Hypernatural numbers, idempotent ultrafilters, and a proof of Rado's theorem

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Introduction

In combinatorics of numbers one can find deep and fruitful interactions among diverse *non-elementary* methods, namely:

- Ergodic theory
- Fourier analysis
- Topological dynamics
- Algebra in the space of ultrafilters $\beta\mathbb{N}$

Also the methods of nonstandard analysis have recently started to give contributions in this area, and they seem to have interesting potentialities.

Theorem (R.Jin 2000)

If A and B are sets of integers with positive upper Banach density, then $A + B$ is piecewise syndetic.

(A set is *piecewise syndetic* if it has bounded gaps on arbitrarily large intervals. The *Banach density* is a refinement of the upper asymptotic density.)

Jin's result raised the attention of several researchers (including V.Bergelson, H. Furstenberg, and B. Weiss), who tried to find other proofs expressed into more familiar terms, and to improve on it.

Recently, M. Beiglböck found a really nice *ultrafilter proof* of Jin's theorem.

- Ultrafilters are essential in constructing models of nonstandard analysis (which in fact can be characterized as *limit ultrapowers*).
- Every point ξ in the hyper-extension ${}^*X$ corresponds to an ultrafilter $\mathfrak{U}_\xi$ on X.

Plan of the talk

- A quick introduction to nonstandard analysis.
- Hyper-natural numbers $\xi \in {}^*\mathbb{N}$ of nonstandard analysis as representatives of ultrafilters on $\mathbb{N}$.
- A peculiar manageable "algebra" in the nonstandard setting to manipulate linear combinations of idempotent ultrafilters.
- Use of that "algebra" in the study of partition regularity problems.

As examples of applications, I will sketch nonstandard proofs of:

- Ramsey Theorem
- Rado's Theorem
- Milliken-Taylor Theorem

This technique also applies in the study of partition regularity of *non-linear* equations.

Nonstandard Analysis, hyper-quickly

Nonstandard analysis is essentially grounded on the following two properties:

1. Every object X can be extended to an object ${}^*X$.
2. ${}^*X$ is a sort of "weakly isomorphic" copy of X, in the sense that it satisfies exactly the same "elementary properties" as X.

E.g., ${}^*\mathbb{R}$ is an *ordered field* that properly extends the real line $\mathbb{R}$. The two structures $\mathbb{R}$ and ${}^*\mathbb{R}$ cannot be distinguished by any "elementary property".

Here we shall focus on ${}^*\mathbb{N}$, which is the positive part of the *discretely ordered ring* ${}^*\mathbb{Z}$.

Star-map

To every mathematical object X is associated its *hyper-extension* (or *nonstandard extension*) ${}^*X$.

$$X \longmapsto {}^*X$$

So, ${}^*\mathbb{N}$ is the set of hyper-natural numbers, ${}^*\mathbb{R}$ is the set of hyper-real numbers, *etc.*

(It is assumed that ${}^*r = r$ for all numbers $r \in \mathbb{R}$, and the non-triviality condition $A \subsetneq {}^*A$ for all infinite $A \subseteq \mathbb{R}$.)

Transfer principle

If $P(x_1, \ldots, x_n)$ is any property expressed in "elementary terms", then

$$P(A_1, \ldots, A_n) \Longleftrightarrow P({}^*A_1, \ldots, {}^*A_n)$$

P is expressed in "elementary terms" if it is written in the *first-order language of set theory* (everything is expressed by only using the *equality* and the *membership* relations).

Moreover, quantifiers must be used in the *bounded forms*:

$$\text{“}\forall x \in A \;\; P(x, \ldots)\text{”} \quad \text{and} \quad \text{“}\exists x \in A \;\; P(x, \ldots)\text{”}.$$

All basic set properties: "$A \subseteq B$", "$C = A \cup B$", "$C = A \cap B$", "$C = A \setminus B$" *etc.*, can be directly expressed in elementary terms.

Besides, it is a well-known fact that virtually *all* mathematical objects can be "coded" as sets (*e.g.*, an *ordered pair* (a, b) can be defined as a *Kuratowski pair* $\{\{a\}, \{a, b\}\}$; a *function* can be defined as a suitable *set of ordered pairs*; *etc.*).

As a consequence, virtually *all* mathematical properties can be expressed in elementary terms.

E.g., by *transfer*, the following are easily proved.

1. $A \subseteq B \Leftrightarrow {}^*A \subseteq {}^*B$.
2. ${}^*(A \cup B) = {}^*A \cup {}^*B$
3. ${}^*(A \cap B) = {}^*A \cap {}^*B$
4. ${}^*(A \setminus B) = {}^*A \setminus {}^*B$
5. ${}^*(A \times B) = {}^*A \times {}^*B$
6. $f : A \to B \Leftrightarrow {}^*f : {}^*A \to {}^*B$
7. The function f is 1-1 $\Leftrightarrow$ the function ${}^*f$ is 1-1, *etc.*

Moreover:

- ${}^*\{x \in X \mid P(x, A_1, \ldots, A_n)\} = \{x \in {}^*X \mid P(x, {}^*A_1, \ldots, {}^*A_n)\}$

By *transfer*, ${}^*\mathbb{R}$ is an *ordered field* where the sum and product operation are the hyper-extensions of the binary functions $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$; and the order relation is the hyper-extension ${}^*\{(a, b) \in \mathbb{R} \times \mathbb{R} \mid a < b\}$.

Moreover:

- The *hyper-rational numbers* ${}^*\mathbb{Q}$ are dense in ${}^*\mathbb{R}$.
- Every $\xi \in {}^*\mathbb{R}$ has an *integer part*, *i.e.* there exists a unique hyper-integer $\nu \in {}^*\mathbb{Z}$ such that $\nu \leq \xi < \nu + 1$.

and so forth.

As a proper extension of the reals, the hyper-real field ${}^*\mathbb{R}$ contains infinitesimal numbers $\varepsilon \neq 0$ such that:

$$-\frac{1}{n} < \varepsilon < \frac{1}{n} \quad \text{for all } n \in \mathbb{N}$$

as well as infinite numbers

$$|\Omega| > n \quad \text{for all } n \in \mathbb{N}.$$

Note that ${}^*\mathbb{R}$ is *not* Archimedean, and hence it is *not* complete (the bounded set of infinitesimals does not have a least upper bound).

The hyper-integers

By *transfer*, one can easily show that the hyper-integers ${}^*\mathbb{Z}$ are a *discretely ordered ring* whose positive part are the hyper-natural numbers ${}^*\mathbb{N}$.

$$
{}^*\mathbb{N} \;=\; \{\underbrace{1, 2, \ldots, n, \ldots}_{\text{finite numbers}} \quad \underbrace{\ldots, N-2, N-1, N, N+1, N+2, \ldots}_{\text{infinite numbers}}\}
$$

Hyper-integers can be used as a convenient setting for the study of certain *density properties* and certain aspects of *additive number theory*.

Nonstandard characterizations

We now recall a few basic notions of "largeness" for sets of integers, along with their *nonstandard* characterizations:

Definition

A is thick if for every k there exists x such that $[x, x+k] \subseteq A$.

Definition (Nonstandard)

A is thick if there exists an *infinite interval* $[\nu, \mu] \subseteq {}^*A$.

Definition

A is syndetic if there exists $k \in \mathbb{N}$ such that every interval $[x, x+k] \cap A \neq \emptyset$. (That is, if A^c is *not* thick.)
Equivalently, there exists a finite F such that $F + A = \mathbb{Z}$.

Definition (Nonstandard)

A is syndetic if ${}^*A$ has only *finite gaps*,
i.e. ${}^*A \cap I \neq \emptyset$ for every infinite interval I.

Definition

A is piecewise syndetic if $A = B \cap C$ where B is *thick* and C is *syndetic*.
Equivalently, there exists a finite F such that $F + A$ is thick.

Definition (Nonstandard)

A is piecewise syndetic if ${}^*A$ has only *finite gaps* on some *infinite interval*.

Theorem

Piecewise syndetic sets are partition regular.

Nonstandard proof.

Let A be piecewise syndetic. By induction, it is enough to check the property for 2-partitions $A = C_1 \cup C_2$.

Pick an infinite interval I where ${}^*A$ has only finite gaps.

If ${}^*C_1$ has only finite gaps in I then C_1 is piecewise syndetic.

Otherwise, there exists an infinite interval $J \subseteq I$ such that $J \cap {}^*C_1 = \emptyset$. But then $J \cap {}^*C_2 = J \cap {}^*A$ has only finite gaps, and hence C_2 is piecewise syndetic.

Hyper-natural numbers as ultrafilters

Every hyper-natural number $\xi \in {}^*\mathbb{N}$ generates an ultrafilter on $\mathbb{N}$:

$$\mathfrak{U}_\xi = \{A \subseteq \mathbb{N} \mid \xi \in {}^*A\}$$

If ${}^*\mathbb{N}$ is $\mathfrak{c}^+$-saturated, then every ultrafilter is generated by some number $\xi \in {}^*\mathbb{N}$ (actually, by at least $\mathfrak{c}$-many).

In some sense, in the nonstandard setting every ultrafilter becomes a *principal* ultrafilter!

Differently from the usual approach to nonstandard analysis, we work in a suitable framework where one can take the hyper-extension of any object.
In particular, one can iterate hyper-extensions and consider:

- The hyper-hypernatural numbers ${}^{**}\mathbb{N}$
- The hyper-extension ${}^{*}\xi \in {}^{**}\mathbb{N}$ of an hyper-natural number $\xi \in {}^{*}\mathbb{N}$, and so forth.

WARNING: The foundational aspects require some attention.

By *transfer*, one proves that:

- The natural numbers are an initial segment of the hyper-natural numbers: $\mathbb{N} < {}^*\mathbb{N} \setminus \mathbb{N}$
- The hyper-natural numbers are an initial segment of the hyper-hyper-natural numbers: ${}^*\mathbb{N} < {}^{**}\mathbb{N} \setminus {}^*\mathbb{N}$; *etc.*

A nonstandard proof of Ramsey theorem

As a warm-up application of iterated hyper-extensions, let us see a nonstandard proof of Ramsey theorem for pairs.

Theorem (Ramsey 1928)

Given a finite colouring $[\mathbb{N}]^2 = C_1 \cup \ldots \cup C_r$ of the pairs of natural numbers, there exists an infinite H whose pairs are monochromatic: $[H]^2 \subseteq C_i$.

We have the finite coloring $[{}^{**}\mathbb{N}]^2 = {}^{**}C_1 \cup \ldots \cup {}^{**}C_r$.

Pick an infinite $\xi \in {}^*\mathbb{N}$. Then $\{\xi, {}^*\xi\} \in {}^{**}C_i$ for some i.

$\xi \in \{x \in {}^*\mathbb{N} \mid \{x, {}^*\xi\} \in {}^{**}C_i\} = {}^*\{x \in \mathbb{N} \mid \{x, \xi\} \in {}^*C_i\} = {}^*A$.

Pick $a_1 \in A$, so $\{a_1, \xi\} \in {}^*C_i$.

Then $\xi \in {}^*\{x \in \mathbb{N} \mid \{a_1, x\} \in C_i\} = {}^*B_1$.

$\xi \in {}^*A \cap {}^*B_1 \Rightarrow A \cap B_1$ is infinite: pick $a_2 \in A \cap B_1$ with $a_2 > a_1$.

$a_2 \in B_1 \Rightarrow \{a_1, a_2\} \in C_i$.

$a_2 \in A \Rightarrow \{a_2, \xi\} \in {}^*C_i \Rightarrow \xi \in {}^*\{x \in \mathbb{N} \mid \{a_2, x\} \in {}^*C_1\} = {}^*B_2$.

$\xi \in {}^*A \cap {}^*B_1 \cap {}^*B_2 \Rightarrow$ we can pick $a_3 \in A \cap B_1 \cap B_2$ with $a_3 > a_2$.

$a_3 \in B_1 \cap B_2 \Rightarrow \{a_1, a_3\}, \{a_2, a_3\} \in C_i$, and so forth.

The infinite set $H = \{a_n \mid n \in \mathbb{N}\}$ is such that $[H]^2 \subset C_i$.

Definition

We say that two elements $\alpha, \beta \in {}^*\mathbb{N}$ are *u-equivalent*, and write $\alpha \underset{u}{\sim} \beta$, when they generate the same ultrafilter:

$$\mathfrak{U}_\alpha = \mathfrak{U}_\beta.$$

For $n \in \mathbb{N}$

1. $\alpha + n \underset{u}{\sim} \alpha' + n$,
2. $\alpha - n \underset{u}{\sim} \alpha' - n$,
3. $n \cdot \alpha \underset{u}{\sim} n \cdot \alpha'$,
4. $\alpha / n \underset{u}{\sim} \alpha' / n$, provided α is divisible by n.

Denote by ${}^{k*}X$ the k-iterated hyper-extension of a set X.

If $\nu \in {}^{k*}\mathbb{N}$, we extend the notion of generated ultrafilter by putting

$$\mathfrak{U}_\nu \;=\; \{A \subseteq \mathbb{N} \mid \nu \in {}^{k*}A\}.$$

The *u*-equivalence relation is extended accordingly to all pairs of numbers in the following union

$${}^{\star}\mathbb{N} \;=\; \bigcup_{k\in\mathbb{N}} {}^{k*}\mathbb{N}.$$

Remark 1. The above definitions are coherent.

Remark 2. The maps $X \longmapsto {}^{k*}X$ and the map $X \longmapsto {}^{\star}X$ are nonstandard embeddings, *i.e.* they satisfy the *transfer principle*.

Proposition

Let $\alpha, \beta \in {}^*\mathbb{N}$ *and* $A \subseteq \mathbb{N}$. *Then*

$$A \in \mathfrak{U}_\alpha \oplus \mathfrak{U}_\beta \iff \alpha + {}^*\beta \in {}^{**}A$$

Proof.

Consider the set $\widehat{A} = \{n \in \mathbb{N} \mid A - n \in \mathfrak{U}_\beta\}$ and its hyper-extension:

$${}^*\widehat{A} = {}^*\{n \in \mathbb{N} \mid \beta + n \in {}^*A\} = \{\gamma \in {}^*\mathbb{N} \mid {}^*\beta + \gamma \in {}^{**}A\}$$

Then the following equivalences yield the thesis:

$$A \in \mathfrak{U}_\alpha \oplus \mathfrak{U}_\beta \iff \widehat{A} \in \mathfrak{U}_\alpha \iff \alpha \in {}^*\widehat{A} \iff {}^*\beta + \alpha \in {}^{**}A.$$

Corollary

For every $\xi_0, \xi_1, \ldots, \xi_k \in {}^*\mathbb{N}$, and for every $a_0, a_1, \ldots, a_k \in \mathbb{N}$, the ultrafilter $a_0\mathfrak{U}_{\xi_0} \oplus a_1\mathfrak{U}_{\xi_1} \oplus \ldots \oplus a_k\mathfrak{U}_{\xi_k}$ is generated by $a_0\xi_0 + a_1{}^*\xi_1 + \ldots + a_k{}^{k*}\xi \in {}^{(k+1)*}\mathbb{N}$.

Particularly relevant for applications is the class of idempotent ultrafilters: $\mathcal{U} \oplus \mathcal{U} = \mathcal{U}$.

Proposition

The ultrafilter $\mathfrak{U}_\xi$ is idempotent if and only if $\xi \underset{u}{\sim} \xi + {}^*\xi$.

Remark. In general sums in ${}^*\mathbb{N}$ are not coherent with *u*-equivalence, *i.e.* it can be the case that $\alpha \underset{u}{\sim} \alpha'$ and $\beta \underset{u}{\sim} \beta'$, but $\alpha + \beta \underset{u}{\not\sim} \alpha' + \beta'$.
However, one has that $\alpha + {}^*\beta \underset{u}{\sim} \alpha' + {}^*\beta'$ in ${}^{**}\mathbb{N}$, as they generate the same ultrafilter $\mathfrak{U}_\alpha \oplus \mathfrak{U}_\beta = \mathfrak{U}_{\alpha'} \oplus \mathfrak{U}_{\beta'}$.

Definition

The *u-equivalence* $\underset{u}{\approx}$ between strings of integers is the smallest equivalence relation such that:

- The empty string $\varepsilon \underset{u}{\approx} \langle 0 \rangle$.
- $\langle a \rangle \underset{u}{\approx} \langle a, a \rangle$ for all $a \in \mathbb{Z}$.
- $\underset{u}{\approx}$ is coherent with *concatenations*, *i.e.*

$$\sigma \underset{u}{\approx} \sigma' \text{ and } \tau \underset{u}{\approx} \tau' \implies \sigma^\frown\tau \underset{u}{\approx} \sigma'^\frown\tau'.$$

Example:

$$\langle 1,1,3,7,0,7,7,0,4 \rangle \underset{u}{\approx} \langle 1,0,3,3,7,4,4,4 \rangle \underset{u}{\approx} \langle 1,3,7,4 \rangle$$

The following characterization will be used to handle linear combinations of idempotent ultrafilters.

Theorem

Let $\mathfrak{U}_\xi$ be idempotent. Then the following are equivalent:

1. $a_0\xi + a_1{}^*\xi + \ldots + a_k \cdot {}^{k*}\xi \underset{u}{\sim} b_0\xi + b_1{}^*\xi + \ldots + b_h \cdot {}^{h*}\xi$
2. $\langle a_0, a_1, \ldots, a_k\rangle \underset{u}{\approx} \langle b_0, b_1, \ldots, b_h\rangle$.

Theorem (Bergelson-Hindman 1990)

Let $\mathcal{U}$ be an idempotent ultrafilter. Then every set $A \in 2\mathcal{U} \oplus \mathcal{U}$ contains a 3-term arithmetic progression.

Pick $\xi \in {}^*\mathbb{N}$ such that $\mathcal{U} = \mathfrak{U}_\xi$ and consider the following 3-AP:

- $\nu = 2\xi + 0 + {}^{**}\xi \;\rightsquigarrow\; \langle 2, 0, 1 \rangle$
- $\mu = 2\xi + {}^*\xi + {}^{**}\xi \;\rightsquigarrow\; \langle 2, 1, 1 \rangle$
- $\lambda = 2\xi + 2{}^*\xi + {}^{**}\xi \;\rightsquigarrow\; \langle 2, 2, 1 \rangle$

Notice that $\nu \underset{u}{\sim} \mu \underset{u}{\sim} \lambda$ and the generated ultrafilter

$$\mathcal{W} = \mathfrak{U}_\nu = \mathfrak{U}_\mu = \mathfrak{U}_\lambda = \mathcal{U}_{2\xi + {}^*\xi} = \mathfrak{U}_{2\xi} \oplus \mathfrak{U}_\xi = 2\mathcal{U} \oplus \mathcal{U}$$

If $A \in \mathcal{W}$ then a 3-AP in A is proved to exist by backward *transfer*:

$$\exists\, \nu, \mu, \lambda \in {}^{***}A \text{ s.t. } \mu - \nu = \lambda - \mu > 0.$$

We now elaborate on this example and generalize.

Definition

A function $F(x_1,\ldots,x_n)$ is injectively partition regular (IPR) on $\mathbb{N}$ if for every finite coloring of $\mathbb{N}$ there exist distinct monochromatic elements $a_1,\ldots,a_n$ such that $F(a_1,\ldots,a_n)=0$.

Theorem

A function $F(x_1,\ldots,x_n)$ is IPR on $\mathbb{N}$ iff there exists distinct numbers $\xi_1,\ldots,\xi_n \in {}^\star\mathbb{N}$ which are u-equivalent to each other and ${}^\star F(\xi_1,\ldots,\xi_n)=0$.

Rado's Theorem

We now give a nonstandard ultrafilter proof of the following version of Rado's theorem.

Theorem

Let $c_1X_1 + \ldots + c_nX_n = 0$ be a diophantine equation with $n \geq 3$. If $c_1 + \ldots + c_n = 0$ then there exist $a_1, \ldots, a_{n-1} \in \mathbb{N}$ such that for every idempotent $\mathcal{U}$, the ultrafilter

$$\mathcal{V} = a_1\mathcal{U} \oplus \ldots \oplus a_{n-1}\mathcal{U}$$

witnesses that the equation is IPR (i.e. for every $A \in \mathcal{V}$ there exist distinct $a_1, \ldots, a_n \in A$ with $c_1a_1 + \ldots + c_ka_k = 0$.)

Let $\mathcal{U} = \mathfrak{U}_\xi$ be any idempotent ultrafilter.

For simplicity, denote by $u_1 = \xi$, $u_2 = {}^*\xi$, $u_3 = {}^{**}\xi$, *etc.*

Let $a_1, \ldots, a_{n-2}$ be arbitrary integers, and consider the following elements in ${}^\star\mathbb{N} = \bigcup_k {}^{k*}\mathbb{N}$:

$$
\begin{array}{lllllllllllllll}
\zeta_1 & = & a_1 u_1 & + & a_1 u_2 & + & a_2 u_3 & + & a_3 u_4 & + & \ldots & + & a_{n-2} u_{n-1} & + & a_{n-1} u_n \\
\zeta_2 & = & a_1 u_1 & + & 0 & + & a_2 u_3 & + & a_3 u_4 & + & \ldots & + & a_{n-2} u_{n-1} & + & a_{n-1} u_n \\
\zeta_3 & = & a_1 u_1 & + & a_2 u_2 & + & 0 & + & a_3 u_4 & + & \ldots & + & a_{n-2} u_{n-1} & + & a_{n-1} u_n \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\zeta_{n-1} & = & a_1 u_1 & + & a_2 u_2 & + & a_3 u_3 & + & \ldots & + & a_{n-2} u_{n-2} & + & 0 & + & a_{n-1} u_n \\
\zeta_n & = & a_1 u_1 & + & a_2 u_2 & + & a_3 u_3 & + & \ldots & + & a_{n-2} u_{n-2} & + & a_{n-1} u_{n-1} & + & a_{n-1} u_n
\end{array}
$$

$\zeta_1 \underset{u}{\sim} \zeta_2 \underset{u}{\sim} \ldots \underset{u}{\sim} \zeta_n$ all generate the same ultrafilter:

$$\mathcal{V} \; = \; a_1\mathcal{U} \oplus a_2\mathcal{U} \oplus \ldots \oplus a_{n-1}\mathcal{U}$$

Now, $c_1\zeta_1 + \ldots + c_n\zeta_n = 0$ if and only if the coefficients a_i fulfill the following conditions:

$$\begin{cases} (c_1 + c_2 + \ldots + c_n) \cdot a_1 = 0 \\ c_1 \cdot a_1 + (c_3 + \ldots + c_n) \cdot a_2 = 0 \\ (c_1 + c_2) \cdot a_2 + (c_4 + \ldots + c_n) \cdot a_3 = 0 \\ \qquad\qquad \vdots \\ (c_1 + c_2 + \ldots + c_{n-3}) \cdot a_{n-3} + (c_{n-1} + c_n) \cdot a_{n-2} = 0 \\ (c_1 + c_2 + \ldots + c_{n-2}) \cdot a_{n-2} + c_n \cdot a_{n-1} = 0 \\ (c_1 + c_2 + \ldots + c_n) \cdot a_{n-1} = 0 \end{cases}$$

The first and the last equations are trivially satisfied because of the hypothesis $c_1 + c_2 + \ldots + c_n = 0$.

The remaining $n-2$ equations are satisfied by (infinitely many) suitable $a_1, \ldots, a_{n-1} \in \mathbb{N}$, that can be explicitly given in terms of the c_i.

Since all the $a_i \neq 0$, the numbers ζ_is are mutually distinct, and we can apply the nonstandard characterization of IPR.

Remark

This technique can be extended and applied to study of partition regularity of *non-linear* equations.

For a nonempty set X, consider its *finite sums*:

$$FS(X) = \left\{ \sum_{x \in F} x \,\middle|\, F \text{ nonempty finite subset of } X \right\}.$$

A fundamental result in combinatorics is

Theorem (Hindman 1974)

For every finite coloring of $\mathbb{N}$ there exists an infinite X such that all sums in $FS(X)$ are monochromatic.

The original proof consisted in really intricate combinatorial arguments.

"Anyone with a very masochistic bent is invited to wade through the original combinatorial proof." (Neil Hindman)

The very next year, Galvin and Glazer found an elegant (and much simpler) proof by using an idempotent ultrafilter.

In fact, if $\mathcal{U}$ is idempotent then every $A \in \mathcal{U}$ includes a set $FS(X)$ where X is infinite.

Milliken-Taylor Theorem

Theorem

Let $\mathcal{U}$ be any idempotent ultrafilter, and let $a_1, \ldots, a_k \in \mathbb{N}$. For every $A \in a_1\mathcal{U} \oplus \ldots \oplus a_k\mathcal{U}$ there exists an infinite

$$X = \{x_1 < x_2 < \ldots < x_n < \ldots\}$$

such that for every sequence $I_1 < \ldots < I_k$ of nonempty finite sets

$$\sum_{i \in I_1} a_1 x_i + \ldots + \sum_{i \in I_k} a_k x_i \in A.$$

One example: let $A \in 7\mathfrak{U}_\xi \oplus 5\mathfrak{U}_\xi$ where $\mathfrak{U}_\xi$ idempotent.

We want to find $X = \{x_1 < x_2 < \ldots < x_n < \ldots\}$ such that for every nonempty finite sets of indexes $I < J$, one has

$$\sum_{i \in I} 7x_i \; + \; \sum_{j \in J} 5x_j \; \in A.$$

We start by picking x_1 such that

1. $7x_1 + 5\xi \in {}^*A$
2. $7x_1 + 7\xi + 5{}^*\xi \in {}^{**}A$

This is possible because $7\xi + 5{}^*\xi \underset{u}{\sim} 7\xi + 7{}^*\xi + 5{}^{**}\xi$, and so

- $7\xi + 5{}^*\xi \in {}^{**}A \Rightarrow \xi \in {}^*\{n \in \mathbb{N} \mid 7n + 5\xi \in {}^*A\}$
- $7{}^*\xi + 5{}^{**}\xi \in {}^{***}A \Rightarrow \xi \in {}^*\{n \in \mathbb{N} \mid 7n + 7\xi + 5{}^*\xi \in {}^{**}A\}$

and hence $\{n \in \mathbb{N} \mid 7n + 5\xi \in {}^*A \;\&\; 7n + 7\xi + 5{}^*\xi \in {}^{**}A\} \neq \emptyset$.

We now look for $x_2 > x_1$ such that

1. $7x_1 + 5x_2 \in A$
2. $7x_1 + 7x_2 + 5\xi \in {}^*A$
3. $7x_1 + 7x_2 + 7\xi + 5{}^*\xi \in {}^{**}A$
4. $7x_2 + 5\xi \in {}^*A$
5. $7x_2 + 7\xi + 5{}^*\xi \in {}^{**}A$

This is possible because

- $7x_1 + 5\xi \in {}^*A \Rightarrow \xi \in {}^*\{n \mid 7x_1 + 5n \in A\} = {}^*B_1$
- $7x_1 + 7\xi + 5{}^*\xi \in {}^{**}A \Rightarrow \xi \in {}^*\{n \mid 7x_1 + 7n + 5\xi \in {}^*A\} = {}^*B_2$
- $7x_1 + 7\xi + 7{}^*\xi + 5{}^{**}\xi \in {}^{***}A \Rightarrow$
  $\xi \in {}^*\{n \mid 7x_1 + 7n + 7\xi + 5{}^*\xi \in {}^{**}A\} = {}^*B_3$
- $7\xi + 5{}^*\xi \in {}^{**}A \Rightarrow \xi \in {}^*\{n \mid 7n + 5\xi \in {}^*A\} = {}^*B_4$
- $7x_1 + 5\xi \in {}^*A \Rightarrow \xi \in {}^*\{n \mid 7x_1 + 5n \in A\} = {}^*B_5$

$$\xi \in \bigcap_{s=1}^{5} {}^*B_s = {}^*\left(\bigcap_{s=1}^{5} B_s\right)$$

implies that $B_1 \cap \ldots \cap B_5$ is an infinite set, and any $x_2 > x_1$ in that intersection has the desired properties.

We then inductively iterate the process to find the set $X = \{x_1 < x_2 < \ldots < x_n < \ldots\}$ we are looking for.

Note that $X \subseteq \{k \in \mathbb{N} \mid (A - 7k)/5 \in \mathcal{U}\} \in \mathcal{U}$.

Conclusions

- Certain *ultrafilter techniques* can be conveniently accommodated in the nonstandard setting.
  In fact, there is a natural way of identifying ultrafilters with the *hyper-natural* numbers ${}^*\mathbb{N}$ of nonstandard analysis.
- The resulting "algebra" is suitable to study partition regularity problems, also in the case of *non-linear* equations.
- Several "nonstandard" directions are still to be explored.

THANK YOU

for your attention