

Initial Embeddings in the Surreal Numbers of Models of $\mathcal{T}_{\text{an}}(\text{exp})$

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Outline

- 1 Conway's Field of Surreal Numbers
- 2 \mathcal{T}_{an} and $\mathcal{T}_{\text{an}}(\text{exp})$
- 3 Initial Sets and Initial Embeddings

Definition of Surreal Numbers

A **surreal number** is a function $x : \alpha \rightarrow \{\downarrow, \uparrow\}$, where α is an ordinal number. Equivalently, a surreal number is a (possibly infinite) sequence of \downarrow and \uparrow . **No** is the class of all surreal numbers.

Definition (Linear order)

Let $a : \alpha \rightarrow \{\downarrow, \uparrow\}$ be a surreal number. Define $a(i) := 0$ for every $i \geq \alpha$. The order is the lexicographic one, with the rule $\downarrow < 0 < \uparrow$.

$<$ is a linear ordering on **No**.

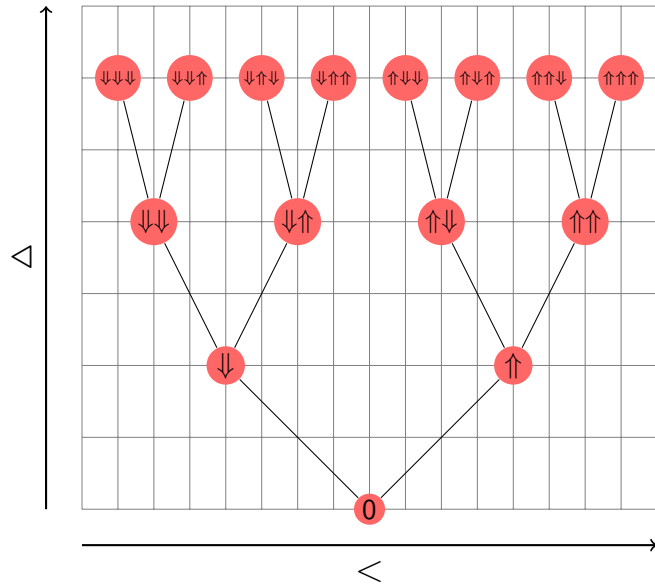
Definition (Simplicity relation)

Let x and y be surreal numbers. We say that x is **simpler** than y , in symbols $x \triangleleft y$, iff, as a function, x is restriction of y to an initial segment.

\triangleleft is a set-like **well-founded** partial ordering on **No**.

1. Invented by Conway.
2. \downarrow is on the left, and \uparrow on the right.

Tree of Surreal Numbers



1. 0 is the empty sequence, i.e. the only surreal number with domain 0.
2. $x \triangleleft y$ (x is **strictly simpler** than y) means $x \trianglelefteq y$, but $y \not\triangleleft x$.
3. $\mathbf{No}^{<\omega}$ is the class of tuples on \mathbf{No} .
4. Since \trianglelefteq is well-founded, we can make induction on it.
5. The simplicity relation on $\mathbf{No}^{<\omega}$ extends the one on \mathbf{No} .

Simplicity Relation on Tuples

Definition

Given $\vec{x} \in \mathbf{No}^n$ and $\vec{y} \in \mathbf{No}^m$, \vec{x} is simpler than \vec{y} , $\vec{x} \trianglelefteq \vec{y}$, iff there exists a function $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, m\}$ such that

- $\forall k \leq n \quad x_k \trianglelefteq y_{\sigma(k)}$;
- if there exists $j \neq k$ such that $\sigma(j) = \sigma(k)$, then $x_k \triangleleft y_{\sigma(k)}$.

\trianglelefteq is a set-like **well-founded** quasi-ordering on $\mathbf{No}^{<\omega}$.

A sub-class $S \subseteq \mathbf{No}$ is **convex** iff

$$\forall x, y \in S \forall z \in \mathbf{No} (x < z < y \Rightarrow z \in S).$$

Theorem

Every non-empty **convex** sub-class of \mathbf{No} contains a unique **simplest** element.

Recursive Definitions on No

Given x_1^L, x_2^L, \dots and $x_1^R, x_2^R, \dots \in \mathbf{No}$, we write

$$x := \{ x_1^L, x_2^L, \dots \mid x_1^R, x_2^R, \dots \}$$

if $x \in \mathbf{No}$ is the simplest element such that $\forall i, j \quad x_i^L < x < x_j^R$.

Definition

Let A and B be families of functions on \mathbf{No} . B is **defined recursively** over A iff, for every $f \in B$ and $\vec{x} \in \text{Dom}(f)$, there exist families $F^L, F^R \subseteq A \times B^{<\omega} \times \mathbf{No}^{<\omega}$ s.t.

$$f(\vec{x}) = \left\{ h^L(\vec{x}, \vec{x}'_0, g_1^L(\vec{x}'_1), \dots, g_m^L(\vec{x}'_m)) \mid h^R(\vec{x}, \vec{x}''_0, g_1^R(\vec{x}''_1), \dots, g_m^R(\vec{x}''_m)) \right\},$$

where we let $(h^L; g_1^L, \dots, g_m^L; \vec{x}'_0, \dots, \vec{x}'_m)$ vary in F^L , with the conditions

- 1 $\vec{x}'_k \triangleleft \vec{x} \quad k = 0, \dots, m$,
- 2 the involved functions are defined;

and similarly for the right half.

1. A function on **No** is a map $f : D \rightarrow \mathbf{No}$ such that D is a sub-class of \mathbf{No}^n for some n .
2. **Idea:** A is a family of functions we already “know” how to compute. Given $\vec{x} \in \mathbf{No}^n$, assume that we have already computed $g(\vec{y})$ for every $g \in B$ and $\vec{y} \triangleleft \vec{x}$. Then, $f(\vec{x})$ is the simplest element in some interval, whose end-points are computed using the functions in A and the values of $g(\vec{y})$ we already know.

1. $-$ is the symmetric function on **No**, exchanging left with right.
2. We will use the notation x^L and x^R for variables that vary in **No** with the conditions $x^L < x < x^R$ and $x^L, x^R \triangleleft x$.
3. With the above structure, **No** is a *real closed field*. The neutral element of $+$ is 0.
4. We will say more later on the functions ω^x , \exp and \log .

Examples

A list of pairs (f, A) , such that $\{f\}$ is defined recursively over A

$$\textcircled{1} f(x) := -x, A := \{\text{id}\}$$

$$-x = \{ -x^R \mid -x^L \}, \quad \text{where } x^L < x < x^R \text{ and } x^L, x^R \triangleleft x$$

$$\textcircled{2} f(x, y) := x + y, A := \{\text{id}\}$$

$$x + y = \{ x^L + y, x + y^L \mid x^R + y, x + y^R \}$$

$$\textcircled{3} f(x, y) := x \cdot y, A \text{ is the set of functions generated (via composition) by } \{+, -\}$$

$$xy = \{ x^L y + xy^L - x^L y^L, x^R y + xy^R - x^R y^R \mid x^L y + xy^R - x^L y^R, x^R y + xy^L - x^R y^L \}$$

$$\textcircled{4} f(x) := \omega^x, A := \mathbb{Q}[X] \text{ is the set of polynomials with rational coefficients}$$

$$\textcircled{5} f := \exp, A := \mathbb{Q}(X) \text{ is the set of rational functions with rational coefficients}$$

$$\textcircled{6} f \text{ is the logarithmic function restricted to } \omega^{\mathbf{No}}, A \text{ is the set of functions generated by } \mathbb{Q}(X) \text{ and the functions of the form } x \mapsto x^{1/n}$$

Power Series on No

- $(\mathbf{No}, <, +, -, \cdot, 0, 1)$ is a **real closed field**, containing \mathbb{R} , where $1 := \uparrow$.
- In a natural way, **No** is isomorphic to $\mathbb{R}((\mathbf{No}))$.
- ω is the “basis” of the power series structure. Namely, ω^x denotes the sum with support $\{x\}$.
- On **No** it is also defined an exponential function \exp .

Every field of generalised power series $K := \mathbb{R}((G))$ with real coefficients has an analytic structure: any real analytic function f converges on the infinitesimal elements of K .

1. $\mathbb{R}(\mathbf{No})$ is the field of **generalised power series** with real coefficients and exponent in \mathbf{No} itself.
2. Since all the power series in \mathbf{No} have support a set (and not a proper class), we should really write $\mathbf{No} \simeq \mathbb{R}((\mathbf{No}))_{\mathbf{On}}$.
3. The function ω^x is *not* the “exponentiation with base ω ”. Namely, $\omega^x \neq \exp(x/\log(\omega))$.
4. However, ω^x is an injective order-preserving group homomorphism. It is *not* surjective.

Recursive Definitions of Analytic Functions

- Let $f(\vec{x}) := \sum_{I \in \mathbb{N}^n} r_I \vec{x}^I$ be a real power series, $\vec{x} := (x_1, \dots, x_n)$.
- f defines an analytic function on the infinitesimal elements of \mathbf{No}^n .
- Define $f_m(\vec{x}, \vec{y}) := \sum_{|I| \leq m} \frac{1}{i_1! \cdots i_n!} \frac{\partial^{|I|} f(\vec{y})}{\partial y_1^{i_1} \cdots \partial y_n^{i_n}} (\vec{x} - \vec{y})^I$,
the m -truncation of the Taylor expansion of $f(\vec{x})$ at \vec{y} .

Then, for every $\vec{x} \in \mathbf{No}^n$ infinitesimal,

$$f(\vec{x}) = \left\{ r_0 - \varepsilon, f_m(\vec{x}, X') - \varepsilon |\vec{x} - X'|^m \mid r_0 + \varepsilon, f_m(\vec{x}, X') + \varepsilon |\vec{x} - X'|^m \right\},$$

where $r_0 := f(0)$, ε varies among the positive real numbers, $X' \in \mathbf{No}^n$ among the infinitesimal tuples such that $X' \triangleleft \vec{x}$, and $m \in \mathbb{N}^*$.

Corollary

The family of analytic functions on \mathbf{No} (restricted to infinitesimal elements) is recursively defined over $\mathbb{Q}[X]$.

Definitions

$\mathcal{L}_0 := (<, +, -, \cdot, 0, 1)$ language of ordered fields.

\mathcal{L}_{an} Expansion of \mathcal{L}_0 . For every f **real analytic** function, defined on a neighbourhood of $[-1, 1]^n$, add an n -ary function symbol f^* .

\mathbb{R}_{an} Interpret each symbol f^* by the real function with values

$$\begin{cases} f(\vec{x}) & \text{if } |\vec{x}| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

\mathcal{T}_{an} Theory of \mathbb{R}_{an} (in the language \mathcal{L}_{an}).

exp is the exponential function on the reals;

$\mathcal{L}_{\text{an}}(\text{exp})$ denotes the expansions of \mathcal{L}_{an} with the function symbol exp ;

$\mathbb{R}_{\text{an}}(\text{exp})$ denotes the expansions of \mathbb{R}_{an} with the function exp ;

$\mathcal{T}_{\text{an}}(\text{exp})$ is the theory of $\mathbb{R}_{\text{an}}(\text{exp})$.

Properties

Theorem

\mathcal{T}_{an} and $\mathcal{T}_{\text{an}}(\text{exp})$ are ***o-minimal theories***, and admit ***elimination of quantifiers*** in the languages $\mathcal{L}_{\text{an}}(-1)$ and $\mathcal{L}_{\text{an}}(\text{exp}, \log)$ respectively.

Assume that

- M is a model of \mathcal{T}_{an} ;
- N is a real closed sub-field and an \mathcal{L}_{an} -sub-structure of M .

Then, N is an **elementary** \mathcal{L}_{an} -sub-structure of M .

Assume that N is an $\mathcal{L}_{\text{an}}(\text{exp}, \log)$ -sub-structure of $M \models \mathcal{T}_{\text{an}}(\text{exp})$.

Then, N is an elementary $\mathcal{L}_{\text{an}}(\text{exp})$ -sub-structure of M .

No and $\mathcal{T}_{an}(exp)$

- **No** is a real closed field, and \mathbb{R} is a sub-field of **No**.
- **No** $\cong \mathbb{R}((\mathbf{No}))$.
- Every field of generalised power series $\mathbb{R}((G))$, with G **divisible** group, is a model of \mathcal{T}_{an} .
- **No** is a **saturated** model of $\mathcal{T}_{an}(exp)$, and $\mathbb{R}_{an}(exp)$ is an elementary sub-structure of **No**.

1. S^A is the union of $S, A(S), A(S \cup A(S)), \dots$

Initial Sets and Closure

Definition

A sub-class $S \subseteq \mathbf{No}$ is **initial** iff

$$\forall x \in S \forall y \in \mathbf{No} \quad y \triangleleft x \Rightarrow y \in S.$$

A map $\Phi : V \rightarrow \mathbf{No}$ is initial iff its image is.

Definition

Let $S \subseteq \mathbf{No}$, and f be a function on **No**. S is **closed** under f iff $f(S^n) \subseteq S$. S is closed under a family A of functions on **No** iff S is closed under each $f \in A$. The **closure** of S under A , denoted by S^A , is the smallest sub-class of **No** containing S and closed under A .

Theorem

Let A and B be families of functions on **No, such that B is **recursively defined** over A . Suppose that, for every $U \subseteq \mathbf{No}$ initial sub-class, U^A is also initial. Then, for every $S \subseteq \mathbf{No}$ initial sub-class, $S^{A \cup B}$ is also initial.**

Initial Sub-structures

Let S be an initial sub-class of **No**. The following sub-classes of **No** are also initial:

- the sub-group
 - the sub-ring
 - the sub-field
 - the real closure of the sub-field
 - the sub-model of \mathcal{T}_{an}
 - the sub-model of $\mathcal{T}_{an}(exp)$
- } generated by S inside **No**.

1. All the cases are proved using the “main theorem”, except for the one about the real closure, where other techniques are needed.

Initial embeddings

Let M be either

- a real closed field,
- a model of \mathcal{T}_{an} ,
- a model of $\mathcal{T}_{\text{an}}(\text{exp})$.

Then, there exists an **initial** embedding of M in **No**, which preserves the field, the \mathcal{L}_{an} , or the $\mathcal{L}_{\text{an}}(\text{exp})$ -structure respectively.

Proof.

By induction.

- 1 The prime sub-model of T in **No** is initial. (where T is any of the above theories).
- 2 Extend the embedding in an initial way, 1 element at a time, using the previous theorem, plus the \mathfrak{o} -minimality of T .
- 3 The domain of a maximal initial embedding is all of M .

□

1. P. Ehrlich proved a similar theorem, but only for \mathcal{T}_{an} .
2. **Idea:** Models of \mathcal{T}_{an} admits a **truncation closed** embeddings into fields of generalised power series, and in particular in **No**.
Give conditions for such an embedding to be **initial**.
3. Since **No** is a saturated model of $\mathcal{T}_{\text{an}}(\text{exp})$, we knew already that every model of $\mathcal{T}_{\text{an}}(\text{exp})$ admits an elementary embedding in **No**. Here we proved that we can also find an embedding which is also *initial*.

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