha) = *g(a)$. This is impossible, because (by proposition 2) $*g(a) = g(a) \in N$ while $\alpha \notin N$. $\dashv$

For each subset $A \subseteq N$, let $\chi_A : N \rightarrow \{0, 1\}$ be its characteristic function. That is, $\chi_A(n) = 1$ if $n \in A$ and $\chi_A(n) = 0$ otherwise.

Theorem 6.

$\mathcal{U} = \{ A \subseteq N : \chi_A(\alpha) = 1 \}$ is an ultrafilter over N that contains no finite sets.

Proof. In the following, recall that 0 and 1 are identified with the emptyset $\emptyset$ and with the singleton $\{\emptyset\}$, respectively. Let A be a given subset of N and denote by A^c its complement $N \setminus A$. For all $n \in N$, $\chi_A(n) \cup \chi_{A^c}(n) = \{\emptyset\}$ and $\chi_A(n) \cap \chi_{A^c}(n) = \emptyset$. Thus $\chi_A(\alpha) \cup \chi_{A^c}(\alpha) = \{\emptyset\}$ and $\chi_A(\alpha) \cap \chi_{A^c}(\alpha) = \emptyset$. Since every characteristic function takes its ideal value in the set $\{\emptyset, \{\emptyset\}\}$, the above equalities imply

- For every subset $A \subseteq N$,
 (i) $\chi_A(\alpha) = 1 \Rightarrow \chi_{A^c}(\alpha) = 0$; (ii) $\chi_A(\alpha) = 0 \Rightarrow \chi_{A^c}(\alpha) = 1$.

Now, let us check the properties of filter. Assume $A, B \in \mathcal{U}$, i.e. $\chi_A(\alpha) = \chi_B(\alpha) = 1$. By property (i), $\chi_{A^c}(\alpha) = \chi_{B^c}(\alpha) = 0$. Notice that $\chi_{(A^c \cup B^c)}(n) = \chi_{A^c}(n) \cup \chi_{B^c}(n)$ for all $n \in N$, thus $\chi_{(A^c \cup B^c)}(\alpha) = \chi_{A^c}(\alpha) \cup \chi_{B^c}(\alpha) = \emptyset \cup \emptyset = 0$. Since $A \cap B = (A^c \cup B^c)^c$, by (ii) it follows that $\chi_{A \cap B}(\alpha) = 1$, i.e. $A \cap B \in \mathcal{U}$.

Now assume $A \supseteq B \in \mathcal{U}$. Then $\chi_A(n) \supseteq \chi_B(n)$ for all $n \in N \Rightarrow \chi_A(\alpha) \supseteq \chi_B(\alpha) = 1$, hence $\chi_A(\alpha) = 1$. The ultrafilter property is given by the above (i) and (ii). Finally, let A be a finite subset of N . Then $\chi_A(n) = c_\emptyset(n)$ for all but a finite number of n , and by proposition 3, $\chi_A(\alpha) = c_\emptyset(\alpha) = 0$. $\dashv$

Proposition 7. *Let $\varphi, \psi : N \rightarrow A$ be two sequences. Then*

(i) $\varphi(\alpha) = \psi(\alpha) \Leftrightarrow \{n \in N : \varphi(n) = \psi(n)\} \in \mathcal{U}$.

(ii) $\varphi(\alpha) \in \psi(\alpha) \Leftrightarrow \{n \in N : \varphi(n) \in \psi(n)\} \in \mathcal{U}$.

Proof. (i). Denote $\Lambda = \{n \in N : \varphi(n) = \psi(n)\}$. Assume first $\Lambda \in \mathcal{U}$. Let ξ be the sequence such that $\xi(n) = A$ if $n \in \Lambda$ and $\xi(n) = 0$ otherwise. Notice that $\xi = f \circ \chi_\Lambda$ where $f : \{0, 1\} \rightarrow \{0, A\}$ is the function with $0 \mapsto 0$ and $1 \mapsto A$. $\chi_\Lambda(\alpha) = 1 = c_1(\alpha) \Rightarrow \xi(\alpha) = (f \circ \chi_\Lambda)(\alpha) = (f \circ c_1)(\alpha) = c_A(\alpha) = *A$. For all $n \in N$, $[\{\varphi(n)\} \setminus \{\psi(n)\}] \cap \xi(n) = 0$, thus $[\{\varphi(\alpha)\} \setminus \{\psi(\alpha)\}] \cap \xi(\alpha) = 0$. But $\varphi(\alpha) \in *A \Rightarrow \{\varphi(\alpha)\} \setminus \{\psi(\alpha)\} = 0 \Rightarrow \varphi(\alpha) = \psi(\alpha)$. Vice versa, assume $\varphi(\alpha) = \psi(\alpha)$. For all $n \in N$, $[\{\varphi(n)\} \setminus \{\psi(n)\}] \cup \chi_\Lambda(n) \neq 0$, so $[\{\varphi(\alpha)\} \setminus \{\psi(\alpha)\}] \cup \chi_\Lambda(\alpha) \neq 0$. Since $\{\varphi(\alpha)\} \setminus \{\psi(\alpha)\} = 0$ by the hypothesis, we conclude $\chi_\Lambda(\alpha) \neq 0$, i.e. $\Lambda \in \mathcal{U}$. (ii). Denote $\Gamma = \{n \in N : \varphi(n) \in \psi(n)\}$. If $\varphi(\alpha) \in \psi(\alpha)$, then $\varphi(\alpha) = \xi(\alpha)$ with ξ a sequence such that $\xi(n) \in \psi(n)$ for all $n \in N$. By (i), $\Lambda = \{n \in N : \varphi(n) = \xi(n)\} \in \mathcal{U}$, thus also $\Gamma \in \mathcal{U}$ because $\Gamma \supseteq \Lambda$. Vice versa, assume $\Gamma \in \mathcal{U}$. Let ξ and ζ be the sequences where $\xi(n) = \{\varphi(n)\} \cap \psi(n)$ and $\zeta(n) = \{\varphi(n)\}$ for all $n \in N$. Notice that $\Lambda = \{n \in N : \xi(n) = \zeta(n)\} \in \mathcal{U}$ because $\Lambda \supseteq \Gamma \in \mathcal{U}$. Then, by (i), $\xi(\alpha) = \zeta(\alpha)$, i.e. $\{\varphi(\alpha)\} \cap \psi(\alpha) = \{\varphi(\alpha)\} \Leftrightarrow \varphi(\alpha) \in \psi(\alpha)$. $\dashv$

Notice that $\mathcal{U}$ gives a nonatomic finitely additive two-valued measure $\mu : \mathcal{P}(N) \rightarrow \{0, 1\}$ defined for all subsets of N . Thus we could restate (i) in the above proposition as follows: “Two sequences have the same ideal value if and only if they are equal almost everywhere (with respect to μ)”. Similarly (ii). For any set A , let $A^N/\mathcal{U}$ be the *ultrapower* of A modulo the ultrafilter $\mathcal{U}$. Let π_A be the function such that, for every sequence $\varphi : N \rightarrow A$, $\pi_A(\varphi(\alpha)) = [\varphi] \in A^N/\mathcal{U}$ is the $\mathcal{U}$ -equivalence class of φ modulo $\mathcal{U}$.

Representation Theorem 8.

For every set A , the function $\pi_A : *A \rightarrow A^N/\mathcal{U}$ is an $\in$ -isomorphism and the following diagram commutes. ⁷

$$\begin{array}{ccc} & A^N/\mathcal{U} & \\ d_A \nearrow & & \searrow \pi_A \\ A & \xrightarrow{*} & *A \end{array}$$

Moreover, for all $\in$ -formulas $\sigma(x_1, \dots, x_n)$ and elements $a_1, \dots, a_n \in A$

$$\langle A, \in \rangle \models \sigma(a_1, \dots, a_n) \Leftrightarrow \langle *A, \in \rangle \models \sigma(*a_1, \dots, *a_n)$$

⁷ d_A denotes the diagonal embedding of A into its ultrapower.

Proof. By the previous proposition, $\varphi(\alpha) = \psi(\alpha) \Leftrightarrow \{n \in N : \varphi(n) = \psi(n)\} \in \mathcal{U} \Leftrightarrow [\varphi] = [\psi]$. Thus π_A is 1-1, and also it is trivially onto. Besides, $\varphi(\alpha) \in \psi(\alpha) \Leftrightarrow \{n \in N : \varphi(n) \in \psi(n)\} \in \mathcal{U} \Leftrightarrow A^N/\mathcal{U} \models [\varphi] \in [\psi]$ and π_A is an $\in$ -homomorphism. The commutativity of the diagram is immediate. In fact, for every $a \in A$, $\pi_A(*a) = \pi(c_a(\alpha)) = [c_a] = d_A(a)$, where c_a is the sequence constantly equal to a . Finally, the equivalence for $\in$ -formulas σ is given by Los theorem on ultrapowers.⁸ $\dashv$

We are now ready to prove the two fundamental principles of nonstandard analysis, namely the *transfer* principle and the *saturation* property. As in the superstructure approach, by definition we say that a set x is *internal* if $x \in *A$ for some A . Equivalently, x is internal if $x = \varphi(\alpha)$ is the ideal value of some sequence φ . Notice that the class I of internal elements is transitive, i.e. elements of internal sets are internal.

Theorem 9. *Countable Saturation.*

Let $\mathcal{F}$ be a countable family of internal sets. If every finite subfamily of $\mathcal{F}$ has nonempty intersection, then $\mathcal{F}$ has nonempty intersection.

Proof. Let $\mathcal{F} = \{F_n : n \in N\}$ and take A with $F_n \in *A$ for all n . For each $n \in N$, choose $a_n \in \bigcap_{i=0}^n F_i$, and consider the function $\varphi : N \rightarrow A$ where $n \mapsto a_n$. For each $k \in N$, $\{n \in N : \varphi(n) \in F_k\} = \{n \in N : n \geq k\}$ is a cofinite set, hence an element of $\mathcal{U}$. By the Representation theorem, $A^N/\mathcal{U} \models [\varphi] \in d_A(F_k) \Leftrightarrow \langle *A, \in \rangle \models \pi_A^{-1}([\varphi]) \in F_k$. Since this holds for each $k \in N$, we have found an element $a = \pi_A^{-1}([\varphi]) \in \bigcap \mathcal{F}$. $\dashv$

Theorem 10. *Transfer Principle.*

*For every bounded $\in$ -formula $\sigma(x_1, \dots, x_n)$ and elements $a_1, \dots, a_n$:*⁹

$$\sigma(a_1, \dots, a_n) \Leftrightarrow \sigma^I(*a_1, \dots, *a_n)$$

Proof. Take A a transitive set containing $a_1, \dots, a_n$ and consider the $\in$ -isomorphism $\pi_A : *A \rightarrow A^N/\mathcal{U}$ and the diagonal embedding $d_A : A \rightarrow A^N/\mathcal{U}$ as given by the Representation theorem. It is easily seen that also $*A$ is transitive. Since bounded formulas are absolute between transitive classes¹⁰, then $\sigma(a_1, \dots, a_n) \Leftrightarrow \langle A, \in \rangle \models \sigma(a_1, \dots, a_n) \Leftrightarrow \langle *A, \in \rangle \models \sigma(*a_1, \dots, *a_n) \Leftrightarrow \sigma^I(*a_1, \dots, *a_n)$. $\dashv$

Finally, the next theorem gives a foundational justification to ZFC[α].

Theorem 11.

ZFC[α] is equiconsistent with ZFC.

⁸See [CK] §4.1.

⁹Recall that a formula σ is *bounded* if every quantifier occurs in bounded form $\forall x \in I$ (i.e. $\forall x x \in I \rightarrow \dots$) or $\exists x \in I$ (i.e. $\exists x x \in I \wedge \dots$). By σ^I we denote the I -relativization of σ , that is the formula obtained from σ by replacing each quantifier $\forall x$ and $\exists x$ with its restricted form $\forall x \in I$ (i.e. $\forall x x \in I \rightarrow \dots$) and $\exists x \in I$ (i.e. $\exists x x \in I \wedge \dots$), respectively.

¹⁰See [JE] §14.

Proof. If M is a model of $\text{ZFC}[\alpha]$, then its wellfounded part $\mathbf{WF}^M = \{m \in M : M \models \text{"}m \text{ is wellfounded"}\}$ is a model of ZFC . Vice versa, let us assume the consistency of ZFC . We need to find a model of $\text{ZFC}[\alpha]$. In the following, we will work in ZFBC , an equiconsistent variant of Zermelo-Fraenkel set theory where a bijection $\mathbf{C} : \mathbf{ON} \rightarrow \mathbf{V}$ between the class of ordinals and the universal class of all sets is available (thanks to a new binary relation C considered in the language), and where regularity is replaced by Boffa's *superuniversality* axiom. Superuniversality is a strong anti-foundation axiom which implies the existence of a transitive collapse for every (not necessarily wellfounded) extensional structure.¹¹ The construction below closely follows arguments in the proof of the Extension Principle of [BH]. Thus we give here only a sketch and refer the reader to that paper for details.¹² The universal class $\mathbf{V}$ is an increasing union of transitive sets indexed by ordinals. For instance, $\mathbf{V} = \bigcup \{W_\alpha : \alpha \in \mathbf{ON}\}$ where W_α is the transitive closure of the set $\{\mathbf{C}(\beta) : \beta < \alpha\}$. Now fix a nontrivial ultrafilter D over N . As a consequence of superuniversality, one gets transitive collapses $\pi_\alpha : W_\alpha^N/D \rightarrow T_\alpha$ such that for all $\alpha \leq \beta$, π_α is the restriction of π_β to W_α^N/D :

$$\begin{array}{ccc} W_\alpha^N/D & \xrightarrow{\pi_\alpha} & \langle T_\alpha, \in \rangle \\ \downarrow & & \downarrow \\ W_\beta^N/D & \xrightarrow{\pi_\beta} & \langle T_\beta, \in \rangle \end{array}$$

The class $\pi \doteq \bigcup \{\pi_\alpha : \alpha \in \mathbf{ON}\}$ is a transitive collapse for the D -ultrapower of the universe $\mathbf{V}^N/D = \{[f] : f \text{ is a function with } \text{dom}(f) = N\}$. We remark that in order to make each equivalence class $[f]$ a set (and not a proper class) some caution is needed. In ZFC , this problem is overcome by Scott's trick. In our nonwellfounded context (where a rank is not available) one can define $[f]$ as the least function in the well-ordering of the universe (given by $\mathbf{C}$) which is D -equivalent to f . Doing so, the collection $\{[f] : f \text{ is a function with } \text{dom}(f) = N\}$ is actually the extension of a formula in ZFBC . Now, let $\pi([1_N])$ be the interpretation of the constant symbol α , where $1_N : N \rightarrow N$ denotes the identity map on N . For all N -sequences f , let us interpret $f(\alpha) = \pi([f])$. Notice that the nonstandard embedding $*$ corresponds to the bounded elementary embedding $(\bigcup_{\alpha \in ON} \pi_\alpha \circ d_\alpha) : \mathbf{V} \rightarrow \mathbf{V}$, where $d_\alpha : W_\alpha \rightarrow W_\alpha^N/D$ are the diagonal embeddings. All axioms of $\text{ZFC}[\alpha]$ directly follow from properties of the ultrapower construction. $\dashv$

¹¹ZFBC was first proposed as a foundational framework for the practice of nonstandard methods by D. Ballard and K. Hrbáček in [BH] (see that paper for a precise formulation). That theory was also adopted by the author for the construction of superstructure-like nonstandard universes in [D1]. A proof of the consistency of superuniversality is in [Bo].

¹²See also [BD], where a stronger theory than $\text{ZFC}[\alpha]$ is given a faithful interpretation in ZFC .

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