ALPHA-THEORY: AN ELEMENTARY AXIOMATICS FOR NONSTANDARD ANALYSIS

VIERI BENCI AND MAURO DI NASSO

Abstract. *The methods of nonstandard analysis are presented in elementary terms by postulating a few natural properties for an infinite "ideal" number* α. *The resulting axiomatic system, including a formalization of an interpretation of Cauchy's idea of infinitesimals, is related to the existence of ultrafilters with special properties, and is independent of ZFC. The Alpha-Theory supports the feeling that technical notions such as superstructure, ultrapower and the transfer principle are definitely not needed in order to carry out calculus with actual infinitesimals.*

Introduction

In early stages of calculus, up to the first years of the nineteenth century, infinitesimal quantities were widely used to develop many of the classic results of analysis. Afterwards, at the end of nineteenth century, a severe foundational criticism led to the current ϵ-δ formalization due to Weierstrass, and infinitesimal and infinite numbers were banned from calculus.

Nonstandard analysis was introduced by A. Robinson [44, 45] in the early sixties. By using the machinery of model theory, a branch of mathematical logic, he succeeded in providing the actual use of infinitesimal numbers with rigorous foundations, thus giving a solution to a century-old problem. Since then, the methods of nonstandard analysis have been successfully applied and have led to new results in such diverse fields of mathematics as functional analysis, measure and probability theorons, additive number theory, stochastic analysis, hydromechanics, etc. (See [1] for a broad collection of reviews. See also [34] for nonstandard methods applied to additive number theory.)

Unfortunately, very soon the formalism of Robinson's original presentation appeared too technical to many, and not directly usable by those mathematicians without a good background in logic. Over the last forty years, many attempts have been made in order to simplify the foundational matters and popularize nonstandard analysis by means of "easy to grasp" presentations. Most notably, the pioneering work by W.A.J. Luxemburg [40], where a direct use of ultrapowers was made; the *superstructure approach* [46] presented by A. Robinson jointly with E. Zakon within a set-theoretic framework; the elementary axiomatics [35] given by H.J. Keisler in the seventies (the relative textbook [36] was actually adopted for calculus classes with good results in several countries); the algebraic presentation of hyperreals by W.S. Hatcher [29], then extended by the authors to the full generality of nonstandard analysis [4, 5]; and finally the recent "gentle" introduction by W. Henson [31].

2000 *Mathematics Subject Classification.* 26E35 Nonstandard analysis; 03E65 Other hypotheses and axioms; 03C20 Ultraproducts and related constructions.

Currently, most practitioners work in the setting of the *superstructure approach* or follow Nelson's *Internal Set Theory* IST [42]. In this paper, we present a new alternative approach to nonstandard analysis aimed at showing that technical notions such as superstructure, ultrafilter, ultrapower, bounded formula and the transfer principle, are not needed to rigorously develop calculus with infinitesimals. We proceed axiomatically. Precisely, the *Alpha-Theory* consists of a few natural properties postulated for an "ideal" natural number α, to be thought as "infinitely large".

Historically, the idea of adjoining a new number that, in some precise sense, behaves like a very large natural number, goes back to the pre-Robinsonian work by C. Schmieden and D. Laugwitz [48]. They considered a new symbol Ω and postulated that if a given "formula" is true for all sufficiently large natural numbers, then the "formula" is true for Ω as well. The resulting theory has a constructive flavour, and it inspired various "non-classical" presentations of nonstandard analysis.[1]

Although well suited to provide a foundational justification for the use of infinitesimals and to develop large parts of calculus (see [39]), the Ω-approach soon revealed inadequate for applications in classical mathematics. Most notably, the crucial drawback is that the nonstandard reals contain zero divisors and are only a partially ordered ring.

In a few years, with the turn of the sixties, that approach was superseded by the more powerful "Robinsonian" analysis. However, it seems correct to say that the Ω-approach was a turning point in the treatment of the idea of infinitesimals. Even if there was no direct influence on nonstandard analysis as practiced today, we think it was not by chance that in the seminal paper [44], Robinson himself acknowledged the work by Schmieden and Laugwitz as a "recent and rather successful effort of developing a calculus of infinitesimals".

In some sense, the Alpha-Theory can be seen as a convenient strengthening of the Ω-approach. On the one hand, it preserves the intuitive appeal of postulating a few elementary properties which are preserved by $\mathbb{N}$-sequences when extended to the "ideal" point at infinity; on the other hand, has the full strength of Robinsonian nonstandard analysis. Besides, the Alpha-Theory seems to be the right framework to formalize an idea of an infinitesimal number which has some philosophical and historical relevance. According to J. Cleave [16, 17], Cauchy's conception of infinitesimals can be described as "values of null real sequences at infinity". Now, the very idea of values of sequences at infinity is what the Alpha-Theory is aimed to formalize. The *Cauchy's infinitesimals principle*, stating that every infinitesimal number is the value at infinity of some infinitesimal sequence, is naturally formulated in our context. We shall also consider the stronger version postulating that every nonzero infinitesimal is the value at infinity of some monotone sequence.

As for the foundational matters, the Alpha-Theory without Cauchy's principles admits a faithful interpretation in *Zermelo-Fraenkel Set Theory* ZFC. Thus, informally we can say that any theorem in "ordinary mathematics" which is proved by the Alpha-Theory is true.

Inspired by the criticism by N. Cutland, C. Kessler, E. Kopp and D. Ross [19] on the mathematical content of the Cleave's interpretation mentioned above, we also investigate

[1]See e.g. the overview [43] by E. Palmgren. A recent *non*-nonstandard treatment of calculus was presented by J.M. Henle [30]. Interesting articles on the constructive viewpoint on nonstandard analysis can be found in the recent book [49].

the foundational strength of Cauchy's infinitesimals principles and prove the following results: The Alpha-Theory plus Cauchy's principle is equiconsistent with ZFC plus "there exists a *P-point*", and the Alpha-Theory plus the *strong* Cauchy's principle is equiconsistent with ZFC plus "there exists a *selective ultrafilter*" (we recall that the existence of either a P-point or a selective ultrafilter is independent of ZFC). Informally, this means on the one hand that Cauchy's principles are "sound" (i.e. they do not bring contradictions), on the other hand they allow proving theorems that "ordinary mathematics" cannot prove. (Clearly, the blanket assumption is the consistency of ZFC).

In this paper, we focused on the elementary nature of our approach, and in one Appendix we also included a small number of basic examples taken from calculus, so to give a taste of nonstandard methods "in action". All the technical parts are relegated to the last Section, which contains a formal presentation of the Alpha-Theory as a first-order theory, and the consistency proofs.

Some additional comments for "nonbeginners".
The Alpha-Theory has the full strength of Robinsonian nonstandard analysis with countable saturation. In fact, both the *transfer principle* for all bounded formulas and the *countable saturation property* are theorems of the Alpha-Theory. According to the popular plan of this paper, in the exposition, as well as in the Appendix "A Little Calculus", the transfer principle is never mentioned. Rather, on a case to case basis, we prove directly only those instances that are actually needed to obtain the desired results.

Also from a foundational point of view, the Alpha-Theory is aimed to simplify matters. In the superstructure approach, two different universes are considered, namely the *standard model* and the *nonstandard model*. In our context there is a single universe, the "usual" world of mathematics where all axioms of ZFC except foundation are assumed. Moreover, contrarily to IST and other nonstandard set theories that have been presented in the literature (see [21] for a survey), external sets are available without restrictions, and the nonstandard embedding $*$ is defined for *all* sets. As a consequence, $*$ can be iterated to get things like the hyper-hyperreal numbers $(\mathbb{R}^*)^*$, or the star-transforms of external objects, such as Loeb measures, etc. (E.g., the double nonstandard iterations used by V.A. Molchanov [41] to study compactifications in topology, can be naturally formalized in the framework of the Alpha-Theory.)

Acknowledgements

The authors thank Marco Forti for valuable discussions, and thank Jerry Keisler for useful remarks on a previous version of the paper.

1. The Alpha-Theory.

Our approach is based on the existence of a new mathematical object, namely α. We can think of α as a new ideal natural number added to $\mathbb{N}$, in a similar way as the imaginary unit i can be seen as a new ideal number added to the real numbers. By adopting the notion of *numerosity* introduced in [6], an interpretation of the ideal number α can be given as the numerosity of the set of natural numbers (see [7]). However, in order to help the intuition, it is enough to think of α as a "very large" natural number.

Before going into the axioms, we remark that the blanket assumption is that *all* usual principles of "ordinary mathematics" are assumed. Informally, we can say that by adopting our theory, one can construct sets and functions according to the "usual" practice of mathematics, with no restrictions whatsoever.[2]

1.1. **The five axioms.**
According to the usual foundational framework of mathematics, namely *Zermelo-Fraenkel* set theory ZFC, every object of the mathematical universe is a set. Clearly, the philosophical plausibility of such an assumption is highly disputable, but the foundational success of pure set theory is due to the fact that virtually all entities of mathematics can actually be coded as sets. For instance, an ordered pair $\langle a,b\rangle$ can be identified with the Kuratowski pair $k=\{\{a\},\{a,b\}\}$ (so that the two elements a,b are obtained as elements of elements of k, the first element of the ordered pair being the one that belongs to both elements of k); an equivalence relation $\sim$ can be identified with the set of ordered pairs $\{\langle a,b\rangle \mid a\sim b\}$ that satisfy it; a function $f:A\to B$ can be identified with its graph $\Gamma(f)=\{\langle a,b\rangle \mid f(a)=b\}$, etc. As for numbers, one can consider the *von Neumann natural numbers*: $0=\emptyset$, $n+1=n\cup\{n\}$ (so that each natural number is the set of its predecessors and the order relation is given by the membership). One can define the integers (and the rationals) as ordered pairs of naturals (of integers, resp.) identified modulo suitable equivalence relations. One can then define the real numbers as equivalence classes of Cauchy sequences, the complex numbers as a quotient of ordered pairs of reals, etc.

However, in practice it is often convenient to have *atoms* available, i.e. primitive objects which are not sets. For instance, it is more natural to many mathematicians to think of numbers as individuals rather than sets. So, to simplify matters, in the following we assume a set of atoms $\mathcal{A}$, which includes all real numbers $\mathbb{R}\subset\mathcal{A}$. When talking about *sets* we shall always mean nonatoms, i.e. objects that are identified with the collection of their elements. In particular, there can be only one set with no elements, namely the empty set $\emptyset$. When talking about *elements* or *entities*, we shall mean objects of the universe in general, i.e. either atoms or sets.[3] In the sequel, by *sequence* we shall always mean a function whose domain is the set of natural numbers.

The use of α is governed by the following five axioms.

α**1**. *Extension Axiom.*
For every sequence φ there is a unique element $\varphi[\alpha]$, called the "ideal value of φ" or the "value of φ at infinity".

The next axiom gives a natural coherence property with respect to compositions.

α**2**. *Composition Axiom.*
If φ and ψ are sequences and if f is any function such that compositions $f\circ\varphi$ and $f\circ\psi$ make sense, then

$$\varphi[\alpha]=\psi[\alpha]\Rightarrow (f\circ\varphi)[\alpha]=(f\circ\psi)[\alpha]$$

So, if two sequences takes the same value at infinity, by composing them with any given function, we again obtain sequences with the same value at infinity.

[2] A formalization of what we mean by "usual principles of mathematics" (i.e. our underlying set theory) is given in the last Section 6 where we concentrate on the technical and foundational matters.

[3] We remark that our theory can also be formulated in a purely set theoretic context (i.e. where no atoms exist). See [24].

α**3**. *Number Axiom.*
If $c_r : n \mapsto r$ is the constant sequence with value $r \in \mathbb{R}$, then $c_r[\alpha] = r$. If $1_{\mathbb{N}} : n \mapsto n$ is the identity sequence on $\mathbb{N}$, then $1_{\mathbb{N}}[\alpha] = \alpha \notin \mathbb{N}$.

The first part of this axiom simply says that those sequences which constantly equal a real number, have the expected ideal values. The latter condition says that α is obtained as the value at infinity of the identity sequence, and that α is actually a *new* number. Notice that $1_{\mathbb{N}}$ provides a first example of a sequence $\varphi : \mathbb{N} \to A$ such that $\varphi[\alpha] \notin A$.

The next axiom supplies other examples of sequences which have the "expected" ideal values.

α**4**. *Pair Axiom.*
For all sequences φ, ψ and ϑ:

$$\vartheta(n) = \{\varphi(n), \psi(n)\} \text{ for all } n \;\Rightarrow\; \vartheta[\alpha] = \{\varphi[\alpha], \psi[\alpha]\}$$

The next (and last) axiom says what elements are there in ideal values.

α**5**. *Internal Set Axiom.*
If ψ is a sequence of atoms, then $\psi[\alpha]$ is an atom. If $c_{\emptyset} : n \mapsto \emptyset$ is the sequence with constant value the empty set, then $c_{\emptyset}[\alpha] = \emptyset$. If ψ is a sequence of nonempty sets, then

$$\psi[\alpha] = \{\varphi[\alpha] \mid \varphi(n) \in \psi(n) \text{ for all } n\}$$

The name *Internal Set Axiom* is justified because our values at infinity $\psi[\alpha]$ correspond to the *internal sets* in the usual terminology of nonstandard analysis (see Section 4.1).

As an immediate consequence of this axiom, the membership relation is preserved at the ideal values. That is, if $\varphi(n) \in \psi(n)$ for all n, then $\varphi[\alpha] \in \psi[\alpha]$. Besides, when ψ a sequence of nonempty sets, all elements of $\psi[\alpha]$ are obtained in this way. That is, they all are values at infinity of sequences which are pointwise members of ψ.

An interesting example is the following. Suppose φ is a two-valued sequence, say $\varphi : \mathbb{N} \to \{-1, 1\}$. Then its ideal value makes no surprise, i.e. either $\varphi[\alpha] = -1$ or $\varphi[\alpha] = 1$ (but in general it cannot be decided which is the case). This fact directly follows from the *pair axiom* and the *internal set axiom.*

Although well-suited for the "working" mathematician, we remark that the above five axioms are only given in a "semi-formal" fashion. Indications for a rigorous formalization as sentences of a suitable first-order language are given in Section 6.

1.2. **First consequences of the axioms.**

In this subsection, we list a collection of basic and natural properties of the values at infinity, which directly follow from the axioms. In order to make the exposition as smooth as possible and go quickly to more interesting topics, we shall postpone the proofs to Appendix B.

Let ξ and ζ be two sequences. In the following, for the sake of simplicity we shall write $\xi(n) = \zeta(n)$ to mean that such an equality holds for all $n \in \mathbb{N}$. Similarly, we shall write $\xi(n) \neq \zeta(n)$, $\xi(n) \in \zeta(n)$, $\xi(n) \subseteq \zeta(n)$ etc., to mean that those relationships hold for all n.

The first Proposition shows that all basic set operations (except the powerset) are preserved at infinity.

Proposition 1.1. *Assume that φ, ψ are sequences of nonempty sets.*[4] *Then the following hold:*

(1) *Difference:* $\varphi(n) \neq \psi(n) \Rightarrow \varphi[\alpha] \neq \psi[\alpha]$ *and* $\varphi(n) \notin \psi(n) \Rightarrow \varphi[\alpha] \notin \psi[\alpha]$;
(2) *Subset:* $\varphi(n) \subseteq \psi(n) \Rightarrow \varphi[\alpha] \subseteq \psi[\alpha]$;
(3) *Union:* $\vartheta(n) = \varphi(n) \cup \psi(n) \Rightarrow \vartheta[\alpha] = \varphi[\alpha] \cup \psi[\alpha]$;
(4) *Intersection:* $\vartheta(n) = \varphi(n) \cap \psi(n) \Rightarrow \vartheta[\alpha] = \varphi[\alpha] \cap \psi[\alpha]$;
(5) *Setminus:* $\vartheta(n) = \varphi(n) \setminus \psi(n) \Rightarrow \vartheta[\alpha] = \varphi[\alpha] \setminus \psi[\alpha]$;
(6) *Ordered pair:* $\vartheta(n) = \langle \varphi(n), \psi(n) \rangle \Rightarrow \vartheta[\alpha] = \langle \varphi[\alpha], \psi[\alpha] \rangle$;
(7) *Cartesian product:* $\vartheta(n) = \varphi(n) \times \psi(n) \Rightarrow \vartheta[\alpha] = \varphi[\alpha] \times \psi[\alpha]$.

Clearly, properties 3, 4, 6 and 7 also hold for finite unions, finite intersections, finite n-tuples and finite Cartesian products. Similarly, the *pair axiom* is extended to finite sets.

In the next Proposition, some further basic properties of the ideal values are considered. For instance, it will be excluded the possibility of a sequence of sets whose ideal value is an atom.

Proposition 1.2.

(1) *If φ is a sequence of atoms (or sets, or nonempty sets) then $\varphi[\alpha]$ is an atom (a set, a nonempty set, respectively);*
(2) *If the value at infinity $\varphi[\alpha]$ is an atom (or a set, or a nonempty set) then there exists a sequence $\psi(n)$ of atoms (of sets, of nonempty sets, respectively) such that $\psi[\alpha] = \varphi[\alpha]$;*
(3) *Elements of values at infinity are values at infinity.*

As a straight consequence, the results in Proposition 1.1 can be extended to *all* sequences φ, ψ of (possibly empty) sets.

The next result formalizes the intuition that changing finitely many values does not affect the values at infinity.

Proposition 1.3.

(1) *If $\varphi(n) = \psi(n)$ eventually (i.e. for all but finitely many n), then $\varphi[\alpha] = \psi[\alpha]$;*
(2) *If $\varphi(n) \neq \psi(n)$ eventually, then $\varphi[\alpha] \neq \psi[\alpha]$.*

We remark that the above implications cannot be reversed.

The underlying idea of the next result is that if two sequences take the same value at infinity, then the collection of indices where they agree is large enough to detect whether any other two sequences have the same ideal values or not. The same fact holds for the set of indices where two given sequences are different, in case they have different values at infinity.

Proposition 1.4. *Suppose $\varphi[\alpha] = \psi[\alpha]$ (or $\varphi[\alpha] \neq \psi[\alpha]$) and let $\Lambda = \{n \mid \varphi(n) = \psi(n)\}$ ($\Lambda = \{n \mid \varphi(n) \neq \psi(n)\}$, respectively). Then, for all sequences ξ, ζ:*

(1) *$\xi(n) = \zeta(n)$ (or $\xi(n) \neq \zeta(n)$) for all $n \in \Lambda \Rightarrow \xi[\alpha] = \zeta[\alpha]$ ($\xi[\alpha] \neq \zeta[\alpha]$, respectively);*
(2) *$\xi(n) \in \zeta(n)$ (or $\xi(n) \notin \zeta(n)$) for all $n \in \Lambda \Rightarrow \xi[\alpha] \in \zeta[\alpha]$ ($\xi[\alpha] \notin \zeta[\alpha]$, respectively).*

[4] No hypotheses on the values $\varphi(n), \psi(n)$ are needed in items 1 and 6.

Thanks to these properties, some simplifying assumptions can be made. For instance, when considering two sequences with $\varphi[\alpha] \neq \psi[\alpha]$, we can directly assume without loss of generality that $\varphi(n) \neq \psi(n)$ for all n.[5] Similarly, if $\varphi[\alpha] = \psi[\alpha]$, we can directly assume that $\varphi(n) = \psi(n)$ for all n.

2. The Star-Operator

We now introduce a notion of "idealization" for any entity A, namely the *nonstandard extension* or *star-transform* A^*, and present some of the most relevant sets of nonstandard analysis. The proofs are straightforward, so we only give a few hints and leave details to the reader.

The fundamental tool of nonstandard analysis is the *transfer principle*. Roughly speaking, it states that an "elementary property" is satisfied by mathematical objects $a_1, \ldots, a_k$ if and only if it is satisfied by their star-transforms $a_1^*, \ldots, a_k^*$. In order to avoid the use of technical notions from mathematical logic, in the exposition we shall keep the notion of "elementary property" at an informal level, and prove directly only those instances of the *transfer principle* that will be needed.[6]

2.1. **Star-transforms of sets, functions, relations.**

Definition 2.1. For any entity A, its *nonstandard extension* or *star-transform* is $A^* = c_A[\alpha]$, the value at infinity taken by the constant sequence $c_A : n \mapsto A$.

By the *number axiom*, $r^* = r$ for all $r \in \mathbb{R}$. We remark that the similar property does not hold for every atom. For instance, $\alpha^* \neq \alpha$ because $1_{\mathbb{N}}(n) = n \neq \alpha$ for all n. Notice that, if A is a nonempty set, by the *internal set axiom*, A^* is precisely the set of ideal values of A-valued sequences:

$$A^* = \{\varphi[\alpha] \mid \varphi : \mathbb{N} \to A\}$$

As an immediate consequence of Proposition 1.1, the star-operator preserves all basic operations of sets (except the powerset).

Proposition 2.2. *For all A, B, the following hold:*[7]

1. $A = B \;\Leftrightarrow\; A^* = B^*$;
2. $A \in B \;\Leftrightarrow\; A^* \in B^*$;
3. $A \subseteq B \;\Leftrightarrow\; A^* \subseteq B^*$;
4. $\{A, B\}^* = \{A^*, B^*\}$;
5. $\langle A, B\rangle^* = \langle A^*, B^*\rangle$;
6. $(A \cup B)^* = A^* \cup B^*$;
7. $(A \cap B)^* = A^* \cap B^*$;
8. $(A \setminus B)^* = A^* \setminus B^*$;
9. $(A \times B)^* = A^* \times B^*$.

[5]Otherwise, let $\Lambda = \{n \mid \varphi(n) \neq \psi(n)\}$ and define $\varphi'(n) = \varphi(n)$ if $n \in \Lambda$ and $\varphi'(n) \neq \psi(n)$ if $n \notin \Lambda$. Then $\varphi'(n) \neq \psi(n)$ for all n and $\varphi'[\alpha] = \varphi[\alpha]$.

[6]A precise definition of what is meant by "elementary property", and a formal statement and proof of the transfer principle, are given in the last Section 6.

[7]When we write $A \in B$ in 2, it is implicitly assumed that B is a set. Similar implicit assumptions will be made throughout the paper.

The above properties are generalized in a straightforward manner to n-tuples, finite intersections, finite unions and finite Cartesian products.

A binary relation $\mathcal{R}$ on a set A is identified with the set $\{\langle a,a'\rangle \in A\times A \mid \mathcal{R}(a,a')\}$ of ordered pairs which satisfy it. Thus its star-transform $\mathcal{R}^*\subseteq A^*\times A^*$ is a binary relation on A^*. Similarly for n-place relations. Since a function $f:A\to B$ is identified with its graph $\{\langle a,b\rangle\in A\times B\mid f(a)=b\}$, its nonstandard extension $f^*\subseteq A^*\times B^*$. The following result guarantees that f^* is actually (the graph of) a function.

Proposition 2.3. *Let $f:A\to B$ be a function. Then its star-transform $f^*:A^*\to B^*$ is a function such that, for every sequence $\varphi:\mathbb{N}\to A$,*

$$f^*(\varphi[\alpha])=(f\circ\varphi)[\alpha]$$

Moreover, f is 1-1 (or onto) if and only if f^* is 1-1 (onto, respectively).

Proof. Notice that $\zeta=\langle\xi,\eta\rangle\in f^*$ if and only if $\zeta=\zeta[\alpha]$ for some sequence $\zeta(n)=\langle\varphi(n),f(\varphi(n))\rangle\in f$. Now, let $\zeta'=\zeta'[\alpha]$ where $\zeta'(n)=\langle\varphi'(n),f(\varphi'(n))\rangle'\in f$. If $\varphi[\alpha]=\varphi'[\alpha]$, then by the *composition axiom*, $(f\circ\varphi)[\alpha]=(f\circ\varphi')[\alpha]$. This proves that f^* is a function and that $f^*(\varphi[\alpha])=(f\circ\varphi)[\alpha]$. The properties of being 1-1 or onto are also readily seen to be preserved by under star-transforms. □

For every sequence $\varphi:\mathbb{N}\to A$:

$$\varphi^*(\alpha)=\varphi^*(1_{\mathbb{N}}[\alpha])=(\varphi\circ 1_{\mathbb{N}})[\alpha]=\varphi[\alpha]$$

Thus, from this point on, we stop using square brackets for values at infinity, and directly write $\varphi^*(\alpha)$ instead of $\varphi[\alpha]$. Following the practice in nonstandard analysis, when confusion is unlikely we shall also drop the $*$ symbol from star-transforms.

2.2. **The hyperreal line.**

Definition 2.4. The set of *hyperreal numbers* is the star-transform $\mathbb{R}^*$ of the set of real numbers.

$$\mathbb{R}^*=\{\varphi(\alpha)\mid\varphi:\mathbb{N}\to\mathbb{R}\}$$

By the *number axiom*, all real numbers are hyperreal numbers. We remark that the inclusion $\mathbb{R}\subset\mathbb{R}^*$ is proper (this will be shown in the sequel). To simplify matters, we shall often abuse notation, and write e.g.

$$\sin\frac{2}{\alpha},\ \arctan(\alpha),\ \frac{3+4\alpha}{7+\alpha},\ \log\alpha,\ e^\alpha \text{ and } \alpha!$$

to mean the hyperreal numbers obtained as the ideal values of the real sequences

$$\left\{\sin\frac{2}{n}\right\}_n,\ \left\{\arctan\frac{2}{n}\right\}_n,\ \left\{\frac{3+4n}{7+n}\right\}_n,\ \{\log n\}_n,\ \{e^n\}_n \text{ and } \{n!\}_n$$

respectively.

The sum and product operations on $\mathbb{R}$ are binary functions; so their star-transforms, which will be denoted by the same symbols $+$ and $\cdot$, are binary functions on $\mathbb{R}^*$. Similarly for the sum inverse function $-:\mathbb{R}\to\mathbb{R}$ and the product inverse function $(\ )^{-1}:\mathbb{R}\setminus\{0\}\to\mathbb{R}\setminus\{0\}$. By the *composition axiom*, for all real sequences φ,ψ the following hold:

(1) $\varphi(\alpha)+\psi(\alpha)=\xi(\alpha)$ where ξ is the sequence $\xi(n)=\varphi(n)+\psi(n)$;
(2) $\varphi(\alpha)\cdot\psi(\alpha)=\xi(\alpha)$ where ξ is the sequence $\xi(n)=\varphi(n)\cdot\psi(n)$;
(3) $-\varphi(\alpha)=\xi(\alpha)$ where ξ is the sequence $\xi(n)=-\varphi(n)$.

As for the product inverse, some caution is needed. In fact, when $\varphi(\alpha) \neq 0$, it could well be $\varphi(n) = 0$ for some n. However, thanks to Proposition 1.4, the following is well-posed.

- If $\varphi(\alpha) \neq 0$ then $\varphi(\alpha)^{-1} = \xi(\alpha)$ where ξ is *any* sequence such that $\xi(n) = 1/\varphi(n)$ for all n with $\varphi(n) \neq 0$.

An ordering on $\mathbb{R}^*$ is given by the nonstandard extension $<^*$ of the ordering on $\mathbb{R}$. The following is an alternative equivalent definition.

- $\xi \prec \zeta \Leftrightarrow \zeta - \xi \in (\mathbb{R}_+)^*$, where $\mathbb{R}_+$ is the set of positive reals.

That $\prec$ is actually the nonstandard extension of the ordering on $\mathbb{R}$ can be seen by checking that:

$$\xi \prec \zeta \text{ if and only if } \langle \xi, \zeta \rangle \in \{\langle x, y \rangle \in \mathbb{R} \times \mathbb{R} \mid x < y\}^*$$

In the sequel, we shall directly use the symbol $<$ instead of $\prec$ or $<^*$.

The next Proposition gives a useful sufficient (but not necessary) condition to check the order relation between two given hyperreal numbers.

Proposition 2.5. *Let φ, ψ be real sequences, and assume that $\varphi(n) < \psi(n)$ eventually. Then $\varphi(\alpha) < \psi(\alpha)$.*

Proof. Let $\vartheta(n) = \varphi(n)$ if $\varphi(n) < \psi(n)$, and $\vartheta(n) = \psi(n) - 1$ otherwise. Then $\vartheta(n) < \psi(n)$ for all n and so $\langle \vartheta(\alpha), \psi(\alpha) \rangle \in \{\langle x, y \rangle \in \mathbb{R} \times \mathbb{R} \mid x < y\}^*$. Since $\vartheta(n) = \varphi(n)$ for all but finitely many n, it follows that $\vartheta(\alpha) = \varphi(\alpha)$ and the claim is proved. □

We are now ready for the first crucial fact in nonstandard analysis.

Theorem 2.6. *The hyperreal number system $\langle \mathbb{R}^*, +, \cdot, 0, 1, < \rangle$ is an ordered field.*

Proof. Straightforward. As an example, let us show the commutativity property of addition. Let $\varphi(\alpha), \psi(\alpha) \in \mathbb{R}^*$. By definition, $\varphi(\alpha) + \psi(\alpha) = \vartheta(\alpha)$ and $\psi(\alpha) + \varphi(\alpha) = \vartheta'(\alpha)$ where $\vartheta(n) = \varphi(n) + \psi(n)$ and $\vartheta'(n) = \psi(n) + \varphi(n)$. Since trivially $\vartheta(n) = \vartheta'(n)$ for all n, we conclude that $\varphi(\alpha) + \psi(\alpha) = \psi(\alpha) + \varphi(\alpha)$. □

2.3. **The hypernatural numbers.**
Similarly to the hyperreal numbers, we give the following

Definition 2.7. The set of *hypernatural* numbers is the star-trasform of the set of natural numbers:

$$\mathbb{N}^* = \{\varphi(\alpha) \mid \varphi : \mathbb{N} \to \mathbb{N}\}$$

The hypernatural numbers play a central role in the practice of nonstandard analysis.

Proposition 2.8.

(1) *The natural numbers are a proper initial segment of the hypernatural numbers. That is, $\mathbb{N} \subset \mathbb{N}^*$ and for every $\xi \in \mathbb{N}^*$, $\xi < n \in \mathbb{N} \Rightarrow \xi \in \mathbb{N}$;*
(2) *The hypernatural numbers are unbounded in the hyperreal line. That is, for every $\zeta \in \mathbb{R}^*$ there exists $\xi \in \mathbb{N}^*$ with $\xi > \zeta$;*
(3) *For every $\xi \in \mathbb{N}^*$, there are no hypernatural numbers ζ strictly between ξ and $\xi + 1$, i.e. such that $\xi < \zeta < \xi + 1$.*

Proof. 1. That $\mathbb{N} \subset \mathbb{N}^*$ directly follows from the *number axiom*. The inclusion is strict because $\alpha \in \mathbb{N}^*$ but $\alpha \notin \mathbb{N}$. Now let $\zeta : \mathbb{N} \to \mathbb{N}$ be such that $\zeta(\alpha) < n$. By Proposition 1.4 we can assume without loss of generality that $\zeta(m) < n$ for all m. Then

$\zeta(m) \in \{0,1,\ldots,n-1\}$ implies that $\zeta(\alpha) \in \{0,1,\ldots,n-1\}^* = \{0,1,\ldots,n-1\}$, hence the claim.
2. Let $\zeta = \zeta(\alpha)$. For each n, pick $\xi(n) \in \mathbb{N}$ with $\xi(n) > \zeta(n)$. Then $\xi = \xi(\alpha) \in \mathbb{N}^*$ and $\xi > \zeta$.
3. Similar to the previous ones. □

Notice that $\alpha \in \mathbb{N}^*$ is greater than all natural numbers (hence $\alpha \notin \mathbb{R}$). In fact, for every $k \in \mathbb{N}$, $1_{\mathbb{N}}(n) > k$ eventually, hence $\alpha = 1_{\mathbb{N}}(\alpha) > c_k(\alpha) = k$. This fact is consistent with the underlying idea of the Alpha-Theory that α is an "infinitely large" (hyper)natural number.

Similarly as $\mathbb{R}^*$ and $\mathbb{N}^*$, one introduces also the collections of *hyperintegers* $\mathbb{Z}^*$ and of *hyperrationals* $\mathbb{Q}^*$, namely the star-trasforms of the sets of integer and rational numbers, respectively. Clearly $\mathbb{N}^* \subset \mathbb{Z}^* \subset \mathbb{Q}^* \subset \mathbb{R}^*$.

We remark that, as a consequence of our axioms, all "elementary properties" of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are inherited by $\mathbb{N}^*, \mathbb{Z}^*, \mathbb{Q}^*, \mathbb{R}^*$.[8] As an example, the reader may want to prove the following.

Proposition 2.9.

(1) *For every hyperreal ζ there exists a unique hyperinteger ξ with $\xi \le \zeta < \xi + 1$;*
(2) *The hyperrational numbers are dense in the hyperreals. That is, for every $\zeta < \eta$ in $\mathbb{R}^*$, there exists $\xi \in \mathbb{Q}^*$ with $\zeta < \xi < \eta$.*

2.4. Hyperfinite sets.

Besides the considered sets of hyper-numbers, another fundamental collection in nonstandard analysis is the following.

Definition 2.10. A *hyperfinite set* is the ideal value of a sequences of finite sets.

The importance of hyperfinite sets lies in the fact that they retain all nice "elementary properties" of finite sets. As an example, let us prove the following.

Proposition 2.11. *Every nonempty hyperfinite subset of $\mathbb{R}^*$ has a greatest element.*

Proof. Let χ be a nonempty hyperfinite subset of $\mathbb{R}^*$. Then $\chi = \chi(\alpha)$ for some sequence $\{\chi(n)\}_n$ of nonempty subsets of $\mathbb{R}$. For every n, let $\mu(n) = \max \chi(n)$. It is easily verified that $\mu(\alpha) = \max \chi$. □

Some applications of hyperfinite sets will be given in Appendix A.

3. Introducing nonstandard calculus

In this Section we present the fundamental properties of the hyperreal numbers, and introduce the basics of nonstandard calculus. A short selection of topics from calculus is contained in the Appendix A, so as to give the flavor of nonstandard methods and, in particular, of the Alpha-theory. Once the reader gets acquainted with our approach, they can try to "translate" (if needed) the usual nonstandard proofs found in the literature into our context. Good references for a complete course of calculus by nonstandard methods are the textbooks [36, 20, 51, 52] and the recent [28].

[8] At this stage the notion of "elementary property" can only be taken at an informal level. A precise definition by using the formalism of first-order logic is given in Section 6.

3.1. **Infinitely small and infinitely large numbers.**
A characteristic feature of nonstandard calculus is that the intuitive notions of a "small" number and a "large" number can be formalized as actual objects of the hyperreal line.

Definition 3.1. A hyperreal number $\xi \in \mathbb{R}^*$ is *bounded* or *finite* if $-r < \xi < r$ for some positive $r \in \mathbb{R}$. We say that ξ is *unbounded* or *infinite* if it is not bounded. ξ is *infinitesimal* if $-r < \xi < r$ for all positive reals r.

Clearly, the inverse of an infinite number is infinitesimal and vice versa, the inverse of a nonzero infinitesimal number is infinite. A first example of an infinitesimal is given by $\frac{1}{\alpha}$, the ideal value of the sequence $\left\{\frac{1}{n}\right\}_n$. Other examples of infinitesimals are the following:

$$\sin\left(\frac{1}{\alpha}\right), \ \frac{\alpha}{7+\alpha^3} \text{ and } \log(1-\frac{1}{\alpha}).$$

All infinitesimals and all real numbers are bounded. However there are bounded hyperreals that are neither infinitesimal nor real. Examples are the following:

$$5+\frac{1}{\alpha}, \ 7+\sin\alpha, \ \frac{5+\alpha}{7+2\alpha} \text{ and } \log(6-\frac{1}{\alpha^3}).$$

In nonstandard analysis, the use of infinitesimal and infinite numbers completely replaces the use of limits. As a consequence, all the basic notions of calculus are simplified and brought closer to your intuition. Next, we itemize a number of simple properties which are the counterparts of the usual theorems about infinitesimal and infinite sequences that are considered in calculus courses.

Proposition 3.2.

(1) *If ξ and ζ are finite, then $\xi+\zeta$ and $\xi\cdot\zeta$ are finite;*
(2) *If ε and η are infinitesimals, then $\varepsilon+\eta$ is infinitesimal;*
(3) *If ε is infinitesimal and ξ is finite, then $\varepsilon\cdot\xi$ is infinitesimal;*
(4) *If ω is infinite and ξ is not infinitesimal, then $\omega\cdot\xi$ is infinite;*
(5) *If $\varepsilon \neq 0$ is infinitesimal and ξ is not infinitesimal, then ξ/ε is infinite;*
(6) *If ω is infinite and ξ is finite, then ξ/ω is infinitesimal.*

Permitting a witticism, we can say that the existence of infinitesimals contradicts the *"American dream"*. In fact, if someone is born poor (an infinitesimal ε), then even if they work hard to improve their condition and become $\varepsilon+\varepsilon$ or $\varepsilon+\varepsilon+\varepsilon$ or $n\varepsilon$ etc., they will always remain poor (infinitesimal). Thanks to infinitesimals, it is possible to formalize a notion of "closeness".

Definition 3.3. We say that two hyperreal numbers ξ and η are *infinitely close* if $\xi-\eta$ is infinitesimal. In this case we write $\xi \sim \eta$.

It is easily seen that $\sim$ is an equivalence relation.

3.2. **The "shadow" theorem.**
Let us first characterize the notions of *least upper bound* and *greatest lower bound.*

Proposition 3.4. *Let A be a nonempty subset of real numbers, and let $l \in \mathbb{R}$. Then:*

(1) $\sup A = l$ *if and only if $l \geq A$ (i.e. $l \geq a$ for all $a \in A$) and $l \sim \xi$ for some $\xi \in A^*$;*
(2) $\inf A = l$ *if and only if $l \leq A$ and $l \sim \xi$ for some $\xi \in A^*$;*

(3) $\sup A = +\infty$ *[or* $\inf A = -\infty$*] if and only if there exists an infinite* $\xi \in A^*$ *which is positive [negative, respectively].*

Proof. Assume $\sup A = l$. For every n there is $\xi(n) \in A$ with $l - 1/n \leq \xi(n) \leq l$. Then $\xi = \xi(\alpha) \in A^*$ is such that $l \sim \xi$. Vice versa, let $l \geq A$. If $l \neq \sup A$, then there is n with $l - 1/n > a$ for all $a \in A$. It follows that $l - \xi < 1/n$ for all $\xi \in A$, against the hypothesis. Property 2 is similar. As for Property 3, assume there is a sequence $\xi(n)$ of elements of A with $\xi(\alpha)$ positive infinite. By Proposition 2.5, for each $r \in \mathbb{R}$, it is $\xi(n) > r$ for infinitely many n. In particular, there is some $a \in A$ with $a > r$ and so A is unbounded above. Vice versa, for each n pick $\xi(n) \in A$ with $\xi(n) > n$. Then $\xi(\alpha) \in A^*$ is positive infinite (in fact, $\xi(\alpha) > \alpha$). The case $\inf A = -\infty$ is proved in the same manner. □

In a course of nonstandard calculus, the above properties are directly given as the definitions of sup and inf.

The next Theorem gives a picture of the hyperreal line and allows a "canonical" representation of the bounded hyperreal numbers. Its proof makes an essential use of the completeness property of real numbers.

Theorem 3.5. *(Shadow Theorem)*
Every finite hyperreal number ξ *is infinitely close to a unique real number* r*, called the shadow of* ξ*. Symbolically* $r = sh(\xi)$*.*

Proof. Let $A = \{a \in \mathbb{R} \mid a < \xi\}$. By the hypothesis A is bounded above, and so we can consider its supremum $\sup A = r \in \mathbb{R}$. By the previous Proposition, there is $\zeta \in A^*$ with $\zeta \sim r$. We claim that $r \sim \xi$. By contradiction, assume $|r - \xi| > x$ for some positive real number x. If $r < \xi - x$ then $r + x < \xi \Rightarrow r + x \in A$ and so $r + x \leq r$, which is absurd. Otherwise, let $r > \xi + x$ and consider a sequence $\zeta(n)$ of elements of A with $\zeta(\alpha) = \zeta$. For every n, $\zeta(n) \in A \Rightarrow \zeta(n) < \xi < r - x \Rightarrow \zeta < r - x$, contradicting the hypothesis $\zeta \sim r$. As for the uniqueness, if $r \sim \xi \sim r'$ then $r - r' \sim 0$. Since the only infinitesimal real number is 0, we conclude that $r = r'$. □

Thus, any bounded hyperreal number ξ is uniquely represented as the sum $r + \varepsilon$ of a real number $r = sh(\xi)$ and an infinitesimal number $\varepsilon = \xi - x$. Such a "canonical" representation is called *normal form*. The following are two examples of normal forms:

$$\frac{5+\alpha}{7+2\alpha} = \frac{1}{2} + \frac{3}{14+4\alpha}, \ \log(6 - \frac{1}{\alpha^3}) = \log 6 + \log(1 - \frac{1}{6\alpha^3})$$

The notion of a shadow is extended to every hyperreal number, by setting $sh(\xi) = +\infty$ if ξ is positive unbounded, and $sh(\xi) = -\infty$ if ξ is negative unbounded. With this notation, we can summarize the characterizations given in the previous Section as follows. Let $l \in \mathbb{R} \cup \{\pm\infty\}$, then:

- $\sup A = l$ if and only if $l \geq A$ and $sh(\xi) = l$ for some $\xi \in A^*$;
- $\inf A = l$ if and only if $l \leq A$ and $sh(\xi) = l$ for some $\xi \in A^*$.

Any field $\mathbb{F}$ contains a natural copy of the natural numbers $\mathbb{N}$ given by $1,\ 1+1,\ 1+1+1$, etc. Recall the *Archimedean property* (the "American dream" mentioned immediately after Proposition 3.2).

Definition 3.6. An ordered field $\mathbb{F}$ is *Archimedean* if for every $x \in \mathbb{F}$ there exists $n \in \mathbb{N}$ with $-n < x < n$.

In other words, $\mathbb{F}$ is nonarchimedean if it has infinite elements or, equivalently, if it contains infinitesimals. We already saw that the hyperreal line $\mathbb{R}^*$ is an example of a nonarchimedean field extending $\mathbb{R}$. The definitions of infinitesimal and infinite numbers, as well as the basic results itemized in Proposition 3.2, also make sense for any nonarchimedean field extending $\mathbb{R}$. However we remark that nonstandard analysis is way stronger than nonarchimedean analysis. The main point is that *any* given real function can be extended to a function on the hyperreals (namely, its *star-transform*) in such a way that all "elementary properties" are preserved. It is in fact possible to extend rational functions to non-archimedean fields $\mathbb{F} \supset \mathbb{R}$, but there is no general way of extending all the transcendental ones.

3.3. Ideal values at α and the notion of limit.

In some sense, one could say that, similarly to "classic" calculus, our Alpha-calculus is grounded on the notion of a limit. This idea could be misleading.

It is certainly true that there are relationships between the limit of a real sequence $\{\varphi(n)\}_n$ and its ideal value $\varphi(\alpha)$. Precisely, it is easily proved that if $\lim_{n\to\infty} \varphi(n) = l \in \mathbb{R} \cup \{\pm\infty\}$, then $sh(\varphi(\alpha)) = l$. Vice versa, if $sh(\varphi(\alpha)) = l$ then $\lim_{k\to\infty} \varphi(n_k) = l$ for some subsequence $\{\varphi(n_k)\}_k$.

However, we remark that limits and values at infinity have relevant differences. First, *any* sequence takes a value at infinity. All we can prove is that the shadow of the value at infinity must be a limit point of the sequence, but our axioms cannot decide which one is taken. *Ad hoc* axioms could settle some of these situations. For instance, we could consistently postulate that the infinite hypernatural α is even. In this case, the alternating sequence $\{(-1)^n\}_n$ takes the value $(-1)^\alpha = 1$ at infinity. Another diversity is that there are plenty of sequences with the same limit but with different (though infinitely close) ideal values.[9]

With respect to this, a simplifying assumption could be that if a hyperreal number ξ is infinitely close to some real number r, then ξ is the value at infinity of some sequence which converges to r in the classic sense. A discussion of this assumption will be the content of Section 5.

4. Other Nonstandard Topics.

Although our approach as outlined so far allows a complete development of an elementary course in calculus (see Appendix A), there are general tools in nonstandard analysis such as the *countable saturation property* and the *overflow* and *underflow* principles, that are needed to deal with more advanced topics. In this Section, we concentrate on such principles.

4.1. Internal sets.

Definition 4.1. An entity is *internal* if it is the ideal value of some sequence. An entity is *external* if it is not internal.

In particular, all nonstandard extensions are internal. The *number axiom* implies that all real numbers, as well as the ideal number α, are internal. Notice that the collection of internal sets is *transitive*, i.e. elements of internal sets are internal.[10] In the next

[9] Recall that $\varphi(n) \neq \psi(n)$ for all n implies $\varphi(\alpha) \neq \psi(\alpha)$.

[10] See condition 3 of Proposition 1.2.

Proposition, we collect some basic facts whose proofs are straightforward (see Proposition 1.1).

Proposition 4.2. *If A, B are internal, then also $\{A, B\}$, $\langle A, B\rangle$, $A \cup B$, $A \cap B$, $A \setminus B$ and $A \times B$ are internal. Moreover, if A is an internal set and f is an internal function, then both $f(A) \doteq \{f(a) \mid a \in A\}$ and $f^{-1}(A) \doteq \{x \mid f(x) \in A\}$ are internal.*

For any set B, the collection of internal subsets of B^* is precisely the star-transform of the collection of subsets of B. That is,

$$\wp(B)^* = \{x \subseteq B^* \mid x \text{ is internal }\}.$$

We remark that $\wp(B)^*$ is a proper subset of $\wp(B^*)$ whenever B is infinite. For instance, it is shown in the next subsection that all infinite subsets which are countable must be external.

We already mentioned that in the usual approach to nonstandard analysis, a fundamental tool is the *transfer principle*, stating (informally) that all "elementary properties" of some given entities $a_1, \ldots, a_k$ transfer to their nonstandard extensions $a_1^*, \ldots, a_k^*$. Thus, as consequence of the above equality, we have that all "elementary properties" of *subsets* of a given set A transfer to *internal subsets* of A^* (but not to *external* subsets). As an example, let us see the following.

Proposition 4.3. *Every nonempty internal subset of $\mathbb{N}^*$ has a least element.*

Proof. Let $B \subseteq \mathbb{N}^*$ be internal and nonempty. Then $B = \varphi(\alpha)$ for some sequence $\{\varphi(n)\}_n$ of nonempty subsets of $\mathbb{N}$. For every n, let $\psi(n) = \min \varphi(n)$. Then the ideal value $\psi(\alpha)$ is the least element of B. ☐

In particular, the collection $\mathbb{N}^*_\infty$ of infinite hypernatural numbers is an *external* subset of $\mathbb{N}^*$ (it is nonempty *without* a least element). Also $\mathbb{N}$ is external, otherwise $\mathbb{N}^* \setminus \mathbb{N} = \mathbb{N}^*_\infty$ would be internal.

Usually, one proves that a given subset $X \subset A^*$ is external by showing that it does not satisfy some "elementary property" of subsets of A. For instance, since all nonempty subset of $\mathbb{R}$ has a least upper bound, we conclude that the collection $o \subset \mathbb{R}^*$ of infinitesimals is external (it has no l.u.b.). Also the collection $\mathbb{R}^*_\infty$ of infinite hyperreals is external, otherwise, by Proposition 4.2, $o = \{1/\xi \mid \xi \in \mathbb{R}^*_\infty\}$ would be internal, etc.

4.2. **Saturation property.**

Besides transfer, the other fundamental tool of nonstandard analysis is *saturation*, an intersection property for internal sets.

Theorem 4.4. *(Countable Saturation Principle)*
Let $\{A_k \mid k \in \mathbb{N}\}$ be a countable family of internal sets with the finite intersection property (FIP), i.e. such that any finite intersection $A_1 \cap \cdots \cap A_n \neq \emptyset$ is nonempty. Then the entire family has a nonempty intersection: $\bigcap_k A_k \neq \emptyset$.

Proof. For every k, let $\{\varphi_k(n)\}_n$ be a sequence with $\varphi_k(\alpha) = A_k$. For any fixed n, we pick an element $\psi(n) \in \varphi_1(n) \cap \cdots \cap \varphi_n(n)$ if the latter intersection is nonempty. Otherwise, we pick $\psi(n) \in \varphi_1(n) \cap \cdots \cap \varphi_{n-1}(n)$ if the latter intersection is nonempty, and continue in this manner until you get an element. In case $\varphi_1(n) = \emptyset$ then we let $\psi(n)$ be an arbitrary element. As a consequence of this, if $n \geq k$ and $\varphi_1(n) \cap \cdots \cap \varphi_k(n) \neq \emptyset$ then $\psi(n) \in \varphi_1(n) \cap \cdots \cap \varphi_k(n)$. Now, $\varphi_1(\alpha) \cap \cdots \cap \varphi_k(\alpha) \neq \emptyset$ by hypothesis, and if $n \in \{m \geq k \mid \varphi_1(m) \cap \cdots \cap \varphi_k(m) \neq \emptyset\}$ then $\psi(n) \in \varphi_1(n) \cap \cdots \cap \varphi_k(n)$. Hence $\psi(\alpha) \in \varphi_1(\alpha) \cap \cdots \cap \varphi_k(\alpha)$. As this holds for every k, the proof is completed. □

As a straightforward application of the saturation principle, one has an alternative proof of the existence of infinitesimal numbers: The countable family of internal sets $\{(0, 1/n)^* \mid n \in \mathbb{N}\}$ trivially satisfies the FIP, and its nonempty intersection $\bigcap_n (0, 1/n)^*$ is precisely the collection of positive infinitesimals.

Proposition 4.5. *Any infinite countable set is external.*

Proof. By contradiction, assume that the infinite countable set $A = \{a_n \mid n \in \mathbb{N}\}$ is internal. Then, for each n, the nonempty set $A_n = A \setminus \{a_1, \ldots, a_n\}$ is internal as well. The countable family $\{A_n \mid n \in \mathbb{N}\}$ trivially has the FIP, and by saturation its intersection is nonempty, a contradiction. □

The countable saturation is crucial to carry out the *Loeb measure* construction, currently one of the major sources of applications of nonstandard methods (see e.g. [18]). Stronger forms of saturation, namely κ-saturation for uncountable cardinals κ, are essential for a nonstandard study of topological spaces with uncountable bases.[11]

4.3. **Overflow and underflow phenomena.**
The following *overflow* and *underflow* principles are frequently used in the practice of nonstandard analysis.

Proposition 4.6.

(1) *(Overflow). If an internal set $A \subseteq \mathbb{N}^*$ contains arbitrarily large natural numbers then it contains also some infinite hypernatural number.*
(2) *A subset $B \subseteq \mathbb{N}$ contains arbitrarily large numbers if and only if its nonstandard extension B^* contains some infinite hypernatural number.*
(3) *If an internal set $A \subseteq \mathbb{N}^*$ contains all natural numbers k greater than some $n \in \mathbb{N}$, then it also contains all hypernatural numbers ξ with $n < \xi < \zeta$ for some infinite hypernatural ζ.*
(4) *A subset $B \subseteq \mathbb{N}$ contains all numbers from some point on if and only if its nonstandard extension B^* contains all infinite hypernatural numbers.*
(5) *(Underflow). If an internal set $A \subseteq \mathbb{N}^*$ contains arbitrarily small infinite hypernatural numbers, then it also contains arbitrarily large natural numbers.*

Proof. 1. For every natural number n, the following set is internal:

$$A_n = \{a \in A \mid a > n\} = (\mathbb{N}^* \setminus \{0, 1, \ldots, n\}) \cap A$$

By the hypothesis, $\{A_n \mid n \in \mathbb{N}\}$ has the FIP and, by saturation, the intersection $\bigcap_n A_n$ is nonempty and consists of the infinite hypernaturals in A.
2. Since $B \subseteq B^*$, one direction is trivially proved by applying 1. Vice versa, B does not contain arbitrarily large natural numbers if and only if $B \subseteq \{0, 1, \ldots, n\}$ for some natural number n. In this case, trivially $B^* = B$ does not contain infinite hypernaturals.
3. Assume that there are infinite hypernaturals which does not belong to A, otherwise the claim is trivial. Then $A' = (\mathbb{N}^* \setminus \{0, 1, \ldots, n\}) \setminus A$ is a nonempty internal subset of $\mathbb{N}^*$. Let $\zeta = \min A'$ be its least element. By the hypothesis, $A' \cap \mathbb{N} = \emptyset$, hence ζ is infinite,

[11]The Alpha-Theory can be generalized so to accomodate all nonstandard arguments which use a prescribed level κ of saturation. See [24].

and every hypernatural number ξ with $n < \xi < \zeta$ belongs to A.
4. Notice that B contains all natural numbers from some point on if and only if its complement $B' = \mathbb{N} \setminus B$ does not contain arbitrarily large natural numbers. Then apply 2 to B'.
5. If by contradiction A does not contain arbitrarily large natural numbers, i.e. if $A \cap \mathbb{N} \subseteq \{0, 1, \ldots, n\}$ for some n, then its complement $A' = \mathbb{N}^* \setminus A$ contains all natural numbers from some point on. So by 3, A' contains all infinite hypernatural numbers up to some $\zeta \in \mathbb{N}^*$, contradicting the hypothesis. □

Similar principles hold for internal collections of hyperreal numbers. E.g., if an internal set $A \subseteq \mathbb{R}^*$ contains arbitrarily large infinitesimal numbers, then it also contains some positive real number, etc. A straight consequence of the above Proposition is a nonstandard characterizations of the notions of "infinitely often" and "eventually".

Proposition 4.7. *Let φ, ψ be two real sequences. Then,*

(1) *("Infinitely often") $\forall k\, \exists n > k\ \varphi(n) = \psi(n)$ (or $\varphi(n) \neq \psi(n)$, or $\varphi(n) > \psi(n)$) if and only if there exists an infinite hypernatural ω with $\varphi^*(\omega) = \psi^*(\omega)$ ($\varphi^*(\omega) \neq \psi^*(\omega)$, or $\varphi^*(\omega) > \psi^*(\omega)$, respectively).*
(2) *("Eventually") $\exists k\, \forall n > k\ \varphi(n) = \psi(n)$ (or $\varphi(n) \neq \psi(n)$, or $\varphi(n) < \psi(n)$) if and only if for all infinite hypernaturals ω, $\varphi^*(\omega) = \psi^*(\omega)$ ($\varphi^*(\omega) \neq \psi^*(\omega)$, or $\varphi^*(\omega) < \psi^*(\omega)$, respectively).*

Proof. Notice that if $A = \{n \in \mathbb{N} \mid \varphi(n) = \psi(n)\}$ then $A^* = \{\xi \in \mathbb{N}^* \mid \varphi^*(\xi) = \psi^*(\xi)\}$, and similarly with $B = \{n \in \mathbb{N} \mid \varphi(n) \neq \psi(n)\}$ and $C = \{n \in \mathbb{N} \mid \varphi(n) < \psi(n)\}$. Then apply the overflow principles. □

5. Cauchy's Infinitesimals Principles.

In his famous textbooks published in the first half of the nineteenth century, Cauchy made use of infinitely small quantities described as "variables converging to zero". What Cauchy actually meant by "variable" was not made clear in those books, and has been repeatedly disputed in the recent historical literature.[12] In particular, J. Cleave [16] proposed an interpretation in terms of nonstandard analysis. By concentrating on a hyperreal line $\mathbb{R}^*$ obtained as a suitable quotient of the set of real sequences, he proposed to describe "Cauchy's infinitesimals" as the equivalence classes of infinitesimal sequences.[13]

The Alpha-Theory provides an axiomatic framework where this idea of an infinitesimal can be accommodated in a natural way.

Cauchy's Infinitesimals Principle (CIP). *Every infinitesimal number is the value at infinity of some infinitesimal sequence.*

As a consequence of this principle, convergent sequences are enough to obtain all hyperreal numbers.

Proposition 5.1. *Cauchy's infinitesimals principle CIP is equivalent to the following property: Every hyperreal $\xi \in \mathbb{R}^*$ is the ideal value $\xi = \xi(\alpha)$ of some real sequence $\{\xi(n)\}_n$ such that $\lim_{n\to\infty} \xi(n) = sh(\xi)$.*

[12] See e.g. [16], [38], [26], [27], [17] and [19].
[13] By infinitesimal sequence we mean a sequence which converges to zero in the usual "standard" sense. See the the next Section 6 for a mathematical discussion of Cleave's interpretation.

Proof. One direction is trivial. Vice versa, assume first that ξ is bounded, hence $\xi = r+\varepsilon$ where $r = sh(\xi) \in \mathbb{R}$ and ε is infinitesimal. By CIP, $\varepsilon = \varepsilon(\alpha)$ for some infinitesimal sequence $\{\varepsilon(n)\}_n$. Let $\xi(n) = r + \varepsilon(n)$. Then $\xi(\alpha) = \xi$ and $\lim_{n\to\infty} \xi(n) = sh(\xi)$. Now let ξ be positive unbounded (if ξ is negative unbounded the proof is similar). Pick an infinitesimal sequence $\{\varepsilon(n)\}_n$ such that $\varepsilon(\alpha) = \frac{1}{\xi} \sim 0$. Without loss of generality we can assume $\varepsilon(n) > 0$ for all n. Then $\xi(n) = 1/\varepsilon(n)$ is a sequence such that $\xi(\alpha) = \xi$ and $\lim_{n\to\infty} \xi(n) = +\infty = sh(\xi)$. □

Next, we formulate a stronger version of the above principle, which in our opinion more closely corresponds to Cauchy's conception of infinitesimals.

Strong Cauchy's Infinitesimals Principle (SCIP). *Every nonzero infinitesimal number is the value at infinity of some monotone sequence.*

In other words, the positive infinitesimals are ideal values of decreasing infinitesimal sequences and the negative infinitesimals are ideal values of increasing infinitesimal sequences. The next result provides two equivalent formulations (the proof is similar to the previous Proposition).

Proposition 5.2. *The strong Cauchy's infinitesimals principle SCIP is equivalent to either of the following conditions:*

(i) *For every hyperreal $\xi \in \mathbb{R}^*$ there exist an increasing sequence $\{\xi(n)\}_n$ and a decreasing sequence $\{\xi'(n)\}_n$ such that $sh(\xi) = \lim_{n\to\infty} \xi(n)$ if $sh(\xi) \geq \xi$, and $sh(\xi) = \lim_{n\to\infty} \xi'(n)$ if $sh(\xi) \leq \xi$;*

(ii) *For every infinite hypernatural $\xi \in \mathbb{N}^*$ there exists an increasing sequence $\{\xi(n)\}_n$ of natural numbers such that $\xi(\alpha) = \xi$.*

In the next Section, we will show that SCIP holds if and only if the hyperreals $\mathbb{R}^*$ are isomorphic to the ultrapower of $\mathbb{R}$ modulo a *selective* ultrafilter. We remark that nonstandard models originated by such ultrafilters have interesting special properties, that have been recently used in the study of probability measures (see e.g. [8] and references therein).

We conclude the exposition by giving a picture of the "strength" of the Alpha-Theory $\alpha\mathbf{1} - \alpha\mathbf{5}$, and of the Cauchy's Principles. The proofs are given in the next Section 6 where the technical parts and the foundational matters are relegated.

- Any theorem in "ordinary mathematics" is proved by the Alpha-Theory if and only if it is "true";[14]
- By assuming the Alpha-Theory, we cannot prove nor disprove Cauchy's Infinitesimals Principle;
- Assume the Alpha-Theory and Cauchy's Infinitesimals Principle. Then we cannot prove nor disprove the Strong Cauchy's Infinitesimals Principle;
- The Alpha-Theory plus the Strong Cauchy's Infinitesimals Principle is a "sound" system, i.e. it does not bring to contradictions.[15]

[14] This statement is an "informal" formulation of Theorem 6.4.

[15] Clearly, we assume ZFC to be consistent. The proofs of the latter three itemized properties is in Theorem 6.10.

6. The Foundations

In this Section we assume the reader has a background in mathematical logic. In particular, we assume familiarity with the formalism of first-order logic, and a knowledge of the notions of ultrafilter, ultrapower, model of set theory, internal model, relative consistency, etc. For notation and basic results in model-theory and set-theory we refer to [13], [33] and [37].

6.1. The formal definition of the Alpha-Theory.

In order to avoid a direct use of the formalism of first-order logic, the axioms $\alpha\mathbf{1},\ldots,\alpha\mathbf{5}$ given in Section 1.1, were expressed in an "informal" language. For instance, the sentence: "For every sequence φ there is a unique element $\varphi[\alpha]$" cannot be formalized as a first-order formula where α is a constant symbol.

Next, we give indications for a rigorous definition of the Alpha-Theory in the language $\mathcal{L}=\{\in,\mathcal{A},J\}$, namely the language of set theory with a set $\mathcal{A}$ of atoms ($\mathcal{A}$ is a constant symbol of $\mathcal{L}$), augmented with a new binary relation symbol J.

We first postulate that J is a function defined on the class of all sequences.[16]

J1. *Extension Axiom.* If φ is a sequence, then there exists a unique x such that $J(\varphi,x)$. Vice versa, if $J(\varphi,x)$ holds for some x, then φ is a sequence.

Now, having in mind that $\varphi[\alpha]$ denotes the *unique* element x such that $J(\varphi,x)$, it is clear how the remaining axioms $\alpha\mathbf{2}-\alpha\mathbf{5}$ can be formalized as $\in$-J-sentences.

J2. *Composition Axiom.*
If φ and ψ are sequences and if f is any function such that compositions $f\circ\varphi$ and $f\circ\psi$ make sense, then

$$\forall x\ \left[\,(\,J(\varphi,x)\wedge J(\psi,x)\,)\ \rightarrow\ \exists y\ (\,J(f\circ\varphi,y)\wedge J(f\circ\psi,y)\,)\right]$$

J3. *Number Axiom.*
Let $r\in\mathbb{R}\subset\mathcal{A}$. If $c_r:n\mapsto r$ is the constant sequence with value r, then: "$\forall x\ (\,J(c_r,x)\rightarrow x=r\,)$". If $1_{\mathbb{N}}:n\mapsto n$ is the identity sequence on $\mathbb{N}$, then "$\forall x\ (\,J(1_{\mathbb{N}},x)\rightarrow x\notin\mathbb{N}\,)$".

J4. *Pair Axiom.*
For all sequences φ,ψ and ϑ such that $\vartheta(n)=\{\varphi(n),\psi(n)\}$ for all n:

$$\forall x\,\forall y\,\forall z\ \left[\,(\,J(\varphi,x)\wedge J(\psi,y)\wedge J(\vartheta,z)\,)\ \rightarrow\ z=\{x,y\}\,\right].$$

In the next axiom, we use the notation "$\psi\mathrm{E}\varphi$" as a shorthand for "ψ and φ are sequences and $\psi(n)\in\varphi(n)$ for all $n\in\mathbb{N}$".

J5. *Internal Set Axiom.*
If ψ is a sequence of atoms, then "$\forall x\ J(\varphi,x)\rightarrow x\in\mathcal{A}$". If $c_\emptyset$ is the constant sequence with value the empty set, then "$J(c_\emptyset,\emptyset)$". If φ is a sequence of nonempty sets, then:

$$\forall x\ \left[\,J(\varphi,x)\rightarrow\forall y\ \ (\,y\in x\leftrightarrow\exists\psi\ (\,\psi\mathrm{E}\varphi\wedge J(\psi,y)\,)\,)\right].$$

Once $J1,\ldots,J5$ are formalized according to the indications given above, the Alpha-Theory is introduced in a "rigorous" way as follows.

Definition 6.1. The *Alpha-Theory* is the first-order theory in the language $\mathcal{L}=\{\in,\mathcal{A},J\}$, where $\in$ and J are binary relation symbols and $\mathcal{A}$ is a constant symbol, and whose set of axioms consists of:

[16] One cannot consider J as a function symbol because it is a *partial* function defined only on sequences.

- All axioms of *Zermelo-Fraenkel set theory* with atoms ZFCA, with the only exception of the axiom of foundation. The separation and replacement schemata are also considered for formulas containing the J-symbol;[17]
- The five axioms **J1**, ..., **J5** as given above.

6.2. **The transfer principle.**
Before going to the foundational matters, in this subsection we show that the Alpha-Theory proves the *transfer principle*, hence it actually has the full strength of "Robinsonian" nonstandard analysis.

Informally, the *transfer principle* states that a given "elementary property" holds for entities $a_1, \ldots, a_k$ if and only if the same property holds for the corresponding star-transforms $a_1^*, \ldots, a_k^*$. The notion of "elementary property" is made precise by using the language of mathematical logic. Precisely, by *elementary property* it is meant a property that can be formalized as a *bounded formula* in the first-order language of set theory.[18]

Theorem 6.2. *The Alpha-Theory proves the transfer principle. That is, for every bounded formula $\sigma(x_1, \ldots, x_k)$ in the first order language $\mathcal{L} = \{\in\}$ of set theory, and for every $a_1, \ldots, a_k$, $\sigma(a_1, \ldots, a_k) \Leftrightarrow \sigma(a_1^*, \ldots, a_k^*)$.*

Proof. We shall prove a more general fact, which is a version of Łos Theorem of ultrapowers (see [13] §4.1). Precisely, we claim that if $\varphi_1, \ldots, \varphi_k$ are sequences, and $\sigma(x_1, \ldots, x_k)$ is a bounded formula, then

$$\sigma\left(\varphi_1[\alpha], \ldots, \varphi_k[\alpha]\right)) \;\Leftrightarrow\; \alpha \in \{n \mid \sigma\left(\varphi_1(n), \ldots, \varphi_k(n)\right)\}^*$$

This will yield the thesis. In fact, by definition, $a_j^* = c_{a_j}(\alpha)$ where c_{a_j} is the sequence with constant value a_j. In particular, the set $\{n \mid \sigma(c_{a_1}(n), \ldots, c_{a_k}(n))\}$ either equals $\mathbb{N}$ or the empty set, depending on whether $\sigma(a_1, \ldots, a_k)$ holds or fails. Thus:

$$\sigma(a_1, \ldots, a_k) \;\Leftrightarrow\; \alpha \in \{n \mid \sigma(c_{a_1}(n), \ldots, c_{a_k}(n))\}^*\,.$$

We proceed by induction on the complexity of formulas. We first prove that

$$\varphi_1(\alpha) = \varphi_2(\alpha) \;\Leftrightarrow\; \alpha \in \Lambda^* \text{ where } \Lambda = \{n \mid \varphi_1(n) = \varphi_2(n)\}\,.$$

By definition, $\alpha \in \Lambda^*$ if and only if $\alpha = \chi(\alpha)$ for some sequence $\chi : \mathbb{N} \to \Lambda$. Now, $\chi(\alpha) = 1_{\mathbb{N}}(\alpha)$ implies that $\varphi_1(\alpha) = (\varphi_1 \circ 1_{\mathbb{N}})(\alpha) = (\varphi_1 \circ \chi)(\alpha)$, and similarly $\varphi_2(\alpha) = (\varphi_2 \circ \chi)(\alpha)$. But then $\varphi_1(\alpha) = \varphi_2(\alpha)$ because trivially $\varphi_1 \circ \chi = \varphi_2 \circ \chi$.

Vice versa, $\alpha \notin \Lambda^*$ if and only if $\alpha \in \Gamma^*$ where $\Gamma = \{n \mid \varphi_1(n) \neq \varphi_2(n)\}$. Then we proceed similarly as before, by noticing that $(\varphi_1 \circ \chi)(\alpha) \neq (\varphi_2 \circ \chi)(\alpha)$ because $\varphi_1(\chi(n)) \neq \varphi_2(\chi(n))$ for all n.

In the same manner, the following is also proved:

$$\varphi_1(\alpha) \in \varphi_2(\alpha) \;\Leftrightarrow\; \alpha \in \Lambda^* \text{ where } \Lambda = \{n \mid \varphi_1(n) \in \varphi_2(n)\}\,.$$

The disjunction and the negation steps are straightforward. We are left to consider the existential quantifier. Assume first that

$$\exists x \in \varphi_0[\alpha]\; \sigma(x, \varphi_1[\alpha], \ldots, \varphi_k[\alpha]).$$

---

[17]This formalizes the blanket assumption of "all principles of ordinary mathematics". We refer to [37] for a presentation of Zermelo-Fraenkel set theory ZFC. See e.g. [33] §21 for ZFCA, the version of ZFC with a set A of atoms.

[18]Recall that, by definition, a formula σ is *bounded* if every quantifier occurs in the bounded form $\forall x \in y$ (i.e. $\forall x\;\; x \in y \to \ldots$) or $\exists x \in y$ (i.e. $\exists x\, x \in y \wedge \cdots$). See [13] §4.4.

Then there is a sequence $\{\psi(n)\}_n$ such that

$$\psi(n) \in \varphi_0(n) \text{ for all } n \text{ and } \sigma(\psi(\alpha), \varphi_1(\alpha), \ldots, \varphi_k(\alpha)).$$

By the inductive hypothesis, $\alpha \in \{n \mid \sigma(\psi(n), \varphi_1(n), \ldots, \varphi_k(n))\}^*$, which is included in $\{n \in \mathbb{N} \mid \exists x \in \varphi_0(n)\ \sigma(x, \varphi_1(n), \ldots, \varphi_k(n))\}^*$.

Vice versa, there is a sequence $\psi : \mathbb{N} \to \{n \mid \exists x \in \varphi_0(n)\ \sigma(x, \varphi_1(n), \ldots, \varphi_k(n))\}$ with $\psi(\alpha) = \alpha$. Then for all n, $\exists x \in \varphi_0(\psi(n))\ \sigma(x, \varphi_1(\psi(n)), \ldots, \varphi_k(\psi(n)))$, hence

$$\alpha \in \{n \mid \exists x \in \varphi_0(\psi(n))\ \sigma(x, \varphi_1(\psi(n)), \ldots, \varphi_k(\psi(n)))\}^* = \mathbb{N}^*.$$

By the inductive hypothesis,

$$\exists x \in \varphi_0(\psi(\alpha))\, \sigma(x, \varphi_1(\psi(\alpha)), \ldots, \varphi_k(\psi(\alpha)))$$

and the claim follows because $\psi(\alpha) = \alpha$. □

6.3. **A faithful interpretation of ZFC in the Alpha-Theory.**
We now concentrate on the consistency matters. As for the soundness of the Alpha-Theory, a superstructure was constructed in [3] which provides a model for the weakened version where the infinity axiom and the replacement schema are dropped but where foundation is assumed. A general foundational justification as a nonstandard set theory was given in [23]. In this subsection we prove that ZFC is *faithfully interpretable* in the Alpha-Theory. Namely, we show that any $\in$-sentence σ is a theorem of ZFC if and only if its relativization σ^{WF} to the class of well-founded sets is a theorem of the Alpha-Theory.[19]

A convenient framework to construct models of the Alpha-Theory is the so-called *Zermelo-Fraenkel-Boffa* set theory ZFBC, which was first used for the foundations of nonstandard methods by D. Ballard and K. Hrbàček [2]. Roughly, ZFBC is a non-wellfounded variant of ZFC where the axiom of foundation is replaced by Boffa's *superuniversality* axiom postulating the existence of transitive collapses for all extensional structures, and where a *global choice axiom* is also assumed in the form of a well-ordering of the universe.[20]

A central role in the arguments to follow is played by the family:

$$\mathcal{U}_\alpha \doteq \{A \subseteq \mathbb{N} \mid \alpha \in A^*\}$$

It is readily seen that $\mathcal{U}_\alpha$ is a nonprincipal ultrafilter.

Theorem 6.3.

(i) *ZFBC proves the following: For each nonprincipal ultrafilter D on $\mathbb{N}$, a function J_D can be defined on the class of all sequences in such a way that the internal model $\mathfrak{M}_D = \langle V, \in, J_D\rangle$ is a model of the Alpha-Theory and $\mathfrak{M}_D \models$ "$\mathcal{U}_\alpha = D$".*[21]

(ii) *Let $\mathfrak{A}$ be a countable model of ZFC and assume that*

$$\mathfrak{A} \models \text{“}D \text{ is a nonprincipal ultrafilter on } \mathbb{N}\text{”}.$$

Then $\mathfrak{A}$ is the well-founded part of some model $\mathfrak{N}_D$ of the Alpha-Theory such that $\mathfrak{N}_D \models$ "$\mathcal{U}_\alpha = D$".[22]

---

[19]On the notion of interpretability, see e.g. [32] Ch. V.

[20]See [2] for a precise formulation.

[21]V denotes the universal class of all sets. We remark that the correspondance $D \mapsto J_D$ is definable, i.e. there exists a formula $\vartheta(x, y, z)$ such that $J_D(x) = y$ if and only if ZFBC proves $\vartheta(x, y, D)$.

[22]We say that $\mathfrak{A}$ is the well-founded part of $\mathfrak{N}$ to mean that $\mathfrak{A}$ is the submodel of $\mathfrak{N}$ whose universe is the set $\{x \in \mathfrak{N} \mid \mathfrak{N} \models \text{“}x \text{ is wellfounded ”}\}$.

Proof. The proof consists of known arguments in nonstandard set theory (see e.g. [2, 22, 23]). We outline here the main steps.
(i) The theory ZFBC allows to formalize the following construction: Let d be the diagonal embedding of the universe V into its ultrapower $V^{\mathbb{N}}/D$ modulo D, and let π be the first transitive collapse of $V^{\mathbb{N}}/D$ in the well-ordering of the universe. Then $J_D = \pi \circ d$ yields the thesis.
(ii) By putting together a classic result by U. Felgner [25] on global choice, with Boffa's construction of models of superuniversality [11], we obtain the following property: Every countable model $\mathfrak{A}$ of ZFC is the wellfounded part of some model $\mathfrak{B}$ of ZFBC. Let $D \in \mathfrak{B}$ be such that $\mathfrak{B} \models$ "D is a nonprincipal ultrafilter on $\mathbb{N}$". Now work inside $\mathfrak{B}$. By applying the above construction (i), we obtain a model $\mathfrak{N}_D \doteq (\mathfrak{M}_D)^{\mathfrak{B}}$ of the Alpha-Theory with same universe and same membership relation as $\mathfrak{B}$, and such that $\mathfrak{N}_D \models$ "$\mathcal{U}_\alpha = D$". □

For any $\in$-sentence σ, let us denote by σ^{WF} its *relativization* to the class WF of wellfounded sets.[23] We are now ready to prove the main result.

Theorem 6.4. *The translation $t : \sigma \mapsto \sigma^{WF}$ is a faithful interpretation of ZFC in the Alpha-Theory. That is, a sentence σ in the language of set theory is a theorem of ZFC if and only if σ^{WF} is a theorem of the Alpha-Theory.*

Proof. If σ is not a theorem of ZFC, then there is a model $\mathfrak{A}$ of ZFC where $\mathfrak{A} \models \neg\sigma$. We can assume that $\mathfrak{A}$ is countable (otherwise apply the downward Löwenheim-Skolem Theorem). Pick $D \in \mathfrak{A}$ such that $\mathfrak{A} \models$ "D is a nonprincipal ultrafilter over $\mathbb{N}$", and consider the model $\mathfrak{N}_D$ as given by (ii) of the previous Theorem. Since $\mathfrak{A}$ is the wellfounded part of $\mathfrak{N}_D$, then

$$\mathfrak{A} \models \neg\sigma \;\Leftrightarrow\; \mathfrak{N}_D \models \neg\sigma^{WF},$$

hence σ^{WF} is not a theorem of the Alpha-Theory. (Notation used above is not ambiguous because $\neg(\sigma^{WF})$ is same as $(\neg\sigma)^{WF}$.)

Vice versa, if σ^{WF} is not a theorem of the Alpha-Theory, we can pick a model $\mathfrak{M}$ of the Alpha-Theory where $\mathfrak{M} \models \neg\sigma^{WF}$. Let $\mathfrak{A}$ be the wellfounded part of $\mathfrak{M}$. Then $\mathfrak{A}$ is a model of ZFC and clearly $\mathfrak{A} \models \neg\sigma$, and we conclude that σ is not a theorem of ZFC. □

This latter Theorem formalizes the first property itemized at the end of the previous Section 5: Any theorem in "ordinary mathematics" is proved by the Alpha-Theory if and only if it is "true".

6.4. **The strength of Cauchy's Infinitesimals Principles.**
Following I. Lakatos [38], J. Cleave proposed in [16] an interpretation of Cauchy's conception of infinitesimals in the context of nonstandard analysis. Precisely, by considering models of the hyperreal line given by ultrapowers $\mathbb{R}^{\mathbb{N}}/D$, he proposed to interpret Cauchy's infinitesimals as the D-equivalence classes of those real sequences that converge to zero. In Section 5, we have formalized this idea as the *Cauchy's Infinitesimals Principle* CIP. We have also formulated the stronger version SCIP where one restrics to monotone null sequences.

In their paper [19], N. Cutland, C. Kessler, E. Kopp and D. Ross discussed the mathematical content of Cleave's interpretation. Most notably, they pointed out that the

[23] I.e., every quantifier $\exists x \ldots$ occurring in σ is replaced by $\exists x$ ("x is wellfounded" $\wedge \ldots$); and every quantifier $\forall x \ldots$ is replaced by $\forall x$ ("x is wellfounded" $\rightarrow \ldots$).

assumption of an ultrapower where *every* infinitesimal is originated by some infinitesimal sequence, requires a *P-point*, an ultrafilter with additional properties whose existence is independent of ZFC. The similar foundational issues arise in our context.

In the following, the strength of our two Cauchy's Infinitesimals Principles will be made precise, by showing their relation with two special kinds of ultrafilters, namely the *P-points* and the *selective ultrafilters*.[24] Let us recall the definitions.

Definition 6.5. A non-principal ultrafilter D on $\mathbb{N}$ is a *P-point* (a *selective ultrafilter*) if every function on $\mathbb{N}$ becomes finite-to-one or constant (1-1 or constant, respectively) if restricted to some suitable set in D.

Not every ultrafilter is a P-point and, trivially, every selective ultrafilter is a P-point. It is known that the existence of such special ultrafilters is independent of ZFC. Given the *continuum hypothesis* (or even *Martin's Axiom*, a strictly weaker assumption) selective ultrafilters exist, as well as P-points that are not selective (see e.g. [15]). On the other hand, there are models of ZFC with no P-points.[25]

We shall need the following facts.

Proposition 6.6. *Let D be a non-principal ultrafilter on $\mathbb{N}$. Then:*

1. *D is a P-point if and only if every infinitesimal in the ultrapower $\mathbb{R}^{\mathbb{N}}/D$ is the D-equivalence class of some infinitesimal sequence;*
2. *D is selective if and only if every infinitesimal in the ultrapower $\mathbb{R}^{\mathbb{N}}/D$ is the D-equivalence to some infinitesimal monotone sequence.*

Proof. A proof of item (1) can be found e.g. in [15] (see also [19]).

Let us turn to (2). First, recall the following characterization: An ultrafilter D is selective if and only if every function $\varphi : \mathbb{N} \to \mathbb{N}$ becomes increasing if restricted to a suitable subset in D.[26] Let $\varepsilon \in \mathbb{N}^{\mathbb{N}}/D$ be an infinitesimal. Without loss of generality we can assume that $\varepsilon > 0$ and that $\varepsilon = [f]$ is the D-equivalence class of a function $f : \mathbb{N} \to \mathbb{R}$ such that $0 < f(n) < 1$ for all n. For each n, define $\varphi(n) = k$, where k is such that $\frac{1}{k+1} \leq f(n) < \frac{1}{k}$. By the above property, there is an increasing $\varphi' : \mathbb{N} \to \mathbb{N}$ with $\Lambda = \{n \mid \varphi(n) = \varphi'(n)\} \in D$. If $g : \mathbb{N} \to \mathbb{R}$ is any decreasing function such that $g(n) = f(n)$ for all $n \in \Lambda$, then clearly $\varepsilon = [g]_D$ and $\{g(n)\}_n$ is infinitesimal.

Vice versa, let $\varphi : \mathbb{N} \to \mathbb{N}$ be a function which is not D-equivalent to any constant function. Let $f(n) \doteq \frac{1}{\varphi(n)}$ (we can assume that φ has no zeros). Notice that $\varepsilon \doteq [f]$ is a positive infinitesimal (for each $k > 0$, $\{n \mid f(n) < 1/k\} = \{n \mid \varphi(n) > k\} \in D$). By the hypothesis, there exists a decreasing $g : \mathbb{N} \to \mathbb{R}$ such that $\Lambda = \{n \mid f(n) = g(n)\} \in D$. In particular, φ becomes 1-1 function if restricted to $\Lambda \in D$. □

The nonstandard extension of any given set is isomorphic to the ultrapower modulo $\mathcal{U}_\alpha$.

Theorem 6.7. *The Alpha-Theory proves the following: For any nonempty set A, let $A^{\mathbb{N}}/\mathcal{U}_\alpha$ be the ultrapower of A modulo $\mathcal{U}_\alpha$. Then the function $K_A : A^* \to A^{\mathbb{N}}/\mathcal{U}_\alpha$ which*

[24] To the authors' knowledge, P-points were first introduced by W. Rudin [47], while selective ultrafilters, under the name of *ultrafiltres absolu*, were introduced in the pioneering papers [14, 15] by G. Choquet. For properties of these ultrafilters, see e.g. [12, 9, 10] and references therein.

[25] This latter fact was proved by S. Shelah [50] Ch. VI §4.

[26] A proof of this equivalence can be found e.g. in [6].

maps every element $\varphi(\alpha) \in A^*$ *to the* $\mathcal{U}_\alpha$*-equivalence class* $[\varphi] \in A^{\mathbb{N}}/\mathcal{U}_\alpha$*, is an isomorphism.*

Proof. It is enough to show that for all sequences φ, ψ:

- $\varphi(\alpha) = \psi(\alpha) \Leftrightarrow \{n \mid \varphi(n) = \psi(n)\} \in \mathcal{U}_\alpha \Leftrightarrow [\varphi] = [\psi]$;
- $\varphi(\alpha) \in \psi(\alpha) \Leftrightarrow \{n \mid \varphi(n) \in \psi(n)\} \in \mathcal{U}_\alpha \Leftrightarrow \mathbb{N}^{\mathbb{N}}/D \models$ "$[\varphi] \in [\psi]$".

Notice that the first item also shows that the definition of K_A is well-posed, i.e., it does not depend on the representative chosen in the $\mathcal{U}_\alpha$-equivalence class. Let $\Lambda \doteq \{n \mid \varphi(n) = \psi(n)\}$ (or $\Lambda \doteq \{n \mid \varphi(n) \in \psi(n)\}$). Trivially, $\Lambda = \{n \mid 1_{\mathbb{N}}(n) \in \Lambda\}$, and the thesis follows from Proposition 1.4. □

Corollary 6.8. *The Alpha-Theory proves the following:*

(i) *CIP holds if and only if* $\mathcal{U}_\alpha$ *is a P-point;*
(ii) *SCIP holds if and only if* $\mathcal{U}_\alpha$ *is a selective ultrafilter.*

Proof. By the previous Theorem, the map $K_{\mathbb{R}} : \varphi(\alpha) \mapsto [\varphi]$ yields an isomorphism between $\mathbb{R}^*$ and the ultrapower $\mathbb{R}^{\mathbb{N}}/\mathcal{U}_\alpha$. Then use the characterizations given in Proposition 6.6. □

As a straight consequence, the proof of Theorem 6.4 can also be employed to include Cauchy's Infinitesimals Principles.

Theorem 6.9. *The translation* $t : \sigma \mapsto \sigma^{WF}$ *is a faithful interpretation of ZFC plus "there exists a P-point" (or plus "there exists a selective ultrafilter") in the Alpha-Theory plus CIP (plus SCIP, respectively).*

Finally, we can justify the statements itemized at the end of the previous Section.

Theorem 6.10.

(i) *The Alpha-theory does not prove CIP;*
(ii) *The Alpha-theory plus CIP does not prove SCIP;*
(iii) *The Alpha-theory plus SCIP is consistent with ZFC.*

Proof. (i) Take a countable model $\mathfrak{A}$ of ZFC, and pick $D \in \mathfrak{A}$ such that $\mathfrak{A} \models$ "D is an ultrafilter on $\mathbb{N}$ which is *not* a P-point", and consider the structure $\mathfrak{N}_D$ given by Theorem 6.3 (ii). Then $\mathfrak{N}_D$ is a model the Alpha-Theory with $\mathfrak{N}_D \models$ "$D = \mathcal{U}_\alpha$ is *not* a P-point", hence $\mathfrak{N}_D \models$ " CIP fails" by Corollary 6.8.
(ii) Take a countable model $\mathfrak{A}$ of ZFC plus the continuum hypothesis, and let $D \in \mathfrak{A}$ be such that $\mathfrak{A} \models$ "D is a P-point which is not selective". Then $\mathfrak{N}_D$ is a model of the Alpha-Theory where CIP holds but SCIP fails.
(iii) Pick a countable model $\mathfrak{A}$ of ZFC where there is a selective ultrafilter D. Then $\mathfrak{N}_D$ is a model of the Alpha-Theory plus SCIP. □

Appendix A. A Little Calculus

In this Appendix we present a little selection of classic results from calculus, so to give the flavour of nonstandard methods and, in particular, of the Alpha-Theory in action. In the last subsection, we introduce the notion of Alpha-integral, a generalization of the Riemann integral that has the pedagogical advantage of making sense for *all* real functions.

A.1. **Continuity.**
In the current ϵ-δ formalization, introduced by Weierstrass at the end of the nineteenth century, infinitesimal and infinite numbers are banned from calculus, and a central role is given to the notion of limit instead. ¿From a historical point of view, many nonstandard definitions are very close, and sometimes identical, to the original ones as adopted by Leibniz, Euler, Cauchy and others.

A first example is the notion of continuity. Informally, sometimes it is said that a function is continuous at x_0 if $f(x)$ is "close" to $f(x_0)$ whenever x is "close" to x_0. If in the latter statement we replace "close" by "infinitely close", what we obtain is precisely the nonstandard definition.

Definition A.1. $f : A \to \mathbb{R}$ be a function, and suppose that A contains a neighborhood of x_0. Then f is *continuous* at x_0 if for every $\xi \in A^*$, $\xi \sim x_0 \Rightarrow f^*(\xi) \sim f(x_0)$.

The equivalence with the "standard" definition of continuity is a corollary of the following characterization of limits.

Proposition A.2. *Let $f : A \to \mathbb{R}$ be a function, $x_0, l \in \mathbb{R}$, and assume that A is a neighborhood of x_0. Then the following are equivalent:*

(i) *For every real number $\epsilon > 0$, there exists a real $\delta > 0$, such that for all reals x, $x_0 - \delta < x < x_0 + \delta \Rightarrow l - \epsilon < f(x) < l + \epsilon$;*
(ii) *If $\varepsilon \sim 0$ then $f^*(x_0 + \varepsilon) \sim l$.*

Proof. By contradiction, assume there is a positive real $r > 0$ such that for all positive $n \in \mathbb{N}$ there exists an element $\xi(n) \in \mathbb{R}$ with $|\xi(n) - x_0| < 1/n$ and $|f(\xi(n)) - l| > r$. At infinity, we have $|\xi(\alpha) - x_0| < 1/\alpha$ and $|f^*(\xi(\alpha)) - l| > r$. In particular $\xi(\alpha) \sim x_0$ but $f^*(\xi(\alpha)) \not\sim l$.

Vice versa, for every positive $a \in \mathbb{R}$, let

$$X_a = \{x \in \mathbb{R} \mid |x - x_0| < a\} \text{ and } Y_a = \{x \in \mathbb{R} \mid |f(x) - l| < a\}$$

Fix a positive $r \in \mathbb{R}$. By hypothesis there exists a positive $\delta \in \mathbb{R}$ such that $X_\delta \subseteq Y_r$, hence $X^*_\delta \subseteq Y^*_r$. If $\varepsilon \sim 0$, then clearly $x_0 + \varepsilon \in X^*_\delta = \{\xi \in \mathbb{R}^* \mid |\xi - x_0| < \delta\}$, and so $x_0 + \varepsilon \in Y^*_r$, i.e. $|f^*(x_0 + \varepsilon) - l| < r$. As this is true for all positive $r \in \mathbb{R}$, it follows that $f^*(x_0 + \varepsilon) \sim l$. □

The view of many working mathematicians in nonstandard analysis is that infinitesimal numbers do in fact exist and that the notion of a limit is just an awkward way to indirectly talk about infinitesimals without explicitly mentioning them. In fact, the very notion of limit is banned from a calculus by nonstandard methods, and all the classical ϵ-δ definitions are reformulated in *simpler* terms.[27]

We remark that no proofs in the style of the previous Proposition are needed when introducing nonstandard calculus to freshmen (simply, there are no "standard" definition to compare to!).

The nonstandard counterparts of basic theorems on limits are the following properties of shadows.

Proposition A.3. *For every bounded hyperreals ξ and ζ:*

(1) *$sh(\xi + \zeta) = sh(\xi) + sh(\zeta)$;*

[27]Here "in simpler terms" has the precise meaning of "by using a smaller number of quantifiers".

(2) *$sh(\xi-\zeta)=sh(\xi)-sh(\zeta)$;*
(3) *$sh(\xi\cdot\zeta)=sh(\xi)\cdot sh(\zeta)$;*
(4) *If ζ is not infinitesimal, then $sh(\frac{\xi}{\zeta})=\frac{sh(\xi)}{sh(\zeta)}$.*

If we adopt the usual conventional algebra on $\mathbb{R}\cup\{\pm\infty\}$, i.e. if we agree that

$$\begin{aligned}(+\infty)+(+\infty) &= +\infty\ ;\ (-\infty)+(-\infty)=-\infty\ ;\\ (+\infty)\cdot(+\infty) &= +\infty\ ;\ (+\infty)\cdot(-\infty)=-\infty\ ;\quad \text{etc.}\end{aligned}$$

then the above Proposition can be extended to infinite numbers, with only exceptions of the following *indeterminate forms*:

$$+\infty-\infty\ ;\ \infty\cdot 0\ ;\ \frac{\infty}{\infty}\ ;\ \frac{0}{0}.$$

In these cases, it is readily seen that the resulting shadows could be any element of $\mathbb{R}\cup\{\pm\infty\}$ (the same old story as with "classic" limits...).

Example A.4. Let us prove that the function $f(x)=x^2$ is continuous at all $x\in\mathbb{R}$. This is easy, because for any given $x\in\mathbb{R}$, $f^*(x+\varepsilon)-f^*(x)=(x+\varepsilon)^2-x^2=2x\varepsilon+\varepsilon^2$ is infinitesimal for every $\varepsilon\sim 0$.

Example A.5. Let us consider the function

$$f(x)=\begin{cases}\frac{1}{q} & \text{if } x=\frac{p}{q}\in\mathbb{Q} \text{ is a reduced fraction with } p\geq 0\ ;\\ 0 & \text{if } x\in\mathbb{R}\setminus\mathbb{Q}.\end{cases}$$

We show that f is continuous at all $x\in\mathbb{R}\setminus\mathbb{Q}$ and discontinuous at all $x\in\mathbb{Q}$. Let $x\in\mathbb{R}\setminus\mathbb{Q}$ and $\xi\sim x$. If $\xi\in\mathbb{R}^*\setminus\mathbb{Q}^*$ is *hyperirrational*, then $f^*(\xi)=0=f(x)$ and trivially $f^*(\xi)\sim f(x)$. If $\xi=\sigma/\tau$ is a reduced *hyperrational* with $\sigma\geq 0$, we claim that τ is infinite. Otherwise, if σ is bounded, then $\xi=\sigma/\tau\in\mathbb{Q}$ and $\xi\sim x\Rightarrow\xi=x\in\mathbb{Q}$. If σ is unbounded, then also $\xi=\sigma/\tau\in\mathbb{Q}^*$ is unbounded. Both cases contradict the hypotheses. Thus $f^*(\xi)=1/\tau\sim 0=f(x)$.

The other case $x\in\mathbb{Q}$ is easier. It is enough to show that there is a *hyperirrational* number ξ with $\xi\sim x$ (hence $f^*(\xi)=0\not\sim f(x)$). But this directly follows from the density property of the irrational numbers (see Proposition 2.9).

Definition A.6. Let $f:A\to\mathbb{R}$ be a real function, and suppose that A contains a neighborhood of x_0. Then f is *lower semicontinuous* (or *upper semicontinuous*) at $x_0\in A$ if for every $\xi\in A^*$, $\xi\sim x_0\Rightarrow f(x_0)\leq sh(f^*(\xi))$ ($f(x_0)\geq sh(f^*(\xi))$, respectively).

Trivially, a function is continuous at a given point if and only if it is both lower and upper semicontinuous. Two basic results of calculus are the following.

Theorem A.7. *(Extreme Value)*
Let $f:[a,b]\to\mathbb{R}$ be a lower semicontinuous function (or upper semicontinuous function) on a bounded and closed interval. Then f attains a minimal value (a maximal value, respectively) in $[a,b]$.

Proof. Let $l=\inf\{f(x)\mid x\in[a,b]\}$ (possibly $l=-\infty$). By the nonstandard characterization of greatest lower bound, there is $\xi\in[a,b]^*$ with $sh(f^*(\xi))=l$. Since $[a,b]$ is closed and bounded, $sh(\xi)=x_0\in[a,b]$. Then, by lower semicontinuity, $f(x_0)\leq sh(f^*(\xi))=l$, hence $l\neq-\infty$ and f attains its minimal value at x_0. The case when f is upper semicontinuous is treated similarly. ☐

Theorem A.8. *(Intermediate Value)*
Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(a) < 0$ and $f(b) > 0$. Then there exists $x_0 \in (a, b)$ with $f(x_0) = 0$.

Proof. For each (positive) $n \in \mathbb{N}$, let

$$A(n) = \left\{a + \nu \tfrac{b-a}{n} \mid \nu = 0, \ldots, n-1\right\}$$

be the finite set that partitions $[a, b]$ into n intervals of equal length. Let $\xi(n) = \max\{x \in A(n) \mid f(x) < 0\}$ and $\zeta(n) = \xi(n) + \frac{b-a}{n}$. Then clearly $\zeta(n) \in [a, b]$ and $f(\zeta(n)) \geq 0$. Now consider the ideal values $\xi = \xi(\alpha)$ and $\zeta = \zeta(\alpha)$. $\zeta(n) - \xi(n) = \frac{1}{n}(b - a) \Rightarrow \zeta - \xi \sim 0$ and so ξ and ζ have the same shadow $x_0 \in [a, b]$. By continuity, $f^*(\xi) \sim f(x_0) \sim f^*(\zeta)$. Now, $f(\xi(n)) > 0$ for all $n \Rightarrow f^*(\xi) > 0$ and $f(\zeta(n)) \leq 0$ for all $n \Rightarrow f^*(\zeta) \leq 0$. We conclude that $f(x_0) = 0$. □

Distinguishing between continuity and uniform continuity is usually not an easy matter to first year students. To this end, nonstandard definitions seem to be much easier to grasp.

Definition A.9. A real function $f : A \to \mathbb{R}$ is *uniformly continuous* if for all $\xi, \zeta \in A^*$, $\xi \sim \zeta \Rightarrow f^*(\xi) \sim f^*(\zeta)$.

Thus uniform continuity is checked by comparing values attained at all pairs of numbers in (the star-transform of) the domain which are infinitely close. Notice that for continuity one restricts to those pairs where one of the two points is real. For instance, $f(x) = x^2$ is *not* uniformly continuous on $\mathbb{R}$. E.g., if ε is a positive infinitesimal, then $1/\varepsilon \sim 1/\varepsilon + \varepsilon$ but $f^*(1/\varepsilon) \not\sim f(1/\varepsilon + \varepsilon)$ because $(1/\varepsilon + \varepsilon)^2 - (1/\varepsilon)^2 = 2 + \varepsilon^2$ is *not* infinitesimal.
The nonstandard proof of the following theorem is almost a triviality.

Theorem A.10. *(Cantor)*
Every continuous function $f : [a, b] \to \mathbb{R}$ on a closed and bounded interval is uniformly continuous.

Proof. If $\xi, \zeta \in [a, b]^*$ are infinitely close, then $sh(\xi) = sh(\zeta) = x_0 \in [a, b]$. By continuity, $f^*(\xi) \sim f(x_0) \sim f^*(\eta)$. □

A.2. **Derivatives.**
Let f be a real function defined on a neighborhood of x_0.

Definition A.11. We say that f has *derivative* at x_0 if there exists a real number $f'(x_0)$ such that for all non-zero infinitesimals ε,

$$\frac{f^*(x_0 + \varepsilon) - f(x_0)}{\varepsilon} \sim f'(x_0)$$

Equivalently, f has derivative $f'(x_0)$ at x_0 if for every infinitesimal ε there exists an infinitesimal τ such that $f^*(x_0 + \varepsilon) = f(x_0) + f'(x_0)\varepsilon + \tau\varepsilon$.

Example A.12. The derivative of $f(x) = x^2$ is easily computed by noticing that for any given $x \in \mathbb{R}$ and for every non-zero infinitesimal ε,

$$\frac{(x + \varepsilon)^2 - x^2}{\varepsilon} = 2x + \varepsilon \sim 2x$$

Another central theorem of calculus is the following

Theorem A.13. *(Fermat)*
Assume that the function $f : (a,b) \to \mathbb{R}$ has a derivative at all $x \in (a,b)$. If f attains a greatest or a lowest value at some point x_0, then $f'(x_0) = 0$.

Proof. Assume that f attains a lowest value at x_0 (if $f(x_0)$ is the greatest value, the proof is similar). Fix an infinitesimal $\varepsilon > 0$. Clearly both $x_0 - \varepsilon, x_0 + \varepsilon \in (a,b)^*$. Then $f(x_0) \leq f^*(x_0 + \varepsilon)$ and $f(x_0) \leq f^*(x_0 - \varepsilon)$ and so

$$f'(x_0) \sim \frac{f(x_0+\varepsilon) - f(x_0)}{\varepsilon} \geq 0 \text{ and } f'(x_0) \sim \frac{f(x_0-\varepsilon) - f(x_0)}{-\varepsilon} \leq 0$$

We conclude that $f'(x_0) = 0$. □

A.3. **Hyperfinite sums.**
Before going to the nonstandard definition of Alpha-integral, we need to introduce a typical nonstandard tool, namely the hyperfinite sums.

Recall that in subsection 2.4, we defined a a hyperfinite set of hyperreals as the ideal value of some sequence of finite subsets of $\mathbb{R}$.

Definition A.14. If $\chi = \chi(\alpha)$ is a hyperfinite set of hyperreal numbers, then its *hyperfinite sum* $\sum_{x \in \chi} x = Sum_\chi(\alpha)$ is the ideal value of the sequence of finite sums $Sum_\chi(n) = \sum_{x \in \chi(n)} x$.

It is easily checked that this definition does not depend on the choice of the sequence $\{\chi(n)\}_n$, but only on its value at infinity. Now let $\{a_n\}_n$ be a sequence of real numbers, and let $\omega = \omega(\alpha), \tau = \tau(\alpha)$ be two hypernatural numbers with $\omega < \tau$. Without loss of generality, we assume $\omega(n) < \tau(n)$ for all n. Then,

$$\sum_{i=\omega}^{\tau} a_i = s_\omega^\tau(\alpha) \text{ where } s_\omega^\tau(n) = \sum_{i=\omega(n)}^{\tau(n)} a_i.$$

Hyperfinite sums retain all nice "elementary properties" of finite sums. Let us see an application.

Example A.15. Let us prove that $\lim_{n\to\infty} \sqrt[n]{n} = 1$. Let ω be an infinite hypernatural number. If we write $\sqrt[\omega]{\omega} = 1 + \varepsilon$, we have to show that $\varepsilon \sim 0$. Now, clearly $\varepsilon > 0$ and by the nonstandard version of Newton's binomial expansion (which follows easily from the standard version), we have:

$$\omega = (1+\varepsilon)^\omega = \sum_{i=0}^{\omega} \binom{\omega}{i} \varepsilon^i > \binom{\omega}{2} \varepsilon^2 = \frac{\omega(\omega-1)}{2}\varepsilon^2 \ \Rightarrow\ \varepsilon < \sqrt{\frac{2}{\omega - 1}} \sim 0.$$

The following definitions are equivalent to the "standard ones".[28]

Definition A.16. Let $l \in \mathbb{R} \cup \{\pm\infty\}$, and let $\{a_n\}_n$ be a sequence of real numbers. Then $\sum_{n=0}^{\infty} a_n = l$ if for every infinite hypernatural ω, the hyperfinite sum $\sum_{i=0}^{\omega} a_i \sim l$.

Theorem A.17. *(Cauchy's criterion)*
$\sum_{n=0}^{\infty} a_n = l \in \mathbb{R}$ if and only if for all infinite hypernaturals $\omega < \tau$, the hyperfinite sum $\sum_{i=\omega}^{\tau} a_i \sim 0$.

[28]The proofs that the nonstandard definitions given here are in fact equivalent to the usual ones, can be found in the textbooks mentioned at the beginning of Section 3.

Proof. One direction is trivial. Vice versa, assume by contradiction that there are $\omega < \tau$ infinite hypernaturals such that $\sum_{i=\omega}^{\tau} a_i = \xi \not\sim 0$. Then $\sum_{i=0}^{\omega} \not\sim \sum_{i=0}^{\omega} a_i + \xi = \sum_{i=0}^{\tau} a_i$. □

A.4. **The Alpha-integral.**
In this Section we introduce a notion of integral which generalizes the Riemann integral. It has a simple definition and has the advantage of making sense for any real function.

For positive natural numbers n, let us consider the *n-grid*:

$$\mathbb{H}(n) = \left\{ \pm\frac{k}{n} \mid k = 0, 1, \dots, n^2 \right\} \subset \mathbb{Q}$$

which partitions $[-n, n]$ into n^2 intervals of equal width $1/n$. Thus, for every real number $r \in [-n, n)$, there exist $x, x' \in \mathbb{H}(n)$ with $x \le r < x'$ and $x' - x = 1/n$.

Definition A.18. The *hyperfinite grid* $\mathbb{H} \doteq \mathbb{H}(\alpha)$ is the ideal value of the sequence of n-grids $\{\mathbb{H}(n)\}_n$. Equivalently:

$$\mathbb{H} = \left\{ \pm\frac{\nu}{\alpha} \mid \nu = 0, 1, \dots, \alpha^2 \right\} \subset \mathbb{Q}^*$$

The notion of a hyperfinite grid is a useful tool in nonstandard analysis (see for instance [36]). The fundamental properties are that $\mathbb{H}$ is hyperfinite, and that for every bounded hyperreal number ξ, there exists $\zeta \in \mathbb{H}$ with the same shadow (in fact, there are $\zeta, \zeta' \in \mathbb{H}$ with $\zeta \le \xi < \zeta'$ and $\zeta' - \zeta = 1/\alpha \sim 0$).

Definition A.19. Let A be a subset of $\mathbb{R}$ and let $f : A \to \mathbb{R}$ be a function. The *Alpha-integral* of f on A is the shadow of the following hyperfinite sum:[29]

$$\int_A f(x)\, d_\alpha x = sh\left(\frac{1}{\alpha} \cdot \sum_{\xi \in \mathbb{H} \cap A^*} f^*(\xi) \right)$$

Notice that

$$\int_A f(x)\, d_\alpha x = sh\left(S_A(\alpha)\right) \in \mathbb{R} \cup \{\pm\infty\} \text{ where } S_A(n) = \frac{1}{n} \cdot \sum_{x \in \mathbb{H}(n) \cap A} f(x)$$

If $A = (a, b)$ is an open interval, we shall adopt the usual notation $\int_a^b f(x)\, d_\alpha x$.

Thus the Alpha-integral $\int_a^b f(x)\ d_\alpha x$ is the shadow of a hyperfinite sum that provides an approximation to the (oriented) area determined by the graph of f. The approximation is obtained by considering a hyperfinite sequence of points in $(a, b)^*$ placed at the constant infinitesimal pace of $1/\alpha$. We remark that this closely corresponds to the intuitive idea of an integral.

We stress the fact that the Alpha-integral $\int_a^b f(x)\ d_\alpha x$ is defined for *all* functions. In fact, while the sequence $S_a^b(n) = \frac{1}{n} \cdot \sum_{x \in \mathbb{H}(n) \cap (a,b)} f(x)$ may not have a limit in the classic sense, its ideal value $S_a^b(\alpha)$ is always defined.

It is easily seen from the definitions that if the function f is Riemann integrable then $\lim_{n\to\infty} S_a^b(n)$ exists and coincides with the Alpha-integral.[30] Thus the Alpha-integral actually generalizes Riemann integral. Certainly, it cannot replace the Lebesgue integral because it does not have the nice properties required in advanced applications, but it can

[29] In case $\mathbb{H} \cap A^* = \emptyset$, put $\int f(x)\, d_\alpha x = 0$.

[30] Recall that if a real sequence $\{\varphi(n)\}_n$ has limit $l \in \mathbb{R} \cup \{\pm\infty\}$, then $\varphi(a) \sim l$ (see subsection 3.3).

be an useful tool in several situations. For instance, it has the advantage that can be easily introduced in elementary calculus courses.

Example A.20. Let us compute the Alpha-integral of $f(x) = x^2$ over $(0,1)$.

$$\int_0^1 x^2 \, d_\alpha x \quad = \quad sh\left(\frac{1}{\alpha}\sum_{\xi\in\mathbb{H}\cap(0,1)^*}\xi^2\right) = sh\left(\frac{1}{\alpha}\sum_{\nu=1}^{\alpha-1}\left(\frac{\nu}{\alpha}\right)^2\right) =$$

$$sh\left(\frac{1}{\alpha^3}\sum_{\nu=1}^{\alpha-1}\nu^2\right) \quad = \quad sh\left(\frac{1}{\alpha^3}\cdot\frac{\alpha(\alpha-1)(2\alpha-1)}{6}\right) = \frac{1}{3}.$$

The Alpha-integral allows also to compute improper integrals.

Example A.21. Let us compute $\int_1^{+\infty} \frac{1}{x^2}\, d_\alpha x$. By definition,

$$\int_1^{+\infty} \frac{1}{x^2}\, d_\alpha x = sh\left(\frac{1}{\alpha}\sum_{\nu=\alpha+1}^{\alpha^2}\frac{1}{(\nu/\alpha)^2}\right) = sh\left(\alpha\cdot\sum_{\nu=\alpha+1}^{\alpha^2}\frac{1}{\nu^2}\right).$$

Now,

$$\alpha\cdot\sum_{\nu=\alpha+1}^{\alpha^2}\frac{1}{\nu^2} \geq \alpha\cdot\sum_{\nu=\alpha+1}^{\alpha^2-1}\frac{1}{\nu(\nu+1)} = \alpha\cdot\sum_{\nu=\alpha+1}^{\alpha^2-1}\left(\frac{1}{\nu}-\frac{1}{\nu+1}\right) = \frac{\alpha}{\alpha+1}-\frac{1}{\alpha}\sim 1.$$

On the other direction,

$$\alpha\cdot\sum_{\nu=\alpha+1}^{\alpha^2}\frac{1}{\nu^2} \leq \alpha\cdot\sum_{\nu=\alpha+1}^{\alpha^2}\frac{1}{\nu(\nu-1)} = \alpha\cdot\sum_{\nu=\alpha+1}^{\alpha^2}\left(\frac{1}{\nu-1}-\frac{1}{\nu}\right) = 1-\frac{1}{\alpha}\sim 1.$$

We conclude that $\int_0^1 \frac{1}{x^2}\, d_\alpha x = 1$.

In the next example, we consider an improper integral over an unbounded domain.

Example A.22. Let us consider $\int_0^{+\infty} e^{-x}\, d_\alpha x$. If we denote by $\delta = 1/\alpha$, we have

$$\int_0^{+\infty} e^{-x}\, d_\alpha x = sh\left(\delta\cdot\sum_{\nu=1}^{\alpha^2}e^{-\delta\nu}\right)$$

Now,

$$\delta\cdot\sum_{\nu=1}^{\alpha^2}e^{-\delta\nu}\sim\delta\cdot\sum_{\nu=0}^{\alpha^2-1}e^{-\delta\nu} = \delta\cdot\frac{1-e^{-\alpha}}{1-e^{-\delta}}\sim\frac{\delta}{1-e^{-\delta}}\sim 1$$

and so $\int_0^{+\infty} e^{-x}\, d_\alpha x = 1$.

Appendix B. The proofs of results in subsection 1.2

Proof of Proposition 1.1.
3. If $\xi\in\varphi[\alpha]$ then $\xi=\xi[\alpha]$ for some sequence $\xi(n)\in\varphi(n)\subseteq\vartheta(n)$, hence $\xi[\alpha]\in\vartheta[\alpha]$ by the *internal set axiom.* Similarly if $\xi\in\psi[\alpha]$ then $\xi\in\vartheta[\alpha]$. This proves one inclusion. Vice versa, let $\xi\in\vartheta[\alpha]$, hence $\xi=\xi[\alpha]$ where $\xi(n)=\varphi(n)\cup\psi(n)$. Define the sequence

$$\zeta(n)=\begin{cases}\varphi(n) & if\ \ \xi(n)\in\varphi(n)\\ \psi(n) & if\ \ \xi(n)\in\psi(n)\setminus\varphi(n)\end{cases}$$

Notice that $\xi(n) \in \zeta(n) \Rightarrow \xi[\alpha] \in \zeta[\alpha]$. Moreover, $\zeta(n) \in \{\varphi(n), \psi(n)\}$, and so, by the *pair axiom*, $\zeta[\alpha] = \varphi[\alpha]$ or $\zeta[\alpha] = \psi[\alpha]$. We conclude that $\xi \in \varphi[\alpha] \cup \psi[\alpha]$.
2. Notice that $\varphi(n) \subseteq \psi(n) \Leftrightarrow \psi(n) = \varphi(n) \cup \psi(n)$ and then apply Property 3.
6. By definition, the ordered pair $\langle a, b\rangle$ is the Kuratowski pair $\{\{a\}, \{a, b\}\}$. Thus $\vartheta(n) = \{\xi(n), \zeta(n)\}$ where $\xi(n) = \{\varphi(n)\}$ and $\zeta(n) = \{\varphi(n), \psi(n)\}$, and the claim is obtained by the *pair axiom*.
7. If $\xi \in \vartheta[\alpha]$, then $\xi = \xi[\alpha]$ where $\xi(n) \in \varphi(n) \times \psi(n)$. Then $\xi(n) = \langle\beta(n), \gamma(n)\rangle$ where $\beta(n) \in \varphi(n)$ and $\gamma(n) \in \psi(n)$. We conclude that $\xi[\alpha] = \langle\beta[\alpha], \gamma[\alpha]\rangle \in \varphi[\alpha] \times \psi[\alpha]$. Vice versa, if $\xi = \langle\beta, \gamma\rangle \in \varphi[\alpha] \times \psi[\alpha]$, then $\beta = \beta[\alpha]$ and $\gamma = \gamma[\alpha]$ where $\beta(n) \in \varphi(n)$ and $\gamma(n) \in \psi(n)$. Hence $\xi = \xi[\alpha]$ where $\xi(n) = \langle\beta(n), \gamma(n)\rangle$.
1. Let $\varphi'(n) = \langle\varphi(n), n\rangle$ and $\psi'(n) = \langle\psi(n), n\rangle$. By the hypothesis, $\operatorname{ran}\varphi'$ and $\operatorname{ran}\psi'$ are disjoint sets. Thus we can pick a function f such that $f(x) = 0$ if $x \in \operatorname{ran}\varphi'$ and $f(x) = 1$ if $x \in \operatorname{ran}\psi'$. If by contradiction $\varphi[\alpha] = \psi[\alpha]$, then by Property 6, $\varphi'[\alpha] = \psi'[\alpha]$, and by the *composition axiom*, $(f \circ \varphi')[\alpha] = (f \circ \psi')[\alpha]$. This is impossible because $(f \circ \varphi')[\alpha] = c_0[\alpha] = 0$ while $(f \circ \psi')[\alpha] = c_1[\alpha] = 1$. The second part also follows by noticing that $\varphi(n) \notin \psi(n) \Leftrightarrow \{\varphi(n)\} \cup \psi(n) \neq \psi(n)$.
5. Let $\xi \in \vartheta[\alpha]$. Then $\xi = \xi[\alpha]$ where $\xi(n) \in \varphi(n) \setminus \psi(n)$. Now, $\xi(n) \in \varphi(n) \Rightarrow \xi \in \varphi[\alpha]$; and $\xi(n) \notin \psi(n) \Rightarrow \xi \notin \psi[\alpha]$. Vice versa, let $\xi = \xi[\alpha]$ with $\xi(n) \in \varphi(n)$ but $\xi \notin \psi[\alpha]$. Define:

$$\zeta(n) = \begin{cases} \vartheta(n) & \textit{if } \xi(n) \notin \psi(n) \\ \psi(n) & \textit{if } \xi(n) \in \psi(n) \end{cases}$$

Notice that $\xi(n) \in \zeta(n) \Rightarrow \xi \in \zeta[\alpha]$. By the *pair axiom*, either $\zeta[\alpha] = \vartheta[\alpha]$ or $\zeta[\alpha] = \psi[\alpha]$, and so either $\xi \in \vartheta[\alpha]$ or $\xi \in \psi[\alpha]$. As $\xi \notin \psi[\alpha]$ by hypothesis, it follows that $\xi \in \vartheta[\alpha]$.
4. Notice that $\vartheta(n) = \vartheta_1(n) \setminus \vartheta_2(n)$ where $\vartheta_1(n) = \varphi(n) \cup \psi(n)$; $\vartheta_2(n) = \vartheta_3(n) \cup \vartheta_4(n)$; $\vartheta_3(n) = \varphi(n) \setminus \psi(n)$ and $\vartheta_4(n) = \psi(n) \setminus \varphi(n)$. Then repeatedly apply Properties 3 and 5. □

Proof of Proposition 1.2.
1 and 2. Let $\varphi_1(n) = \varphi(n)$ if $\varphi(n)$ is an atom, $\varphi_1(n) = 0$ otherwise; and let $\varphi_2(n) = \varphi(n)$ if $\varphi(n)$ is a nonempty set, $\varphi_2(n) = \{0\}$ otherwise. By the *internal set axiom*, $\varphi_1[\alpha]$ is an atom and $\varphi_2[\alpha]$ is a nonempty set. Notice that $\varphi(n) \in \{\varphi_1(n), \varphi_2(n), c_\emptyset(n)\}$, so $\varphi[\alpha] \in \{\varphi_1[\alpha], \varphi_2[\alpha], c_\emptyset[\alpha]\}$ and we only have the following three possibilities. i) $\varphi[\alpha]$ is an atom and $\varphi[\alpha] = \varphi_1[\alpha]$; ii) $\varphi[\alpha]$ is a nonempty set and $\varphi[\alpha] = \varphi_2[\alpha]$; iii) $\varphi[\alpha] = c_\emptyset[\alpha] = \emptyset$. We are left to show that if $\varphi(n)$ is a set (i.e. a nonatom) for all n, then also $\varphi[\alpha]$ is a set. This trivially follows by noticing that $\varphi(n) \in \{\varphi_2(n), c_\emptyset(n)\} \Rightarrow$ either $\varphi[\alpha] = \varphi_2[\alpha]$ or $\varphi[\alpha] = \emptyset$.
3. Let $\xi \in \varphi[\alpha]$. Since $\varphi[\alpha]$ is a nonempty set, $\varphi[\alpha] = \psi[\alpha]$ for some sequence ψ of nonempty sets. By the *internal set axiom*, then $\xi = \xi[\alpha]$ for some sequence $\xi(n) \in \psi(n)$. □

Proof of Proposition 1.3.
1. Let $\{n \mid \varphi(n) \neq \psi(n)\} = \{n_1, \dots, n_k\}$. Fix m with $\varphi(m) = \psi(m)$ and consider

$$\zeta(n) = \begin{cases} n & \textit{if } \varphi(n) \neq \psi(n) \\ m & \textit{if } \varphi(n) = \psi(n) \end{cases}$$

By the Pair Axiom, either $\zeta[\alpha] = \alpha$ or $\zeta[\alpha] = m$. Now, $\zeta(n) \in \{n_1, \dots, n_k, m\} \Rightarrow \zeta[\alpha] \in \{n_1, \dots, n_k, m\}$, hence $\zeta[\alpha] = m$ (recall that $\alpha \notin \mathbb{N}$). Let A be the range of φ and define $\xi(m) = \emptyset$ and $\xi(n) = A$ for $n \neq m$. Then $\{\varphi(n)\} \setminus \{\psi(n)\} \subseteq (\xi \circ \zeta)(n) \Rightarrow$

$\{\varphi[\alpha]\}\setminus\{\psi[\alpha]\} \subseteq (\xi\circ\zeta)[\alpha]$. But $\zeta[\alpha] = c_m[\alpha]$ and so, by the *composition axiom*, $(\xi\circ\zeta)[\alpha] = (\xi \circ c_m)[\alpha] = c_\emptyset[\alpha] = \emptyset$. We conclude that $\varphi[\alpha] = \psi[\alpha]$.
2. By the hypothesis $\{\varphi(n)\} \setminus \{\psi(n)\} = \{\varphi(n)\}$ for all but finitely many n. Then $\{\varphi[\alpha]\} \setminus \{\psi[\alpha]\} = \{\varphi[\alpha]\}$ and so $\varphi[\alpha] \neq \psi[\alpha]$. □

Proof of Proposition 1.4.
Assume first that $\varphi[\alpha] = \psi[\alpha]$ and $\Lambda = \{n \mid \varphi(n) = \psi(n)\}$. Define $\vartheta(n) = \{\xi(n)\} = \{\zeta(n)\}$ if $n \in \Lambda$ and $\vartheta(n) = \emptyset$ otherwise. Now, $(\{\varphi(n)\} \setminus \{\psi(n)\}) \cup \vartheta(n) \neq \emptyset \Rightarrow (\{\varphi[\alpha]\} \setminus \{\psi[\alpha]\}) \cup \vartheta[\alpha] = \vartheta[\alpha] \neq \emptyset$ and so it must be $\vartheta[\alpha] = \{\xi[\alpha]\} = \{\zeta[\alpha]\}$, hence $\xi[\alpha] = \zeta[\alpha]$. The other properties follows by noticing the following facts: $\xi(n) \neq \zeta(n) \Leftrightarrow \{\xi(n)\} \cap \{\zeta(n)\} = \emptyset$, $\xi(n) \in \zeta(n) \Leftrightarrow \{\xi(n)\} \cap \zeta(n) = \{\xi(n)\}$, and $\xi(n) \notin \zeta(n) \Leftrightarrow \{\xi(n)\} \cap \zeta(n) = \emptyset$, respectively.
If $\varphi[\alpha] \neq \psi[\alpha]$ and $\Lambda = \{n \mid \varphi(n) \neq \psi(n)\}$, let $\vartheta(n) = \{\varphi(n)\} \cap \{\psi(n)\}$. Then $\vartheta[\alpha] = \{\varphi[\alpha]\} \cap \{\psi[\alpha]\} = \emptyset = c_\emptyset[\alpha]$, $\{n \mid \vartheta(n) = c_\emptyset(n)\} = \Lambda$, and the claim directly follows from the above proofs. □

References

[1] L.O. Arkeryd, N.J. Cutland and C.W. Henson, eds., *Nonstandard Analysis - Theory and Applications*, NATO ASI Series C **493**, Kluwer Academic Publishers, 1997.
[2] D. Ballard and K. Hrbàček, *Standard foundations for nonstandard analysis*, J. Symb. Logic, **57** (1992), 471–478.
[3] V. Benci, *A construction of a nonstandard universe*, in Advances in Dynamical Systems and Quantum Physics, S. Albeverio et al. eds., World Scientific, 1995, 207–237.
[4] V. Benci, *An algebraic approach to nonstandard analysis*, in Calculus of Variations and Partial Differential Equations, G. Buttazzo, A. Marino and M.K.V. Murthy, eds., Springer-Verlag, 1999, 285–326.
[5] V. Benci and M. Di Nasso, *A ring homomorphism is enough to get nonstandard analysis*, Bull. Belg. Math. Soc. (accepted for publication).
[6] V. Benci and M. Di Nasso, *Numerosities of labelled sets: a new way of counting*, Advances in Math., **173** (2003), 50–67.
[7] V. Benci and M. Di Nasso, book in preparation.
[8] M. Benedikt, *Ultrafilters which extend measures*, J. Symb. Logic, **63** (1998), 638–662.
[9] A. Blass, *The Rudin-Keisler ordering of P-points*, Trans. Am. Math. Soc., **179** (1973), 145–166.
[10] A. Blass, *Ultrafilters: where topological dynamics = algebra = combinatorics*, Topology Proc., **18** (1993), 33–56.
[11] M. Boffa, *Forcing et négation de l'axiome de fondement*, Mém. Acad. Sc. Belg., tome XL, fasc. 7, 1972.
[12] D. Booth, *Ultrafilters on a countable set*, Ann. of Math. Logic, **2** (1970), 1–24.
[13] C.C. Chang and H.J. Keisler, *Model Theory* (3rd edition), North-Holland, 1990.
[14] G. Choquet, *Construction d'ultrafiltres sur* ℕ, Bull. Sc. Math., **92** (1968), 41–48.
[15] G. Choquet, *Deux classes remarquables d'ultrafiltres sur* ℕ, Bull. Sc. Math., **92** (1968), 143–153.
[16] J. Cleave, *Cauchy, convergence and continuity*, Brit. J. Phil. Sci., **22** (1971), 27–37.
[17] J. Cleave, *The concept of 'variable' in nineteenth century analysis*, Brit. J. Phil. Sci., **30** (1979), 266–278.
[18] N. Cutland, *Loeb Measures in Practice: Recent Advances*, Lecture Notes in Mathematics **1751**, Springer, 2000.
[19] N.J. Cutland, C. Kessler, E. Kopp and D. Ross, *On Cauchy's notion of infinitesimal*, Brit. J. Phil. Sci., **39** (1988), 375–378.
[20] M. Davis, *Applied Nonstandard Analysis*, John Wiley & Sons, 1977.
[21] M. Di Nasso, *On the foundations of nonstandard mathematics*, Math. Japonica, **50** (1999), 131–160.
[22] M. Di Nasso, *An axiomatic presentation of the nonstandard methods in mathematics*, J. Symb. Logic, **67** (2002), 315–325.

[23] M. Di Nasso, *Nonstandard analysis by means of ideal values of sequences*, in [49], 63–73.
[24] M. Di Nasso, *Elementary embeddings of the universe in a nonwellfounded context*, in preparation. [Abstract: Bull. Symb. Logic, **7** (2001), 138.]
[25] U. Felgner, *Comparison of the axioms of local and universal choice*, Fund. Math., **71** (1971), 43–62.
[26] G. Fisher, *Cauchy and the infinitesimally small*, Historia Math., **5** (1978), 313–338.
[27] G. Fisher, *Cauchy's variables and orders of the infinitesimally small*, Brit. J. Phil. Sci., **30** (1979), 361–365.
[28] R. Goldblatt, *Lectures on the Hyperreals - An Introduction to Nonstandard Analysis*, Springer, Graduate Texts in Mathematics **188**, 1998.
[29] W.S. Hatcher, *Calculus is algebra*, Am. Math. Monthly, **98** (1982), 362–370.
[30] J.M. Henle, *Non-nonstandard analysis: real infinitesimals*, Math. Intelligencer, **21** (1999), 67–73.
[31] C.W. Henson, *A gentle introduction to nonstandard extensions*, in [1], 1–49.
[32] W. Hodges, *Model Theory*, Cambridge University Press, 1993.
[33] T. Jech, *Set Theory*, Academic Press, 1978.
[34] R. Jin, *Applications of nonstandard analysis in additive number theory*, Bull. Symb. Logic, **6** (2000), 331–341.
[35] H.J. Keisler, *Foundations of Infinitesimal Calculus*, Prindle, Weber and Schmidt, 1976.
[36] H.J. Keisler, *Elementary Calculus*, Prindle, Weber and Schmidt, 1976.
[37] K. Kunen, *Set Theory. An Introduction to Independence Proofs*, North-Holland, 1980.
[38] I. Lakatos, *Cauchy and the continuum: the significance of non-standard analysis for the history and philosophy of mathematics*, in Mathematics, Science and Epistemology, J. Worrall and G. Curries eds., Philosophical Papers vol. **2**, Cambridge University Press, 1978, 43–60.
[39] D. Laugwitz, *The Theory of Infinitesimals. An Introduction to Nonstandard Analysis*, Accademia Nazionale dei Lincei, Roma, 1980.
[40] W.A.J. Luxemburg, *Non-standard Analysis*, Lecture Notes, California Institute of Technology, Pasadena, 1962.
[41] V.A. Molchanov, *The use of double nonstandard enlargements in topology*, Siberian Math. J., **30** (1989), 397–402. [Translated from Sibirskii Matem. Z., **30** (1989), 64–71.]
[42] E. Nelson, *Internal set theory. A new approach to nonstandard analysis*, Bull. Amer. Math. Soc., **83** (1977), 1165–1198.
[43] E. Palmgren, *Developments in constructive nonstandard analysis*, Bull. Symb. Logic, **4** (1998), 223–271.
[44] A. Robinson, *Non-standard analysis*, Proc. Roy. Acad. Amsterdam, **64** (= Indag. Math., **23**) (1961), 432–440.
[45] A. Robinson, *Non-standard Analysis*, North-Holland, 1966.
[46] A. Robinson and E. Zakon, *A set-theoretical characterization of enlargements*, in: Applications of Model Theory to Algebra, Analysis and Probability, W.A.J. Luxemburg ed., Holt, Rinehart and Winston, 1969, 109–122.
[47] W. Rudin, *Homogeneity problems in the theory of Čech compactifications*, Duke Math. J., **23** (1956), 409–419.
[48] C. Schmieden and D. Laugwitz, *Eine erweiterung der infinitesimalrechnung*, Math. Zeitschr., **69** (1958), 1–39.
[49] P. Schuster, U. Berger and O. Horst, eds., *Reuniting the Antipodes - Constructive and Nonstandard Views of the Continuum*, Synthese Library **306**, Kluwer Academic Publishers, 2001.
[50] S. Shelah, *Proper Forcing*, Lecture Notes in Mathematics **940**, North-Holland, 1982.
[51] K.D. Stroyan and W.A.J. Luxemburg, *Introduction to the Theory of Infinitesimals*, Academic Press, 1976.
[52] K.D. Stroyan and J.M. Bayod, *Foundations of Infinitesimal Stochastic Analysis*, North-Holland, 1986.

Dipartimento di Matematica Applicata "U. Dini", Università di Pisa, Italy.
E-mail address: benci@dma.unipi.it

Dipartimento di Matematica "L. Tonelli", Università di Pisa, Italy.
E-mail address: dinasso@dm.unipi.it