Remarks on multiple recurrent points

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- We recall the notion of a recurrent (uniformly recurrent, multiple recurrent) point in an arbitrary dynamical system X over a monoid S.

- Uniformly recurrent points do exist, and "usually" they are recurrent.

- Multiple recurrent points do not always exist.

- All of these points can be characterized by the arithmetic on βS and its action on X.

- We show that under a condition of equicontinuity (of the action of S on X), multiple recurrent points do exist.

Contents

A. Dynamical systems and (uniformly) recurrent points

We assume that S is a *monoid*, i.e. a semigroup $(S, \cdot)$ with an identity 1_S.

The most important example here is the monoid $(\omega, +)$, where $\omega = \{0, 1, 2, \ldots\}$.

- Definition

  A *dynamical system (DS)* over S is a structure (X, m), X a compact Hausdorff space,

  $$m : S \times X \to X$$

  a continuous *action (operation)* of S on X.
  We write

  $$sx = s \cdot x = m(s, x)$$

  for $s \in S$ and $x \in X$ and view m as a left multiplication by S on X; then
  – $s(tx) = (st)x$
  – $1_s \cdot x = x$
  – the map $x \mapsto sx$ (from X to X) is continuous, for every $s \in S$.

Discrete systems

- Special case

  A *discrete* DS is a system over the monoid $(\omega, +)$.

  It is determined by the continuous map $t : X \to X$ defined by $t(x) = 1 \cdot x$, because:

  $$0 \cdot x = x,\ \ 1 \cdot x = t(x),\ \ 2 \cdot x = 1 \cdot (1 \cdot x) = t^2(x), \ldots$$

  $$n \cdot x = t^n(x).$$

  We usually consider (X, t) as the discrete DS.

Return sets

- Definition

  In a DS X over S, we define, for $x \in X$ and $U \subseteq X$, the *return set* of x to U

$$R(x, U) = \{s \in S : sx \in U\}.$$

If $x \in U$, then trivially, $1_S \in R(x, U)$.

Recurrent points

- Definition

  $x \in X$ is *recurrent* if $R(x, U) \neq \{1_S\}$ holds for every neighbourhood U of x.

So in a discrete DS (X, t), x is recurrent iff for every neighbourhood U of x, there is some $n \geq 1$ such that $t^n x \in U$.

- Example

  Let S a compact topological group, operating on itself by left multiplication.
  For $x \in X$ and $U \subseteq X$, $R(x, U) = Ux^{-1}$.

  – If S is discrete (i.e.finite), then *no* $x \in S$ is recurrent.

  – Otherwise *every* $x \in S$ is recurrent.

Syndetic sets

- Definition

  $A \subseteq S$ is *syndetic* if there is some finite $e \subseteq S$ such that

  $$S = \bigcup_{x \in e} x^{-1}A$$

  (where $x^{-1}A = \{s \in S : xs \in A\}$). I.e. if S is covered by finitely many (backwards left) translates of A.

- Example

  In $(\omega, +)$, A is syndetic iff there is some $k \geq 1$ such that A intersects every interval of lenghth k, in ω.

Uniformly recurrent points

- Definition

  A point x of X is *uniformly recurrent* if for every neighbourhood U of x, the return set $R(x, U)$ is syndetic.

These points have a very pleasing characterization.

- Definition

  $Y \subseteq X$ is a *subsystem* of X if it is closed, non-empty, and $\{sy : y \in Y, s \in S\} \subseteq Y$.
  It is a *minimal subsystem* of X if it has no proper subsystem.

By Zorn's lemma, every DS has a minimal subsystem.

- **Theorem**

  $x \in X$ is uniformly recurrent iff $x \in M$, for some minimal subsystem M of X.
  (Hence uniformly recurrent points do exist.)

- Remark

  If S is not a finite group, then all uniformly recurrent points in DSs over S are recurrent.

B. βS and its operation on X. Characterizing (uniformly) recurrent points

We assume aquaintance with the following constructions.

- For an arbitrary set S, βS is the set of all ultrafilters on S, a compact Hausdorff space under the Stone topology. We identify S with a subset of βS.

- For $(S, \cdot)$ a semigroup, the multiplication of S extends to βS in such a way that the functions $x \mapsto sx$ (for $s \in S$) and $x \mapsto xq$ (for $q \in \beta S$) are continuous.

βS as a dynamical system over S

- A standard example

  The compact space βS is a DS over S, under the multiplication of points in βS with elements of S from the left – the universal dynamical system over S.

p-Limits

For a compact Hausdorff space X, $(x_s)_{s\in S}$ a family of points in X and $p \in \beta S$, the p-*limit* of $(x_s)_{s\in S}$ is the unique point

$$x = p - \lim_{s\in S} x_s$$

such that for every neighbourhood U of x there is some $A \in p$ such that $\{x_s : s \in A\} \subseteq U$.

$p \cdot x \in X$, for $x \in X$ and $p \in \beta S$

We apply the p-limit construction to a DS X over S:

- Definition

  For $p \in \beta S$ and $x \in X$, put

$$p \cdot x = px = p - \lim_{s \in S} sx.$$

– The map $p \mapsto px$ is continuous, for fixed $x \in X$.
– But $x \mapsto px$ is not necessarily continuous, for fixed $p \in \beta S$.
in particular, $(p, x) \mapsto px$ is not (jointly) continuous.
– The function $(p, x) \mapsto px$ defines, in fact, an action of βS on X (which extends the action of S), but X is not a DS over βS.

Using the $p \cdot x$ construction

- **Theorem**

  A point $x \in X$ is recurrent
  $\Leftrightarrow \quad px = x$ holds for some $p \in \beta S$, $p \neq 1_S$
  $\Leftrightarrow \quad ex = x$ holds for some $e \in \beta S$ satisfying $e^2 = e$ (an *idempotent* of βS).

- Example

  Assume $e = e^2 \in \beta S, e \neq 1_S$ (such an e exists if S is not a finite group) and $y \in X$. Then $x = ey$ satisfies $ex = x$, so x is recurrent.

C. A combinatorial property of syndetic sets

The following consequence of the Hales-Jewett theorem is the principal tool used below to prove existence of multiple recurrent points.

- **Theorem**

  Let $(S, \cdot)$ be a commutative semigroup, $A \subseteq S$ syndetic and e a finite subset of S. Then there are $s \in S$ and $d \geq 1$ such that

  $$\{sa^d : a \in e\} \subseteq A.$$

The van der Waerden property of syndetic sets

- Special case (van der Waerden)

  Let A be a syndetic subset of the semigroup $(\omega, +)$ and $k \geq 1$. Then there are $s \in \omega$ and $d \geq 1$ such that

  $$\{s, s+d, s+2d, \ldots, s+kd\} \subseteq A.$$

  I.e. A includes an arithmetic progression of length k.

Large subsets of S

- Remark

  In fact, these results hold for sets $A \subseteq S$ with a weaker property, the *piecewise syndetic* ones.

  There are other notions of largeness for subsets of a semigroup S (*thick*, *central*, *IP*), which will not be used in this survey.

A consequence for uniformly recurrent points

- Consequence

  Assume X is a DS over a commutative monoid S, and $x \in X$ is uniformly recurrent. Let $e \subseteq S$ be finite (and without loss of generality, $1_S \in e$); let U be a neighbouthood of x. Then there are $y \in U$ and $d \geq 1$ such that $\{a^d y : a \in e\} \subseteq U$.

Proof. The return set $A = R(x, U) = \{t \in S : tx \in U\}$ is syndetic; so pick $s \in S$ and $d \geq 1$ satisfying

$$\{sa^d : a \in e\} \subseteq A.$$

Then $y = sx$ is as required: for $a \in e$, we have

$$sa^d x = a^d sx = a^d y \in U.$$

In particular for $a = 1_S$, we have $1_S \cdot y = y \in U$.

The point of this proof is that there is some $d \geq 1$ such that

$$\{sa^d : a \in e\} \subseteq R(x, U).$$

D. Multiple recurrent points

- Definition

  Let X be a dynamical system over S and $e \subseteq S$. A point x of X is *e-recurrent* if for every neighbourhood U of x, there is some $d \geq 1$ such that $\{a^d \cdot x : a \in e\} \subseteq U$, i.e.

  $$\{a^d : a \in e\} \subseteq R(x, U).$$

  x is *multiple recurrent* if it is e-recurrent for every finite $e \subseteq S$.

Such points do not necessarily exist, even in minimal dynamical systems:

Simple remarks on e-recurrent points

- Remark

  If x is e-recurrent and $a \in e$, then x is recurrent in the discrete dynamical system (X, m_a) where $m_a(y) = a \cdot y$.
  Hence $x \in aX = m_a[X]$.
  – So an e-recurrent point x is a *common* recurrent point of the discrete systems (X, m_a), $a \in e$.
  – If $a, b \in e$ and $aX \cap bX = \emptyset$, then no point of X is e-recurrent.
  We will see that this situation is ruled out by commutativityof S.

Characterizing multiple recurrent points

- Notation

  For $x \in X$, $a \in S$, and $p \in \beta\omega$, we define

$$a^p x = p - \lim_{n \in \omega} a^n x.$$

- Proposition (S. K.)

  $x \in X$ is multiple recurrent iff there is $p \neq 1_S \in \beta\omega$ such that $a^p x = x$ holds for every $a \in S$.
  I.e. the recurrence of x with respect to all $m_a, a \in S$, is certified by a *common* $p \in \beta\omega$.

Existence results for multiple recurrent points

The following classical result is a precursor of the Balcar-Kalašek-Williams theorem below.

- **Theorem** (the Multiple Birkhoff Recurrence Theorem)

  Assume X is a compact metric space and F is a commuting finite set of continuous maps from X to itself. Then there exists a point x such that for every neighbourhood U of x there is $d \geq 1$ such that $\{f^d(x) : f \in F\} \subseteq U$.

The best known existence theorem for multiple recurrent points:

- **Theorem** (B.Balcar, P. Kalašek, S. W. Williams)

  Let S be commutative and X a minimal dynamical system. Moreover assume that S is countable and X has a countable base. Then the set MR of multiple recurrent points of X is dense in X.

  (A countable set $\{U_i : i \in I\}$ of dense open subsets of X is constructed such that $\bigcap_{i\in I} U_i \subseteq MR$. And $\bigcap_{i\in I} U_i$ is dense, by Baire's theorem.)

A discrete DS without e-recurrent points, $e = \{0, 1, 2\}$

- Example (Balcar, Kalasek)

  We consider $S = (\omega, +)$ and $X = \beta\omega$, the universal system over S, here $t(x) = x + 1$.
  – For $k \in \omega$ and $p \in X$, we put

  $$k \cdot p = p - \lim_{n \in \omega} kn.$$

  (Attention: $2 \cdot p \neq p + p$, in general).
  – Then for $x \in X$ and $a \in \omega$,

  $$a^p x = a \cdot p + x,$$

  where $+$ is the semigroup operation on $\beta\omega$ induced by addition on ω.

- Example (continued)

  – Put $e = \{0, 1, 2\}$, a finite subset of ω.
  Then $x \in \beta\omega$ is e-recurrent iff there is $p \neq 0$ in $\beta\omega$ satisfying

  $$x = p + x = 2 \cdot p + x.$$

  But it can be shown that this equation is unsolvable in $\beta\omega$.
  (Here S is commutative and countable, but $X = \beta\omega$ does not have a countable base.)

E. Equicontinuity and multiple recurrent points

- Notation

  For X a topological space, $C = (U_i)_{i \in I}$ a cover of X and $f : X \to X$, write

  $$f^{-1}C = (f^{-1}[U_i])_{i \in I},$$

  a cover of X (the preimage of C under f).
  So f is continuous iff for every open C, also $f^{-1}C$ is open iff for every open C, there is an open cover D such that $D \leq f^{-1}C$, i.e. D refines $f^{-1}C$.

- Definition

  A family $(f_k)_{k \in K}$ of functions from X to X is *equicontinuous* iff for every open cover C of X there is an open cover D satisfying

  $$D \leq f_k^{-1}C, \text{ for every } k \in K.$$

- Example

  Let (X, d) be a compact metric space and F a family of functions from X to X such that

  $$d(fx, fy) \leq d(x, y)$$

  holds for all $f \in F$ and all $x, y \in X$. Then F is equicontinuous.

- Example

  Assume P is compact Hausdorff and $f : P \times X \to X$ is (jointly) continuous. Then the family $(f_p)_{p \in P}$ where $f_p(x) = f(p, x)$ is equicontinuous.

- Proposition (S.K., probably folklore)

  Assume X is compact Hausdorff. The family $(f_k)_{k \in K}$ is equicontinuous iff there is a compact Hausdorff space $P \supseteq K$ and a (jointly) continuous function $f : P \times X \to X$ such that $f_p(x) = f(p, x)$ holds for every $k \in K$.

  E.g., put $P = \beta K$, K discrete, and

$$f(p, x) = p - \lim_{k \in K} f_k(x).$$

- **Theorem** (S.K.)

  Assume S is a commutative monoid, X a dynamical system over S and for every $a \in S$, the family $(m_{a^n})_{n\in\omega}$ (where $m_{a^n}(x) = a^n \cdot x$) is equicontinuous. Then every uniformly recurrent point of X is multiple recurrent.

  Sketch of proof.
  Let x be uniformly recurrent. Put

  $I = \{(e, U) : e \subseteq S \text{ finite}, 1_S \in e, U \text{ a neighbourhood of } x\}$,

  a directed set under

  $$(e, U) \leq (f, V) \Leftrightarrow e \subseteq f \text{ and } V \subseteq U.$$

  Let q an ultrafilter over I containing the sets

  $$A_i = \{j \in I : i \leq j\}.$$

- *Sketch of proof, continued.*

  – For $i = (e, U) \in I$, pick $x_i \in X$ and $1 \leq n_i \in \omega$ such that

$$\{a^{n_i} x_i : a \in e\} \subseteq U.$$

Then $q-\lim_{i \in I} x_i = x$; put $p = q-\lim_{i \in I} n_i$ (in $\beta\omega$). For every $a \in S$, we have $a^p x = x$ because

$$\mu_a : \beta\omega \times X \to X, \;\; \mu_a(r, x) = a^r x$$

is (jointly) continuous and hence commutes with q-limits.

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