An Aristotelian notion of size*

Vieri Benci, Marco Forti
Dipart. di Matem. Applicata "U. Dini"
Università di Pisa, Italy.
{benci,forti}@dma.unipi.it

Mauro Di Nasso
Dipart. di Matem. "L. Tonelli"
Università di Pisa, Italy.
dinasso@dm.unipi.it

Abstract

The naïve idea of "size" for collections seems to obey both to *Aristotle's Principle*: "the whole is greater than its parts" and to *Cantor's Principle*: "1-to-1 correspondences preserve size". Notoriously, Aristotle's and Cantor's principles are incompatible for infinite collections. Cantor's theory of cardinalities weakens the former principle to "the part is not greater than the whole", but the outcoming cardinal arithmetic is very unusual. It does not allow for inverse operations, and so there is no direct way of introducing *infinitesimal* numbers. (Sizes are added by means of disjoint unions and multiplied by means of disjoint unions of equinumerous collections.)

Here we maintain Aristotle's principle, halving instead Cantor's principle to "equinumerous collections are in 1-1 correspondence". In this way we obtain a very nice arithmetic: in fact, our "numerosities" may be taken to be *nonstandard integers*. These numerosities appear naturally suited to *sets of ordinals*, but they depend, for generic sets, on a "labelling" of the universe by ordinals. The problem of finding a canonical way of attaching numerosities to all sets seems to be worth of further investigation.

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Introduction

The everyday practice of counting finitely many objects seems to suggest that the notion of "size" of collections be ruled by two general principles:

AP (Aristotle's Principle)

$$\textit{If } A \textit{ is a } \text{proper } \textit{subcollection of } B \textit{ then } \mathfrak{s}(A) < \mathfrak{s}(B),$$

and

CP (Cantor's Principle)

$$\mathfrak{s}(A) = \mathfrak{s}(B) \textit{ if and only if } A \textit{ is in } 1\text{–}1 \textit{ correspondence with } B.$$

Here $\mathfrak{s}$ denotes *size*, and $<$ refers to the *natural comparison of sizes.* This ordering may originate from the ancient general principle that sizes of *homogeneous objects* are arranged in a *linear ordering.*

The same intuition brings us to introduce the operation of *addition of sizes*, as corresponding to *disjoint union of collections*:

SP (Sum Principle)

$$\textit{If } A \cap B = \emptyset, \textit{ then } \mathfrak{s}(A) + \mathfrak{s}(B) = \mathfrak{s}(A \cup B).$$

Similarly, the natural idea of *multiplication* as "iterated addition of equals" suggests to relate *product of sizes* to union of disjoint collections of "equinumerous" sets, according to the principle

PP (Product Principle) *Let the elements of A be pairwise disjoint sets.*

$$\textit{If } \mathfrak{s}(a) = \mathfrak{s}(B) \textit{ for all } a \in A, \textit{ then } \mathfrak{s}(A) \cdot \mathfrak{s}(B) = \mathfrak{s}(\bigcup A).$$

Notice that the *Cartesian product* $A \times B$ can naturally be obtained as the union of an *A-indexed family of pairwise disjoint B-indexed sets.* So, by assuming Cantor's principle CP, the product principle can take the suggestive form

CPP (Cartesian Product Principle) $\mathfrak{s}(A) \cdot \mathfrak{s}(B) = \mathfrak{s}(A \times B)$.

But now we meet an insuperable barrier. Historically the fundamental principles AP and CP revealed incompatible for infinite collections, long before the celebrated Galileo's remark that there should be simultaneously "equally many" and "much less" perfect squares than natural numbers. The impact of this inconsistency cannot be overestimated: let us only mention that it led Leibniz (an inventor of infinitesimal analysis!) to assert the *impossibility of infinite numbers.*

Cantor relaxed AP to

$$A \subseteq B \Longrightarrow \mathfrak{s}(A) \leq \mathfrak{s}(B),$$

and developed its beautiful theory of cardinalities. Cantor's cardinal arithmetic provides an excellent treatment of infinitely large numbers, but its algebraic properties are trivialized by the well-known awkward property

$$\mathfrak{a} + \mathfrak{b} = \mathfrak{a} \cdot \mathfrak{b} = \max\left(\mathfrak{a}, \mathfrak{b}\right)$$

whenever $\mathfrak{a}$ is infinite and $\mathfrak{b} \neq 0$. As a consequence sum and product cannot admit inverse operations.[1] In particular, this algebra does not produce "infinitely small" numbers, hence it does not provide a *natural* introduction of "infinitesimal analysis". (History repeats itself: Leibniz stressed the usefulness of ideal elements such as infinitesimal numbers, but declared the impossibility of actual infinite numbers. Cantor, in turn, asserted the existence of *transfinite* numbers, but strongly negated that of *actual infinitesimal* numbers!)

In order to save the ancient principle that "the whole is greater than its parts", we decide to maintain only *one half* of Cantor's Principle, namely

HCP (Half Cantor's Principle)

If $\mathfrak{s}(A) = \mathfrak{s}(B)$*, then* A *is in* 1–1 *correspondence with* B.

The main interest in a notion of size preserving AP lies in the fact that it allows the corresponding "numbers" to behave well with respect to addition and multiplication, satisfying the usual algebraic properties of natural numbers. Here we shall directly assume that our class $\mathcal{N}$ of numbers ("numerosities") is included in the *non-negative part of an ordered ring* $\mathcal{A}$.

Unfortunately, if we want to measure *all sets*, we have to abandon the general principle PP that the size of the *union of a disjoint family of equinumerous sets* equals the size of the family times the size of any of its members. Take for instance the collection of iterated singletons

$$A = \{\{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \ldots\}:$$

[1] It is worth mentioning that even the basic principle of *comparability of cardinalities* had to wait a few decades before Zermelo gave it satisfying axiomatic grounds.

then $\bigcup A = \{\emptyset\} \cup A,$ and so $\mathfrak{s}(\bigcup A) = 1 + \mathfrak{s}(A) > \mathfrak{s}(A) \cdot 1 = \mathfrak{s}(A)$.[2]

Similarly, if we put

$$b_0 = a,\ b_{n+1} = (a, b_n),\ B = \{\, b_n \mid n \in \mathbb{N}\,\},$$

then $\{a\} \times B$ is a proper subset of B, and so, according to Aristotle's principle, even the weakest form of CPP fails: $\mathfrak{s}(\{a\} \times B) < \mathfrak{s}(\{a\}) \cdot \mathfrak{s}(B)$.[3]

In Subsection 4.2, we shall consider a few "stratified" multiplicative principles, which do not lead to contradiction. However we prefer to ground our basic Definition 1.1 upon a "flat" product, apparently more appropriate when considering sets of ordinals. This "flat" version has the same "arithmetical" consequences of the original version. Moreover it can be viewed as a weakening of the general product principle PP, suitable for overcoming the obstacles originated by the well known phenomena of "absorption" affecting ordinal arithmetic.

The paper is organized as follows. In Section 1 we specialize the general principles above, and we give the axioms for our "Aristotelian" notion of size (*numerosity*) for sets of ordinals. In Section 2 we construct a model of these axioms, by means of "finite approximations": in these models numerosities are in fact *nonstandard integers*. In Section 3, we outline a possible "labelling" of the universe by ordinals, and we get a corresponding "Aristotelian" size of all sets. Final remarks and open questions can be found in Section 4.

In general, we refer to [4] for all the set-theoretical notions and facts used in this paper, and to [3] for definitions and facts concerning ultrapowers, ultrafilters, and nonstandard models.

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1 Measuring sets of ordinals

In [1], an "Aristotelian" notion of size, called "numerosity", has been presented for countable "labelled" sets, starting from the following observation.

[2] Of course we take it for granted that *finite sets* receive their usual size (= *number of elements*), in particular $\mathfrak{s}(\emptyset) = 0$ and $\mathfrak{s}(\{x\}) = 1$ for all x.

[3] In order to maintain CPP, one could choose a sort of "Axiom der Beschränkung", following the trick commonly used to avoid non-wellfounded sets. Examples of such restrictions are exploited in [2].

Very often, in measuring the "size" of a given set, one first splits it into parts to be counted separately, and then one computes the size of the given set as the "ultimate value" of the sequence of partial sums. (Obviously such a sequence is eventually constant for finite sets.) In order to apply this procedure to an infinite set, one partitions it into *finite* parts, by assigning to each element a "label" in such a way that only finitely many elements have the same label. In [1] natural numbers are taken as labels, and so only countable sets can be considered.

Definition ([1], Def. 1.1) A *labelled set* is a pair $\mathbf{A} = \langle A, \ell_A \rangle$ where A is a set (the *domain* of $\mathbf{A}$) and $\ell_A : A \to \mathbb{N}$ (the *labelling function* of $\mathbf{A}$) is finite-to-one.[4]

For infinite sets, the "numerosity" so obtained depends on the chosen labelling, and so it is not completely satisfactory as a measure of size. However *sets of natural numbers* can be canonically labelled by the identical labelling function, and so their numerosities may be considered "natural".

Aiming to measure the size of arbitrary sets, we enlarge the collection of labels to the class of *all ordinals*. As a consequence, *every set of ordinals* comes out with its natural identical labelling. We devote this section to axiomatize the notion of size for sets of ordinals, without any explicit mentioning of labels. Following [1], we call *numerosities* the values taken by the size function. We follow here the common practice of modern set theory, and we consider the so called *Von Neumann ordinals*, i.e. each ordinal *is precisely the set of all smaller ordinals*. Thus, by assigning a numerosity to every ordinal, we obtain an order isomorphism of the class *Ord* of ordinals into $\mathcal{N}$, the class of numerosities.

We want a "good arithmetic" of numerosities, and we specify this requirement by assuming that numerosities belong to (the nonnegative part of) an *ordered ring*. Given the characteristic absorption properties of ordinal arithmetic, we may not (and we do not want to) make the above mentioned order-isomorphism into a *semiring* isomorphism. More generally, we cannot assume that size is insensitive to *arbitrary translations and homotheties*. However we would like that the only constraints be given by unavoidable clashings of the translated or homothetical transforms of the original set. In particular, when τ is "sufficiently compact and large", we shall ensure that the size of A be equal to those of its translates $\{\tau\} + A = \{\tau + \alpha \mid \alpha \in A\}$ and of its homotheticals $\{\tau\} \cdot A = \{\tau\alpha \mid \alpha \in A\}$.

[4] I.e., for any given $n \in \mathbb{N}$, there are only finitely many $a \in A$ such that $\ell_A(a) = n$. So only countable sets can be labelled.

Finally, since the ordinary Cartesian product of sets of ordinals is no more a set of ordinals, we shall adopt a natural "flat" notion of product, namely

$$A \otimes_\tau B = \{\tau\beta + \alpha \mid \alpha \in A, \beta \in B\}.$$

In doing so, we identify the ordered pair (α, β) with the ordinal $\tau\beta + \alpha$. By taking $A = B = \tau$, the Cartesian square $\tau \times \tau$ is squashed down upon the horizontal axis Ord, preserving the order inside each horizontal segment, as well as the vertical ordering of the segments (so the *antilexicographical* ordering is preserved). Notice that $\tau \otimes_\tau \gamma$ is equal to the *ordinal product* $\tau\gamma$ for all γ. This procedure is suitable for stating a "flat product principle", provided that τ be "sufficiently compact and large". Taking into account that even $\tau \otimes_\tau 2$ presents absorption if τ is not of the form $\omega^\alpha n$, an appropriate class of ordinals can be isolated as follows. Call an infinite ordinal τ a *tile* if $\tau = \omega^\alpha$ for some $\alpha > 0$. Call an infinite ordinal θ *arithmetically closed* or shortly an *atom* if $\alpha\beta + \gamma < \theta$ for all $\alpha, \beta, \gamma < \theta$. According to this definition, θ is an atom if and only if there exists β such that $\theta = \omega^{\omega^\beta}$. By considering the Cantor normal form of the exponent $\alpha = \omega^{\alpha_1} h_1 + \omega^{\alpha_2} h_2 + \ldots + \omega^{\alpha_m} h_m$ of the tile $\tau = \omega^\alpha$, one obtains a unique representation of τ as an "ordered monomial" $\theta_1^{h_1}\theta_2^{h_2}\ldots\theta_m^{h_m}$, where $\theta_1 = \omega^{\omega^{\alpha_1}} > \theta_2 = \omega^{\omega^{\alpha_2}} > \ldots > \theta_m = \omega^{\omega^{\alpha_m}}$ are atoms. Alternatively, τ has a unique representation as an *ordinal power* θ^γ, with *arithmetically closed basis* $\theta = \theta_m$, and *non-limit exponent* $\gamma = \omega^{\delta_1} h_1 + \omega^{\delta_2} h_2 + \ldots + h_m$, where $\alpha_i = \alpha_m + \delta_i$. The atom θ is called the *basis of* τ, and we say that τ is a θ*-tile*. So, when τ is a θ-tile, in order to avoid absorption in the product $\tau \otimes_\tau \delta$ one has to take $\delta < \theta^\omega$.

Grounding on the general discussion above, we formulate our fundamental definition.

Definition 1.1 Let $\mathcal{W} = \mathcal{P}(Ord)$ be the class of all sets of ordinals, and let $\mathcal{A}$ be an *ordered* (class-) *ring*.[5] A *numerosity function* (for $\mathcal{W}$) is a function $\mathfrak{n} : \mathcal{W} \to \mathcal{A}$ taking *nonnegative* values, and satisfying the following conditions:

(hcp) if $\mathfrak{n}(X) = \mathfrak{n}(Y)$ then $|X| = |Y|$;

(sp) if $X \cap Y = \emptyset$ then $\mathfrak{n}(X) + \mathfrak{n}(Y) = \mathfrak{n}(X \cup Y)$;

[5] Clearly $\mathcal{A}$ has to be a *proper class* in order to provide numerosities for all sets in $\mathcal{W}$. The way of dealing with this problem will be considered in Section 4. By now, the reader can replace Ord by any regular cardinal.

(fpp) if τ is a θ-tile, then $\mathfrak{n}(X)\cdot\mathfrak{n}(Y)=\mathfrak{n}(X\otimes_\tau Y)$ for all $X\subseteq\tau$ and all $Y\subseteq\delta<\theta^\omega$;

(up) $\mathfrak{n}(\{\alpha\})=1$ for every ordinal α.

It is interesting to remark that the four conditions above suffice to derive Aristotle's Principle, as well as interesting geometrical and arithmetical properties of numerosities.

1.1 Aristotle's Principle

$$\mathfrak{n}(A)<\mathfrak{n}(B) \textit{ for all proper subsets } A\subset B$$

In fact, let β be an element of the complement $B\setminus A$. Then

$$\mathfrak{n}(B)=\mathfrak{n}(A)+\mathfrak{n}(\{\beta\})+\mathfrak{n}(B\setminus(A\cup\{\beta\}))\geq\mathfrak{n}(A)+1>\mathfrak{n}(A),$$

because numerosities belong to the *nonnegative part of an ordered ring.*

1.2 Finite sets

It should be clear that the "unit property" (up) is aimed to assign the *natural size* to all finite sets of ordinals. Namely, if $\mathbb{N}$ is identified, as usual, with the subsemiring of $\mathcal{A}$ generated by 1, then $\mathfrak{n}(F)=|F|(=$ *number of elements* of $F)$ for all finite F.

1.3 Translation invariance

$$\mathfrak{n}(A)=\mathfrak{n}(\{\tau\beta\}+A) \textit{ for all tiles } \tau, \textit{ all } A\subseteq\tau\cdot n, \textit{and all } \beta.$$

Proof We proceed by induction on β.
If $\beta=m<\omega$, put $A=\cup_{i=0}^{n-1}(\{\tau i\}+A_i)$, with $A_i\subseteq\tau$. Then

$$\mathfrak{n}(\{\tau m\}+A)=\sum_{i=0}^{n-1}\mathfrak{n}(\{\tau(i+m)\}+A_i)=$$
$$=\sum_{i=0}^{n-1}\mathfrak{n}(A_i\otimes_\tau\{i+m\})=\sum_{i=0}^{n-1}\mathfrak{n}(A_i\otimes_\tau\{i\})=\mathfrak{n}(A).$$

If $\beta=\omega^\gamma$ is a tile, then $A\subseteq\tau\beta$, and $\mathfrak{n}(\{\tau\beta\}+A)=\mathfrak{n}(A\otimes_{\tau\beta}\{1\})=\mathfrak{n}(A)$.

If $\beta>\omega$ is not a tile, then $\beta=\omega^\gamma(m+1)+\delta$, with $0\leq m$, $\delta<\omega^\gamma<\beta$, and $\omega^\gamma m+\delta<\beta$. Hence $\{\tau(\omega^\gamma m+\delta)\}+A\subseteq\tau\omega^\gamma(m+1)$, and, by applying the induction hypothesis twice,

$$\mathfrak{n}(\{\tau\beta\}+A)=\mathfrak{n}(\{\tau\omega^\gamma\}+(\{\tau(\omega^\gamma m+\delta)\}+A))=\mathfrak{n}(\{\tau(\omega^\gamma m+\delta)\}+A)=\mathfrak{n}(A).$$

□

1.4 Homothety invariance

$$\mathfrak{n}(A) = \mathfrak{n}(\{\tau\} \cdot A) \textit{ for all } \theta\textit{-tile } \tau \textit{ and all } A \subseteq \delta < \theta^{\,\omega}.$$

Proof Immediate, since $\mathfrak{n}(\{\tau\} \cdot A) = \mathfrak{n}(\{0\} \otimes_\tau A) = \mathfrak{n}(A)$, by (fpp).

□

1.5 Cantor normal forms

Let $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$*. Then*

$$\sum_{i=1}^{n} \mathfrak{n}(\omega^{\alpha_i}) = \mathfrak{n}(\sum_{i=1}^{n} \omega^{\alpha_i}) \quad \textit{and} \quad \prod_{i=1}^{n} \mathfrak{n}(\omega^{\omega^{\alpha_i}}) = \mathfrak{n}(\omega^{\sum_{i=1}^{n} \omega^{\alpha_i}}).$$

(So the arithmetic of numerosities preserves the *Cantor normal forms.*)

Proof We proceed by induction on n.

We have $\sum_{i=1}^{n} \omega^{\alpha_i} = \omega^{\alpha_1} \cup (\{\omega^{\alpha_1}\} + \sum_{i=2}^{n} \omega^{\alpha_i})$, hence

$$\mathfrak{n}(\sum_{i=1}^{n} \omega^{\alpha_i}) = \mathfrak{n}(\omega^{\alpha_1}) + \mathfrak{n}(\{\omega^{\alpha_1}\} + \sum_{i=2}^{n} \omega^{\alpha_i}) = \mathfrak{n}(\omega^{\alpha_1}) + \mathfrak{n}(\sum_{i=2}^{n} \omega^{\alpha_i})$$

by translation invariance, since $\sum_{i=2}^{n} \omega^{\alpha_i} \subseteq \omega^{\alpha_1} \cdot n$. The first equality follows by induction hypothesis.

In order to prove the second equality, put $\theta_i = \omega^{\omega^{\alpha_i}}$. Then $\omega^{\sum \omega^{\alpha_i}} = \theta_1 \theta_2 \ldots \theta_n = \theta_1 \otimes_{\theta_1} (\theta_2 \ldots \theta_n)$. Hence $\mathfrak{n}(\omega^{\sum \omega^{\alpha_i}}) = \mathfrak{n}(\theta_1) \cdot \mathfrak{n}(\theta_2 \ldots \theta_n)$ by (fpp), since $\theta_2 \ldots \theta_n < \theta_1^{\omega}$, and the second equality follows by induction.

□

1.6 Natural ordinal arithmetic

The class *Ord* becomes the non-negative part of a discretely ordered ring when endowed with the so called *natural sum* $\oplus$ and *natural product* $\otimes$ of ordinals. Essentially, these operations are obtained by considering Cantor normal forms as *formal polynomials* in ω. More precisely, let

$$\alpha = \omega^{\gamma_1} a_1 + \ldots + \omega^{\gamma_n} a_n, \quad \beta = \omega^{\gamma_1} b_1 + \ldots + \omega^{\gamma_n} b_n,$$

with $\gamma_1 > \ldots > \gamma_n \geq 0$ and $0 \leq a_i, b_i < \omega$. Then

$$\alpha \oplus \beta = \omega^{\gamma_1}(a_1 + b_1) + \ldots + \omega^{\gamma_n}(a_n + b_n), \text{ and } \alpha \otimes \beta = \bigoplus_{i,j=1}^{n} \omega^{\gamma_i \oplus \gamma_j} a_i b_j.$$ [6]

[6] Equivalently, $\oplus$ and $\otimes$ are the ring operations on ordinals when they are viewed as *Conway's surreal numbers*.

We have

$$\mathfrak{n}(\alpha \oplus \beta) = \mathfrak{n}(\alpha) + \mathfrak{n}(\beta), \ \textit{and} \ \mathfrak{n}(\alpha \otimes \beta) = \mathfrak{n}(\alpha) \cdot \mathfrak{n}(\beta) \ \textit{for all} \ \alpha, \beta.$$

Proof By considering the Cantor normal forms of α, β, we obtain from the preceding equalities that $\mathfrak{n}(\alpha \oplus \beta) = \mathfrak{n}(\alpha) + \mathfrak{n}(\beta)$ and $\mathfrak{n}(\omega^{\alpha \oplus \beta}) = \mathfrak{n}(\omega^{\alpha}) \cdot \mathfrak{n}(\omega^{\beta})$. Then, by distributivity, we obtain $\mathfrak{n}(\alpha \otimes \beta) = \mathfrak{n}(\alpha) \cdot \mathfrak{n}(\beta)$.

□

2 Measuring size by finite approximation

As already remarked in 1.2 above, any *numerosity function* $\mathfrak{n}$ extends the usual counting of finite sets. In [1] the size of *countable labelled sets* was measured by viewing them as *increasing unions of suitable sequences of finite subsets* ("finite approximations"). This way of counting suggests that a model of Aristotelian size can be obtained by considering appropriate families of "finite approximations". Let us give the following general definition, which will be basic in the sequel:

Definition 2.1 Let $\mathcal{C}$ be a class, let $\mathcal{W} = \mathcal{P}(\mathcal{C})$ be the class of all subsets of $\mathcal{C}$, and let $\mathcal{I}$ be a *directed* class.[7] A map $\varphi : \mathcal{W} \times \mathcal{I} \to \mathcal{W}$ is a *finite approximation* if the following conditions are fulfilled for all $X, Y \in \mathcal{W}$ and all $i, j \in \mathcal{I}$

(FA1) $\varphi(X, i)$ is a finite subset of X;

(FA2) for all $x \in X$ there exists $i \in \mathcal{I}$ such that $x \in \varphi(X, i)$;

(FA3) if $i \leq j$, then $\varphi(X, i) \subseteq \varphi(X, j)$;

(FA4) $\varphi(X \cup Y, i) = \varphi(X, i) \cup \varphi(Y, i)$;

The *counting function* $\Phi : \mathcal{W} \to \mathbb{N}^{\mathcal{I}}$ *associated* to the finite approximation φ is defined by $\Phi(X)(i) = |\varphi(X, i)|$ for all $X \in \mathcal{W}$ and all $i \in \mathcal{I}$.

When no ambiguity can arise, we write shortly X_i for $\varphi(X, i)$, so that $\Phi(X)(i) = |X_i|$.

The numerosity functions we shall consider in the sequel are generated by suitable finite approximations. Namely, we fix $\mathcal{W} = \mathcal{P}(Ord)$, since we are considering only sets of ordinals, and we take $\mathcal{A}$ to be a *homomorphic image* of the ring $\mathbb{Z}^{\mathcal{I}}$.

[7] We could restrict $\mathcal{W}$ to contain only *special* parts of $\mathcal{C}$, provided the necessary closure properties be granted.

Theorem 2.2 *Let $\mathcal{W} = \mathcal{P}(Ord)$, let $\varphi : \mathcal{W} \times \mathcal{I} \to \mathcal{W}$ be a finite approximation, and let $\Phi : \mathcal{W} \to \mathbb{N}^{\mathcal{I}}$ be the associated counting function. Given a prime ideal $\mathfrak{p}$ of the ring $\mathbb{Z}^{\mathcal{I}}$, let $\pi : \mathbb{Z}^{\mathcal{I}} \to \mathcal{A}$ be the canonical projection onto the quotient $\mathcal{A} = \mathbb{Z}^{\mathcal{I}}/\mathfrak{p}$, and let $\mathcal{U}$ be the ultrafilter associated to $\mathfrak{p}$.*[8]

Put $\mathfrak{n} = \pi \circ \Phi$: then the function $\mathfrak{n}$ satisfies the property (sp) *of numerosity functions. Moreover*

- $\mathfrak{n}$ *satisfies* (hcp) *if and only if*

$$C_{XY} = \{i \in \mathcal{I} \mid |X_i| < |Y_i|\} \in \mathcal{U}$$

whenever $|X| < |Y|$;

- $\mathfrak{n}$ *satisfies* (fpp) *if and only if*

$$P^{\tau}_{XY} = \{i \in \mathcal{I} \mid |X_i| \cdot |Y_i| = |(X \otimes_{\tau} Y)_i|\} \in \mathcal{U}$$

for all θ-tiles τ such that $X \subseteq \tau$ and $Y \subseteq \delta < \theta^{\omega}$;

- $\mathfrak{n}$ *satisfies* (up) *if and only if*

$$C_{\alpha} = \{i \in \mathcal{I} \mid \{\alpha\}_i = \{\alpha\}\} \in \mathcal{U}$$

for all α. In particular (up) *follows from* (hcp)*, since $C_{\alpha} = C_{\emptyset\{\alpha\}}$.*

Hence $\mathfrak{n}$ is a numerosity function if and only if $\mathcal{U}$ contains both families

$$\mathcal{C} = \{C_{XY} \mid |X| < |Y|\} \text{ and } \mathcal{P} = \{P^{\tau}_{XY} \mid \tau \text{ a } \theta\text{-tile}, X \subseteq \tau, Y \subseteq \delta < \theta^{\omega}\}$$

and in this case the range $\mathfrak{n}(\mathcal{W}) = \mathcal{N}$ is canonically isomorphic to a subsemiring of the ultrapower $\mathbb{N}^{\mathcal{I}}_{\mathcal{U}}$.

Proof The ideal $\mathfrak{p}$ being prime, the quotient ring is discretely ordered by the relation $\pi(\xi) < \pi(\eta)$ if and only if $\{i \mid \xi(i) < \eta(i)\} \in \mathcal{U}$.

By (FA1), disjoint sets have disjoint approximations and so (sp) follows from (FA4). Similarly, each of the conditions (hcp), (fpp), and (up) is equivalent to the fact that all the corresponding subsets of $\mathcal{I}$ belong to $\mathcal{U}$.

[8] Regarding our loose treatment of proper classes, see Section 4. When $\mathcal{I}$ is a set, recall that any prime ideal of the ring $\mathbb{Z}^{\mathcal{I}}$ contains exactly one element of each pair of complementary idempotents $\epsilon_I, \epsilon_{\mathcal{I}\setminus I}$, where $\epsilon_I : \mathcal{I} \to \{0,1\}$ is defined by $\epsilon_I(i) = 0 \iff i \in I$. Hence the *zero-sets* of the idempotents in $\mathfrak{p}$ constitute an *ultrafilter* $\mathcal{U}$ over $\mathcal{I}$, which is nonprincipal if and only if $\mathfrak{p}$ is nonprincipal. Conversely, to each ultrafilter $\mathcal{U}$ is associated a corresponding prime ideal. Moreover two elements of $\mathbb{Z}^{\mathcal{I}}$ belong to the same class modulo $\mathfrak{p}$ if and only if they agree on a set in $\mathcal{U}$. Therefore the quotient ring $\mathbb{Z}^{\mathcal{I}}/\mathfrak{p}$ is isomorphic to the *ultrapower* $\mathbb{Z}^{\mathcal{I}}_{\mathcal{U}}$, and so it is a discretely ordered ring.

$\square$

Now we take $\mathcal{I}$ to be the class $\mathcal{P}_{fin}(Ord)$ of all finite sets of ordinals, partially ordered by inclusion, and we define the finite approximation $\varphi : \mathcal{W} \times \mathcal{I} \to \mathcal{W}$ by intersection:

$$\varphi(A, i) = A \cap i, \textit{ for all } A \in \mathcal{W} \textit{ and all } i \in \mathcal{I}$$

By the above theorem, in order to get a numerosity function, we need an ultrafilter $\mathcal{U}$ over $\mathcal{I}$ such that, for all $A, B \in \mathcal{W}$,

- $|A| < |B| \Longrightarrow C_{AB} = \{i \in \mathcal{I} \mid |A \cap i| < |B \cap i|\} \in \mathcal{U}$;
- for all θ-tiles τ, all $A \subseteq \tau$, and all $B \subseteq \delta < \theta^{\omega}$,
  $$P^{\tau}_{AB} = \{i \in \mathcal{I} \mid |A \cap i| \cdot |B \cap i| = |(A \otimes_{\tau} B) \cap i|\} \in \mathcal{U}.$$

We call *hyperfine* such an ultrafilter $\mathcal{U}$. (Notice that hyperfine ultrafilters are *fine* in the usual sense, since $C_{\emptyset\{\alpha\}} = \{i \in \mathcal{I} \mid \alpha \in i\}$.)

Now we show that the family of sets

$$\mathcal{F} = \{C_{AB} \mid |A| < |B|\} \cup \{P^{\tau}_{AB} \mid \tau \text{ a } \theta\text{-tile}, A \subseteq \tau, B \subseteq \delta < \theta^{\omega}\}$$

has the Finite Intersection Property. The core of our construction lies in the following technical lemma.

Lemma 2.3 *Let $\theta_1 > \ldots > \theta_m = \omega$ be atoms, and let $X_1, \ldots, X_n$ be sets of ordinals. Then, for every positive $k \in \mathbb{N}$, there exists a finite set of ordinals I such that*

(*i*) *every finite X_u is included in I;*

(*ii*) $|I \cap X_u| > |I \cap \bigcup\{X_v \mid |X_v| < |X_u|\}|$ *for every infinite X_u;*

(*iii*) *for all $s \leq m$ and all $h_t \leq k$, if $X_u \subseteq \theta_1^{h_1} \ldots \theta_s^{h_s}$ and $X_v \subseteq \theta_s^k$, then*

$$(X_u \cap I) \otimes_{\theta_1^{h_1} \ldots \theta_s^{h_s}} (X_v \cap I) = (X_u \otimes_{\theta_1^{h_1} \ldots \theta_s^{h_s}} X_v) \cap I.$$

Proof For infinite α, consider the Cantor normal form $\alpha = \sum \omega^{\alpha_i} a_i$. Call *atom of* α any atom θ_{ij} appearing in the representation $\omega^{\alpha_i} = \prod \theta_{ij}^{h_{ij}}$ of some tile as "ordered monomial".[9] Call *degree of* α the greatest integer appearing as a coefficient a_i, or as an exponent h_{ij}. (For finite α, let $deg(\alpha) = \alpha$.) If

[9] Cfr. the discussion preceding Definition 1.1.

X is a set of ordinals, let $at(X)$ be the set of all atoms of (elements of) X, and let $span(X)$ be the set of all α such that $at(\{\alpha\}) \subseteq at(X)$. Clearly

- if X is finite, then $at(X)$ is finite, and $|span(X)| = \aleph_0$;

- if X is countably infinite, then $|span(X)| = \aleph_0$, and $at(X) \leq \aleph_0$;

- $|X| = |at(X)| = |span(X)|$ whenever any of them is uncountable.

Now let $span_d(X)$ be the set of all $\alpha \in span(X)$ with $deg(\alpha) < d$. Then

$$|span_d(X)| = \begin{cases} d^{d^{|at(X)|}} & \text{if } at(X) \text{ is finite,} \\ |at(X)| & \text{otherwise.} \end{cases}$$

Put $\kappa_0 = \aleph_0$, and let $\kappa_1 < \ldots < \kappa_r$ be the enumeration of the sizes of all uncountable X_u. Put $\nu_i = \kappa_i$ if κ_i is regular, and $\nu_i = \kappa_{i-1}^+$ otherwise.

Let $Y_i = \bigcup\{X_u \mid |X_u| \leq \kappa_i\}$ and $Z_i = \bigcup\{X_u \mid |X_u| < \kappa_i\}$. Put $\Theta = \{\theta_1, \ldots, \theta_m\}$, $A_i = \Theta \cup at(Y_i)$ and $B_i = \Theta \cup at(Z_i)$. Fix $i > 0$ and let $|X_u| = \kappa_i$. For each finite $F \subseteq B_i$ put

$$X_u^F = \{\alpha \in X_u \mid at(\{\alpha\}) \cap B_i \subseteq F\}.$$

So X_u is the directed union of all these sets X_u^F. The finite subsets of B_i are κ_{i-1} many, while $|X_u| = \kappa_i \geq \nu_i$, which is regular and greater than κ_{i-1}. Hence there exists a finite $F_u \subseteq B_i$ such that $|X_u^{F_u}| \geq \nu_i$. Moreover, there exists an integer d_u such that in $X_u^{F_u}$ there are ν_i many elements of degree less than d_u.

Now take $F_i = \bigcup\{F_u \mid |X_u| = \kappa_i\}$ and $d_i = \max\{d_u \mid |X_u| = \kappa_i\}$. Then

$$|X_u \cap span_{d_i}(F_i \cup (A_i \setminus B_i))| \geq \nu_i \text{ for all } X_u \text{ with } |X_u| = \kappa_i.$$

Similarly, there are an integer d_0 and a finite subset $F_0 \subseteq A_0$ such that $Z_0 \cup \Theta \subseteq span_{d_0}(F_0)$, and

$$|X_u \cap span_{d_0}(F_0)| > |Z_0| \text{ for all } X_u \text{ with } |X_u| = \aleph_0 \text{ (if any).}$$

Now let k be fixed. Put $d = \max\{2k, d_0, \ldots, d_r\}$ and $F = \bigcup_{i=0}^r F_i$. Inductively on $i = 1, \ldots, r$ choose finite subsets $G_i \subseteq A_i \setminus B_i$ in such a way that

$$|X_u \cap span_{d_i}(F_i \cup G_i)| > d^{d^{|H_{i-1}|}} \text{ for all } X_u \text{ with } |X_u| = \kappa_i,$$

where $H_s = F \cup \bigcup_{j=1}^s G_j$ (and $H_0 = F$).

Put $I = span_d(H_r)$. Notice that, if $\Theta \subseteq A$, $s \leq m$, $h_t \leq k$ for $1 \leq t \leq s$, and $\tau = \theta_1^{h_1} \ldots \theta_s^{h_s}$, then $span_d(A)$ is "closed under flat τ-products", i.e.

$(\tau \cap span_d(A)) \otimes_\tau (\theta_s^k \cap span_d(A)) \subseteq span_d(A)$. Thus condition (iii) is satisfied, since $\Theta \subseteq H_r$. Condition (i) holds since $F_0 \subseteq H_r$.

In order to verify that also condition (ii) is fulfilled one has only to remark that $Z_i \cap I \subseteq span(B_i) \cap I \subseteq span_d(H_{i-1})$, and so $|Z_i \cap I| \leq d^{d^{|H_{i-1}|}} < |X_u \cap I|$ for all X_u with $|X_u| = \kappa_i$.

□

We are now ready to prove

Theorem 2.4 *The family of sets*

$$\mathcal{F} = \{C_{AB} \mid |A| < |B|\} \cup \{P_{AB}^{\tau} \mid \tau \text{ a } \theta\text{-tile}, A \subseteq \tau, B \subseteq \delta < \theta^{\omega}\}$$

has the Finite Intersection Property. Therefore hyperfine ultrafilters over $\mathcal{I} = \mathcal{P}_{fin}(Ord)$ *exist.*

Proof Given $F_1, \ldots, F_l \in \mathcal{F}$, let $k \in \mathbb{N}$ and $\theta_1 > \theta_2 > \ldots > \theta_m = \omega$ be atoms such that whenever $F_t = P_{A_t B_t}^{\tau_t}$, then

- $\tau_t = \theta_1^{h_{1t}} \theta_2^{h_{2t}} \ldots \theta_m^{h_{mt}}$, with $0 \leq h_{st} \leq k$ for all $s \leq m, t \leq l$, and
- $B_t \subseteq \theta_s^k$ where s is the largest index such that $h_{st} > 0$.

Let $X_1, \ldots, X_n$ be an enumeration of the sets A_t, B_t appearing in any $F_t = P_{A_t B_t}^{\tau_t}$ or $C_{A_t B_t}$. Clearly $X_1, \ldots, X_n$ and $\theta_1, \ldots, \theta_m$ fulfil the hypotheses of Lemma 2.3. Pick a set I satisfying the conditions (i)–(iii) of that lemma with respect to the fixed value of k. Then, by conditions (i) and (ii), the set I belongs to every F_t of type $C_{A_t B_t}$. Moreover, by (iii) and by the choice of k, $(A_t \cap I) \otimes_{\tau_t} (B_t \cap I) = (A_t \otimes_{\tau_t} B_t) \cap I$ for every F_t of the form $P_{A_t B_t}^{\tau_t}$. Hence $I \in \bigcap_{t \leq l} F_t$, and we are done.

□

As an immediate corollary we obtain that the counting functions associated to finite approximations provide numerosities that can be viewed as *hypernatural numbers.*

Corollary 2.5 *There exist numerosity functions* $\mathfrak{n} : \mathcal{P}(Ord) \to \mathcal{A}$, *where* $\mathcal{A}$ *is a class of hyperintegers in the sense of nonstandard analysis.*

Notice that the function Φ takes on only "monotone" values, i.e. $\Phi(X)(i) \leq \Phi(X)(j)$ for all X, whenever $i \subseteq j$. So in general the semiring of numerosities $\mathcal{N} = \mathfrak{n}(\mathcal{W})$ does not exhaust the semiring ${}^*\mathbb{N} = \mathbb{N}_{\mathcal{U}}^{\mathcal{I}}$.

3 Labelling the universe

We devote this section to find a finite approximation for the class of all sets, suitable for assigning them a good Aristotelian size.

We fix an injective map ψ from all infinite sets into the ordinals, and we assume for convenience that sets of rank α are mapped into the ordinal interval $(\beth_\alpha, \beth_{\alpha+1})$. We assign a "label" $\ell(x) \in \mathcal{P}_{fin}(Ord)$ to every set x in the following way.

- For each *ordinal* α put $\ell(\alpha) = \{\alpha\}$, and
- for x *infinite*, $x \notin Ord$, put $\ell(x) = \{\psi(x)\}$.

So *infinite sets* are viewed as *urelements*: their labels do not depend on their elements. On the contrary, finite sets are labelled according to the labels of their elements. We only have to avoid that the same label be assigned to *infinitely many* sets, e.g. to all iterated singletons of x, i.e. to $\{x\}$, $\{\{x\}\}$, $\{\{\{x\}\}\}$, *etc.*

Define the *depth* $d(x)$ of the set x as the maximal length of a descending $\in$-chain starting from x and not including ordinals nor infinite sets.[10] Now, for *finite* $x \notin \omega$, proceed inductively on depth:

- if $d(x) \leq 2$, then put $\ell(x) = \bigcup_{y \in x} \ell(y)$;
- if $d(x) = d > 2$, then put $\ell(x) = (d-2) \cup \bigcup_{y \in x} \ell(y)$.

We have "neutralized" depths 1 and 2 in order to better accomodate finite sets of ordinals and Kuratowski pairs. In fact, every *finite set of ordinals* is its own label, and, when adopting Kuratowski pairs, $\ell(x, y) = \ell(x) \cup \ell(y)$ whenever x, y have depth 0. (In particular, the ordered pair of ordinals (α, β) is labelled by $\{\alpha, \beta\}$).

We are now ready to define a *finite approximation of the whole universe* $\varphi : V \times \mathcal{P}_{fin}(Ord) \to V$ depending in a natural way on the given labelling, namely

$$\varphi(X, i) = X_i = \{x \in X \mid \ell(x) \subseteq i\}$$

One can directly check that all conditions (FA1)-(FA4) are fulfilled. Now we can obtain a numerosity function defined on all sets, that satisfies the

[10] So ordinals and infinite sets have depth 0. That depth is finite follows from König's lemma, because such $\in$-chains can naturally be arranged into a finitary tree without infinite branches, by foundation, which so has finite height. Notice that a finite set x has depth $d + 1$ if and only if the maximal depth of its members is d.

properties (hcp),(sp) and (up). To this end, we follow the construction of the preceding section: we choose a hyperfine ultrafilter $\mathcal{U}$ on $\mathcal{I} = \mathcal{P}_{fin}(Ord)$ and we define the numerosity $\mathfrak{n}(X)$ as the canonical projection of $\Phi(X)$ modulo the corresponding prime ideal $\mathfrak{p}$ of $\mathbb{Z}^{\mathcal{I}}$.

Besides the flat product principle (fpp) for sets of ordinals, we obtain also that other interesting product principles hold for all sets: see Subsection 4.2 for a short list. As it was expected, the equality

$$\varphi(X \times Y, i) = \varphi(X, i) \times \varphi(Y, i),$$

does not hold in general, but we managed to have it valid for *sets of depth* 0, in particular for *sets of ordinals*. Thus these sets satisfy the Cartesian Product Principle (CPP) of the introduction.

If one could label all sets by ordinals in a "natural" way, then the numerosity assigned to every set according to the procedure presented above would be completely satisfactory. To be sure, such a *natural* labelling seems highly unlikely, unless one assumes some "restricting" axiom, such as $V = L$.

4 Final remarks and open questions

It is now time to suggest how we take into account the fact that, *prima facie*, our notions essentially involve proper classes, and so they might present foundational problems. Since $\mathcal{I}, \mathcal{U}, \mathcal{N}, \mathcal{A}$, along with Ord, $\mathcal{P}(Ord)$, and V, are all *proper classes*, our theory of size cannot be directly formalized in ZFC. Instead an appropriate framework may be a class theory such as Gödel-Bernays' GB. Remark that *global choice*, and not simply Zermelo's axiom, is needed in labelling all sets, and, more important, in our construction of a numerosity function. Another formalization could be given within a modified Zermelo-Fraenkel set theory, where the language is extended with a symbol for a global choice function. A simpler alternative could be to restrict our notion of size to sets in the universe V_κ for some inaccessible cardinal κ.

Working inside any of the above mentioned theories, we can formalize the following construction. Fix a closed unbounded class of cardinals K. For each $\kappa \in K$ pick a prime ideal $\mathfrak{p}_\kappa$ of the ring $\mathbb{Z}^{\mathcal{P}_\omega(\kappa)}$, in such a way that the corresponding quotient rings $\mathcal{A}_\kappa = \mathbb{Z}^{\mathcal{P}_\omega(\kappa)}/\mathfrak{p}_\kappa$ be conveniently embedded into each other. (As mentioned above, this step requires global choice.) Namely, start with a first hyperfine prime ideal $\mathfrak{p}_{\kappa_0}$ for $\kappa_0 = \min K$, and, at successor steps, choose a hyperfine prime ideal $\mathfrak{p}_{\kappa_{\alpha+1}}$ including the extension of $\mathfrak{p}_{\kappa_\alpha}$ to $\mathbb{Z}^{\mathcal{P}_\omega(\kappa_{\alpha+1})}$. I.e. $\mathfrak{p}_{\kappa_{\alpha+1}}$ contains all those functions $\zeta : \mathcal{P}_\omega(\kappa_{\alpha+1}) \to \mathbb{Z}$

such that ζ restricted to $\mathcal{P}_\omega(\kappa_\alpha)$ belongs to $\mathfrak{p}_{\kappa_\alpha}$, and $\zeta(c) = \zeta(c \cap \kappa_\alpha)$ for all $c \in \mathcal{P}_\omega(\kappa_{\alpha+1})$. At limit step λ take $\mathfrak{p}_{\kappa_\lambda} = \bigcup_{\alpha<\lambda} \mathfrak{p}_{\kappa_\alpha}$. At the end the "union" $\mathcal{A}$ of the $\mathcal{A}_\kappa$s is a *discretely ordered proper class-ring*, and in fact a *limit ultrapower* of $\mathbb{Z}$, namely the direct limit of the ultrapowers $\mathbb{Z}^{\mathcal{P}_\omega(\kappa)}_{\mathcal{U}_\kappa}$ ($\mathcal{U}_\kappa$ is the ultrafilter associated to the prime ideal $\mathfrak{p}_\kappa$). So $\mathcal{A}$, and hence $\mathcal{N}$, is a *proper class of hyperintegers*.

In the remaining part of this section we sketch a list of issues concerning our notion of size that, in our opinion, would deserve further investigation. Given the novelty of this topic, most arguments are the subject matter of current research (see [2]), and therefore we simply hint a few problems that we submit to the attention of interested researchers.

4.1 Difference

A most wanted property is the natural completion of Aristotle's Principle, namely

(Diff) (Difference Principle)

$$\mathfrak{s}(A) > \mathfrak{s}(B) \Longleftrightarrow \exists C \;\; \mathfrak{s}(A) = \mathfrak{s}(B) + \mathfrak{s}(C)$$

or (almost) equivalently

$$\mathfrak{s}(A) > \mathfrak{s}(B) \Longleftrightarrow \exists B' \subset A \;\; \mathfrak{s}(B') = \mathfrak{s}(B)$$

This property is assumed in [1], where only *countable* sets are considered, and it is proved consistent there, thanks to *selective ultrafilters* over $\mathbb{N}$. The consistency of (Diff) in this general setting is dubious, but the question is still open.

4.2 Product principles

In Section 3, the labelling of finite sets has been chosen so as to obtain

$$\ell(\{x, y\}) = \ell(\{x\}) \cup \ell(\{y\}) \;\text{ and }\; \ell(x, y) = \ell(x, x) \cup \ell(y, y).$$

As a consequence, several "stratified" or "homogeneous" product principles can be proved. These modified versions share the same "arithmetical" consequences of the original Cartesian Product Principle (CPP). They have the advantage of being insensitive to the particular coding of pairs as sets (or atoms) one is adopting. Let us cite a couple of them:

(DiagPP) ("Diagonal Product Principle")

$$\mathfrak{n}(X \times Y) = \mathfrak{n}(\{(x,x) \mid x \in X\}) \cdot \mathfrak{n}(\{(y,y) \mid y \in Y\}),$$

and

(FibPP) ("Fiber Product Principle")

$$\mathfrak{n}(X \times Y) = \mathfrak{n}(X \times \{y\}) \cdot \mathfrak{n}(\{x\} \times Y).$$

A third principle, which seems to us even more appropriate, is the following

(SDP) ("Singleton-Doubleton Principle") If $X \cap Y = \emptyset$, then

$$\mathfrak{n}(\{\{x,y\} \mid x \in X, y \in Y\}) = \mathfrak{n}(\{\{x\} \mid x \in X\}) \cdot \mathfrak{n}(\{\{y\} \mid y \in Y\}).$$

Clearly the pathological sets cited in the introduction would be not equinumerous with their *diagonals*, or *fibers*, or *singleton-sets*, but one can easily isolate the subclass of "Cantorian" sets satisfying these additional conditions (e.g., all sets whose elements have *bounded depth* are Cantorian in this sense).

4.3 Power

The use of *finite approximations* has an interesting consequence on the arithmetic of numerosities, namely

$$2^{\mathfrak{n}(X)} = \mathfrak{n}(\mathcal{P}_{fin}(X)).$$

(The power $2^{\mathfrak{n}(X)}$ is well-defined, since in this case numerosities are hyperintegers.) The problem of finding models where power gives instead the size of the *whole powerset* seems to require a quite different approach.

4.4 Continuous approximation

The numerosity functions presented here originate from the general idea of "finite approximation". In measuring sets of ordinals, a specific idea of "continuous approximation" could appear even more natural and appealing. Thus one is led to consider the following

(ContApp) (Continuous Approximation Property)

For limit λ: $\forall \alpha < \lambda \ \ \mathfrak{s}(X \cap \alpha) \leq \mathfrak{s}(Y \cap \alpha) \Longrightarrow \mathfrak{s}(X \cap \lambda) \leq \mathfrak{s}(Y \cap \lambda)$

Again, this property is assumed in [1], where it is proved consistent, thanks to the fact that the *sole limit ordinal* in play was ω, and so all "approximations" were *finite*. If restricted to an appropriate class of limit ordinals, the consistency of (ContApp) for all sets of ordinals can be proved by constructing suitable *limit ultrapowers* (see [2]). However, in these models, the Half Cantor's Principle (HCP) fails, and so the resulting numerosities seem to better correspond to a measure of *density* rather than of *size*. We are inclined to conjecture the inconsistency of the principle (HCP) with (ContApp), even when restricted to an unbounded class of cardinals.

References

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