Decomposition of Primes in Nonmaximal Orders

Ilaria DEL CORSO
Roberto DVORNICICH
Denis SIMON

Abstract

Let \( R \) be a Dedekind ring with quotient field \( K \), and let \( F(x) \in R[x] \) be an irreducible primitive polynomial. Let \( \alpha \) be a root of \( F \) and \( L = K(\alpha) \). If \( F \) is monic, then \( R[\alpha] \) is a monogenic \( R \)-order of \( L \). If \( F \) is not monic, then one can construct an \( R \)-order \( R_F \) of \( L \) that suitably generalizes \( R[\alpha] \). We show that, even if \( F \) is not monic, the classical invariants of \( R_F \) can be derived directly from the polynomial \( F \), precisely as in the case of monogenic orders.

1 Introduction

Let \( F(x) \in \mathbb{Z}[x] \) be a monic irreducible polynomial, \( \alpha \) be a root of \( F \) and \( L = \mathbb{Q}(\alpha) \). Then the ring \( \mathbb{Z}[\alpha] \) is a subgroup of finite index \( \text{Ind}(F) = [\mathcal{O}_L : \mathbb{Z}[\alpha]] \) of the ring of integers \( \mathcal{O}_L \) of \( L \), given by the formula \( \text{Disc}(L) \cdot \text{Ind}(F)^2 = \text{Disc}(F) \). Decomposing \( p\mathbb{Z}[\alpha] \) into primary ideals is an easy task, and a theorem of Kummer says that, if \( p \) is a prime not dividing \( \text{Ind}(F) \), then the factorization of \( p\mathcal{O}_L \) can be derived directly from the decomposition of \( p\mathbb{Z}[\alpha] \). Also, Dedekind’s criterion allows to test whether or not \( p \) divides \( \text{Ind}(F) \) and to enlarge \( \mathbb{Z}[\alpha] \) when it does. Of course, the best possible situation occurs when \( \mathcal{O}_L \) is monogenic (i.e., there exists \( \alpha \in \mathcal{O}_L \) such that \( \mathcal{O}_L = \mathbb{Z}[\alpha] \)) or, at least, when the index of the field \( K \) (i.e., the greatest common divisor of \( \text{Ind}(F) \) when \( F \) runs over all minimal polynomials of integral generators of \( L \)) is equal to 1. Unfortunately, this is not always the case, but one can decide if \( p \) divides the index of \( L \) in terms of the factorization type of \( p\mathcal{O}_L \) (see for instance [7, Ch. 4, Theorem 4.13]).

In this paper we deal with problems of similar type in the more general case of an irreducible polynomial \( F \) which is primitive without being necessarily monic, and replacing \( \mathbb{Z} \) and \( \mathbb{Q} \) by any Dedekind ring \( R \) and its quotient field \( K \). In this situation D. Simon [5] constructed an order \( R_F \) of \( \mathcal{O}_L \) which generalizes the order \( R[\alpha] \) when \( F \) is monic (it turns out that \( R_F = R[\alpha] \cap R[\alpha^{-1}] \), as shown in Proposition 2 below), and that continues to satisfy the index rule \( \text{Disc}(L) \cdot [\mathcal{O}_L : R_F]^2 = \text{Disc}(F) \). We show that, even in the case when \( F \) is not monic, the classical invariants of \( R_F \) can be derived from the polynomial \( F \), precisely as in the case of monogenic orders.

More precisely, in Section 3 we give the explicit primary decomposition of the ideals \( pR_F \), where \( p \) is a prime ideal of \( R \) (Theorem 1). In Section 4 we generalize the Dedekind criterion for
p-maximality to the ring $R_F$ (Theorem 2). In Section 5 we generalize Kummer’s theorem to the case when $p$ does not divide the index of $R_F$ in $\mathcal{O}_L$ (Theorem 3).

As an application, we introduce a generalized index for $L$, namely the greatest common divisors of the indexes $[\mathcal{O}_L : R_F]$ where $F$ runs over all primitive irreducible polynomials such that $L$ is generated over $K$ by a root $F$, and we show how to decide whether a prime $p$ divides this generalized index in terms of the factorization type of $p\mathcal{O}_L$ (Proposition 10). On one hand, it turns out that this generalized index does not give theoretical advantages respect to the classical index, except for one particular case (see Remark 3). On the other hand, our results show that the same kind of information that one obtains from an integral generator can be also obtained from a non-integral one, thereby giving some computational advantage.

## 2 Notation and basic properties

Let $R$ be a Dedekind ring and $K$ its field of fractions. Let $F(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ be a primitive irreducible polynomial with coefficients in $R$ such that $a_0 \neq 0$. We denote also by $F$ the homogenized polynomial $F(x, y) = a_0x^n + a_1x^{n-1}y + \cdots + a_ny^n$. Define

$$
T_0 = a_0, \quad T_1 = a_0x + a_1, \quad \ldots, \quad T_{n-1} = a_0x^{n-1} + \cdots + a_{n-1}.
$$

Let $L$ be the field $K[x]/(F(x))$. We denote by $\alpha$ the image of $x$ in this projection ($\alpha$ is a root of $F$). We consider

$$R_F = R \oplus T_1(\alpha)R \oplus \cdots \oplus T_{n-1}(\alpha)R.$$

This $R$-module is an order in $L$ (see [5]). In particular, it is contained in the maximal order $\mathcal{O}_L$ of $L$.

Let $\mathfrak{B} = T_0R_F + T_1(\alpha)R_F + \cdots + T_{n-1}(\alpha)R_F$. This is an invertible ideal of $R_F$ (see [6]), and an $R$-basis of it is given by $\mathfrak{B} = T_0R \oplus T_1(\alpha)R \oplus \cdots \oplus T_{n-1}(\alpha)R$. Similarly, let $\mathfrak{A} = \alpha\mathfrak{B} = \alpha T_0R_F + \alpha T_1(\alpha)R_F + \cdots + \alpha T_{n-1}(\alpha)R_F$. We have $\mathfrak{A} = \alpha T_0R \oplus \alpha T_1(\alpha)R \oplus \cdots \oplus \alpha T_{n-1}(\alpha)R$. In [6] it was proved that $\mathfrak{A} + \mathfrak{B} = R_F$. From this we get $(\alpha) = \mathfrak{A}\mathfrak{B}^{-1}$ and therefore $\mathfrak{A}$ and $\mathfrak{B}$ are the numerator and the denominator of $\alpha$.

Since $R$ is a Dedekind domain and $F$ is primitive, the following proposition holds (see for instance [4, Cor.28.4]):

**Proposition 1** $F$ divides a polynomial $U \in R[x]$ in $K[x]$ if and only if $F$ divides $U$ in $R[x]$.

It follows, in particular, that $R[x]/(F(x)) \cong R[\alpha]$. Moreover, we have the following characterization of $R_F$:

**Proposition 2**

$$R_F = R[\alpha] \cap R[\alpha^{-1}].$$

**Proof.** We have $T_1(\alpha) = a_0\alpha^1 + \cdots + a_i = -(a_{i+1}\alpha^{-1} + \cdots + a_n\alpha^{-n})$, so the inclusion $\subseteq$ is clear. To prove the converse inequality, let $P, Q \in R[x]$ be such that $P(\alpha) = Q(\alpha^{-1})$, and let $m = \deg P$.

Let $P = c_0x^m + \cdots + c_m$. We first prove that, if $m > 0$, then $a_0 \mid c_0$. In fact, letting $Q'(x) = x^{\deg Q}Q(x)$, we have that the integer polynomial $U = x^{\deg Q}P - Q'$ vanishes at $\alpha$, and therefore $F$ divides $U$ in $K[x]$; by Proposition 1, $F$ divides $U$ also in $R[x]$, whence $a_0 \mid c_0$. 

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We prove that $P \in R_F$ by induction on $m$. If $m = 0$, the inclusion is trivial. For $m > 0$, let $c_0 = a_0b_0$. If $m \leq n - 1$, then $P(\alpha) - b_0T_m(\alpha) \in R[\alpha] \cap R[\alpha^{-1}]$ and $P - b_0T_m$ has degree $< m$, so the inclusion follows by the induction hypothesis. Similarly, if $m \geq n$, then $P(\alpha) = P(\alpha) - b_0\alpha^{m-n}F(\alpha)$ and $P - b_0x^{m-n}F$ has degree $< m$, so the inclusion follows again.

**Proposition 3** For $d \geq 0$, let $R_d[x, y]$ denote the $R$–module of homogeneous polynomial of degree $n$ with coefficients in $R$. We have

$$\mathfrak{B}^d\{b(\alpha, 1) \mid b \in R_d[x, y]\} = R_F$$

**Proof.** The first inclusion is clear since $\mathfrak{B}\alpha = \mathfrak{A} \subset R_F$. For the converse it is enough to observe that for $b = x^d$ and $b = y^d$ we obtain that $\mathfrak{A}^d$ and $\mathfrak{B}^d$ are in this product. Since $\mathfrak{A}$ and $\mathfrak{B}$ are coprime, we have the conclusion. 

**Remark 1** Given an algebraic number $\alpha$ over $K$ of degree $n$, it is not always possible to find an irreducible primitive polynomial of degree $n$ and with coefficients in $R$ which has $\alpha$ as a root. However, if $R$ is a principal ideal domain, it is straightforward to see that an irreducible primitive polynomial of degree $n$ vanishing at $\alpha$ exists, and one can always reduce to this case by localizing $R$ at a prime ideal. When $R$ is not principal, and no irreducible primitive polynomial of degree $n$ exists, one can also use the generalized definitions of $R_F$, $\mathfrak{A}$ and $\mathfrak{B}$ given by D. Simon in [6], and we suspect that, with these definitions, almost all the results of this paper remain valid.

**Example 1** Let $R = \mathbb{Q}[\sqrt{10}]$ and let $\alpha$ be a root of $x^2 + \frac{\sqrt{10}}{2}$. The factorizations in $R$ of the ideals (2) and $(\sqrt{10})$ are $(2) = p^2$ and $(\sqrt{10}) = pq$ where $p = (2, \sqrt{10})$ and $q = (5, \sqrt{10})$ are not principal. Any quadratic polynomial $F \in R[x]$ vanishing at $\alpha$ has the form $F = cx^2 + c\sqrt{10}$, with $(c) = pI$ where $I$ is a proper ideal such that $pI$ is principal. The ideal generated by the coefficients of $F$ must therefore be exactly $(pI, qI) = I \neq 1$.

**Proposition 4**

$$\text{disc } R_F = \text{disc } F$$

**Proof.** We have

$$\left( \begin{array}{c} 1 \\ T_1(\alpha) \\ \vdots \\ T_{n-1}(\alpha) \end{array} \right) = \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ \ast & a_0 & 0 & 0 \\ \ast & \ast & \ast & 0 \\ \ast & \ast & \ast & a_0 \end{array} \right) \left( \begin{array}{c} 1 \\ \alpha \\ \ldots \\ \alpha^{n-1} \end{array} \right),$$

whence, letting $\sigma_1, \ldots, \sigma_n$ be the embeddings of $K$ in some algebraic closure $\bar{K}$ of $K$,

$$\text{disc } R_F = \text{disc } \{1, T_1(\alpha), \ldots, T_{n-1}(\alpha)\} = a_0^{2n-2} \text{disc } \{1, \sigma_1(\alpha), \ldots, \sigma_n(\alpha)\} = a_0^{2n-2} \prod_{i \neq j} (\sigma_i(\alpha) - \sigma_j(\alpha)) = \text{disc } F.$$

□
3 The primary decomposition of prime ideals

Let $p$ be a prime ideal of $R$ (the interesting case is when $p$ divides $a_0$). We use the notation $x \mapsto \bar{x}$ for the reduction mod $p$. Let

$$F(x, y) = \prod_i F_i(x, y)^{e_i}$$

be the factorization into irreducible factors of $F(x, y)$ modulo $p$. Let $f_i$ be the degree of $F_i$. Fix a choice $F_i(x, y)$ for a lift of $F_i(x, y)$ in $R[x, y]$, homogeneous of degree $f_i$. Define

$$p_i = pR_F + \mathfrak{B}^{F_i}(\alpha, 1).$$

It is easily seen that $p_i$ does not change if we multiply $F_i$ by a unit in $R/p$ and is independent of the choice of the lift $F_i$. By Proposition 3, $p_i$ is an integral ideal of $R_F$. We now define

$$q_i = pR_F + \mathfrak{B}^{e_iF_i}(\alpha, 1).$$

Lemma 1 If $F_i \neq uy$ (with $u \in (R/p)^*$) then $p_i + \mathfrak{B} = R_F = q_i + \mathfrak{B}$.

If $F_i = uy$ then $p_i + \mathfrak{A} = R_F = q_i + \mathfrak{A}$. If $F_i = ux$ then $p_i \subset \mathfrak{A}$.

Proof. Assume first that $F_i \neq uy$ (with $u \in (R/p)^*$). Let $c$ be the coefficient of $F_i$ corresponding to $x^{h_i}$. We can assume without loss of generality that $c = 1$. We have

$$p_i + \mathfrak{B} = pR_F + \mathfrak{B}^{F_i}(\alpha, 1) + \mathfrak{B}$$

$$= pR_F + \mathfrak{A}^{F_i} + \mathfrak{B}$$

$$= pR_F + R_F$$

$$= R_F.$$ 

Assume now that $F_i = uy$ (with $u \in (R/p)^*$). We can still assume that $F_i = y$. Hence $p_i = pR_F + \mathfrak{B} \supset \mathfrak{B}$. The proof is similar for the remaining cases. \(\square\)

Proposition 5 Let $I = pR_F + \mathfrak{B}^{b_iF_i}(\alpha, 1)$ with $1 \leq b_i < e_i$. The quotient ring $R_F/I$ is isomorphic to $(R/p)[x]/\bar{F}_i^{b_i}(x, 1)$ when $\bar{F}_i \neq uy$ with $u\alpha$ a unit in $R/p$, and is isomorphic to $(R/p)[y]/\bar{y}^{b_i}$ otherwise. The norm of the ideal $I$ is $N_{R_F/R(I)} = p^{b_i}$. In particular, the $p_i$ are prime ideals, the $q_i$ are $p_i$-primary and their norms are given by $N_{R_F/R(p_i)} = p^{f_i}, N_{R_F/R(q_i)} = p^{e_i f_i}$.

Proof. We consider first the case $F_i \neq uy$. We have $I \supset q_i$. But, by Lemma 1, $q_i + \mathfrak{B} = R_F$, hence $I + \mathfrak{B} = R_F$. Consider the ring homomorphism

$$\phi : R_F \to (R/p)[x]/(\bar{F}_i^{b_i}(x, 1))$$

defined by $\phi(g(\alpha)) = \bar{g}(x)$. Since $R_F \subset R[\alpha] \cong R[x]/(F(x))$ and $\bar{F}_i^{b_i}$ divides $\bar{F}_i$, this map is well defined. As for its kernel, it is clear that $I \subset \ker \phi$. Conversely, let $g \in R[\alpha]$ such that $g(\alpha) \in \ker \phi$. We have thus $g(x) = F_i^{b_i}(x, 1)q(x) + r(x)$ in $R[x]$ with $q(x) \in R[x]$ and $r(x) \in p[x]$. Multiplying by $\mathfrak{B}^{deg g}$ and evaluating at $\alpha$, we get $\mathfrak{B}^{deg g}g(\alpha) = \mathfrak{B}^{f_i b_i}F_i^{b_i}(\alpha, 1) \cdot \mathfrak{B}^{deg g - f_i b_i}q(\alpha) + \mathfrak{B}^{deg g - r}(\alpha)$, which is a relation between ideals of $R_F$. It follows that $\mathfrak{B}^{deg g}g(\alpha) \subset I$ and, since obviously $Ig(\alpha) \subset I$ and $\mathfrak{B}^{deg g} + I = R_F$, we obtain that $g(\alpha) \in I$. We have therefore proved that $\ker \phi = I$. To prove
surjectivity, let $\gamma \in I$ and $\beta \in \mathfrak{B}$ be such that $\gamma + \beta = 1$. We have $\phi(\beta) = 1$ and $\phi(\alpha \beta) = \bar{x}$, which implies that $\phi$ is onto. The other claims for the case $F_i \neq uy$ are now immediate.

Consider now the case $F_i = uy$, where $u$ is a unit in $R/p$. We can assume that $u = 1$. Let $G$ be the reciprocal polynomial of $F$. We know from Proposition 2 that $R_F = R_G$. It is then possible to work with $G$ instead of $F$. The factor $F_i = y$ of $F$ modulo $p$ corresponds to the factor $G_i = x$ of $G$. We can apply the results of the first case to $G$, and the proposition is proved in all cases. □

**Theorem 1** The decomposition of $pR_F$ into primary ideals is given by

$$pR_F = \bigcap_i q_i = \prod_i q_i$$

with $q_i = pR_F + \mathfrak{B}^{e_i f_i} F_i^{e_i}(\alpha, 1)$.

**Proof.** By Proposition 5 we know that the ideals $q_i$ are $p_i$-primary and therefore pairwise coprime, hence $\bigcap_i q_i = \prod_i q_i = \prod(pR_F + \mathfrak{B}^{e_i f_i} F_i^{e_i}(\alpha, 1)) \subset pR_F + \mathfrak{B}^{\sum e_i f_i} \prod F_i^{e_i}(\alpha, 1)$. Since $\sum e_i f_i = n$, we have $pR_F \subset pR_F + \mathfrak{B}^{n} \prod F_i^{e_i}(\alpha, 1)$. From the definition of the $F_i$ we have that $\prod F_i^{e_i}(x, y) - F(x, y) \in pR[x, y]$. Since $F(\alpha, 1) = 0$, by Proposition 3 we get $\mathfrak{B}^{n} \prod F_i^{e_i}(\alpha, 1) \subset pR_F$, and hence $pR_F = \prod q_i$. □

**Corollary 1** The ideals $q_i$ are invertible. For each $i$, the ideal $p_i$ is invertible if and only if $p_i^{e_i} = q_i$.

**Proof.** Since $p$ is invertible, Theorem 1 says immediately that the $q_i$ are invertible.

The identity $p_i^{e_i} = q_i$ implies that $p_i$ is also invertible. Assume now that $p_i$ is invertible. In this case we have $\mathcal{N}_{R_F/R}(p_i) = \mathcal{N}_{R_F/R}(p_i)^{e_i}$ and, by Proposition 5, we get $\mathcal{N}_{R_F/R}(p_i)^{e_i} = p^{e_i f_i} = \mathcal{N}_{R_F/R}(q_i)$. But $p_i^{e_i} \subset q_i$, hence $p_i^{e_i} = q_i$. □

We can now classify the ideals containing $pR_F$ and their inclusion relations.

**Proposition 6** The ideals of $R_F$ containing $pR_F$ are in one-to-one correspondence with the divisors of $F$, where, if $\bar{P}|\bar{F}$, the corresponding ideal is $pR_F + \mathfrak{B}^{\deg P} P(\alpha, 1)$ for any homogeneous lift $P$ of $\bar{P}$.

If $I_1 = pR_F + \mathfrak{B}^{b_1} P_1(\alpha, 1)$ and $I_2 = pR_F + \mathfrak{B}^{b_2} P_2(\alpha, 1)$ where $\bar{P}_1|\bar{F}$ and $\bar{P}_2|\bar{F}$, then:

(i) $I_1 \subset I_2 \iff \bar{P}_2|\bar{P}_1$;

(ii) $I_1 + I_2 = pR_F + \mathfrak{B}^{\deg D} D(\alpha, 1)$, where $D$ is any homogeneous lift of the greatest common divisor $D$ of $P_1$ and $P_2$;

(iii) $I_1 \cap I_2 = pR_F + \mathfrak{B}^{\deg M} M(\alpha, 1)$, where $M$ is any homogeneous lift of the least common multiple $M$ of $P_1$ and $P_2$.

**Proof.** Via projection, the ideals of $R_F$ containing $pR_F$ are in one-to-one correspondence with the ideals of $R_F/pR_F \cong \prod_i R_F/q_i$. The ideals of the last ring are products of ideals of $R_F/q_i$ and, since projection preserves products of ideals, we get that each ideal $I$ containing $pR_F$ must be of the form $\prod q_i$, where $q_i \subset q_i'$ for all $i$.

By Proposition 5, the ideals $q_i'$ containing $q_i$ are of the form $q_i' = pR_F + \mathfrak{B}^{b_i f_i} F_i^{b_i}(\alpha, 1)$ with $0 \leq b_i \leq e_i$. Now, we have clearly $\prod_i(pR_F + \mathfrak{B}^{b_i f_i} F_i^{b_i}(\alpha, 1)) \subset pR_F + \sum_i \mathfrak{B}^{b_i f_i} F_i^{b_i}(\alpha, 1)$. On the other hand,

$$\prod q_i' = \prod_i(pR_F + \mathfrak{B}^{b_i f_i} F_i^{b_i}(\alpha, 1)) \supset pR_F(\sum_i \prod_{j \neq i} q_j') + \prod_i \mathfrak{B}^{b_i f_i} F_i^{b_i}(\alpha, 1) = pR_F + \prod_i \mathfrak{B}^{b_i f_i} F_i^{b_i}(\alpha, 1).$$
Hence the ideals $I$ containing $pR_F$ are exactly those of the form $I = pR_F + \prod_{i} \mathcal{B}^{b_i} F^{b_i}_i(\alpha, 1)$ and, since the products $\prod_{i} F^{b_i}_i$ represent all divisors of $F$, the first statement of the proposition follows.

(i) Let $I_1 = \prod_{i} q_{i,1} = \prod_{i}(pR_F + \mathcal{B}^{b_i} F^{b_i}_i(\alpha, 1))$ and $I_2 = \prod_{i} q_{i,2} = \prod_{i}(pR_F + \mathcal{B}^{b_i} F^{b_i}_i(\alpha, 1))$. We have $I_1 \subset I_2$ if and only if $q_{i,1} \subset q_{i,2}$ for all $i$, as can be seen by localizing at $q_i$; by Proposition 5, this is true if and only if $b_{i,2} \leq b_{i,1}$ for all $i$.

(ii) and (iii) are easy consequences of (i), since $I_1 + I_2$ is the smallest ideal containing $I_1$ and $I_2$ and $I_1 \cap I_2$ is the greatest ideal contained in $I_1$ and $I_2$.

The following elementary proposition appears to be quite useful when we deal with ideals dividing $pR_F$ in the next section.

**Proposition 7** Let $P \in R[x, y]$ be any homogeneous polynomial and let $I = pR_F + \mathcal{B}^{\deg P} P(\alpha, 1)$.

(i) Then the ideal $I$ can be written canonically as $I = pR_F + \mathcal{B}^{\deg P_0} P_0(\alpha, 1)$, where $P_0$ is any homogeneous polynomial satisfying $(\bar{P}, \bar{F}) = P_0$.

(ii) Let $0 \leq b_i \leq e_i$. We have $pR_F + \mathcal{B}^{b_i} F^{b_i}_i(\alpha, 1)|pR_F + \mathcal{B}^{\deg P} P(\alpha, 1)$ if and only if $\bar{F}^{b_i}|\bar{P}$.

**Proof.** (i) Let $I = \prod_{i} q_{i}$, where $q_{i} \supset q_{i}$. Consider the projection $R_F \to R_F/q_i$. Since $R_F/q_i \cong (R/p)[x]/(F^{e_i}_i(x, 1))$ or $R_F/q_i \cong (R/p)[y]/(y^{e_i})$ (see Proposition 5), the ideal $I$ corresponds to $(\bar{P}(x, 1))/(\bar{F}^{e_i}_i(x, 1))$ or to $(\bar{P}(1, y))/(\bar{y}^{e_i})$. Now, $(\bar{P}(x, 1))/(\bar{F}^{e_i}_i(x, 1)) = (\bar{P}(x, 1), \bar{F}^{e_i}_i(x, 1))/(\bar{F}^{e_i}_i(x, 1))$ and $(\bar{P}(1, y))/(\bar{y}^{e_i}) = (\bar{P}(1, y), y^{e_i})/(\bar{y}^{e_i})$, whence the result follows. (ii) is an immediate consequence of (i) and Proposition 6.

Now, we give a proposition and its corollary, which will not be needed in the rest of this paper, but which give a very practical way to find generators for the ideals containing $pR_F$.

**Proposition 8** Let $\mathcal{D}, I$ be integral ideals of $R_F$, with $\mathcal{D}$ invertible. Then there exists an integral ideal $\mathcal{C}$ of $R_F$ in the same ideal class of $\mathcal{D}$ such that $\mathcal{C} + I = 1$. In particular, there exists an integral ideal $\mathcal{C}$ of $R_F$ such that $\mathcal{C} \mathcal{B} = (\beta)$ is principal and $\mathcal{C} + pR_F = 1$.

**Proof.** We have $\mathcal{C} + I = 1$ if and only if $\mathcal{C} + \sqrt{I} = 1$. Write $\sqrt{I} = \cap_{i} r_i = \prod_i r_i$, where $r_i$ are distinct prime ideals of $R_F$. It follows that $\mathcal{C}$ is coprime to $I$ if and only if it is coprime to each of the $r_i$. Since $\mathcal{D}^{-1} = 1$, we can find, for each $i$, an element $x_i \in \mathcal{D}^{-1}$ such that $x_i\mathcal{D} \not\subset r_i$. Now, since the $r_i$ are prime, $\mathcal{D} \cap \bigcap_{j \neq i} r_j$, and hence there exist elements $y_i \not\in r_i, y_i \in \bigcap_{j \neq i} r_j$. Letting $z_i = x_i y_i$, we have $z_i\mathcal{D} \not\subset r_i$ and $z_i\mathcal{D} \subset \bigcap_{j \neq i} r_j$. Finally, $z = \sum z_i$ is an element of $\mathcal{D}^{-1}$ such that $z\mathcal{D}$ is coprime to $r_i$ for all $i$.

**Corollary 2** Let $\beta \in \mathcal{B}$ be such that $(\beta) = \mathcal{C}\mathcal{B}$ where $\mathcal{C}$ is an ideal of $R_F$ with $\mathcal{C} + pR_F = 1$. Then $\beta$ and $\alpha\beta$ are in $R_F$ and $pR_F + \mathcal{B}^{\deg P} P(\alpha, 1) = (p, P(\alpha\beta, \beta))$.

**Proof.** By Theorem 6, $\beta$ and $\alpha\beta$ are in $R_F$. For each ideal $I$ and each non-negative integer $m$, we have $pR_F + I = pR_F + (p + \mathcal{C}^m)I \subset pR_F + \mathcal{C}^m I \subset pR_F + I$, and therefore $pR_F + I = pR_F + \mathcal{C}^m I$. It follows that $pR_F + \mathcal{B}^{\deg P} P(\alpha, 1) = pR_F + \mathcal{C}^{\deg P} \mathcal{B}^{\deg P} P(\alpha, 1) = (p, P(\alpha\beta, \beta))$. □
4 The Dedekind Criterion

Now we generalize [3, Ch. 2.4], to be able to decide whether the order \( R_F \) is \( p \)-maximal or not, and to enlarge it when it is not. The main result is the generalization of the Dedekind Criterion. For this, we need some definitions.

Let \( O \subset O_L \) be an order in \( L \). Then \( O_L/O \) is a finitely generated torsion \( R \)-module. By [3, Thm 1.2.30], there exist unique integral ideals \( \mathfrak{d}_1, \ldots, \mathfrak{d}_r \), with \( 0 \neq \mathfrak{d}_1 \subset \mathfrak{d}_2 \subset \cdots \subset \mathfrak{d}_r \neq R \), such that

\[
O_L/O \cong (R/\mathfrak{d}_1) \oplus \cdots \oplus (R/\mathfrak{d}_r).
\]

The \textit{index–ideal} \( [O_L : O] \) is by definition the product of the ideals \( \mathfrak{d}_i \). When the base ring \( R \) is \( \mathbb{Z} \), this definition coincides with the usual index if we identify an ideal of \( \mathbb{Z} \) with its positive generator.

We say that an order \( O \) in \( L \) is \( p \)-\textit{maximal} if the index–ideal \( [O_L : O] \) is not divisible by \( p \).

The \( p \)-\textit{radical} \( I_p \) of \( O \) at \( p \) is defined as the radical of the ideal \( p \), namely,

\[
I_p = \sqrt{pO} = \{ x \in O \mid \exists m \geq 1 \text{ such that } x^m \in pO \}.
\]

The \( p \)-radical is a useful tool for enlarging an order \( O \) when it is not \( p \)-maximal, as we can see in Zassenhaus’s theorem: (see [3, Prop. 2.4.4]).

**Proposition 9 (Zassenhaus’s theorem)** Set \( O' = \{ x \in L \mid xI_p \subset I_p \} \). Then

1. \( O' \) is an order in \( L \) containing \( O \),
2. \( O' = O \) if and only if \( O \) is \( p \)-maximal,
3. if \( O' \neq O \), then \([O' : O] = p^k \) with \( 1 \leq k \leq n \).

**Theorem 2 (Dedekind Criterion)** Let \( h = \sum_i f_i \) be the degree of \( H_1 = \prod_i F_i \).

1. The \( p \)-radical of \( R_F \) at \( p \) is given by
   \[
   I_p = pR_F + \mathfrak{B}^h H_1(\alpha, 1)
   \]
2. Let \( \xi \in p^{-1} \setminus R \) (i.e. a uniformizer of \( p \)), \( H_2 \) be a lift of \( \tilde{F}/H_1 \) and \( H_3 = \xi(H_1H_2 - F) \in R[x, y] \). Let also \( G \) be the gcd of \( H_1, H_2 \) and \( H_3 \) in \( R/p[x, y] \), and \( g = \deg G \). Finally, let \( U \) be a lift of \( \tilde{F}/G \) in \( R[x, y] \) of degree \( n - g \). Then the order given by Zassenhaus’s theorem starting with \( O = R_F \) is equal to
   \[
   O' = R_F + p^{-1} \mathfrak{B}^{n-g} U(\alpha, 1).
   \]
   We have \([O' : R_F] = p^g \). In particular, \( R_F \) is \( p \)-maximal, if and only if \((H_1, H_2, H_3) = 1 \).

**Proof.** (1) This statement is a consequence of Proposition 6 and of the equality \( I_p = \sqrt{pR_F} = \sqrt{\bigcap_i \mathfrak{q}_i} = \bigcap_i \sqrt{\mathfrak{q}_i} = \bigcap_i p_i \).

(2) Clearly \( pO' \) is an ideal of \( R_F \) containing \( pR_F \), and by Proposition 6 we may write \( pO' = pR_F + \mathfrak{B}^{\deg P} P(\alpha, 1) \) for some homogeneous polynomial \( P \) such that \( \tilde{P} \) is a divisor of \( \tilde{F} \). Since \( pR_F \) is invertible, we have to prove that we can choose \( P \) such that \( \tilde{P} = U \).

From now on we shall follow closely the lines of the proof of [2, Thm 6.1.4]. The ideal \( pO' \) is characterized as the set of elements \( \gamma \) such that \( \gamma \in I_p \) and \( \gamma \mathfrak{B}^{h} H_1(\alpha, 1) \subset I_p \). We remark first that Proposition 6 and part (1) of this theorem give immediately \( pR_F + \mathfrak{B}^{\deg P} P(\alpha, 1) \subset I_p \) if and only if \( H_1 \mid \tilde{P} \). Hence, the polynomial \( P \) is characterized as the smallest one such that \( H_1 \mid \tilde{P} F \) and

\[
\mathfrak{B}^{h} H_1(\alpha, 1)(pR_F + \mathfrak{B}^{\deg P} P(\alpha, 1)) \subset pI_p.
\] (1)
But \( \mathfrak{B}^h H_1(\alpha, 1) p R_F \subset p^2 R_F + \mathfrak{B}^h H_1(\alpha, 1) p R_F = p I_p \), hence (1) is equivalent to
\[
\mathfrak{B}^{h + \deg \frac{P}{1}} H_1(\alpha, 1) P(\alpha, 1) \subset p I_p .
\] (2)

We note that (2) implies that
\[
\mathfrak{B}^{h + \deg \frac{P}{1}} H_1(\alpha, 1) P(\alpha, 1) \subset p R_F ,
\]
and by Proposition 7, we get \( F|H_1 \), that is \( H_2|P \). Let \( P = A_3 H_2 + B_1 \), where \( A_3 \) and \( B_1 \) are homogeneous polynomials respectively in \( R[x, y] \) and \( p[x, y] \). Now, we have \( H_1 P = H_1 H_2 A_3 + H_1 B_1 + F A_3 \), and (2) is equivalent to
\[
\mathfrak{B}^{h + \deg \frac{P}{1}} ((H_1 H_2 - F) A_3 + H_1 B_1)(\alpha, 1) \subset p I_p ,
\]
or after a multiplication by \( \xi \) to
\[
\mathfrak{B}^{h + \deg \frac{P}{1}} (H_3 A_3 + H_1 \xi B_1)(\alpha, 1) \subset I_p .
\] (3)

We use again the fact that \( I_p = p R_F + \mathfrak{B}^h H_1(\alpha, 1) \), which implies that \( \mathfrak{B}^{h + \deg \frac{P}{1}} (H_1 \xi B_1)(\alpha, 1) \subset \mathfrak{B}^h H_1(\alpha, 1) \subset I_p \), and (3) is now equivalent to
\[
\mathfrak{B}^{h + \deg \frac{P}{1}} (H_3 A_3)(\alpha, 1) \subset p R_F + \mathfrak{B}^h H_1(\alpha, 1) .
\] (4)

By Proposition 7, this is equivalent to \( \tilde{H}_1|H_3 \tilde{A}_3 \), or simply to \( \tilde{H}_4|\tilde{A}_3 \) where \( \tilde{H}_4 = \tilde{H}_1/(\tilde{H}_1, \tilde{H}_3) \). Putting together the different conditions, we see that (1) is equivalent to \( H_1 \tilde{H}_2|P \).

Summarizing, the two conditions on \( P \) mean that \( \tilde{P} \) is the least common multiple of \( H_1 \) and \( \tilde{H}_4 \tilde{H}_2 \). Now,
\[
\operatorname{lcm}(\tilde{H}_1, \tilde{H}_4 \tilde{H}_2) = \tilde{H}_4 \operatorname{lcm}((\tilde{H}_1, \tilde{H}_3), \tilde{H}_2) = \frac{\tilde{H}_1}{(\tilde{H}_1, \tilde{H}_3)} \frac{(\tilde{H}_1, \tilde{H}_3) \tilde{H}_2}{(\tilde{H}_1, \tilde{H}_2, \tilde{H}_3)} = \frac{\tilde{F}}{(\tilde{H}_1, \tilde{H}_2, \tilde{H}_3)} = \tilde{U} .
\]

\( \square \)

5 The case when \( p \) is coprime to the index

Let \( \text{Ind}(R_F) = [\mathcal{O}_L : R_F] \) be the index–ideal of \( R_F \) in \( \mathcal{O}_L \) as it is described in section 4.

**Theorem 3** If \( p \) is coprime to \( \text{Ind}(R_F) \), the factorization of \( p R_F \) into prime ideals is given by
\[
p R_F = \prod_i p_i^{f_i} \]
with \( p_i = p R_F + \mathfrak{B}_i F_i(\alpha, 1) \). Moreover,
\[
p \mathcal{O}_L = \prod_i (p_i \mathcal{O}_L)^{f_i}
\]
and \( N_{\mathcal{O}_L/R}(p_i) = p_i^{f_i} \).

**Proof.** Setting \( S = R \setminus p \) we have \( (S^{-1} R)_F = S^{-1} R_F = S^{-1} \mathcal{O}_L \), and therefore \( S^{-1} R_F \) is a Dedekind domain. The decomposition of \( S^{-1} p \) in \( S^{-1} R_F \) is \( S^{-1} p = \bigcap S^{-1} q_i \), where the prime ideals \( S^{-1} p_i \) are invertible and the \( S^{-1} q_i \) are \( S^{-1} p_i \)-primary. By Corollary 1 we get that \( S^{-1} q_i = S^{-1} p_i^{e_i} \) and, contracting these ideals to \( R_F \) and to \( \mathcal{O}_L \), we obtain \( q_i = p_i^{e_i} \) and \( q_i \mathcal{O}_L = (p_i \mathcal{O}_L)^{e_i} \) (see for instance [1, Proposition 4.8]). The factorizations of \( p R_F \) and \( p \mathcal{O}_L \) now follow from Theorem 1.

Finally, \( R_F/p_i \cong \mathcal{O}_L/p_i \mathcal{O}_L \) and the last statement follows from Proposition 5. \( \square \)
6 Applications

In this section, we consider the standard case $K = \mathbb{Q}$ and $R = \mathbb{Z}$, and we write $\mathbb{Z}_F$ for the ring $R_F$.

**Definition 1** The *generalized index* of a number field $L$ is the greatest common divisor of the indices $[O_L : \mathbb{Z}_F]$ as $\alpha$ runs over all generators of $L$ over $\mathbb{Q}$ and $F$ its primitive irreducible minimal polynomial.

The usual definition of the index of $L$ is the same, except that $\alpha$ is restricted to algebraic integers. It is not difficult to see that the generalized index of $L$ is a divisor of the index of $L$. The following proposition is the analogue of [7, Ch. 4, Theorem 4.13].

**Proposition 10** Let $L$ be a number field, $p$ be a rational prime and let

$$pO_L = p_1^{e_1} \cdots p_t^{e_t}$$

be the factorization of $pO_L$ into prime ideals. Let also $f_1$ be the residual degree of $p_i$ for $i = 1, \ldots, t$. Then $p$ does not divide the generalized index of $L$ if and only if there exist distinct irreducible homogeneous polynomials $F_1, \ldots, F_t \in \mathbb{F}_p[x,y]$ with degrees $f_1, \ldots, f_t$, respectively.

**Remark 2** To say that the polynomials $\bar{F}_1$ and $\bar{F}_j$ are distinct means in this context, that there is no unit $u \in \mathbb{F}_p$ such that $\bar{F}_i = u\bar{F}_j$.

**Proof.** If $p$ does not divide the generalized index of $L$, then there exists a generator $\alpha \in L$ such that $p \nmid [O_L : \mathbb{Z}_{F_\alpha}]$ (where $F_\alpha \in \mathbb{Z}[x]$ is a primitive irreducible polynomial having $\alpha$ as a root). Denote again by $F_\alpha$ the homogenized polynomial of $F_\alpha$ with the same degree; by Theorem 3, the factors of $F_\alpha$ modulo $p$ are homogeneous polynomials with the required properties.

To prove the converse, let $F_1, \ldots, F_t$ be homogeneous lifts to $\mathbb{Z}[x,y]$ of $\bar{F}_1, \ldots, \bar{F}_t$.

If none of the $\bar{F}_i$ is equal to $y$ (up to some invertible element mod $p$), then we can apply the standard result to $\bar{F}_i(x,1)$, for example [7, Ch. 4, Theorem 4.13]. In this case, we obtain that $p$ does not divide the index of $L$, and therefore it can not divide the generalized index of $L$ either.

Assume now that $\bar{F}_i(x,1)$ is not a divisor of $L$, and let $F_i$ be an element of $O_L$ such that $\beta \equiv 1 \pmod{p_i^2}$ and $\beta \equiv 1 \pmod{p_i}$ for $i \neq i_0$. For $i = 1, \ldots, t$, $i \neq i_0$, let $\gamma_i$ be an element of $O_L$ such that $\gamma_i \not\equiv 0 \pmod{p_i}$ and

$$F_i(\gamma_i, 1) \equiv 0 \pmod{p_i}.$$  

Such a root of $\bar{F}_i$ does exist since the degree of $\bar{F}_i$ is equal to the residual degree of $p_i$. Furthermore, the polynomial $\bar{F}_i$ is irreducible mod $p_i$, and its discriminant is coprime to $p_i$, hence to $p_i$, hence $\bar{F}_i(\gamma_i, 1) \not\equiv 0 \pmod{p_i}$, and by Hensel Lemma we can also assume that

$$F_i(\gamma_i, 1) \not\equiv 0 \pmod{p_i^2}.$$  

Since $\beta \equiv 1 \pmod{p_i^2}$, we have also

$$F_i(\gamma_i, \beta) \equiv 0 \pmod{p_i}, \quad F_i(\gamma_i, \beta) \not\equiv 0 \pmod{p_i^2}.$$  

Finally, let $\gamma$ be an element of $O_L$ such that $\gamma \equiv \gamma_i \pmod{p_i^2}$ and $\gamma \equiv 1 \pmod{p_i}$.

Let $q_i = (p, F_i(\gamma, \beta))$ (for $i = i_0$, it gives $q_{i_0} = (p, \beta)$). Clearly, $p_i \nmid q_i, p_i^2 \nmid q_i$. Moreover, if $i \neq j, i, j \neq i_0$, then $p_i \nmid q_j$, since otherwise $F_i(\gamma, 1) \equiv 0 \pmod{p_i}$ and $F_j(\gamma, 1) \equiv 0 \pmod{p_i}$, whence $F_i(x, 1)$ and $F_j(x, 1)$, would have a common root, but they are coprime. Similarly, for $i \neq i_0$,
\( p_{i0} \nmid q_i \) (since otherwise \( F_i(\gamma, \beta) \in p_{i0} \), which would imply that \( \gamma \in p_{i0} \)) and \( p_i \nmid q_{i0} \) (since otherwise \( \beta \in p_i \)). It follows that \( q_i = p_i \) for all \( i \).

Let \( F = F_1^{e_1} \cdots F_t^{e_t} \). It is plain that \( F(\gamma, \beta) \equiv 0 \pmod{pO_L} \). On the other hand, let \( \alpha = \gamma/\beta \) and let \( W(x) \) be a primitive irreducible polynomial with integer coefficients such that \( W(\alpha) = 0 \). In particular, denoting by \( W \) again the homogenized polynomial of \( W \) with the same degree, we have \( W(\gamma, \beta) = 0 \), whence \( W(\gamma, \beta) \in p_i^{e_i} \) for all \( i \). Arguing as above, we see that \( p_i \) and \( G(\gamma, \beta) \) are coprime for all \( G \) such that \( (F_i, G) = 1 \), and we get the inequality \( e_i \leq v_{p_i}(W(\gamma, \beta)) = v_{p_i}(F_i(\gamma, \beta))v_F(\overline{W}) \). In any case, we see that \( v_{p_i}(F_i(\gamma, \beta)) \geq 1 \), and if \( e_i \geq 2 \), we have \( v_{p_i}(F_i(\gamma, \beta)) = 1 \) from the previous discussion, which gives \( e_i \leq v_{p_i}(W) \). It follows that \( F_i^{e_i} \mid W \) for all \( i \), whence \( F = W \). Taking into account the degrees, we have in fact that \( F = W \) up to a nonzero constant.

Consider now the order \( Z_W \). We want to show that \( Z_W \cap pO_L = pZ_W \). One inclusion is obvious, so let \( p \in Z_W \cap pO_L \), and write \( p = T(\alpha) \) for some \( T(x) \in \mathbb{Z}[x] \). Homogenizing \( T \) we obtain that \( T(\gamma, \beta) \equiv 0 \pmod{p_i^{e_i}} \) and therefore \( T \) is divisible by \( F_i^{e_i} \), in view of what we have just proved. By the proof of Proposition 5, \( p \in pZ_W + \mathfrak{B}^{e_i}F_i^{e_i}(\alpha, 1) \) for \( i \neq i_0 \) where \( \mathfrak{B} \) is the denominator of \( \alpha \) in \( Z_W \); similarly, interchanging the role of \( \alpha \) and \( \alpha^{-1} \), we obtain that \( p \in pZ_W + \mathfrak{B}^{e_i}y^{e_i} \) as well. Hence \( p \in pZ_W \) by Theorem 1, as wanted.

Finally, the equality \( Z_W \cap pO_L = pZ_W \) means that the inclusion \( Z_W \to O_L \) induces an isomorphism \( Z_W/pZ_W \xrightarrow{\sim} O_L/pO_L \), showing that \( p \nmid [O_L : Z_W] \). \( \Box \)

**Corollary 3** If \( p \) divides the generalized index of a number field \( L \) with \( [L : \mathbb{Q}] = n \), then \( p < n - 1 \).

**Proof.** It is immediate to check that for \( p + 1 \geq n \) and for every \( d \geq 1 \) there are at least \( \left[ \frac{n}{d} \right] \) distinct homogeneous irreducible polynomials in \( F_p[x, y] \) of degree \( d \). \( \square \)

**Remark 3** For \( d > 1 \), the number of irreducible homogeneous polynomials of degree \( d \) in \( F_p[x, y] \) is the same as the number of irreducible polynomials of degree \( d \) in \( F_p[x] \), whereas for \( d = 1 \) there are \( p + 1 \) irreducible linear forms and \( p \) linear polynomials. It follows that a prime \( p \) does not divide the generalized index of \( L \) if and only if: either (i) \( p \) does not divide the index of \( L \) (with the usual definition given in [7]), or (ii) there are exactly \( p + 1 \) primes with residual degree 1 above \( p \).

**Example 2** Consider the cubic field \( L = \mathbb{Q}(\alpha) \) where \( \alpha \) is a root of \( F = 2x^3 + x^2 + 3x + 2 = 0 \). The discriminant of this cubic field is \( \text{Disc } F = \text{Disc } L = -431 \). We have \( F(x, y) = x(x - y)y \pmod{2} \), and this implies that the generalized index of \( L \) is 1, whereas the usual index of \( L \) is divisible by 2 (it is exactly 2 because of the polynomial \( 4F(\frac{x}{2}) \)).

We also derive a necessary criterion for an element \( \alpha \) of a number field to have index 1.

**Proposition 11** Let \( L = \mathbb{Q}(\alpha) \) be a number field and \( p \) a prime number. Let \( F \) be the minimal polynomial of \( \alpha \), and \( \mathfrak{A} \) the numerator of \( \alpha \) (an ideal in \( O_L \)). If one of the following conditions is satisfied, then \( p \) divides the index \( [O_L : Z_F] \):

(i) there is a prime ideal \( p \) above \( p \) with residual degree \( f_p \geq 2 \) such that \( p \mid \mathfrak{A} \),

(ii) there are two different prime ideals \( p_1 \) and \( p_2 \) above \( p \) with residual degree 1 such that \( p_1p_2 \mid \mathfrak{A} \),

(iii) there is a prime ideal \( p \) above \( p \) with ramification index \( e_p \geq 2 \) such that \( p^2 \mid \mathfrak{A} \).

**Proof.** Let \( p \) be a prime, and assume that \( p \) does not divide the index of \( \alpha \). Then from Theorem 3 all prime ideals above \( p \) above are of the form \( p = pO_L + \mathfrak{B}F_1(\alpha, 1) \). But Lemma 1 shows that \( p \) is coprime to \( \mathfrak{A} \), unless \( F_1 = ux \). This proves that at most one prime ideal above \( p \) can divide \( \mathfrak{A} \). In this case, we have \( f_1 = \deg(ux) = 1 \). This prime ideal \( p \) is such that \( p = pO_L + \alpha \mathfrak{B} = pO_L + \mathfrak{A} \). By inspecting valuations, we see that if \( e_p > 1 \), we must have \( v_p(\mathfrak{A}) = 1 \). \( \square \)
Remark 4 Since the index of $\alpha$ does not change under the transformations of $GL_2(\mathbb{Z})$, we can apply this proposition to all the elements of the form $\frac{a\alpha+b}{c\alpha+d}$ with $ad-bc=\pm1$. In particular, we can apply it to $\frac{1}{\alpha}$ or to $\alpha + q$. This shows, for example, that for any integer $q$ and any prime $p$, the numerator and the denominator of $\alpha + q$ can only be divisible by primes $p$ with residual degree $f_p = 1$, and at most one such prime above each prime $p$.

References


