

Definably complete Baire structures and Pfaffian closure

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May 17, 2010

Abstract

We consider definably complete Baire expansions of ordered fields: every definable subset of the domain of the structure has a supremum and the domain can not be written as the union of a definable increasing family of nowhere dense sets. Every expansion of the real field is definably complete and Baire, and so is every o-minimal expansion of a field. Moreover, unlike the o-minimal case, the structures considered form an axiomatizable class. In this context we prove the following version of Wilkie's Theorem of the Complement: given a definably complete Baire expansion K of an ordered field with a family of smooth functions, if there are uniform bounds on the number of definably connected components of quantifier free definable sets, then K is o-minimal. We further generalize the above result, along the line of Speissegger's theorem, and prove the o-minimality of the relative Pfaffian closure of an o-minimal structure inside a definably complete Baire structure.

Key words: Pfaffian functions; Pfaffian closure; definably complete structures; Baire spaces; o-minimality

MSC2000: Primary 58A17; Secondary 03C64, 32C05, 54E52.

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1 Introduction

We recall that a subset A of a topological space X is said to be meager if there exists a collection $\{Y_i : i \in \mathbb{N}\}$ of nowhere dense subsets of X such that $A \subseteq \bigcup_{i \in \mathbb{N}} Y_i$. The Baire Category Theorem implies that every open subset of \mathbb{R} (with the usual topology) is not meager, i.e. \mathbb{R} is a Baire

space.

The notion of Baire space is clearly not first order. Here we consider a similar (*definable*) notion, which instead is preserved under elementary equivalence, and which coincides with the classical notion over the real numbers (this is made precise in Section 2).

The (first order) structures we consider are definably complete expansions of ordered fields. Definable completeness (see Definition 1.5) is a weak version of Dedekind completeness, which is preserved under elementary equivalence. It is shown in [Miller01], [Servi07], [Fratarc08] that, as in the o-minimal case, (a definable version of) most results of elementary real analysis can be proved in every definably complete expansion of an ordered field. However, to obtain less elementary results one would need some more sophisticated machinery, in the direction of Sard's Lemma and Fubini's Theorem. Both of the quoted classical results refer to a notion of smallness (having measure zero), which has no natural translation in our context. We consider instead a topological notion of smallness (being meager), propose a definable version of this notion and carry out a theory of *definably complete Baire structures*, i.e. expansions of ordered fields such that every definable subset of the domain has a supremum and the domain can not be written as the union of a definable increasing family of nowhere dense sets. In this context we prove an analogue to Fubini's Theorem (the Kuratowski-Ulam's Theorem 4.1) and a very restricted form of an analogue to Sard's Lemma (Theorem 8.9). Notice that it is not known whether every definably complete structure is definably Baire.

Once we have developed the basic tools for definably complete Baire structures (Sections 2 to 6), our next task is to give necessary and sufficient conditions, for a definably complete expansion with C^∞ functions of an ordered field, to be o-minimal.

In [Wilkie99], the author proves his Theorem of the Complement: given an expansion \mathcal{R} of the real field with a family of C^∞ functions, if there are bounds (uniform in the parameters) on the number of connected components of quantifier free definable sets, then \mathcal{R} is o-minimal. In particular, thanks to a well known finiteness result in [Khov91], the structure generated by all real Pfaffian functions is o-minimal (see [Khov91] or [Wilkie99] for the definition of Pfaffian functions and examples). In [KM99], the authors generalize Wilkie's Theorem of the Complement (by weakening the smoothness assumption) in a way which allows them to derive the following result (orig-

inally due to Speissegger, see [Speiss99]): the Pfaffian closure of an o-minimal expansion of the real field is o-minimal.

In Section 7 we proceed to generalize the o-minimality results present in [Wilkie99] and [KM99], to a situation where the base field is not necessarily \mathbb{R} ; moreover, we further weaken the assumption of Wilkie’s Theorem of the Complement, by allowing not only functions, but also **admissible correspondences** (roughly, partial multi-valued functions with finitely many values at each point). We deduce that, given a definably complete Baire expansion \mathbb{K} of an ordered field with a family of C^∞ functions, if there are bounds (uniform in the parameters) on the number of definably connected components of quantifier free definable sets, then \mathbb{K} is o-minimal (Theorem 7.7). In Section 8, by using our restricted version 8.9 of Sard’s Lemma, we proceed to prove the analogue to Khovanskii’s finiteness result in the context of definably complete Baire structures (Theorem 8.4). We derive the o-minimality of every definably complete Baire expansion of an ordered field with any family of definable Pfaffian functions (Theorem 8.2).

Finally, in Section 9, we prove that the relative Pfaffian closure of an o-minimal structure \mathbb{K}_0 inside a definably complete Baire expansion \mathbb{K} of \mathbb{K}_0 is o-minimal. This latter result, whose proof is shaped on the one present in [KM99], can be compared with the main result in [Fratarc08] (which can be derived from ours), where instead Speissegger’s method was followed; it is here where our generalization of Wilkie’s Theorem to admissible correspondences is necessary.

In Section 10 we use the above results to find effective bounds for various topological invariants of sets definable in the Pfaffian closure of the fields of reals, and more generally of recursively axiomatized o-minimal expansions of \mathbb{R} .

The results in this article have been submitted for publication. Since we do not have constraints of space, we opted to give more detailed proofs, explanations and examples that would be suitable for a published version.

1.1 Notation

Throughout this paper, \mathbb{K} is a (first-order) structure expanding an ordered field. We use the word “definable” as a shorthand for “definable in \mathbb{K} with parameters from \mathbb{K} ”.

We denote by x, y, z, \dots the points in \mathbb{K}^n . When we want to stress the fact that they are tuples, we write $\bar{x}, \bar{y}, \bar{z}, \dots$, where $\bar{x} = (x_1, \dots, x_n)$,

etc.

For convenience, on \mathbb{K}^m instead of the usual Euclidean distance we will use the equivalent distance

$$d : (x, y) \mapsto \max_{i=1, \dots, m} |x_i - y_i|.$$

For every $\delta > 0$ and $x \in \mathbb{K}^m$, we define

$$\begin{aligned} B^m(x; \delta) &:= \{y \in \mathbb{K}^m : d(x, y) < \delta\}, \\ \overline{B}^m(x; \delta) &:= \{y \in \mathbb{K}^m : d(x, y) \leq \delta\}, \end{aligned}$$

the open and closed “balls” of center x and “radius” δ ; we will drop the superscript m if it is clear from the context.

Notation 1.1. Let $X \subseteq Y \subseteq \mathbb{K}^n$, with Y definable. We write $\text{cl}_Y(X)$ (or simply \overline{X} if Y is clear from the context) for the topological closure of X in Y , $\text{int}_Y(X)$ (or simply $\overset{\circ}{X}$) for the interior part of X in Y , $\text{bd}_Y(X) := \overline{X} \setminus \overset{\circ}{X}$ for the boundary of X (in Y), and $\partial_Y X := \overline{X} \setminus X$ for the frontier of X (in Y).

Notation 1.2. We define $\Pi_n^{m+n} : \mathbb{K}^{m+n} \rightarrow \mathbb{K}^m$ as the projection onto the first m coordinates. If $A \subset \mathbb{K}^{m+n}$ and $x \in \mathbb{K}^m$, we denote by A_x the fibre of A over x , i.e. the set $\{y \in \mathbb{K}^n : (x, y) \in A\}$.

Notation 1.3. Let $\mathbb{K}_+ := \{x \in \mathbb{K} : x > 0\}$.

Definition 1.4. Let $\tilde{\mathbb{R}}$ be the structure on the reals numbers, with a predicate for every subset of \mathbb{R}^n (it will be used for examples).

1.2 Definably complete structures

Definition 1.5. An expansion \mathbb{K} of an ordered field is called *definably complete* if every definable subset of \mathbb{K} has a supremum in $\mathbb{K} \cup \{\pm\infty\}$.

Generalities on definably complete structures can be found in [Servi07], [DMS10, §2] and [Miller01].

Proviso. For the remainder of the article, \mathbb{K} will always be a definably complete structure.

Definition 1.6. $X \subseteq \mathbb{K}^m$ is *definably compact* (d-compact for short) if it is definable, closed in \mathbb{K}^m and bounded.

Proviso 1.7. We order \mathbb{K}^m lexicographically. In this subsection we will denote by N a definable subset of \mathbb{K}^m which is cofinal in the lexicographic ordering.

Lemma 1.8 (Miller). *X is definably compact iff for every $(Y(y))_{y \in N}$ definable decreasing family of closed non empty subsets of X , we have $\bigcap_y Y(y) \neq \emptyset$.*

Definition 1.9. Let $f : N \rightarrow \mathbb{K}^n$ be definable. Define $\text{acc}_{y \rightarrow \infty} f(y)$ (and write for simplicity $\text{acc } f$) to be the set of accumulation points of f ; that is, $x \in \text{acc } f$ iff

$$(\forall r \in \mathbb{K}^m)(\forall \varepsilon \in \mathbb{K}_+)(\exists y > r) y \in N \ \& \ d(f(y), x) < \varepsilon.$$

Lemma 1.10. *If X is definably compact, then for all definable N (satisfying 1.7) and for all $f : N \rightarrow X$ definable we have $\text{acc } f \neq \emptyset$.*

It is not clear if the converse of the above lemma is true.

Definition 1.11. Let $(A(y))_{y \in N}$ be a definable family of non-empty subsets of \mathbb{K}^n . Define $\text{acc}_{y \rightarrow \infty} A(y)$ (and write for simplicity $\text{acc } A$) to be the set of accumulation points of A ; that is, $x \in \text{acc } A$ iff $(\forall r \in \mathbb{K}^m)(\forall \varepsilon \in \mathbb{K}_+)(\exists y > r) y \in N$ and $d(A(y), x) < \varepsilon$.

Note that $\text{acc } A = \bigcap_y \overline{\bigcup_{z \geq y} A(z)}$.

Remark 1.12. Let $(A(t))_{0 < t \in \mathbb{K}}$ be a definable family of subsets of \mathbb{K}^m , and $G := \bigcup_{t > 0} \{t\} \times A(t)$. Then, $\text{acc}_{t \rightarrow 0} A(t) = (\text{cl}_{\mathbb{K}^{m+1}}(G))_0 := \{x \in \mathbb{K}^m : (0, x) \in \overline{G}\}$.

Lemma 1.13. *X is definably compact iff for all A definable family of non-empty subsets of X we have $X \cap \text{acc } A \neq \emptyset$.*

Proof. First assume that X is d-compact. Let $Y(y) := \overline{\bigcup_{z \geq y} A(z)}$. Then $(X \cap Y(y))$ is a definable decreasing family of closed subsets of X . By Lemma 1.8, $\bigcap_y Y(y) \neq \emptyset$, and we are done.

Conversely, assume that X is not d-compact. By Lemma 1.8, there exists a definable decreasing family $Y := (Y(y))_{y \in \mathbb{K}^m}$ of closed subsets of X such that $\bigcap_y Y(y) = \emptyset$. However, since Y is decreasing, $X \cap \text{acc } Y = \bigcap_y Y(y)$, and we are done. \square

Proof of Lemma 1.10. Define $A(y) := \{f(y)\}$. By Lemma 1.13, $\text{acc } A$ is non-empty. Note that $\text{acc } A = \text{acc } f$. \square

Lemma 1.14. *Let $X \subseteq \mathbb{K}^n$ be definably compact. Fix $\varepsilon \in \mathbb{K}_+$. Let $(A(t))_{t \in N}$ be a definable family of subsets of \mathbb{K}^n . The following are equivalent*

1. $\forall x \in X \forall t \in N$ large enough $X \cap B(x; \varepsilon) \subseteq A(t)$;
2. $\forall t \in N$ large enough $X \subseteq A(t)$.

Proof. That (2) implies (1) is clear.

Conversely, assume that (1) is true. Suppose, for contradiction, that (2) is false. Let $D(t) := X \setminus A(t)$. Let $N' := \{t \in N : D(t) \neq \emptyset\}$. Since (2) is false, N' is cofinal in N . Let $C := \text{acc}_{t \in N', t \rightarrow \infty} D(t)$. By Lemma 1.13, $C \neq \emptyset$; let $x \in C$. By (1), if t is large enough, then $X \cap B(x; \varepsilon) \subseteq A(t)$. Choose $t \in N'$ such that $X \cap B(x; \varepsilon) \subseteq A(t)$ and $d(x, D(t)) < \varepsilon$. Let $y \in D(t)$ such that $d(x, y) < \varepsilon$. Since $y \in D(t)$, we have $y \notin A(t)$. Since $y \in X \cap B(x; \varepsilon)$, we have $y \in A(t)$, a contradiction. \square

Lemma 1.15. *Let $C \subset \mathbb{K}^n$ be a nonempty d -compact set, and let $V := \{V(t) : t \in I\}$ be a definable open cover of C . Then, there exists $\delta_0 \in \mathbb{K}_+$ (a Lebesgue number for V and C) such that, for every subset $X \subseteq C$ of diameter smaller than δ_0 , there exists $t \in I$ such that $X \subseteq V(t)$.*

Proof. Suppose for a contradiction that

$$(\forall \delta > 0)(\exists y \in C)(\forall t \in I) B(y; \delta) \not\subseteq V(t).$$

For every $\delta > 0$, define

$$Y(\delta) := \{y \in C : (\forall t \in I) B(y; \delta) \not\subseteq V(t)\}.$$

Note that $(Y(\delta))_{\delta > 0}$ is a definable family of subsets of C , increasing as δ decreases. Let y_0 be an accumulation point for the family $(Y(\delta))_{\delta > 0}$, as $\delta \rightarrow 0$ (which exists by Lemma 1.13).

Let $t_0 \in I$ and $\delta_0 > 0$ such that $B(y_0; 2\delta_0) \subseteq V(t_0)$. Let $\delta_1 \leq \delta_0$ and $y \in Y(\delta_1)$ such that $|y - y_0| < \delta_0$. Therefore, $B(y; \delta_0) \subseteq B(y_0; 2\delta_0) \subseteq V(t_0)$, contradicting the fact that $y \in Y(\delta_1)$. \square

We will often use without further comment the following result:

Lemma 1.16 (Miller). *Let $f : \mathbb{K}^n \rightarrow \mathbb{K}^m$ be a definable continuous function and let $C \subset \mathbb{K}^n$ be d -compact. Then $f(C)$ is d -compact.*

Definition 1.17. A n -dimensional definable embedded \mathcal{C}^N \mathbb{K} -manifold $V \subseteq \mathbb{K}^d$ (which we will simply call n -dimensional \mathbb{K} -manifold) is a definable subset V of \mathbb{K}^d , such that for every $x \in V$ there exists a definable neighbourhood $U(x)$ of x (in \mathbb{K}^d), and a definable \mathcal{C}^N diffeomorphism $f_x : U(x) \simeq \mathbb{K}^d$, such that $U(x) \cap V = f_x^{-1}(\mathbb{K}^n \times \{0\})$.

Remark 1.18. Note that a \mathbb{K} -manifold V can always be written as the intersection of a definable closed set and a definable open set. In fact, let $\delta : V \rightarrow \mathbb{K}_+ \cup \{+\infty\}$ be the definable map

$$\delta(x) := \sup\{r \in \mathbb{K}_+ : \forall s \in \mathbb{K}_+ (s < r \Rightarrow B(x; s) \cap V \text{ is closed in } B(x; s))\}.$$

Let $U := \bigcup_{x \in V} B(x; \delta(x)/2)$; then, $V = \overline{V} \cap U$.

Note moreover that the dimension n of a \mathbb{K} -manifold V is uniquely determined by V , because \mathbb{K}^n and $\mathbb{K}^{n'}$ are locally diffeomorphic iff $n = n'$. If we consider only \mathcal{C}^0 manifolds, it is not clear anymore if the dimension is well defined.

Finally, recall the following definition.

Definition 1.19. A definable set $X \subset \mathbb{K}^n$ is *definably connected* if it can not be expressed as a union of two *definable* nonempty disjoint open sets. A subset $C \subseteq X$ is a *definably connected component* of X if it is a maximal definably connected subset of X .

Note that if X has finitely many definably connected components, then each component of X is definable. Moreover, if \mathbb{K} expands the real field, every definable and (topologically) connected set is also definably connected. The converse could in general not be true. However it is true if \mathbb{K} is o-minimal. For example, it is true for any expansion of the real field by a Pfaffian chain (see Theorem 8.2).

2 Meager sets

Let $X \subseteq Y \subseteq \mathbb{K}^n$, with Y definable.

Definition 2.1. X is *nowhere dense* (in Y) if $\text{int}_Y(\text{cl}_Y(X)) = \emptyset$. X is *definably meager* (in Y) if there exists a *definable increasing* family $(A(t))_{t \in \mathbb{K}}$ of nowhere dense subsets of Y , such that $X \subseteq \bigcup_t A(t)$. We will call the family $(\text{cl}_Y(A(t)))_{t \in \mathbb{K}}$ a *witness* of the fact that X is definably meager. X is *definably residual* (in Y) if $Y \setminus X$ is definably meager.

Notice that, if $(A(t))_{t \in \mathbb{K}}$ is a witness of the fact that X is meager in \mathbb{K}^n , then also the family

$$(\overline{B}^n(0; |t|) \cap A(t))_{t \in \mathbb{K}}$$

is a witness, hence we may always assume that each $A(t)$ is d-compact.

Notice also that we do *not* require that a definably meager set is definable.

The subsets of Y , with the operations Δ (symmetric difference) and \cap , form a commutative ring; the definably meager subsets of Y form an ideal of this ring.

Definition 2.2. Y is *definably Baire* if every non-empty open definable subset of Y is not definably meager (in Y).

Note that if \mathbb{K} has countable cofinality, then X is definably meager (Baire, respectively) in \mathbb{K}^n if X is meager (Baire, respectively) in the usual topological sense. In general, the converse is not true: for instance, if \mathbb{K} is a countable o-minimal structure, then it is definably Baire, but not Baire in the topological sense. However, the two notions coincide for $\tilde{\mathbb{R}}$. In fact, assume that X is a meager subset of \mathbb{R}^n ; therefore, $X = \bigcup_{i \in \mathbb{N}} Y(i)$, where each $Y(i)$ is a nowhere dense subset of \mathbb{R}^n . For each $t \in \mathbb{R}$, define $Z(t) := \bigcup_{i \in \mathbb{N}, i < t} Y(i)$. Then, each $Z(t)$ is nowhere dense, the family $(Z(t))_{t \in \mathbb{R}}$ is increasing and definable in $\tilde{\mathbb{R}}$, and $X = \bigcup_{t \in \mathbb{K}} Z(t)$.

From now on, we will write “meager” for “definably meager”, and “topologically meager” for the usual topological notion, and similarly for “residual” and “Baire”. Moreover, if Y is clear from the context, we will simply say that X is nowhere dense (resp. “meager”, “residual”) instead of “nowhere dense” (resp. “definably meager”, “definably residual”) in Y .

Proposition 2.3. *Let Y be definable, and $\emptyset \neq U \subseteq Y$ be definable and open. Then, U is meager in Y iff it is meager in itself.*

Proof. Suppose U is meager in Y and let $(Y(t))_{t \in \mathbb{K}}$ be a witness of this fact. For every $t \in \mathbb{K}$, define $X(t) := Y(t) \cap U$. Since U is open, $\text{int}_U(X(t)) = \text{int}_Y(Y(t)) \cap U = \emptyset$. Hence, $(X(t))_{t \in \mathbb{K}}$ is a witness of the fact that U is meager in itself.

Vice versa, let $(X(t))_{t \in \mathbb{K}}$ be a witness of the fact that U is meager in itself, and $Y(t) := \text{cl}_Y(X(t))$. We claim that $\text{int}_Y(Y(t)) = \emptyset$. In fact, $\text{int}_Y(Y(t)) = \text{int}_Y((\text{cl}_Y(X(t)) \cap U)) = \text{int}_U(Y(t)) = \text{int}_U(X(t)) = \emptyset$. Hence, $(Y(t))_{t \in \mathbb{K}}$ is a witness of the fact that U is meager in Y . \square

Corollary 2.4. *Let Y be definable, and $\emptyset \neq U \subseteq Y$ be definable and open. Then,*

1. *If U is not meager in itself, then Y is also not meager in itself.*

2. If Y is Baire, then U is also Baire.

Proof. For (1), if Y is meager in itself, then any subset of Y , in particular U , is meager in Y . Since it is open, U is also meager in itself.

Regarding (2), if Y is Baire, let $V \subseteq U$ be non-empty, definable and open in U . Since U is open in Y , V is also open in Y . Hence, by 2.3, V is not meager in itself, and, again by 2.3, V is not meager in U . Therefore, U is Baire. \square

Lemma 2.5. *Let $Y \subseteq \mathbb{K}^m$ be definable. The following are equivalent:*

1. Y is Baire;
2. for all $X \subseteq Y$, if X is meager, then $\overset{\circ}{X} = \emptyset$;
3. every $x \in Y$ has a definable neighbourhood which is Baire;
4. every residual subset of Y is dense;
5. every open definable non-empty subset of Y is not meager in itself;
6. every meager closed definable subset of Y has empty interior.

Proof.

(2 \Rightarrow 1) is obvious.

(1 \Rightarrow 3) is obvious, because Y itself is a Baire neighbourhood of each point.

(3 \Rightarrow 4) Let $X \subseteq Y$ be meager. Suppose, for a contradiction, that U is a non-empty definable subset of X open in Y , and let $x \in U$. Let V be a definable Baire neighbourhood of x , and $W := V \cap U$. By Proposition 2.3, W is Baire, and therefore it is not meager in Y (by the same proposition), which is not possible.

(4 \Rightarrow 2) Let $X \subseteq Y$ be meager. Hence, $Y \setminus X$ is dense, and therefore $\overset{\circ}{X} = \emptyset$.

(1 \Leftrightarrow 5) Use Proposition 2.3.

(1 \Rightarrow 6) Let $C \subseteq Y$ be definable, closed and meager. If $\overset{\circ}{C} \neq \emptyset$, then $\overset{\circ}{C}$ is not meager, and thus C is not meager.

(6 \Rightarrow 1) Let $U \subseteq Y$ be open, definable and meager in Y . Then, \overline{U} is also meager, because $\overline{U} = U \sqcup \text{bd}U$, and $\text{bd}U$ is nowhere dense. Therefore, \overline{U} has empty interior, and therefore U is empty. \square

Remark 2.6. \mathbb{K}^n is Baire iff it is not meager in itself.

Proof. One implication is obvious.

For the other implication, assume that \mathbb{K}^n is not meager in itself, and let $U \subseteq \mathbb{K}^n$ be an open definable subset. If, for a contradiction, U were meager in itself, then we could find an open non empty box $B \subseteq U$. By Proposition 2.3, B is also meager in itself. However, B is definably homeomorphic to \mathbb{K}^n , because \mathbb{K} expands a field, contradicting the hypothesis. \square

The following result is not trivial and will be proved in Section 4.

Proposition 2.7. *If \mathbb{K} is Baire, then for every $m \geq 1$, \mathbb{K}^m is Baire.*

The converse, however, is trivial

Remark 2.8. If \mathbb{K}^m is Baire for some $m \geq 1$, then \mathbb{K} is Baire.

2.1 Baire structures

Definition 2.9. A definably complete structure \mathbb{K} is a *Baire structure* if \mathbb{K} is definably Baire as a definable subset of \mathbb{K} itself, in the sense of Def. 2.2. A theory T is definably complete and Baire if every model of T is a definably complete Baire structure.

Remark 2.10. The fact that \mathbb{K} is Baire can be expressed by a set of first-order sentences: therefore, every \mathbb{K}' elementary equivalent to \mathbb{K} also satisfies the hypothesis. If moreover the language is recursive, this set of sentences is also recursive.

Notice that an ultra-product of definably complete (resp. Baire) structures is also definably complete (resp. Baire); the same cannot be said for “o-minimal” instead of “definably complete”.

Examples 2.11. The following are examples of definably complete Baire structures.

- Every expansion of \mathbb{R} (because \mathbb{R} is Dedekind complete and topologically Baire).
- Every o-minimal expansion of a field. In fact, a nowhere dense definable subset of \mathbb{K} is finite, and definable families of finite sets are uniformly finite; hence, the union of a definable increasing family of nowhere dense sets is finite, and can not coincide with the whole structure.

- Let \mathcal{B} be an o-minimal expansion of a field, let $A \preceq \mathcal{B}$ be a dense substructure. Then the structure \mathcal{B}_A , generated by adding a unary predicate symbol for A , is definably Baire. This follows from the fact that if $X \subseteq \mathcal{B}$ is \mathcal{B}_A -definable, then its topological closure \overline{X} is \mathcal{B} -definable (see [Dries98a, Theorem 4]). Hence, a closed nowhere dense set is finite, and, since \mathcal{B}_A satisfies the Uniform Finiteness property (see [Dries98a, Corollary 4.5]), the union of a definable increasing family of nowhere dense sets is finite. More generally, as shown in [DMS10, §3.5], any definably complete structure satisfying the Uniform Finiteness condition is definably Baire (one can even show that if \mathbb{K} is definably complete, and every definable closed discrete subset of \mathbb{K} is bounded, then \mathbb{K} is Baire).

3 \mathcal{F}_σ -sets

We now consider a class of sets for which it is easy to determine whether they are meager or not. These sets have also been studied in [DMS10], where they are called D_Σ -sets.

Definition 3.1. Let $X \subseteq Y \subseteq \mathbb{K}^n$, with Y definable. X is in \mathcal{F}_σ in Y (we will also say “ X is an \mathcal{F}_σ subset of Y ”, or “ X is \mathcal{F}_σ ”, and drop the reference to Y if it is clear from the context) if X is the union of a definable increasing family of closed subsets of Y , indexed by \mathbb{K} . X is in \mathcal{G}_δ if its complement is an \mathcal{F}_σ .

Lemma 3.2. *Let \mathcal{A} be either the family of \mathcal{F}_σ or the family of \mathcal{G}_δ subsets of some \mathbb{K}^n , for $n \in \mathbb{N}$. Then, each $A \in \mathcal{A}$ is definable. Moreover, \mathcal{A} is closed under finite unions, finite intersections, Cartesian products, and preimages under definable continuous functions. Besides, the following are in \mathcal{A}*

1. *definable closed subsets of \mathbb{K}^n ;*
2. *definable open subsets of \mathbb{K}^n ;*
3. *finite boolean combinations of definable open subsets of \mathbb{K}^n .*

The family of \mathcal{F}_σ subsets is also closed under images under definable continuous functions.

Proof. See [DMS10]. Let A and B be in \mathcal{F}_σ . The fact that $A \cup B$ and $A \times B$ are also in \mathcal{F}_σ is obvious.

Let $A = \bigcup_t A(t)$ and $B = \bigcup_t B(t)$, where $(A(t))_{t \in \mathbb{K}}$ and $(B(t))_{t \in \mathbb{K}}$ are two definable increasing families of closed (and, we may assume, d-compact) sets.

Then, $A \cap B = \bigcup_t (A(t) \cap B(t))$, because $(A(t))_{t \in \mathbb{K}}$ and $(B(t))_{t \in \mathbb{K}}$ are increasing families. Hence, $A \cap B$ is also in \mathcal{F}_σ .

If $f : \mathbb{K}^n \rightarrow \mathbb{K}^{n'}$ is continuous, then $f(A) = \bigcup_t f(A(t))$. For every $t \in \mathbb{K}$, $f(A(t))$ is d-compact, because $A(t)$ is d-compact, and therefore $f(A)$ is in \mathcal{F}_σ . A similar proof works for preimages.

Let $U \subseteq \mathbb{K}^n$ be open and definable, and $C := \mathbb{K}^n \setminus U$. For every $r \in \mathbb{K}_+$, define $U(r) := \{x \in \mathbb{K}^n : d(x, C) \geq r\}$. Note that each $U(r)$ is closed. Since U is open, $U = \bigcup_{r>0} U(r)$, and therefore U is in \mathcal{F}_σ .

If D is a finite boolean combination of open definable subsets of \mathbb{K}^n , then it is a finite union of sets of the form $C_i \cap U_i$, for some definable sets C_i and U_i , such that each C_i is closed and each U_i is open. Hence, D is in \mathcal{F}_σ .

The corresponding results for \mathcal{G}_δ follow immediately by considering the complements. \square

It is not true in general that, if $Y \subseteq \mathbb{K}^n$ is definable, $X \subseteq Y$ is an \mathcal{F}_σ -subset of Y , and $f : Y \rightarrow Y$ is definable and continuous, then $f(X)$ is an \mathcal{F}_σ . The point where the above proof breaks down for $Y \neq \mathbb{K}^n$ is the fact that it is not necessarily true that every \mathcal{F}_σ subset of Y is an increasing definable union of d-compact sets.

Notice that, by Remark 1.18, every \mathbb{K} -manifold is an \mathcal{F}_σ -set.

Remark 3.3. Let $X \subseteq \mathbb{K}^n$. X is an \mathcal{F}_σ iff X is of the form $\Pi_n^{n+m}(Z)$ for some $Z \subseteq \mathbb{K}^{n+m}$ closed and definable.

Proof. The “if” direction follows from Lemma 3.2. For the other direction, let $(X(t))_{t \in \mathbb{K}}$ be a definable increasing family of closed subsets of \mathbb{K}^n , such that $X = \bigcup_{t \in \mathbb{K}} X(t)$. Define $Z := \overline{\bigsqcup_{t \in \mathbb{K}} (X(t) \times \{t\})}$. \square

Notice that, if \mathbb{K} is o-minimal, then every X definable subset of \mathbb{K} is a finite Boolean combination of definable closed sets (because X is a finite union of cells), and therefore X is an \mathcal{F}_σ .

Remark 3.4. If $X \subseteq \mathbb{K}^n$ is meager, then there exists a meager \mathcal{F}_σ -set containing X .

Lemma 3.5. *Let Y be definable and Baire, and $D \subseteq Y$. Assume that D is in \mathcal{F}_σ . Then, D is meager iff $\overset{\circ}{D} = \emptyset$.⁽¹⁾*

Proof. If $\overset{\circ}{D} \neq \emptyset$, then, since Y is Baire, D cannot be meager. Conversely, assume that D is not meager. If D is in \mathcal{F}_σ , then $D = \bigcup_t D(t)$, for some definable increasing family of closed subsets. Since D is not meager, at least one of the $D(t)$, say $D(t_0)$, is not meager. Hence, $\text{int}(D(t_0)) \neq \emptyset$ (otherwise, $D(t_0)$ would be nowhere dense), and therefore $\overset{\circ}{D} \neq \emptyset$. \square

Note that if $X \subseteq \mathbb{R}^n$ is in \mathcal{F}_σ for the structure $\tilde{\mathbb{R}}$, and of Lebesgue measure zero, then X is meager, but the converse is not true.

We now give a local condition which is sufficient to prove that the image of an \mathcal{F}_σ -set under a continuous definable function is meager.

Proposition 3.6. *Let $C \subseteq \mathbb{K}^m \times \mathbb{K}^n$ be in \mathcal{F}_σ , $f : C \rightarrow \mathbb{K}^d$ be definable and continuous. Assume that for every $y \in \Pi_m^{m+n}(C)$ there exists a neighbourhood $V_y \subseteq \mathbb{K}^m$ of y , such that $f((V_y \times \mathbb{K}^n) \cap C)$ is meager. Then, $f(C)$ is meager.*

Proof. If \mathbb{K} is meager in itself, then by Proposition 2.7 there is nothing to prove. Thus, we may assume that \mathbb{K} (and hence \mathbb{K}^d) is Baire.

We proceed by induction on m . The case $m = 0$ is clear, because if $m = 0$, then $V_0 = \mathbb{K}^0$.

Assume that we have already proved the conclusion for $m - 1$ (and every n). We want to prove it for m . First, we consider the case when C is \mathbf{d} -compact. W.l.o.g., $0 \in C$. Remember that, for every $r > 0$ and $y \in \mathbb{K}^m$, $\overline{B}^m(y; r) \subset \mathbb{K}^m$ is the closed hypercube of side $2r$ and center y ; let $S^m(y; r)$ be its boundary. Moreover, define $D(r) := f(C \cap (\overline{B}^m(0; r) \times \mathbb{K}^n))$.

Note that $f(C) = \bigcup_r D(r)$, and that each $D(r)$ is \mathbf{d} -compact. Therefore, to prove that $f(C)$ is meager, it suffices to prove that each $D(r)$ has empty interior. Suppose, for a contradiction, that $f(C)$ is not meager, and let

$$r_0 := \inf\{r > 0 : \text{int}(D(r)) \neq \emptyset\}.$$

Since the $D(r)$ are closed, $r_0 = \inf\{r > 0 : D(r) \text{ is not meager}\}$. We have that $0 < r_0$ by hypothesis, and $r_0 < +\infty$ because $f(C)$ is not meager.

Let $P := \Pi_m^{n+m}(C)$. Since P is \mathbf{d} -compact, if $\mathbb{K} = \mathbb{R}$, we could find $y_1, \dots, y_k \in P$ such that $P \subseteq V_{y_1} \cup \dots \cup V_{y_k}$. In the general situation, we need another argument. Let $5\delta_0$ be a Lebesgue number for the open cover $\{V_y : y \in P\}$ of P (we may also assume that δ_0 is small in comparison with r_0); $\delta_0 > 0$ exists by Lemma 1.15.

⁽¹⁾This is not true for \mathcal{G}_δ sets: for instance, the set of irrational numbers in \mathbb{R} is a \mathcal{G}_δ which is not meager (it is even residual), but has empty interior.

Note that

$$\overline{B}^m(0; r_0 + \delta_0/2) \subseteq \overline{B}^m(0; r_0 - \delta_0/2) \cup \bigcup_{y \in S^m(0; r_0)} \overline{B}^m(y; \delta_0),$$

hence

$$D(r_0 + \delta_0/2) \subseteq D(r_0 - \delta_0/2) \cup \bigcup_{y \in S^m(0; r_0)} f(C \cap (\overline{B}^m(y; \delta_0) \times \mathbb{K}^n)).$$

By definition of r_0 , we know that $D(r_0 + \delta_0/2)$ is not meager, while $D(r_0 - \delta_0/2)$ is meager. Hence, to obtain a contradiction, it suffices to show that $\bigcup_{y \in S^m(0; r_0)} f(C \cap (\overline{B}^m(y; \delta_0) \times \mathbb{K}^n))$ is meager.

Note that $S^m(0; r_0)$ is the finite union of the faces of the closed hypercube $\overline{B}^m(0; r_0)$: hence, we only need to show that for each face S of $S^m(0; r_0)$ the set $D := \bigcup_{y \in S} f(C \cap (\overline{B}^m(y; \delta_0) \times \mathbb{K}^n))$ is meager. W.l.o.g., we can assume that S is the “top” face $\{y \in \overline{B}^m(0; r_0) : y_m = r_0\}$ and we may identify S with $\overline{B}^{m-1}(0; r_0) \times \{r_0\}$.

Define

$$\begin{aligned} \tilde{C} &:= C \cap \bigcup_{y \in S} (\overline{B}^m(y; \delta_0) \times \mathbb{K}^n), \\ \tilde{f} &:= f \upharpoonright \tilde{C}. \end{aligned}$$

Claim. \tilde{C} and \tilde{f} satisfy the hypothesis of the proposition, with $n' = n+1$, $m' = m-1$, and $V'_z = B(z; \delta_0)$.

\tilde{C} is d-compact, and therefore it is in \mathcal{F}_σ . Let $\tilde{P} \subseteq \mathbb{K}^{m-1}$ be the projection of \tilde{C} onto \mathbb{K}^{m-1} ; note that \tilde{P} is d-compact. Fix $z \in \tilde{P}$; by definition, there exists $t \in [r_0 - \delta_0, r_0 + \delta_0]$ such that $y := (z, t) \in \tilde{C}$. Notice that

$$\tilde{C} \cap (V'_z \times \mathbb{K} \times \mathbb{K}^n) \subseteq C \cap (V'_z \times [r_0 - \delta_0, r_0 + \delta_0] \times \mathbb{K}^n) \subseteq C \cap (\overline{B}^m(y; 2\delta_0) \times \mathbb{K}^n).$$

Since $5\delta_0$ is a Lebesgue number for the cover $\{V_y : y \in P\}$ of P , it follows that there exists $y' \in P$ such that $\overline{B}^m(y; 2\delta_0) \subset V_{y'}$. Putting everything together, we have that $\tilde{C} \cap (V'_z \times \mathbb{K}^{n+1}) \subset C \cap (V_{y'} \times \mathbb{K}^n)$ and thus $\tilde{f}(\tilde{C} \cap (V'_z \times \mathbb{K}^{n+1}))$ is meager, which proves the claim.

Therefore, by inductive hypothesis, $\tilde{f}(\tilde{C})$ is meager. However, $D \subseteq \tilde{f}(\tilde{C})$, and we reached a contradiction.

We now treat the general case when C is in \mathcal{F}_σ . Note that C is an increasing union of d-compact sets $C(t)$. For each $t \in \mathbb{K}$, define $D(t) := f(C(t))$: note that each $D(t)$ is d-compact. By the d-compact case, we can conclude that each $D(t)$ is meager, and therefore nowhere dense. Thus, $D = \bigcup_t D(t)$ is meager. \square

Corollary 3.7. *Let $C \subseteq \mathbb{K}^m$ be in \mathcal{F}_σ , and $f : C \rightarrow \mathbb{K}^d$ be definable and continuous. Assume that for every $x \in C$ there exists $V_x \subseteq C$ neighbourhood of x , such that $f(C \cap V_x)$ is meager. Then, $f(C)$ is meager.*

Proof. Apply the proposition to the case $n = 0$. □

With a similar method, one can prove the following.

Lemma 3.8. *Let $C \subseteq \mathbb{K}^m$ be d -compact and $f : C \rightarrow \mathbb{K}^d$ be definable (but not necessarily continuous). Assume that for every $x \in C$ there exists $V_x \subseteq C$ neighbourhood of x , such that $f(C \cap V_x)$ is nowhere dense. Then, $f(C)$ is nowhere dense.*

Corollary 3.9. *Let $W \subseteq \mathbb{K}^m$ be a definable \mathbb{K} -manifold, $C \subseteq W$ be an \mathcal{F}_σ subset of W , and $f : C \rightarrow \mathbb{K}^d$ be definable and continuous. Assume that for every $x \in C$ there exists V_x neighbourhood of x , such that $f(C \cap V_x)$ is meager. Then, $f(C)$ is meager.*

Proof. Since W is a \mathbb{K} -manifold, it is in \mathcal{F}_σ . Since C is \mathcal{F}_σ in W , it is also \mathcal{F}_σ in \mathbb{K}^m . Apply the previous corollary. □

Corollary 3.10. *Let $C \subseteq \mathbb{K}^m$ be an \mathcal{F}_σ . If every $x \in C$ has a neighbourhood V_x such that $C \cap V_x$ is meager, then C is meager.*

Proposition 3.6 and the following results are trivial if \mathbb{K} is o-minimal, since in this case $C \subseteq \mathbb{K}^n$ is meager iff $\dim(C) < n$.

For the topological notions, we know the following facts to be true:

1. Let $C \subseteq \mathbb{R}^n$ and $f : C \rightarrow Y$ (not necessarily continuous). Assume that, for every $x \in C$, there exists $V_x \subseteq C$ neighbourhood of x , such that $f(C \cap V_x)$ is topologically meager. Then, $f(C)$ is topologically meager.
2. Let $C \subseteq Y$. If every $x \in C$ has a neighbourhood V_x , such that $V_x \cap C$ is topologically meager, then C is topologically meager.

The first fact follows from the fact that \mathbb{R}^n is second countable; the second from [Kelley55, Theorem 6.35]. We were able to prove the definable versions only under additional hypothesis (e.g., C in \mathcal{F}_σ); however, these results are strong enough for our applications.

Lemma 3.11. *Let $f : \mathbb{K}^n \rightarrow \mathbb{K}^m$ be definable, and*

$$\mathcal{D}_f := \{\bar{x} \in A : f \text{ is discontinuous at } \bar{x}\}.$$

If the graph of f is an \mathcal{F}_σ set, then \mathcal{D}_f is meager.

Proof. If, for contradiction, \mathcal{D}_f is not meager, then since it is an \mathcal{F}_σ , it contains a non-empty open box B . Therefore, w.l.o.g. we can assume that $\mathcal{D}_f = \mathbb{K}^n$, and that \mathbb{K}^n is Baire. Let $\Gamma(f) = \bigcup_t X(t)$, where $(X(t))_{t \in \mathbb{K}}$ is a definable increasing family of d-compact sets. Let $Y(t) := \Pi_n^{n+m}(X(t))$. Note that each $Y(t)$ is d-compact, and $\mathbb{K}^n = \bigcup_t Y(t)$. Since \mathbb{K}^n is Baire, there exists t_0 such that $Y(t_0)$ contains a non-empty open box B' . Let $B'' \subseteq B'$ be a closed box with non-empty interior, and $g := f \upharpoonright B''$. Note that $\Gamma(g) = X(t_0) \cap (B'' \times \mathbb{K}^m)$; therefore, $\Gamma(g)$ is d-compact, and so, as in the classical case, g is continuous, contradicting the fact that $B'' \subseteq \mathcal{D}_f$. \square

4 The Kuratowski-Ulam's Theorem

The main result of this section is the following theorem.

Theorem 4.1. *Let $D \subseteq \mathbb{K}^{m+n}$. For every $x \in \mathbb{K}^m$, let $D_x := \{y \in \mathbb{K}^n : (x, y) \in D\}$ be the corresponding section of D . Let $T := T^m(D) := \{x \in \mathbb{K}^m : D_x \text{ is meager in } \mathbb{K}^n\}$.*

If D is meager (in \mathbb{K}^{m+n}), then T is residual.

This is a definable version of Kuratowski-Ulam's Theorem [Oxtoby80, Theorem 15.1], which in turn is an analogue of Fubini's Theorem: they both imply that if D is negligible, then D_y is negligible for almost every y ; in Kuratowski-Ulam's Theorem negligible means "meager", while in Fubini's Theorem negligible means "of measure zero".

It is not clear whether in the above theorem D definable implies that T is definable. Note that if \mathbb{K} is o-minimal and D is definable, then T is also definable.

As a corollary, we obtain Proposition 2.7.

Proof of Proposition 2.7. By induction on m . The case $m = 1$ is our assumption on \mathbb{K} . Assume that we already proved that \mathbb{K}^m is Baire: we want to prove that \mathbb{K}^{m+1} is Baire. Suppose not; then \mathbb{K}^{m+1} is meager in itself. If we apply Theorem 4.1 with $n = 1$, we obtain that either \mathbb{K}^m or \mathbb{K} is meager in itself, a contradiction. \square

Definition 4.2. A definable function $f : Y \rightarrow \mathbb{K}$ is *lower semi-continuous* if, for every $x \in Y$, either x is an isolated point of Y , or

$$\liminf_{\substack{x' \rightarrow x \\ x' \in Y}} f(x') \geq f(x).$$

Remark 4.3. Let $C \subseteq \mathbb{K}^{n+1}$ be d-compact. For every $x \in D := \Pi_n^{n+1}(C)$, let $f(x) := \min C_x$. Then, $f : D \rightarrow \mathbb{K}$ is lower semi-continuous.

Lemma 4.4. Let $Y \subseteq \mathbb{K}^n$ be definable, $f : Y \rightarrow \mathbb{K}$ be lower semi-continuous and definable, and $\mathcal{D}_f \subseteq Y$ be the set of points of discontinuity of f . Then, \mathcal{D}_f is meager (in Y).

Proof. See [DMS10, Lemma 2.8(1)]. □

In the above lemma, if $Y = \mathbb{K} = \mathbb{R}$, we can not conclude that \mathcal{D}_f has Lebesgue measure zero. In fact, let $C \subseteq \mathbb{R}$ be closed, with empty interior, and of positive measure, and f be the characteristic function of $\mathbb{R} \setminus C$. Then, $\mathcal{D}_f = C$, and therefore it is of positive measure.

On the other hand, it is always true that if $f : \mathbb{K}^m \rightarrow \mathbb{K}$ is definable, then \mathcal{D}_f is in \mathcal{F}_σ (see [Oxtoby80, Theorem 7.1]).

Proof of Theorem 4.1. If \mathbb{K}^m is meager in itself, then the conclusion is trivially true, because then every subset of \mathbb{K}^m is meager. Hence, we can assume that \mathbb{K}^m is Baire.

CASE 1. $n = 1$ and D is d-compact.

Hence, D has empty interior, and each D_x is also d-compact. Therefore, by Lemma 3.5, $T = \{x \in \mathbb{K}^m : D_x^\circ = \emptyset\}$. Let $E := \mathbb{K}^m \setminus T$. We have to prove that E is meager.

For every $\varepsilon > 0$ let

$$X(\varepsilon) := \{(x, y) \in \mathbb{K}^m \times \mathbb{K} : B^1(y; \varepsilon) \subseteq D_x\}.$$

Let

$$E(\varepsilon) := \pi(X(\varepsilon)) = \{x \in \mathbb{K}^m : D_x \text{ contains a ball of radius } \varepsilon\}.$$

Note that $X(\varepsilon)$ is d-compact, since its complement is the projection of an open set, therefore so is $E(\varepsilon)$. Note that $E = \bigcup_{\varepsilon > 0} E(\varepsilon)$; hence, to prove that E is meager, it suffices to prove that each $E(\varepsilon)$ is nowhere dense. Since each $E(\varepsilon)$ is d-compact, it suffices to prove the following claim.

Claim 1. For every $\varepsilon > 0$, $\text{int}(E(\varepsilon)) = \emptyset$ (see also [DMS10, Lemma 2.8(2)]).

Assume, for a contradiction, that there exists a nonempty open box $U \subseteq E(\varepsilon)$. Define

$$\begin{aligned} f : U &\rightarrow \mathbb{K} \\ x &\mapsto \min\{y \in \mathbb{K} : (x, y) \in X(\varepsilon)\}. \end{aligned}$$

Note that f is lower semi-continuous and definable. By Lemma 4.4, f is continuous outside a meager set $\mathcal{D}_f \subseteq U$. Since \mathbb{K}^m is Baire, $\mathcal{D}_f \neq U$, and therefore there exists $x_0 \in U$ such that f is continuous at x_0 . It is now easy to show that a neighbourhood of $(x_0, f(x_0))$ is contained in D , contradicting the fact that $\dot{D} = \emptyset$.

CASE 2. $n = 1$ and D arbitrary meager subset of \mathbb{K}^m .

Let $(D(p))_{p \in \mathbb{K}}$ be an increasing definable family of d-compact subsets of \mathbb{K}^{m+1} with empty interior, such that $D \subseteq \bigcup_p D(p)$. For each $p \in \mathbb{K}$, let $E(p) := \{x \in \mathbb{K}^m : D(p)_x \text{ is not meager in } \mathbb{K}\}$. By what we have seen above, $E(p) = \bigcup_{\varepsilon > 0} E(p, \varepsilon)$, where $(E(p, \varepsilon))_{\substack{\varepsilon \in \mathbb{K}_+ \\ p \in \mathbb{K}}}$ is a definable family of subsets of \mathbb{K} , increasing in p and decreasing in ε , such that each $E(p, \varepsilon)$ is closed and nowhere dense. Let

$$E' := \bigcup_{\varepsilon, p} E(p, \varepsilon) = \bigcup_p E(p).$$

Claim 2. $\mathbb{K}^m \setminus T \subseteq E'$.

In fact, let $x \notin T$. Thus, D_x is not meager. However, $D_x \subseteq \bigcup_p D(p)_x$. Since $(D(p)_x)_{p \in \mathbb{K}}$ is an increasing definable family of closed subsets of \mathbb{K} , we obtain that there exists p_0 such that $D(p_0)_x$ has non-empty interior. Thus, $x \in E(p_0) \subseteq E'$.

Therefore, it suffices to prove that E' is meager to obtain that T is residual. However, $E' = \bigcup_{p > 0} E(p, 1/p)$, and we are done.

CASE 3. $n > 1$ and D arbitrary meager subset of \mathbb{K}^m . We argue by induction on n .

Suppose that we have already proved the conclusion for n (and for every m). We want to prove the conclusion for $n+1$. First, we will assume that D is in \mathcal{F}_σ . We want to prove that the set $T := T^m(D) := \{x \in \mathbb{K}^m : D_x \text{ is meager}\}$ is residual. Define

$$\begin{aligned} S &:= \mathbb{K}^{m+1} \setminus T^{m+1}(D) := \{(x, y_{n+1}) \in \mathbb{K}^m \times \mathbb{K} : D_{(x, y_{n+1})} \text{ is not meager}\}, \\ R &:= T^m(S) = \{x \in \mathbb{K}^m : S_x \text{ is meager}\}. \end{aligned}$$

Notice that (for the moment) we do not know whether S and R are definable, even assuming that D is in \mathcal{F}_σ .

Claim 3. S is meager.

By inductive hypothesis.

Claim 4. R is residual.

By the case $n = 1$ and the previous claim.

Claim 5. $R \subseteq T$.

Fix $x \in \mathbb{K}^m$. Assume that $x \notin T$. We have to prove that $x \notin R$. Define $F := D_x \subseteq \mathbb{K}^{n+1}$. Note that F is in \mathcal{F}_σ : therefore, since $x \notin T$, $\overset{\circ}{F} \neq \emptyset$. Let $U := U_1 \times U_2$ be a non-empty open box contained in F , $U_1 \subseteq \mathbb{K}$, $U_2 \subseteq \mathbb{K}^m$. For every $y_{n+1} \in U_1$, $D_{(x, y_{n+1})} = F_{y_{n+1}} \supseteq U_2$, and therefore $(x, y_{n+1}) \in S$. Thus, $U_1 \subseteq S_x$, and $x \notin R$.

Hence, T contains a residual set, and therefore it is residual.

For D arbitrary, let $D' \subseteq \mathbb{K}^{m+n}$ be a meager \mathcal{F}_σ containing D . By the previous case, the corresponding set $T' := T^m(D')$ is residual. Since $T' \subseteq T$, we are done. \square

5 Almost open sets

Proviso. In this section we will assume that \mathbb{K} is definably complete and Baire.

Let $Y \subseteq \mathbb{K}^m$ be definable. We have seen that the family of meager subsets of Y is an ideal, hence it defines an equivalence relation on the family of subsets of Y , given by $X \sim X'$ iff $X \Delta X'$ is meager.

Remark 5.1. $X \sim X'$ iff there exists Z meager such that $X \Delta Z = X'$

Proof. Set $Z := X \Delta X'$. \square

Definition 5.2. $X \subseteq Y$ is *almost open* (in Y), or a.o. for short, if X is equivalent to a definable open set.⁽²⁾

Lemma 5.3. Let $Y \subseteq \mathbb{K}^m$ be definable, and A and B be a.o. subsets of Y . Then, $A \cap B$, $A \cup B$ and $Y \setminus A$ are also a.o.. Moreover, \mathcal{F}_σ and \mathcal{G}_δ subsets of Y are a.o..

Finally, if Y_1 and Y_2 are definable, and $A_i \subseteq Y_i$ are a.o. for $i = 1, 2$, then $A_1 \times A_2$ is a.o. in $Y_1 \times Y_2$.

Proof. It is trivial to see that $A \cap B$, $A \cup B$ and $A_1 \times A_2$ are a.o..

Let $A = U \Delta E$, where U is open and definable, and E is meager. Then, $Y \setminus A = (Y \setminus U) \Delta E$. Hence, to prove that $Y \setminus A$ is a.o. it suffices to prove that $C := Y \setminus U$ is a.o.. However, $C = \overset{\circ}{C} \cup \text{bd}(C)$. Since C is closed, $\text{bd}(C)$ is nowhere dense, and *a fortiori* meager, and we are done.

Let $(D(t))_{t \in \mathbb{K}}$ be a definable increasing sequence of closed subsets of Y . We have to prove that $D := \bigcup_t D(t)$ is a.o.. Let $U := \overset{\circ}{D}$ and $E := D \setminus U$. It is enough to prove that E is meager. For every t , let $E(t) := E \cap D(t)$. Note that $D(t) \subseteq U$; therefore, $E(t) \subseteq \text{bd}(D(t))$ is nowhere dense, and we are done. \square

⁽²⁾Almost open sets are called “sets with the property of Baire” in [Oxtoby80].

Consequently, $X \subseteq Y$ is a.o. iff it is equivalent to a definable closed subset of Y .

Remark 5.4. Every meager set is a.o., being equivalent to the empty set. Every residual set is also a.o., being equivalent to the ambient space.

Corollary 5.5. *Let $A \subseteq Y$. The following are equivalent:*

1. A is a.o.;
2. A is of the form $E \Delta F$, for some meager set E and some set F in \mathcal{F}_σ ;
3. A is of the form $G \sqcup E$, for some G in \mathcal{G}_δ and E meager.

Proof. Cf.[Oxtoby80, Theorem 4.4]. (1 \Leftrightarrow 2) and (3 \Rightarrow 1) are obvious. For (1 \Rightarrow 3), let $A = U \Delta E$ for some U open and E meager. Let Q be a meager set in \mathcal{F}_σ containing E , and $G := U \setminus Q$. Note that G is in \mathcal{G}_δ , and

$$U \Delta E = [(U \setminus Q) \Delta (U \cap Q)] \Delta (E \cap Q) = G \Delta [(U \Delta E) \cap Q] = G \sqcup E',$$

where $E' := (U \Delta E) \cap Q$ is meager. \square

The following is a partial converse of Theorem 4.1.

Lemma 5.6. *Let D be an a.o. subset of \mathbb{K}^{m+n} , and $T(D) := \{x \in \mathbb{K}^m : D_x \text{ is meager}\}$. Then, D is meager iff $T(D)$ is residual.*

Proof. The “only if” direction is Theorem 4.1. For the other direction, let U be an open set such that $E := D \Delta U$ is meager. By Theorem 4.1, $T(E)$ is residual. Moreover, since $U_x = D_x \Delta E_x$, we have $T(U) \supseteq T(D) \cap T(E)$, and therefore $T(U)$ is also residual. However, U is open and \mathbb{K}^n is Baire: therefore, $T(U)$ is the complement of the projection of U on \mathbb{K}^m . Since U is open, $T(U)$ is closed. Therefore, $T(U)$ is closed and residual; since \mathbb{K}^m is Baire, $T(U) = \mathbb{K}^m$. Thus, U is empty, and we are done. \square

The hypothesis that D is a.o. in the above lemma is necessary: [Oxtoby80, Theorem 15.5] gives an example of a set $E \subseteq \mathbb{R}^2$ that is not topologically meager, such that no three points of E are collinear.

6 Further results and open problems

Open problem 6.1. It is not known to the authors if there exists a definably complete structure which is not Baire.

Proviso. For the remainder of this section, \mathbb{K} is a definably complete Baire structure.

Open problem 6.2. Let $(Y(t))_{t \in \mathbb{K}}$ be a definable increasing family of meager subsets of \mathbb{K}^m , and let $Y := \bigcup_t Y(t)$. Is Y necessarily meager? In particular, is it necessarily $Y \neq \mathbb{K}^m$?

Notice that, if in addition the $Y(t)$ are closed, then Y is meager, whereas the same conclusion does not necessarily hold if the $Y(t)$ are in \mathcal{F}_σ (actually, since every meager set is contained in a meager \mathcal{F}_σ -set, it is enough to reduce to this situation). Moreover, the above question has positive answer if \mathbb{K} is o-minimal, because then each Y_t has (o-minimal) dimension less than m , and therefore Y has dimension less than m . In fact, if \mathbb{K} is o-minimal, and $Y \subseteq \mathbb{K}^m$ is definable, then Y is meager iff $\dim Y < m$; moreover, if $(Y(t))_{t \in \mathbb{K}_+}$ is a definable family, decreasing in t , then $\bigcup_t Y(t) \subseteq \text{acc}_{t \rightarrow 0} Y(t)$. Thus, the following lemma proves what we want.

Lemma 6.3. *Let \mathbb{K} be an o-minimal structure, $n \leq m \in \mathbb{N}$, and $(Y(t))_{t > 0}$ be a definable family of subsets of \mathbb{K}^m , and $Z := \text{acc}_{t \rightarrow 0} Y(t)$. If, for every $t > 0$, $\dim(Y(t)) \leq n$, then $\dim(Z) \leq n$.*

Proof. Define $W := \bigcup_{t > 0} Y(t) \times \{t\} \subseteq \mathbb{K}^{m+1}$. Note that $Z = (\overline{W})_0 := \{x \in \mathbb{K}^m : (x, 0) \in \overline{W}\}$. Moreover, since $\dim Y(t) \leq n$, we have $\dim W \leq n + 1$. Since $Z \times \{0\} \subseteq \partial W$, we have $\dim Z < \dim W \leq n + 1$. \square

The following is a partial result for the case of a.o. sets.

Lemma 6.4. *Let $Y \subseteq \mathbb{K}^n$ be definable and Baire, $D \subseteq Y$ be a.o. (in Y), and $(Y(t))_{t \in \mathbb{K}}$ be a definable increasing family of closed subsets of Y , such that $Y = \bigcup_t Y(t)$. Then, D is meager in Y iff each $D \cap Y(t)$ is meager (in Y).*

Proof. The “only if” direction is clear.

For the other direction, let $C \subseteq Y$ be closed, such that $E := C \Delta D$ is meager. It suffices to prove that C is meager. For every $t \in \mathbb{K}$, define

$$\begin{aligned} C(t) &:= C \cap Y(t), \\ D(t) &:= D \cap Y(t). \end{aligned}$$

Then, $D(t) \Delta C(t) \subseteq E$. Therefore, since $D(t)$ and E are meager, $C(t)$ is meager and closed. Since Y is Baire, $C(t)$ is nowhere dense, and thus C is meager. \square

6.1 The Sard property

Let $C \subseteq \mathbb{K}^n$ be meager, and $f : \mathbb{K}^n \rightarrow \mathbb{K}^m$ be definable and \mathcal{C}^1 . We want to investigate in which circumstances $f(C)$ is meager. When $\mathbb{K} = \mathbb{R}$, Sard's Lemma implies that $f(C)$ is meager. This suggests the following definition.

Definition 6.5. Fix d, r, m positive natural numbers. Let $V \subseteq \mathbb{K}^d$ be a \mathbb{K} -manifold of dimension n . Let $f : V \rightarrow \mathbb{K}^m$ be a definable \mathcal{C}^r function and Δ_f be the set of singular points of f . If $\Sigma_f := f(\Delta_f)$ is meager in \mathbb{K}^m , then we say that f has the *Sard property*.

Lemma 6.6. If $\mathbb{K} = \tilde{\mathbb{R}}$ and $f : V \rightarrow \mathbb{K}^m$ is as in the above definition, with $r > \max\{0, n - m\}$, then f has the Sard property.

Proof. By Sard's Lemma, Σ_f has Lebesgue measure zero, and therefore it has empty interior. Since Σ_f is in \mathcal{F}_σ , it is also meager. \square

Open problem 6.7. Does every \mathcal{C}^r definable function $f : \mathbb{K}^n \rightarrow \mathbb{K}^m$ (with $r > \max\{0, n - m\}$) have the Sard property?

Remark 6.8. If \mathbb{K} is o-minimal, then every \mathcal{C}^1 definable function $f : V \rightarrow \mathbb{K}^m$ has the Sard property [BO01, Theorem 3.5].

Proposition 6.9. Suppose $f : \mathbb{K}^n \rightarrow \mathbb{K}^m$ has the Sard property, and let $C \subseteq \mathbb{K}^n$ be meager. Then $f(C)$ is meager.

Proof. We may assume that $C \in \mathcal{F}_\sigma$, since C is contained in a meager \mathcal{F}_σ -set. Let $\Lambda := \mathbb{K}^n \setminus \Delta_f$ be the set of regular points of f . Note that Λ is open.

By the Sard property, $f(C \cap \Delta_f)$ is meager. Hence, it suffices to show that $f(C \cap \Lambda)$ is meager. Let $x \in C \cap \Lambda$. Since x is a regular point for f , by the Implicit Function Theorem there exists a neighbourhood V of x such that f is a diffeomorphism on V ; therefore, $f(C \cap V)$ is meager, and, by Corollary 3.7, $f(C \cap \Lambda)$ is meager. \square

The following lemma is a generalization of Lemma 5.6.

Lemma 6.10. *Let $f : \mathbb{K}^n \rightarrow \mathbb{K}^m$ be a C^1 definable function with the Sard property. Let Λ be the set of the regular points of f , and $C \subseteq \mathbb{K}^n$ be almost open. For every $t \in \mathbb{K}^m$, let $F_t := f^{-1}(t)$, $C_t := F_t \cap C$, and $T := \{t \in \mathbb{K}^m : C_t \text{ is meager in } F_t \text{ or } F_t = \emptyset\}$. Then, T is residual iff $C \cap \Lambda$ is meager.*

Proof. If $n < m$, $\Lambda = \emptyset$, and we have a tautology. Assume that $n \geq m$.

Let $x \in C \cap \Lambda$. Since x is a regular point for f , there exists V open neighbourhood of x such that, up to a change of coordinates, $f \upharpoonright V$ is the projection on the first m coordinates $y := (x_1, \dots, x_m)$. For every $y \in T$, the set $C_y \cap V$ is meager in $\{y\} \times \mathbb{K}^{n-m}$.

Hence, if T is residual, then, by Lemma 5.6, $C \cap V$ is meager; therefore, by Corollary 3.7, $C \cap \Lambda$ is meager.

Conversely, assume that $C \cap \Lambda$ is meager; we must prove that T is residual. Since $f(\Delta_f)$ is meager, it suffices to prove that $T(C \cap \Lambda)$ is residual. Therefore, w.l.o.g. we can assume that $C \subseteq \Lambda$. By Kuratowski-Ulam's Theorem 4.1, the set $T(V \cap C) := \{y \in \mathbb{K}^m : C_y \cap V \text{ is meager in } \mathbb{K}^{n-m}\}$ is residual. Therefore, T is residual. \square

In Subsections 7.2 and 8.1 we will produce examples of classes of functions in definably complete Baire structures, which have the Sard property.

7 A theorem of the complement for a class of definably complete Baire structures

In this section we prove a version of Wilkie's Theorem of the Complement [Wilkie99, Theorem 1.9] which holds not only, as the original theorem, for expansions of the real field, but also for definably complete Baire structures. This result will give a sufficient criterium to establish if a given definably complete Baire structure is in fact o-minimal.

We will assume the reader to have familiarity with [Wilkie99] (which, in turn, uses results from [Maxwell98]) and we will adapt the proofs contained therein to our situation. We will occasionally refer to the treatment of Wilkie's Theorem of the Complement given in [BS04], when more suitable to our purposes.

We recall a few definitions (corresponding to [Wilkie99, Definitions 1.1, 1.2, 1.3, 1.6]), adapted to our present situation.

Proviso. We fix for the rest of the article a definably complete Baire structure \mathbb{K} .

Definition 7.1. For $X \subseteq \mathbb{K}^n$ definable, let $cc(X)$ be the number of definably connected components of X (Def. 1.19), and let $\gamma(X)$ be the least $m \in \mathbb{N}$ such that, for every affine set $L \subseteq \mathbb{K}^n$, we have $cc(X \cap L) \leq m$, with the convention that $\gamma(X) = \infty$ if m does not exist.

Definition 7.2. Let $\mathcal{S} = \langle \mathcal{S}_n : n \in \mathbb{N}^+ \rangle$, where \mathcal{S}_n is a collection of definable subsets of \mathbb{K}^n . We say that \mathcal{S} is a *weak structure* (over \mathbb{K}) if \mathcal{S} contains all zero-sets of polynomials with coefficients in \mathbb{K} and is closed under finite intersection, cartesian product and permutation of the variables.

\mathcal{S} is *closed* if for every n and $A \in \mathcal{S}_n$, A is a closed subset of \mathbb{K}^n ; \mathcal{S} is *semi-closed* if for every n and $A \in \mathcal{S}_n$, A can be obtained as the projection onto the first n coordinates of some closed set $B \in \mathcal{S}_{n+k}$, for some suitable k . \mathcal{S} is *o-minimal* if for every n and $A \in \mathcal{S}_n$ we have $\gamma(A) < \infty$.

Definition 7.3. Let \mathcal{S} be an o-minimal weak structure (over \mathbb{K}). The *Charbonnel closure* $\tilde{\mathcal{S}} = \langle \tilde{\mathcal{S}}_n : n \in \mathbb{N}^+ \rangle$ is obtained from \mathcal{S} by closing under the following *Charbonnel operations*: finite union, intersection with affine sets, projection and topological closure⁽³⁾.

We immediately obtain an analogous result to [Wilkie99, Lemmas 1.4, 1.5]:

Theorem 7.4. *If \mathcal{S} is a semi-closed o-minimal weak structure, then its Charbonnel closure $\tilde{\mathcal{S}}$ is a semi-closed o-minimal weak structure.*

The reader can easily check that the proof of the quoted lemmas contained in [Maxwell98, §1] does not use specific properties of \mathbb{R} , and can be reformulated in any definably complete structure (the Baire property is not needed here). Definable completeness is necessary because the fact that a continuous definable function on a closed bounded definable set assumes maximum is used to bound the γ of the topological closure of a set.

Definition 7.5. Let \mathcal{S} be an o-minimal weak structure. We say that \mathcal{S} is *determined by its smooth functions* (DSF) if, given a set $A \in \mathcal{S}_n$, there exist $k \in \mathbb{N}$ and a C^∞ -function $f_A : \mathbb{K}^{n+k} \rightarrow \mathbb{K}$ whose graph lies in $\tilde{\mathcal{S}}$, such that A is the projection onto the first n coordinates of the zero-set of f_A (compare with [Wilkie99, Definition 1.7]).

⁽³⁾The set of operations defined here gives rise to the same closure as the one originally defined by Charbonnel, see [BS04].

The aim is to prove the following version of [Wilkie99, Theorem 1.8].

Theorem 7.6. *Let \mathcal{S} be a semi-closed o-minimal weak structure (over \mathbb{K}), which is DSF. Then $\tilde{\mathcal{S}}$ is closed under complementation and o-minimal.*

The following version of [Wilkie99, Theorem 1.9] will then automatically follow.

Theorem 7.7. *Let \mathbb{K} be a definably complete Baire structure and \mathcal{F} be a family of \mathbb{K} -definable C^∞ functions. Let $\mathbb{K}_{\mathcal{F}}$ be the reduct of \mathbb{K} generated by the field structure and the functions in \mathcal{F} . Then $\mathbb{K}_{\mathcal{F}}$ is o-minimal if and only if $\gamma(A) < \infty$, for every $A \subset \bigcup_{n \geq 1} \mathbb{K}^n$ quantifier free $\mathbb{K}_{\mathcal{F}}$ -definable set.*

In view of Theorem 7.4, to prove Theorem 7.6 it is sufficient to show that, under the hypothesis of the statement, $\tilde{\mathcal{S}}$ is closed under complementation.

In [KM99], the authors generalized Wilkie's Theorem of the Complement [Wilkie99, Theorem 1.9] (by weakening the DSF assumption) in a way which allowed them to derive the o-minimality of the Pfaffian closure of an o-minimal expansion of the real field. Inspired by [KM99], we will also weaken our DSF assumption and prove a more general statement (Theorem 7.35), from which Theorem 7.6 will follow as a corollary. The motivation for giving such a general statement will be clear in Section 9, where we will show an application.

In Subsection 7.1 we give some results on admissible correspondences, which will play a role in the statement of 7.35. In Subsection 7.2 we develop some preliminary results (corresponding to the results in [Wilkie99, §2]) about o-minimal weak structures. In Subsection 7.3 we can finally state our result precisely, and we proceed as in [Wilkie99, §3] and give the key ingredient of the proof (the Theorem of the Boundary 7.37). Finally, in Subsection 7.4 we conclude the proof by adapting Wilkie's Cell Decomposition Theorem (which can be found in [Wilkie99, §4]) to our situation.

7.1 Admissible correspondences

To be able to state exactly the result we want to prove, we need to give some definitions. All the results in this subsection do not need that \mathbb{K} is Baire, but only that it is definably complete.

Definition 7.8. A correspondence $f : \mathbb{K}^n \rightsquigarrow \mathbb{K}^m$ is a definable partial multi-valued function from \mathbb{K}^n to \mathbb{K}^m .

Definition 7.9. Given $1 \leq N \in \mathbb{N}$, a \mathcal{C}^N **admissible correspondence** is a correspondence $f : \mathbb{K}^n \rightsquigarrow \mathbb{K}^m$, satisfying the following conditions. Let $F \subset \mathbb{K}^n \times \mathbb{K}^m$ be the graph of f .

1. F is definable and has a finite number of definably connected components;
2. F is a \mathcal{C}^N closed embedded submanifold of \mathbb{K}^{n+m} , of dimension n ;
3. for every $\bar{x} \in F$, the normal space $N_{\bar{x}}F$ to F at \bar{x} is transversal to the coordinate space \mathbb{K}^n ; equivalently, the restriction to F of the projection map Π_n^{n+m} is a local diffeomorphism between F and \mathbb{K}^n .

For the remainder of this subsection, $f : \mathbb{K}^n \rightsquigarrow \mathbb{K}^m$ is a \mathcal{C}^N admissible correspondence, with graph F .

Definition 7.10. For every $C \subseteq \mathbb{K}^m$, denote by $f^{-1}(C)$ the preimage of C under f , that is $f^{-1}(C) := \{\bar{x} \in \mathbb{K}^n : \exists \bar{y} \in C (\bar{x}, \bar{y}) \in F\}$. Define $V(f) := f^{-1}(\{0\})$. Define the domain of f to be $\text{dom}(f) := f^{-1}(\mathbb{K}^m)$. For every $A \subseteq \mathbb{K}^n$, denote by $f(A) := \{\bar{y} \in \mathbb{K}^m : \exists \bar{x} \in A (\bar{x}, \bar{y}) \in F\}$, the image of A under f . For every $\bar{x} \in \mathbb{K}^n$, we define $f(\bar{x}) := f(\{\bar{x}\})$.

Examples 7.11.

1. Every \mathcal{C}^N function is an admissible correspondence.
2. The correspondence \sqrt{x} is not admissible.
3. Define $g : \mathbb{R} \rightsquigarrow \mathbb{R}$ be the correspondence with graph $G := \{(x, y) \in \mathbb{R} : y = x^2 \vee y = x^2 - 1\}$. g is \mathcal{C}^∞ admissible, it is definable in the real field, but it is not a partial function.
4. Define $g : \mathbb{R} \rightsquigarrow \mathbb{R}$, $g(x) := 1/x$, defined for $x \neq 0$. g is an admissible \mathcal{C}^∞ partial function. The domain of g is not closed, and therefore it is not true that the preimage of a closed set is closed.

Lemma 7.12.

1. For every $C \subseteq \mathbb{K}^m$ d -compact, $f^{-1}(C)$ is closed (in \mathbb{K}^n). In particular, $V(f)$ is closed.
2. For every $U \subseteq \mathbb{K}^m$ open and definable, $f^{-1}(U)$ is open. In particular, $\text{dom}(f)$ is open.

Proof. Let $x \in \overline{f^{-1}(C)}$. We have to prove that $x \in f^{-1}(C)$. Let $D := \overline{(F \cap (\mathbb{K}^n \times C))}_x$. Notice that $D \subseteq \overline{C} = C$, and therefore $D = D \cap C$. Since $x \in \overline{f^{-1}(C)}$, we have that for every U neighbourhood of x there exists $y \in U$, such that $f^{-1}(y) \cap C \neq \emptyset$, i.e. the section $(F \cap (\mathbb{K}^n \times C))_y$ is non-empty. Since C is d-compact, D is non-empty. Since F and C are closed, we have $\overline{F \cap (\mathbb{K}^n \times C)} = F \cap (\mathbb{K}^n \times C)$, and therefore

$$F_x \cap C = (F \cap (\mathbb{K}^n \times C))_x = D.$$

Since $D \neq \emptyset$, we have that $x \in f^{-1}(C)$. \square

Remark 7.13. If F is the graph of an admissible \mathcal{C}^N correspondence, then every definably connected component of F is the graph of an admissible \mathcal{C}^N correspondence. Conversely, if F_1 and F_2 are the graphs of 2 admissible \mathcal{C}^N correspondences, $f_i : \mathbb{K}^n \rightsquigarrow \mathbb{K}^m$, and F_1 and F_2 are disjoint, then $F_1 \cup F_2$ is the graph of an admissible \mathcal{C}^N correspondence.

Lemma 7.14. *Let $g : \mathbb{K}^n \rightarrow \mathbb{K}^m$ be a definable partial function. Then, g is admissible \mathcal{C}^N iff:*

1. *the domain of g is an open set U ;*
2. *$g : U \rightarrow \mathbb{K}^m$ is a \mathcal{C}^N function;*
3. *for every $\bar{x} \in \partial U$,*

$$\lim_{\substack{\bar{y} \rightarrow \bar{x}, \\ \bar{y} \in U}} |g(\bar{y})| = +\infty.$$

We conjecture that, if F is definably connected and $\text{dom}(f) = \mathbb{K}^n$, then, f is a (total and single-valued) function.

The reader can check that the following properties of admissible correspondences hold.

Lemma 7.15.

- *Let $\phi : \mathbb{K}^m \rightarrow \mathbb{K}^m$ be a \mathcal{C}^N , definable diffeomorphism. Then, $\phi \circ f : \mathbb{K}^n \rightsquigarrow \mathbb{K}^m$ is \mathcal{C}^N and admissible.*
- *Let $\theta : \mathbb{K}^n \rightarrow \mathbb{K}^n$ be a \mathcal{C}^N definable diffeomorphism. Then, $f \circ \theta : \mathbb{K}^n \rightsquigarrow \mathbb{K}^m$ is \mathcal{C}^N and admissible.*
- *Let $\theta : \mathbb{K}^n \rightarrow \mathbb{K}^n$ be a \mathcal{C}^N definable function. If $f \circ \theta : \mathbb{K}^n \rightsquigarrow \mathbb{K}^m$ has a finite number of definably connected components, then it is a \mathcal{C}^N admissible correspondence.*

Notice that in the above lemma we can not drop the hypothesis that ϕ is a diffeomorphism, and replace it with the hypothesis that it is a \mathcal{C}^N function, and similarly we cannot drop the additional conditions on θ .

- In fact, if $m = 1$, $n > 1$, and $\phi(x) = x^2$, it might happen that the graph of $\phi \circ f$ self-intersects. For example, let g be defined as in Example 7.11(3). Then the graph of g^2 self-intersects.
- For instance, let \mathbb{K} be an expansion of \mathbb{R} where the sine function is defined; let $f(x) := 1/x$, and $\theta(t) := \sin t$. Then, $f \circ \theta = 1/\sin(t)$ is not admissible.

Lemma 7.16. *Let $m = 1$, and define $g : \mathbb{K}^{n+1} \rightsquigarrow \mathbb{K}$ as $g(\bar{x}, y) := y - f(\bar{x})$. That is, the graph of g is $G := \{(\bar{x}, y, z) \in \mathbb{K}^{n+2} : (\bar{x}, z - y) \in F\}$. Then, g is \mathcal{C}^N and admissible.*

Lemma 7.17. *Given $g : \mathbb{K}^n \rightarrow \mathbb{K}$ a (total and single-valued) \mathcal{C}^N and definable function, define the correspondence $h := \langle f, g \rangle : \mathbb{K}^n \rightsquigarrow \mathbb{K}^{m+1}$; that is, the graph of h is*

$$H := \{(\bar{x}, \bar{y}, z) \in \mathbb{K}^{n+m+1} : (\bar{x}, \bar{y}) \in F \ \& \ z = g(\bar{x})\}.$$

Then, h is \mathcal{C}^N and admissible.

Definition 7.18. For every $(\bar{x}, \bar{y}) \in F$, it makes sense to define $Df(\bar{x}; \bar{y})$, the differential of f at the point (\bar{x}, \bar{y}) (the notational difference with the usual case when f is a function is that here we have to specify at which $\bar{y} \in f(\bar{x})$ we compute Df). As usual, we say that (\bar{x}, \bar{y}) is a regular point for f if $Df(\bar{x}; \bar{y})$ has maximal rank, otherwise (\bar{x}, \bar{y}) is singular. Similarly, $\bar{y} \in \mathbb{K}^n$ is a regular value if, for every $\bar{x} \in f^{-1}(y)$, (\bar{x}, \bar{y}) is a regular point; otherwise, \bar{y} is a singular value.

Moreover, we have a correspondence on \mathbb{K}^n , which assign to every point \bar{x} the values of $Df(\bar{x}; \bar{y})$, as \bar{y} varies in $f(\bar{x})$. This correspondence in general is not admissible, even if $N \geq 2$, because its graph might self-intersect. The following lemma addresses this point.

Lemma 7.19. *Assume that $N \geq 2$.*

- Let $\tilde{D}f$ be the correspondence $\langle f, Df \rangle$ on \mathbb{K}^n . That is, the graph of $\tilde{D}f$ is

$$H := \{(\bar{x}, \bar{y}, \bar{z}) : (\bar{x}, \bar{y}) \in F \ \& \ \bar{z} = Df(\bar{x}; \bar{y})\}.$$

Then, $\tilde{D}f$ is \mathcal{C}^{N-1} and admissible.

- Assume that $n = m + k$, with $k \geq 1$. Fix $1 \leq i_1 < \dots < i_k \leq n$. Then, the correspondence

$$\langle f, \det\left(\frac{\partial(f_1, \dots, f_k)}{\partial(x_{i_1}, \dots, x_{i_k})}\right)^2 \rangle$$

is admissible.

The two previous lemmas are particular cases of the following:

Lemma 7.20. Given $1 \leq M \leq N$, let $g : F \rightarrow \mathbb{K}^k$ be a \mathcal{C}^M function. Let $h := \langle f, g \rangle$; that is, the graph of h is

$$H := \{(\bar{x}, \bar{y}, \bar{z}) : (\bar{x}, \bar{y}) \in F \text{ \& } z = g(\bar{x}, \bar{y})\}.$$

Then, h is \mathcal{C}^M and admissible.

Proof. Since g is continuous, H is closed in $F \times \mathbb{K}^k$. Since F is closed in \mathbb{K}^{n+m} , H is closed in \mathbb{K}^{n+m+k} . \square

Lemma 7.21. For $i = 1, 2$, let $f_i : \mathbb{K}^{n_i} \rightsquigarrow \mathbb{K}^{m_i}$ be an admissible \mathcal{C}^N correspondence, with graph F_i . The, the correspondence $f_1 \times f_2 : \mathbb{K}^{n_1+n_2} \rightsquigarrow \mathbb{K}^{m_1+m_2}$, with graph $F_1 \times F_2$, is an admissible \mathcal{C}^N correspondence.

Definition 7.22. Given a correspondence $g : \mathbb{K}^n \rightsquigarrow \mathbb{K}^m$, we denote by $|g|$ the correspondence $|g| : \mathbb{K}^n \rightsquigarrow \mathbb{K}$, with graph $|G| := \{(x, t) : \exists \bar{y} \in \mathbb{K}^m (\bar{x}, \bar{y}) \in F \text{ \& } |\bar{y}| = t\}$.

Definition 7.23. Given $C \subseteq \mathbb{K}^n$ and $g : \mathbb{K}^n \rightsquigarrow \mathbb{K}$ correspondence with graph G , and $\bar{x} \in C$, we say that g reaches the minimum on C at \bar{x} , if there exists $y \in g(\bar{x})$ such that, for every $(\bar{x}', y') \in G$, if $\bar{x}' \in C$, then $y \leq y'$; moreover, y is the minimum of g on C .

We also define

$$\inf_{\bar{x} \in C} g(\bar{x}) := \inf g(C) \in \mathbb{K} \sqcup \{\pm\infty\}.$$

Notice that $\inf_{\bar{x} \in C} g(\bar{x}) = +\infty$ iff $g(C) = \emptyset$.

Lemma 7.24. Let $f : \mathbb{K}^n \rightsquigarrow \mathbb{K}$ be admissible, and $C \subseteq \mathbb{K}^n$ be definable, such that $f(C)$ is non-empty.

1. If C is d -compact, then $|f|$ reaches the minimum on C ;
2. however, C d -compact does not imply that $|f|$ reaches the maximum on C ;
3. if $\inf_{\bar{x} \in C} f(\bar{x}) \neq -\infty$, then f reaches the minimum on C (and similarly for the maximum).

Proof. The graph $|F|$ of $|f|$ is closed in $C \times [0, +\infty)$, and C is d -compact; hence, $\pi(|F|)$ is closed in $[0, +\infty)$, where $\pi : C \times [0, +\infty) \rightarrow [0, +\infty)$ is the projection onto the second coordinate. \square

7.2 Preliminary results

In this subsection we develop some preliminary results (corresponding to the results in [Wilkie99, §2]) about o-minimal weak structures. We fix for the rest of the subsection a semi-closed o-minimal weak structure \mathcal{S} .

Lemma 7.25. *Let $A \in \tilde{\mathcal{S}}$. Then the following are equivalent:*

1. A has empty interior.
2. \bar{A} has empty interior.
3. \bar{A} is meager.
4. A is meager.

Proof. We first observe that, $\tilde{\mathcal{S}}$ being semi-closed, every set in $\tilde{\mathcal{S}}$ is an \mathcal{F}_σ -set. In particular, by 3.5, the implications $(1 \Rightarrow 4)$ and $(2 \Rightarrow 3)$ are proven.

The implication $(1 \Rightarrow 2)$ can be proven as in [Maxwell98, Lemma 2.7], where we conclude by using 5.6 instead of Fubini's Theorem, and the previous observation.

The other implications are obvious. \square

With similar modification of Maxwell's proof, one can prove the following.

Lemma 7.26. *Let $A \in \tilde{\mathcal{S}}_{n+1}$ and $A \subset \mathbb{K}^n \times \mathbb{K}_+$. If A has empty interior, then $\bar{A}_0 \subset \mathbb{K}^n$ also has empty interior.*

To obtain versions of [Wilkie99, Theorems 2.3 - 2.6] (whose proofs are in [Maxwell98, §4]) which hold in definably complete Baire structures, we need to reprove some of Maxwell's results.

Lemma 7.27. *Let $C \in \tilde{\mathcal{S}}_{n+1}$ have empty interior, and $B_0 \subset \mathbb{K}^n$ be an open box. Then there exist $p \in \mathbb{N}$ and an open box $B \subseteq B_0$ such that for each $\bar{x} \in B$ the fiber $C_{\bar{x}}$ contains exactly p points.*

Proof. The proof can be easily adapted from [Maxwell98, Lemma 3.1], using 5.6, together with the o-minimality of $\tilde{\mathcal{S}}$ and Lemma 7.25. \square

Proposition 7.28. *Let $f : \mathbb{K}^n \rightarrow \mathbb{K}$ such that $\Gamma(f) \in \tilde{\mathcal{S}}_{n+1}$. Let $\mathcal{D}_f := \{\bar{x} \in A : f \text{ is discontinuous at } \bar{x}\}$. Then $\mathcal{D}_f \in \tilde{\mathcal{S}}$ and \mathcal{D}_f has empty interior.*

Proof. The graph of f is in $\tilde{\mathcal{S}}$, and therefore it is an \mathcal{F}_σ ; hence, by Lemma 3.11, \mathcal{D}_f has empty interior.

It remains to show that \mathcal{D}_f is in $\tilde{\mathcal{S}}$. Up to a change of coordinate in the codomain, we can assume that f is bounded. Let $F \subset \mathbb{K}^{n+1}$ be the closure of $\Gamma(f)$. Then, since f is bounded, we have that $\mathcal{D}_f = \{x \in \mathbb{K}^n : |F_x| \geq 2\}$. The latter set is in $\tilde{\mathcal{S}}$. \square

Having Proposition 7.28, the proofs of the results in [Maxwell98, §4] go easily through in our case as well, up to some minor or obvious modifications,⁽⁴⁾ and so does the proof of the following version of [Wilkie99, Theorem 2.7].

Theorem 7.29 (Sard’s Lemma for \mathcal{C}^1 functions in $\tilde{\mathcal{S}}$). *Suppose that $n \geq m \geq 1$ and that $f : U \rightarrow \mathbb{K}^m$ is a \mathcal{C}^1 function in $\tilde{\mathcal{S}}$, where $U \subset \mathbb{K}^n$ is open. Then the set of singular values of f is in $\tilde{\mathcal{S}}$ and has empty interior (hence, it is meager in \mathbb{K}^m).*

Corollary 7.30. *The above statement still holds if f is a \mathcal{C}^1 admissible correspondence whose graph F is in $\tilde{\mathcal{S}}$.*

Proof. By the Implicit Function Theorem and the definition of F , every point $(\bar{x}, \bar{y}) \in F$ has a definable neighbourhood $U = U_1 \times U_2 \subset \mathbb{K}^n \times \mathbb{K}^m$ such that $U \cap F$ is the graph of a \mathcal{C}^1 function $f_U : U_1 \rightarrow U_2$. By reducing U , if necessary, we can ensure that U_1 and U_2 are in $\tilde{\mathcal{S}}$ (in fact, we can assume they are boxes), so that $f_U \in \tilde{\mathcal{S}}$. We can apply Theorem 7.29 to f_U and obtain that the set of its singular values is meager. Now, since the set Σ_f of the singular values of f is given by $\bigcup_U \Sigma_{f_U}$, we can apply Corollary 3.7 to the projection $\pi : \mathbb{K}^n \times \mathbb{K}^m \rightarrow \mathbb{K}^m$ onto the second factor and obtain that Σ_f is meager. As in the proof of Theorem 7.29, it is clear that $\Sigma_f \in \tilde{\mathcal{S}}$. \square

We now turn our attention to [Wilkie99, Corollary 2.9], which provides the main tool for the approximation of the boundary of the projection of a set in $\tilde{\mathcal{S}}$. We need some preliminary lemmas. The following is our version of [Wilkie99, Theorem 2.8], and its proof does not present difficulties.

Lemma 7.31. *Suppose that $n > m \geq 1$, $\bar{a} \in \mathbb{K}^m$, $g : \mathbb{K}^n \rightsquigarrow \mathbb{K}^m$ is a \mathcal{C}^1 admissible correspondence in $\tilde{\mathcal{S}}$, $h : \mathbb{K}^n \rightarrow \mathbb{K}$ is a \mathcal{C}^1 function in $\tilde{\mathcal{S}}$, and*

⁽⁴⁾In the proof of [Maxwell98, Lemma 4.12] the “unbounded case”(iv) will be treated by using, instead of sequences, the function g , defined in [Maxwell98, Lemma 3.2] as $\Gamma(g) := \{(\bar{x}, y) \mid \exists z((\bar{x}, z) \in \Gamma(f) \wedge zy = 1) \vee ((\bar{x}, 0) \in \Gamma(f) \wedge y = 1)\}$, where $f := \frac{\partial f^*}{\partial x_i}$.

that \bar{a} is a regular value of g . Then there are at most finitely many $b \in \mathbb{K}$ such that (\bar{a}, b) is a singular value of the admissible correspondence $\langle g, h \rangle$.

Lemma 7.32. *Let $f : \mathbb{K}^n \rightsquigarrow \mathbb{K}^m$ be a C^1 admissible correspondence, with graph F . Let $b \in \mathbb{K}$ and $1 \leq i \leq n$. Define*

$$\hat{F} := \{(\bar{x}, \bar{y}) \in \mathbb{K}^{n-1} \times \mathbb{K}^m : (x_1, \dots, x_{i-1}, b, x_{i+1}, \dots, x_n, y_1, \dots, y_m) \in F\}.$$

\hat{F} is the graph of a correspondence \hat{f} . Assume that \hat{F} has a finite number of definably connected components. Then, \hat{f} is C^N admissible. Moreover, given $\bar{a} \in \mathbb{K}^m$, if (\bar{a}, b) is a regular value for $\langle f, \pi_i \rangle$ (where $\pi_i : \mathbb{K}^n \rightarrow \mathbb{K}$ is the projection onto the i th coordinate), then \bar{a} is a regular value for \hat{f} .

We can now prove the analogue of [Wilkie99, Corollary 2.9].

Proposition 7.33. *Let $n, k \geq 1$, $f = \langle f_1, \dots, f_k \rangle : \mathbb{K}^{n+k} \rightarrow \mathbb{K}^k$ be an admissible C^1 correspondence in \tilde{S} , and $V := V(f)$. Suppose further that 0 is a regular value of f , and that U is an open ball in \mathbb{K}^n with the property that the set $X := V \cap \pi^{-1}(U)$ is non-empty and bounded, where $\pi := \Pi_n^{n+k}$. Then either (i) $\pi(X) = U$, or (ii) there exists $\eta > 0$ and distinct $i_1, \dots, i_k \in \mathbb{N}$ with $1 \leq i_1, \dots, i_k \leq n+k$ such that $\det\left(\frac{\partial(f_1, \dots, f_k)}{\partial(x_{i_1}, \dots, x_{i_k})}\right)^2(\bar{x}; 0)$ takes all values in the interval $[0, \eta]$ on X .*

Proof. The proof proceeds as in the original [Wilkie99, Corollary 2.9]. Since f is admissible, V is closed in K^{n+k} .

In the case when $\pi(X)$ is finite, as in Wilkie's proof, one shows that there exists Y a definably connected component of V contained in X . Since V is closed and X is bounded, Y is d-compact. Let $1 \leq i_1 < \dots < i_k \leq n+k$, $j \neq i_1, \dots, i_k$, and $z \in Y$ be a point where the map $\bar{x} \mapsto x_j$ is maximal. This clearly implies that $\det\left(\frac{\partial(f_1, \dots, f_k)}{\partial(x_{i_1}, \dots, x_{i_k})}\right)(z; 0) = 0$, and one concludes this case as in [Wilkie99].

In the case when $\pi(X)$ is infinite, let Y be a definably connected component of X (which is definable, since it is an atom of the finite boolean algebra formed by all definable clopen subsets of X). It follows that either $\det\left(\frac{\partial(f_1, \dots, f_k)}{\partial(x_{n+1}, \dots, x_{n+k})}\right)(z; 0) = 0$ for some $z \in Y$, or $\pi(Y) = U$. Using Lemma 7.31 and Lemma 7.32, we conclude as in Wilkie's proof. \square

7.3 Weakening the DSF condition and the theorem of the boundary

Let \mathcal{S} be a semi-closed o-minimal weak structure.

Definition 7.34. \mathcal{S} satisfies DAC^N for all N if for each $A \in \mathcal{S}_n$, there exist $m \geq n$ and $r \geq 1$, such that, for each N , there exists a set $S_N \subseteq \mathbb{K}^m$, which is a *finite union* of sets, each of which is an intersection of at most r sets of the form $V(f_{N,i})$, where each $f_{N,i} : \mathbb{K}^m \rightarrow \mathbb{K}$ is an *admissible* \mathcal{C}^N correspondence in $\tilde{\mathcal{S}}$, and $A = \Pi_n^m(S_N)$

In the above definition, note that:

1. Each set S_N is of the form $S_N = \bigcup_{0 \leq j < k_N} S_{N,j}$ (for some natural number k_N), where each set $S_{N,j}$ is of the form

$$S_{N,j} = \bigcap_{0 \leq i < r} V(f_{N,rj+i}).$$

2. If S_N is an intersection of r sets, each of which is a finite union of sets of the form $V(f_{N,i})$ (where $f_{N,i} : \mathbb{K}^m \rightarrow \mathbb{K}$ are admissible \mathcal{C}^N correspondences in $\tilde{\mathcal{S}}$), then S_N can be rewritten in a way to satisfy the conditions in the above definition (with the same (m, r) , and using the same correspondences $f_{N,i}$).
3. m and r do *not* depend on N ; however, the number of sets forming the union (and therefore the total number of correspondences $f_{N,i}$) might depend on N .
4. We only ask the correspondences $f_{N,i}$ to be in $\tilde{\mathcal{S}}$, not in \mathcal{S} , and only that they are admissible correspondences, instead of total functions. Thus, the condition above is weaker than the one formulated in [KM99], even for $\mathbb{K} = \mathbb{R}$. Moreover, \mathcal{S} satisfying DAC^N for all N does not imply that \mathcal{S} is semi-closed.
5. DSF implies DAC^N for all N .
6. If each $f_{N,i}$ is a (total single-valued) function, we can replace the functions $f_{N,i}$ by a single function f_N , obtained from the $f_{N,i}$ using products and sums of squares; this is the reason why in [KM99] only one function f_N is used (and in [Wilkie99] one \mathcal{C}^∞ function f). However, for general admissible correspondences, we can not conclude that f_N is admissible.

7. Let S be a semi-closed o-minimal weak structure satisfying DAC^N for all N . Then it is harmless to assume S to be closed: if S' is the collection of all closed sets in S , then S' is a closed o-minimal weak structure satisfying DAC^N for all N and moreover the Charbonnel closures of S and S' coincide.

Theorem 7.35 (After [KM99]). *Suppose that S is a semi-closed o-minimal weak structure satisfying DAC^N for all N . Then \tilde{S} is o-minimal, and the smallest structure containing S .*

Before proving the above theorem, we will give a lemma which is useful in applications.

Lemma 7.36. *Let $A_1, \dots, A_l \in \mathcal{S}_n$ satisfy the condition for A in Definition 7.34 (we say that A_i satisfy DAC^N for all N). Then, also every finite positive Boolean combination (PBC) of A_1, \dots, A_l satisfies DAC^N for all N . Hence, if S' is a subset of S , such that:*

- every set in S' satisfies DAC^N for all N , and
- every set in S is a PBC of sets in S' ,

then S satisfies DAC^N for all N , and therefore, by Theorem 7.35, \tilde{S} is an o-minimal structure.

Proof. It is clear that it suffices to prove the following: for every $A^1, A^2 \in \mathcal{S}_n$, if each A^i satisfies DAC^N for all N , then $A^1 \cup A^2$ and $A^1 \cap A^2$ also satisfy DAC^N for all N .

For $i = 1, 2$, let (m_i, r_i) be the DC-complexity of A^i , and let $m := \max(m_1, m_2)$, $r := \max(r_1, r_2)$. For each N and $i = 1, 2$, let

$$S_N^i := S_{N,1}^i \cup \dots \cup S_{N,k_{i,N}}^i,$$

such that each $S_{N,j}^i \in \mathcal{S}_{m_i}$ is an intersection of r_i sets of the form $V(g)$ for some admissible \mathcal{C}^N correspondence $g : \mathbb{K}^{m_i} \rightsquigarrow \mathbb{K}$ in $\tilde{\mathcal{S}}$, and $A^i = \Pi_n^{m_i} S_N^i$. Notice that we can always assume that $m_1 = m_2 = m$ and $r_1 = r_2 = r$. In fact, for each g as above, define $\tilde{g} : \mathbb{K}^m \rightarrow \mathbb{K}$ as $\tilde{g}(x_1, \dots, x_m) := g(x_1, \dots, x_{m_i})$, and substitute $V(g)$ with $V(\tilde{g})$ everywhere.

For the union, notice that $S_N^1 \cup S_N^2 = S_{N,1}^1 \cup \dots \cup S_{N,k_{1,N}}^1 \cup S_{N,1}^2 \cup \dots \cup S_{N,k_{2,N}}^2$, and $A^1 \cup A^2 = \Pi_n^m (S_N^1 \cup S_N^2)$.

For the intersection, let $\Lambda := \{(\bar{x}, \bar{x}') \in \mathbb{K}^m \times \mathbb{K}^m : x_i = x'_i, 1 \leq i \leq n\}$, Notice that

$$A^1 \cap A^2 = \Pi_n^{2m} ((S_N^1 \times S_N^2) \cap \Lambda) = \bigcup_{\substack{j \leq k_{1,N}, \\ j' \leq k_{2,N}}} \Pi_n^{2m} (S_{N,j}^1 \times S_{N,j'}^2) \cap \Lambda).$$

Hence, w.l.o.g. $k_{1,N} = k_{2,N} = 1$, and $S_N^i = \bigcap_{j \leq r} V(f_{N,j}^i)$, for some $f_{N,j}^i : \mathbb{K}^m \rightsquigarrow \mathbb{K}$ admissible \mathcal{C}^N correspondence in $\tilde{\mathcal{S}}$, $i = 1, 2$. Thus, by distributivity, it suffices to treat the case when $S_N^i = V(f_{N,1}^i)$, $i = 1, 2$. Define $\tilde{F}_N^1 := \{(\bar{x}, \bar{x}', \bar{y}) : (\bar{x}, \bar{y}) \in \Gamma(f_{N,1}^1)\}$, and $\tilde{F}_N^2 := \mathbb{K}^m \times \Gamma(f_{N,1}^2)$. Notice that \tilde{F}_N^i is the graph of an admissible \mathcal{C}^N correspondence $f_N^i : \mathbb{K}^m \times \mathbb{K}^m \rightsquigarrow \mathbb{K}$ in $\tilde{\mathcal{S}}$. Moreover, $\Lambda = V(q)$ for some polynomial $q : \mathbb{K}^m \times \mathbb{K}^m \rightarrow \mathbb{K}$. Finally, $(S_N^1 \times S_N^2) \cap \Lambda = V(\tilde{f}_N^1) \cap V(\tilde{f}_N^2) \cap V(q)$, and therefore $A^1 \cap A^2 = \Pi_n^m(V(\tilde{f}_N^1) \cap V(\tilde{f}_N^2) \cap V(q))$. \square

Proviso. We fix for the rest of the section a closed o-minimal weak structure \mathcal{S} satisfying DAC^N for all N .

We will prove the following result, corresponding to [Wilkie99, Theorem 3.1].

Theorem 7.37. *Let $A \in \tilde{\mathcal{S}}_n$ be closed. Then there exists a closed set $B \in \tilde{\mathcal{S}}_n$ such that B has empty interior and $\text{bd}(A) \subseteq B$.*

Notice that, even without the DAC^N hypothesis, the following is true: if A is closed, then $\text{bd}(A)$ has empty interior. The missing information is whether $\text{bd}(A)$ is in $\tilde{\mathcal{S}}$ or not.

We will follow the outline of [Wilkie99, §3], but we will use [BS04] for some definitions and proofs. The two approaches are equivalent, but we find the latter easier to read.

Definition 7.38.

- Given $\bar{x} \in \mathbb{K}^n$, let $|\bar{x}| := \max\{|x_1|, \dots, |x_n|\}$, and $\|\bar{x}\| := \sqrt{x_1^2 + \dots + x_n^2}$. Notice that $\bar{x} \mapsto \|\bar{x}\|^2$ is a \mathcal{C}^∞ function, and so is the function $\bar{x} \mapsto \frac{1}{1+\|\bar{x}\|^2}$.
- Given $A \subseteq \mathbb{K}^n$ and $\varepsilon \in \mathbb{K}_+$, define the ε -neighborhood A^ε of A as the set $\{x \in \mathbb{K}^n \mid \exists y \in A \mid |x - y| < \varepsilon\}$.
- (The quantifier “for all sufficiently small”) We write $\forall^s \varepsilon \phi$ as a shorthand for $(\exists \mu)(\forall \varepsilon < \mu)\phi$, where μ, ε are always assumed to range in \mathbb{K}_+ . If $\bar{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)$, then $\forall^s \bar{\varepsilon}$ is an abbreviation for $\forall^s \varepsilon_1 \dots \forall^s \varepsilon_n$.
- (Sections) Given $S \subseteq \mathbb{K}^n \times \mathbb{K}_+^k$ and given $\varepsilon_1, \dots, \varepsilon_k \in \mathbb{K}_+$, we define $S_{\varepsilon_1, \dots, \varepsilon_k}$ as the set $\{x \in \mathbb{K}^n \mid (x, \varepsilon_1, \dots, \varepsilon_k) \in S\}$.
- Let $A \subseteq \mathbb{K}^n$, $S \subseteq \mathbb{K}^n \times \mathbb{K}_+^k$. S approximates A from below ($S \leq A$) if

$$\forall^s \varepsilon_0 \forall^s \varepsilon_1 \dots \forall^s \varepsilon_k (S_{\varepsilon_1, \dots, \varepsilon_k} \subseteq A^{\varepsilon_0}).$$

- Let $A \subseteq \mathbb{K}^n, S \subseteq \mathbb{K}^n \times \mathbb{K}_+^k$. S approximates A from above on bounded sets ($S \geq A$) if

$$\forall^s \varepsilon_0 \forall^s \varepsilon_1 \dots \forall^s \varepsilon_k (A \cap B(0, 1/\varepsilon_0) \subseteq (S_{\varepsilon_1, \dots, \varepsilon_k})^{\varepsilon_0}).$$

The reader can check the the following holds.

Lemma 7.39. For $i = 0, 1, 2$, let $\bar{\varepsilon}_i$ and \bar{x}_i be tuples of length n_i . Let $\phi_1(\bar{x}_0, \bar{x}_1)$ and $\phi_2(\bar{x}_0, \bar{x}_2)$ be formulae, with the shown free variables (and possibly bounded variables and parameters). Define

$$\phi(\bar{x}_0, \bar{x}_1, \bar{x}_2) := \phi_1(\bar{x}_0, \bar{x}_1) \ \& \ \phi_2(\bar{x}_0, \bar{x}_2).$$

Then,

$$\begin{aligned} \mathbb{K} \models \left((\forall^s \bar{\varepsilon}_0 \forall^s \bar{\varepsilon}_1 \phi_1(\bar{\varepsilon}_0, \bar{\varepsilon}_1)) \ \& \ (\forall^s \bar{\varepsilon}_0 \forall^s \bar{\varepsilon}_2 \phi_2(\bar{\varepsilon}_0, \bar{\varepsilon}_2)) \right) \rightarrow \\ \rightarrow (\forall^s \bar{\varepsilon}_0 \forall^s \bar{\varepsilon}_1 \forall^s \bar{\varepsilon}_2 \phi(\bar{\varepsilon}_0, \bar{\varepsilon}_1, \bar{\varepsilon}_2)). \end{aligned}$$

We will show that, to obtain Theorem 7.37, it is enough to prove the following:

Proposition 7.40. For each $n \in \mathbb{N}$, $A \in \tilde{\mathcal{S}}_n$, and each $N \geq 1$, the following holds:

(Φ_N): There exists $k \geq 1$ (the N -complexity of A) and a set $S \subseteq \mathbb{K}^n \times \mathbb{K}_+^k$ (an $\mathcal{S}(N)$ -approximant of A) which is a finite union of sets of the form

$$\{(\bar{x}, \bar{\varepsilon}) \in \mathbb{K}^n \times \mathbb{K}_+^k : \exists \bar{y} \in \mathbb{K}^{k-1} f(\bar{x}, \bar{y}) \ni \bar{\varepsilon}\}$$

(the $\mathcal{S}(N)$ -constituents of S), where $f : \mathbb{K}^{n+k-1} \rightsquigarrow \mathbb{K}^k$ is admissible, \mathcal{C}^N and in $\tilde{\mathcal{S}}$, such that S both approximates $\text{bd}(\bar{A})$ from above on bounded sets and approximates \bar{A} from below.

A set S of the above form is called an $\mathcal{S}(N)$ -set.

Remark 7.41. Let $0 \leq d < k$, and $f : \mathbb{K}^n \times \mathbb{K}^d \rightsquigarrow \mathbb{K}^k$ be an admissible \mathcal{C}^N correspondence in $\tilde{\mathcal{S}}$. Let $S(f) := \{(\bar{x}, \bar{\varepsilon}) \in \mathbb{K}^n \times \mathbb{K}_+^k : \exists \bar{y} \in \mathbb{K}^d f(\bar{x}, \bar{y}) \ni \bar{\varepsilon}\}$. Then, $S(f)$ is of the form $\{(\bar{x}, \bar{\varepsilon}) \in \mathbb{K}^n \times \mathbb{K}_+^k : \exists \bar{z} \in \mathbb{K}^{k-1} \tilde{f}(\bar{x}, \bar{z}) \ni \bar{\varepsilon}\}$, where $\tilde{f} : \mathbb{K}^n \times \mathbb{K}^{k-1} \rightsquigarrow \mathbb{K}^k$ is an admissible \mathcal{C}^N correspondence in $\tilde{\mathcal{S}}$; in particular, $S(f)$ is an $\mathcal{S}(N)$ -constituent. The graph of \tilde{f} is

$$\tilde{F} := \{(\bar{x}, \bar{z}, \bar{w}) \in \mathbb{K}^n \times \mathbb{K}^{k-1} \times \mathbb{K}^k : (\bar{x}, z_1, \dots, z_d, \bar{w}) \in F\}.$$

The following statement corresponds to [BS04, Lemma 6.7] (which is a remark at the end of [Wilkie99, §4]).

Lemma 7.42. *Given $N \geq 1$, every $\mathcal{S}(N)$ -set has empty interior.*

Proof. It suffices to show that each $\mathcal{S}(N)$ -constituent S has empty interior. S is of the form $\text{Im}(g) \cap (\mathbb{K}^n \times \mathbb{K}_+^k)$, where

$$g : \mathbb{K}^{n+k-1} \rightsquigarrow \mathbb{K}^{n+k} \\ (\bar{x}, \bar{y}) \mapsto (\bar{x}, f(\bar{x}, \bar{y})),$$

for some $f : \mathbb{K}^{n+k-1} \rightsquigarrow \mathbb{K}^k$ admissible, \mathcal{C}^N and in $\tilde{\mathcal{S}}$. Since g is in $\tilde{\mathcal{S}}$, Theorem 7.30 implies that the image of g has empty interior. \square

The following two statements correspond to [Wilkie99, Lemmas 3.3 and 3.4], and their proofs do not present particular difficulties: it is enough to use Lemma 5.6 and Lemma 7.25 instead of Fubini's Theorem and [Wilkie99, Theorem 2.1].

Lemma 7.43. *Let $A \in \tilde{\mathcal{S}}_n$, $S \in \tilde{\mathcal{S}}_{n+k}$. Suppose that S has empty interior and is an $\mathcal{S}(N)$ -approximant for A . Then so is the section $\tilde{S}_0 = \{\bar{x} \in \mathbb{K}^n \mid (\bar{x}, \bar{0}) \in \tilde{S}\} \in \tilde{\mathcal{S}}_n$.*

Lemma 7.44. *If $A \in \tilde{\mathcal{S}}_k$ has empty interior, then $\forall^s \bar{\varepsilon} \bar{\varepsilon} \notin A$.*

The remainder of this subsection is devoted to the proof of Proposition 7.40. Theorem 7.37 follows immediately from the proposition and Lemma 7.43.

The proof of Proposition 7.40 follows the pattern of [BS04, §10]; however, we need to prove some more intermediate steps, due to the fact that we are dealing with several, not just one, correspondences in Definition 7.34.

Lemma 7.45 (Union). *Let $N, r, n \geq 1$, A_1, \dots, A_r be subsets of \mathbb{K}^n , and, for $i = 1, \dots, r$, let $S_i \subseteq \mathbb{K}^n \times \mathbb{K}_+^{k_i}$ be an $\mathcal{S}(N)$ -approximant for A_i . Then, $A := \bigcup_i A_i$ has an $\mathcal{S}(N)$ -approximant.*

Proof. We may suppose that all the A_i have the same N -complexity k ; then, $\bigcup_i S_i$ is an $\mathcal{S}(N)$ -approximant of A . \square

Lemma 7.46. *Let $f : \mathbb{K}^n \rightsquigarrow \mathbb{K}$ be an admissible \mathcal{C}^N correspondence, and define $S := \{(\bar{x}, t) \in \mathbb{K}^n \times \mathbb{K}_+ : |f(\bar{x})| \ni t\}$. Then S approximates $\text{bd}[V(f)]$ from above on bounded sets.*

Proof. Fix $\varepsilon > 0$, and let $V := V(f)$. Let $X := \text{bd}(V) \cap \overline{B(0; 1/\varepsilon)}$, and $Y_t := X \setminus (|f|^{-1}(t)^\varepsilon)$. Note that X and Y_t are \mathbf{d} -compact. Let

$$P := \{t \in \mathbb{K} : t > 0 \ \& \ Y_t \neq \emptyset\}.$$

Assume for contradiction that the conclusion is false. This implies that P has arbitrarily small elements, if we chose ε small enough. Let $\bar{x} \in \text{acc}_{t \rightarrow 0^+} Y_t$ (\bar{x} exists, because each Y_t is contained in the d-compact set X), and $U := B(\bar{x}; \varepsilon/2)$. Note that V is closed (because f is admissible), and that $\bar{x} \in \text{bd}(V)$.

By shrinking ε if necessary, we may assume that there exists $\delta > 0$, such that $F \cap (U \times (-\delta, \delta))$ is the graph of a \mathcal{C}^N function $g : U \rightarrow (-\delta, \delta)$, such that $g(\bar{x}) = 0$. Since $\bar{x} \in \text{bd}(V)$, $|g|$ assumes a positive value γ on U . Since U is definably connected and g is continuous, $|g|$ assumes all values in the interval $[0, \gamma]$ in U . Choose $t_0 \in P$ such that $Y_{t_0} \cap U \neq \emptyset$, and $t_0 < \gamma$. Since $t_0 < \gamma$, $U \cap |g|^{-1}(t_0) \neq \emptyset$; therefore, $U \subseteq |g|^{-1}(t_0)^\varepsilon$, and thus $Y_{t_0} \cap U = \emptyset$, a contradiction. \square

Lemma 7.47 (Zero-set of correspondences). *If $f : \mathbb{K}^n \rightsquigarrow \mathbb{K}$ is admissible, \mathcal{C}^N and in \tilde{S} , then its zero set $V(f)$ has an $\mathcal{S}(N)$ -approximant $S \in \tilde{\mathcal{S}}_{n+2}$.*

Proof. Define the following 2 sets S_+ and S_- :

$$S_\pm := \{(\bar{x}, \varepsilon_1, \varepsilon_2) \in \mathbb{K}^n \times \mathbb{K}_+^2 : 1 + \|\bar{x}\|^2 \leq 1/\varepsilon_1 \text{ \& } f(\bar{x}) \ni \pm \varepsilon_2\},$$

and $S := S_+ \cup S_-$. By Lemma 7.17, $\langle \pm f, \phi \rangle$ are \mathcal{C}^N and admissible, where $\phi : \mathbb{K}^{n+1} \rightarrow \mathbb{K}$, $(\bar{x}, y) \mapsto (1 + \|\bar{x}\|^2 + y^2)^{-1}$ (and in \tilde{S}). Thus, S is an $\mathcal{S}(N)$ -set.

We prove that S approximates $V(f)$ from below, namely

$$\forall^s \varepsilon_0 \forall^s \varepsilon_1 \forall^s \varepsilon_2 S_{\varepsilon_1, \varepsilon_2} \subseteq V(f)^{\varepsilon_0}.$$

Let $K := \{\bar{x} \in \mathbb{K}^n : 1 + \|\bar{x}\|^2 \leq 1/\varepsilon_1\}$, and $H := K \setminus V(f)^{\varepsilon_0}$. Note that K and H are d-compact, and $S_{\varepsilon_1, \varepsilon_2} \subseteq K$.

Claim. $|f|$ has a positive minimum on H , if $f(H)$ is non-empty.

If not, then, by Lemma 7.24, there exists $\bar{x} \in H$ such that $|f(\bar{x})| \ni 0$; however, this means that $\bar{x} \in V(f) \cap H$, contradicting the definition of H .

Thus, if we choose ε_2 smaller than the minimum of $|f|$ on H (or arbitrarily if H is empty), then $S_{\varepsilon_1, \varepsilon_2} \cap H = \emptyset$, and therefore $S_{\varepsilon_1, \varepsilon_2} \subseteq K \cap V(f)^{\varepsilon_0} \subseteq V(f)^{\varepsilon_0}$.

To prove that S approximates $\text{bd}(V(f))$ from above we proceed as in [BS04, Lemma 10.3], using Lemma 7.46 instead of [BS04, Lemma 10.2]. \square

Lemma 7.48 (Projection). *Let $N \geq 1$. If $A \subseteq \mathbb{K}^{n+1}$ has an $\mathcal{S}(N+1)$ -approximant $S \subseteq \mathbb{K}^{n+1} \times \mathbb{K}_+^k$, then there is an $\mathcal{S}(N)$ -approximant $S' \subseteq \mathbb{K}^n \times \mathbb{K}_+^{k+1}$ for $\Pi_n^{n+1} A \subseteq \mathbb{K}^n$.*

The drop in regularity in the above lemma from $N + 1$ to N is due to the fact that the definition of S' involves the derivatives of the functions defining S .

Proof. Define $S_{\varepsilon_1, \dots, \varepsilon_k}$, $S_{\varepsilon_1, \dots, \varepsilon_k}[\varepsilon_{k+1}]$, and $S'_{\varepsilon_1, \dots, \varepsilon_{k+1}}$ as in the proof of [BS04, Lemma 10.6]. More precisely, $S = S^1 \cup \dots \cup S^l$, where each S^i is an $\mathcal{S}(N)$ -constituent. Hence, each S^i is of the form $S^i = F^i \cap (\mathbb{K}^{n+1} \times \mathbb{K}^{k-1} \times \mathbb{K}_+^k)$, where each F^i is the graph of some admissible \mathcal{C}^{N+1} correspondence $f^i : \mathbb{K}^{n+1} \times \mathbb{K}^{k-1} \rightsquigarrow \mathbb{K}^k$. Define, $S_{\bar{\varepsilon}}[\varepsilon_{k+1}] := \bigcup_i S_{\bar{\varepsilon}}^i[\varepsilon_{k+1}]$, where $\bar{\varepsilon} := (\varepsilon_1, \dots, \varepsilon_k)$, and $S_{\bar{\varepsilon}}^i[\varepsilon_{k+1}]$ is the set of points \bar{x} in $(f^i)^{-1}(\bar{\varepsilon})$, such that one of the following conditions is satisfied for some $1 \leq i_1 \leq \dots \leq i_k \leq n + k$:

- either $1 + \|(x_{n+1}, \dots, x_{n+k})\|^2 = 1/\varepsilon_{k+1}$,
- or $\det\left(\frac{\partial f^i}{\partial(x_{i_1}, \dots, x_{i_k})}(\bar{x}; \varepsilon)\right)^2 = \varepsilon_{k+1}$.

Finally, S' is the set whose sections $S'_{\bar{\varepsilon}, \varepsilon_{k+1}} \subseteq \mathbb{K}^n$ are given by:

$$S'_{\bar{\varepsilon}, \varepsilon_{k+1}} := \Pi_n^{n+1} S_{\bar{\varepsilon}}[\varepsilon_{k+1}].$$

By lemmas 7.19 and 7.17, S' is an $\mathcal{S}(N)$ -set.

The fact that S' approximates $\overline{\Pi_n^{n+1} A}$ from below follows as in [BS04, Lemma 10.6].

It remains to prove that S' approximates $\text{bd}(\overline{\Pi_n^{n+1} A})$ from above on bounded sets.

Using Lemma 7.44, it is easy to see that $\forall^s \bar{\varepsilon}$, $\bar{\varepsilon}$ is a regular value of each f^1, \dots, f^k . Fix $\varepsilon_0 > 0$. Let $X := \text{bd}(\overline{A}) \cap \overline{B(0; 1/\varepsilon_0)}$; note that X is d -compact. Let $\bar{x} \in X$, and U be the open ball of center \bar{x} and radius ε_0 . Reasoning as in [BS04, Lemma 10.6], and using Proposition 7.33 instead of [BS04, Lemma 10.4], we see that $\forall^s \varepsilon_1 \dots \forall^s \varepsilon_{k+1} U \subseteq S'_{\varepsilon_1, \dots, \varepsilon_{k+1}}^{\varepsilon_0}$. Using Lemma 1.14, we deduce that $\forall^s \varepsilon_1 \dots \forall^s \varepsilon_{k+1} X \subseteq S'_{\varepsilon_1, \dots, \varepsilon_{k+1}}^{\varepsilon_0}$, which is the conclusion. \square

Lemma 7.49 (Product). *Let $n_1, n_2, k_1, k_2, N \geq 1$. For $i = 1, 2$, let $A_i \in \tilde{\mathcal{S}}_{n_i}$, such that A_i has empty interior (in \mathbb{K}^{n_i}). Assume that each A_i has an $\mathcal{S}(N)$ -approximant $S^i \subset \mathbb{K}^{n_i} \times \mathbb{K}_+^{k_i}$. Then, $A_1 \times A_2$ has an $\mathcal{S}(N)$ -approximant $S \subset \mathbb{K}^{n_1+n_2} \times \mathbb{K}_+^{k_1+k_2}$. Moreover, up to permutation of variables, $S = S_1 \times S_2$.*

Proof. W.l.o.g., each S^i has only one $\mathcal{S}(N)$ -constituent, that is, it is of the form

$$S^i := \{(\bar{x}, \bar{\varepsilon}) \in \mathbb{K}^{n_i} \times \mathbb{K}^{k_i} : \exists \bar{y} \in \mathbb{K}^{k_i-1} f_i(\bar{x}, \bar{y}) \ni \bar{\varepsilon}\},$$

for some \mathcal{C}^N admissible correspondence $f_i : \mathbb{K}^{n_i} \times \mathbb{K}^{k_i-1} \rightsquigarrow \mathbb{K}^{k_i}$, $i = 1, 2$. Define

$$S := \{(\bar{x}_1, \bar{x}_2, \bar{\varepsilon}_1, \bar{\varepsilon}_2) \in \mathbb{K}^{n_1} \times \mathbb{K}^{n_2} \times \mathbb{K}_+^{k_1} \times \mathbb{K}_+^{k_2} : \\ \exists \bar{y}_1 \in \mathbb{K}^{k_1-1} \exists \bar{y}_2 \in \mathbb{K}^{k_2-1} f_1(\bar{x}_1, \bar{y}_1) \ni \bar{\varepsilon}_1 \ \& \ f_2(\bar{x}_2, \bar{y}_2) \ni \bar{\varepsilon}_2\};$$

By Lemma 7.21 and Remark 7.41, S is an $\mathcal{S}(N)$ -set in $\tilde{\mathcal{S}}_{n_1+n_2+k_1+k_2}$. Since each A_i has empty interior, also the $\overline{A_i}$ have empty interiors; therefore, $\text{bd}(\overline{A_i}) = \overline{A_i}$, and we have $A_i \leq S^i$, and $S^i \leq A_i$. By Lemma 7.39, S approximates $A_1 \times A_2$. \square

Lemma 7.50 (Linear intersection). *Given $N, n, k \geq 1$, let $A \in \tilde{\mathcal{S}}_n$ have an $\mathcal{S}(N)$ -approximant $S \subset \mathbb{K}^n \times \mathbb{K}_+^k$, and suppose Y is an $(n-1)$ -dimensional affine subset of \mathbb{K}^n ; suppose further that $\overset{\circ}{A} \cap Y = \emptyset$. Then, there is an $\mathcal{S}(N)$ -approximant $S' \subseteq \mathbb{K}^n \times \mathbb{K}_+^{k+2}$ for $\overline{A} \cap Y$.*

Proof. The proof of [BS04, Lemma 10.8] goes through (with $\mathcal{S}(N)$ -approximants replacing $M(\mathcal{S})$ -approximants), using Lemma 7.17 to ensure that the set S' is indeed an $\mathcal{S}(N)$ -set. \square

Lemma 7.51 (Small intersection). *Let $n, k_1, k_2, N \geq 1$; define $M := N + n$. For $i = 1, 2$, let A_i be closed sets in $\tilde{\mathcal{S}}_{n_i}$. Assume that each A_i has an $\mathcal{S}(M)$ -approximant $S^i \subset \mathbb{K}^n \times \mathbb{K}_+^{k_i}$. Assume moreover that each A_i has empty interior. Then, $A := A_1 \cap A_2$ has an $\mathcal{S}(N)$ -approximant $S \subset \mathbb{K}^n \times \mathbb{K}_+^{3n+k_1+k_2}$.*

Proof. $A = \Pi_n^{2n}((A_1 \times A_2) \cap \Delta)$, where Δ is the diagonal of $\mathbb{K}^n \times \mathbb{K}^n$. By Lemma 7.49, $A_1 \times A_2$ has an $\mathcal{S}(M)$ -approximant in $\mathcal{S}_{2n+k_1+k_2}$. By hypothesis, $A_1 \times A_2$ has empty interior, hence we can apply Lemma 7.50 n times, and therefore $(A_1 \times A_2) \cap \Delta$ has an $\mathcal{S}(M)$ -approximant in $\mathcal{S}_{2n+k_1+k_2+2n}$. Finally, by Lemma 7.48, A has an $\mathcal{S}(M-n)$ -approximant in $\mathcal{S}_{4n+k_1+k_2}$. \square

Lemma 7.52. *Let $f : \mathbb{K}^n \rightsquigarrow \mathbb{K}$ be an admissible \mathcal{C}^N correspondence in $\tilde{\mathcal{S}}$. Let $A := V(f) \times \{0\} \subset \mathbb{K}^{n+1}$. Then, A has a $\mathcal{S}(N)$ -approximant in \mathcal{S}_{n+4} .*

Proof. Define

$$S := \{(\bar{x}, z, \varepsilon_1, \varepsilon_2, \varepsilon_3) \in \mathbb{K}^{n+1} \times \mathbb{K}_+^3 : \\ 1 + \|\bar{x}\|^2 \leq 1/\varepsilon_1 \ \& \ |z|^2 \leq \varepsilon_2 \ \& \ f(\bar{x}) + z \ni \varepsilon_3\}.$$

Notice that S is an $\mathcal{S}(N)$ set (with only one component): in fact, $S = \{(\bar{x}, z, \varepsilon_1, \varepsilon_2, \varepsilon_3) : \exists y_1 y_2 \ 1/(1+\|\bar{x}\|^2) + y_1^2 = \varepsilon_1 \ \& \ z^2 + y_2^2 = \varepsilon_2 \ \& \ f(\bar{x}) + z \ni \varepsilon_3\}$.

Notice also that A has empty interior. We claim that $A \leq S$ and $S \leq A$, proving the conclusion. For fixed $t > 0$, let $K(t) := \{\bar{x} \in \mathbb{K}^n : 1 + \|\bar{x}\|^2 \leq 1/t\}$.

Claim 1. $S \leq A$.

I.e., $\forall^s \varepsilon_0 \forall^s \bar{\varepsilon} S_{\bar{\varepsilon}} \subseteq A^{\varepsilon_0}$, where $\bar{\varepsilon} := (\varepsilon_1, \varepsilon_2, \varepsilon_3)$. Let $(\bar{x}, z) \in S_{\bar{\varepsilon}}$.

Define $I := [-\sqrt{\varepsilon_2}, \sqrt{\varepsilon_2}]$ and $H := K(\varepsilon_1) \setminus V(f)^{\varepsilon_0/4}$. I , H and $K(\varepsilon_1)$ are d-compact, and $S_{\bar{\varepsilon}} \subseteq K(\varepsilon_1) \times I$.

We claim that $|f|$ has a positive minimum on H , if $f(H)$ is non-empty. Otherwise, by Lemma 7.24, there exists $\bar{x} \in H$ such that $f(\bar{x}) \ni 0$, contradicting the definition of H . Let $\delta > 0$ be such minimum (or $\delta = 1$ if $f(H)$ is empty).

If we choose ε_3 smaller than δ , then $\bar{x} \in K(\varepsilon_1) \setminus H$, and therefore $\bar{x} \in V(f)^{\varepsilon_0/4}$. Now choose ε_2 smaller than $\varepsilon_0^2/4$, and obtain

$$(\bar{x}, z) \in V(f)^{\varepsilon_0/4} \times [-\varepsilon_0/4, \varepsilon_0/4] \subseteq A^{\varepsilon_0}.$$

Claim 2. $A \leq S$.

I.e., $\forall^s \varepsilon_0 \forall^s \bar{\varepsilon} A \cap B(0; 1/\varepsilon_0) \subseteq (S_{\bar{\varepsilon}})^{\varepsilon_0}$. Fix $\varepsilon_0 > 0$, and choose $1 > \delta_1 > 0$ such that $B(0; 1/\varepsilon_0) \subseteq K(\delta_1)$, Let $\delta_2 := \varepsilon_0/2$. For any ε_2 such that $0 < \varepsilon_2 < \delta_2$, let $\delta_3 := \varepsilon_2/2$. Finally, choose any ε_3 such that $0 < \varepsilon_3 < \delta_3$. Let $\bar{y} := (\bar{x}, z) \in A \cap B(0; 1/\varepsilon_0)$. We prove that, for ε_0 and $\bar{\varepsilon}$ chosen as above, $\bar{y} \in (S_{\bar{\varepsilon}})^{\varepsilon_0}$. First, notice $z = 0$ and $\bar{x} \in V(f)$. Let $\bar{w} := (\bar{x}, \varepsilon_3)$. Notice that $\text{dist}(\bar{y}, \bar{y}') = \varepsilon_3 < \varepsilon_0$, and that $\bar{w} \in S_{\bar{\varepsilon}}$, and therefore $\bar{y} \in (S_{\bar{\varepsilon}})^{\varepsilon_0}$. Hence,

$$\forall \varepsilon_0 \exists \delta_1 \forall \varepsilon_1 < \delta_1 \exists \delta_2 \forall \varepsilon_2 < \delta_2 \exists \delta_3 \forall \varepsilon_3 < \delta_3 (A \cap B(0; 1/\varepsilon_0) \subseteq (S_{\bar{\varepsilon}})^{\varepsilon_0}). \quad \square$$

Proof of 7.40. First, we prove the case when $A \in \mathcal{S}_n$. Fix $N \geq 1$. Let M be large enough (how large will be clear from the rest of the proof).

By hypothesis, there exists $m \geq n$ and $r \geq 1$, such that $A = \Pi_n^m(S_M)$, for some $S_M \subseteq \mathbb{K}^m$ of the form $S_M = \bigcup_{0 \leq j < k_M} S_{M,j}$ where each set $S_{M,j}$ is of the form

$$S_{M,j} = \bigcap_{0 \leq i < r} V(f_{M,i,j}),$$

and each $f_{M,i,j} : \mathbb{K}^m \rightsquigarrow \mathbb{K}$ is a \mathcal{C}^M admissible correspondence in $\tilde{\mathcal{S}}$.

Let $A_j := \Pi_n^m(S_{M,j})$. If we prove that each A_j satisfies (Φ_N) , then, by Lemma 7.45, A also satisfies (Φ_N) . Therefore, w.l.o.g., $k_M = 1$, i.e. $S_M = \bigcap_{0 \leq i < r} V_{M,i}$, where $V_{M,i} := V(f_{M,i})$ (where each $f_{M,i} : \mathbb{K}^m \rightsquigarrow \mathbb{K}$ is a \mathcal{C}^M admissible correspondence in $\tilde{\mathcal{S}}$). By Lemma 7.47, each $V_{M,i}$ satisfies (Φ_M) . We need to prove that S_M satisfies $(\Phi_{M'})$ (for a suitable M'). If all the $V_{M,i}$ were with empty interior, we could apply Lemma 7.51. Otherwise, for every i , define $W_{M,i} := V_{M,i} \times \{0\} \subset \mathbb{K}^{m+1}$. By Lemma 7.52,

each $W_{M,i}$ has an $\mathcal{S}(M)$ -approximant in \mathcal{S}_{m+4} ; moreover, each $W_{M,i}$ has empty interior, and therefore, by Lemma 7.51, $W_M := \bigcap_i W_{M,i}$ has an $\mathcal{S}(M - (r - 1)(m + 1))$ -approximant in $\mathcal{S}_{(3 \cdot 2^r - 2)m + 4 \cdot 2^r - 5}$. Since $S_M = \Pi_m^{m+1}(W)$, S_M has an $\mathcal{S}(M - rm + m - r)$ -approximant in $\mathcal{S}_{(3 \cdot 2^r - 2)m + 4 \cdot 2^r - 5}$. Finally, by Lemma 7.48, A has an $\mathcal{S}(M - rm + n - r)$ -approximant in $\mathcal{S}_{(3 \cdot 2^r - 2)m + 4 \cdot 2^r - 5}$.

The general case $A \in \tilde{\mathcal{S}}_n$ can be proved as in [BS04, §10], using Lemma 7.48 instead of [BS04, Lemma 10.6], and Lemma 7.50 instead of Lemma 10.8. \square

7.4 Cell decomposition

We can proceed to prove Theorem 7.6 by a cell decomposition argument: for every $A \in \tilde{\mathcal{S}}_n$, the ambient space \mathbb{K}^n can be partitioned into finitely many sets $A_1, \dots, A_N \in \tilde{\mathcal{S}}_n$, such that A (and hence its complement) is the union of some of the A_i s. We follow the outline of [Wilkie99, §4]. The reader can refer to [Wilkie99, Definitions 4.1 and 4.3], where we replace \mathbb{R} by \mathbb{K} .

Our aim is now to prove the analogue of the $\tilde{\mathcal{S}}$ -cell Decomposition Theorem 4.5 in [Wilkie99]. Once established this result, we see that Theorem 7.6 follows easily, as explained in the remarks preceding the proof of [Wilkie99, Theorem 4.5].

There are three points in Wilkie's proof that do use some reasoning whose translation in our context is not readily apparent. We will examine them.

Claim 1. Assume that $A \in \tilde{\mathcal{S}}_n$ has empty interior. For each $i \geq 1$, consider the set

$$A_i := \left\{ \bar{x} \in C : \exists y_1, \dots, y_i (y_1 < \dots < y_i \wedge \bigwedge_{j=1}^i (\bar{x}, y_j) \in A) \right\}.$$

Then each set A_i lies in $\tilde{\mathcal{S}}_n$, and A_N has empty interior in \mathbb{K}^n for some $N \geq 1$.

Proof. We proceed as in [BS04, Lemma 7.8]. The definition of A_i implies immediately that $A_i \in \tilde{\mathcal{S}}_n$.

Let $N := \gamma(A) + 1$, and fix $\bar{x} \in C$. Note that if the fibre $A_{\bar{x}}$ has cardinality greater or equal to N , then it has non-empty interior.

Since A has empty interior, it is meager. Therefore, by Lemma 5.6, the set of those points $\bar{x} \in C$ such that $A_{\bar{x}}$ has non-empty interior is meager. Thus, A_N is meager, and hence it has empty interior. \square

Claim 2. Let C' be an open \tilde{S} -cell compatible with $\overline{A_1}, \dots, \overline{A_N}, \tilde{H}, \tilde{H}_f$ and \tilde{H}_g , and such that $C' \cap \overline{A_1} \neq \emptyset$. Choose $k < N$ maximal such that $C' \cap \overline{A_k} \neq \emptyset$. Then $C' \subseteq A_k$.

Proof. As in Wilkie's proof, we conclude that \tilde{H}, \tilde{H}_f and \tilde{H}_g are disjoint from C' .

We have $C' \subseteq \overline{A_k}$, because $C' \cap \overline{A_k} \neq \emptyset$. Consider a point $\bar{x} \in C'$. Let M be the cardinality of the fibre $A_{\bar{x}}$; note that, by definition of k , $M \leq k$. Let $y_0 := f(\bar{x})$, $y_{M+1} := g(\bar{x})$, and, for $1 \leq i \leq M$, y_i be the i -th point of \mathbb{K} such that $(\bar{x}, y_i) \in A$.

If $1 \leq i \leq M$, since $\bar{x} \notin \tilde{H}$, we may find open neighbourhoods V_i of \bar{x} in \mathbb{K}^n and J_i of y_i in \mathbb{K} , such that for each $\bar{x}' \in V_i$ there is at most one $y' \in J_i$ such that $(\bar{x}', y') \in A$.

Similarly, if $i = 0$ or $i = M+1$ then, since $\bar{x} \notin \tilde{H}_f \cup \tilde{H}_g$, we may choose V_i and J_i such that $(V_i \times J_i) \cap A = \emptyset$. Let $T := \{y \in \mathbb{K} : (\bar{x}, y) \in A \text{ \& } y \notin \bigcup_i J_i\}$, and $T' := \{\bar{x}\} \times T$. Note that T' is a compact subset of C , that A is a closed subset of C disjoint from T' . Hence, the distance between T' and A is some positive number $d > 0$. Let $U := \bigcap_{i=0}^{M+1} V_i \cap \{\bar{x}' \in C' : |\bar{x}' - \bar{x}| < d\}$.

Therefore, for every $\bar{x}' \in U$,

$$|(\{\bar{x}'\} \times \mathbb{K}) \cap A| \leq |(\{\bar{x}\} \times \mathbb{K}) \cap A| = M. \quad (1)$$

We conclude as in [Wilkie99]: as $\bar{x} \in \overline{A_k}$, we may choose $\bar{x}' \in U \cap A_k$ here, from which it follows (using the maximality of k) that $M = k$. Hence $\bar{x} \in A_k$ and the claim is justified. \square

Thus, for each $i = 1, \dots, k$, we may define the function $f_i : C' \rightarrow K$ is \tilde{S} by $f_i(\bar{x}) = y$ iff y is the i -th point of \mathbb{K} such that $(\bar{x}, y_i) \in A$.

Claim 3. Each function f_i is continuous.

Proof. Let $\bar{x} \in C'$. Let U, V_i and J_i be defined as in the proof of the previous claim, for $i = 1, \dots, k$. Let $\bar{x}' \in U$. Note that, since we have equality in (1), then, for every $i = 1, \dots, k$, there is exactly one $y'_i \in J_i$ such that $(\bar{x}', y'_i) \in A$. Note also that $y'_i = f_i(\bar{x}')$. Fix i such that $1 \leq i \leq k$, and fix J neighbourhood of $y_i = f_i(\bar{x})$. In the construction of V_i and J_i , we could have chosen J_i such that $J_i \subseteq J$, and then found a corresponding V_i . Proceeding in the construction, we see that, for every J neighbourhood of $f_i(\bar{x}')$, we can find U neighbourhood of \bar{x} such that $f_i(U) \subseteq J$, which is equivalent to the definition of f_i being continuous at \bar{x} . Since $\bar{x} \in U$ is arbitrary, the claim is proved. \square

8 Pfaffian functions

Khovanskii's results in [Khov91] show that any expansion of the real field with a Pfaffian chain of functions satisfies the hypotheses of [Wilkie99, Theorem 1.9]. Let \mathbb{K} be a definably complete Baire structure. In this section we give an analog of Khovanskii's results, thus providing an example of a class of structures to which Theorem 7.7 applies.

Definition 8.1. Let $f_1, \dots, f_s : \mathbb{K}^n \rightarrow \mathbb{K}$ be definable and \mathcal{C}^1 . We say that (f_1, \dots, f_s) is a *Pfaffian chain* if $\frac{\partial f_i}{\partial x_j} \in \mathbb{K}[\bar{x}, f_1, \dots, f_i]$ for $i = 1, \dots, s$ and $j = 1, \dots, n$. A definable map $F = (F_1, \dots, F_m) : \mathbb{K}^n \rightarrow \mathbb{K}^m$ is *Pfaffian* if $F_1, \dots, F_m \in \mathbb{K}[\bar{x}, f_1, \dots, f_s]$ for some Pfaffian chain (f_1, \dots, f_s) .

Consider polynomials $p_{ij} \in \mathbb{K}[\bar{x}, y_1, \dots, y_i]$, $q_k \in \mathbb{K}[\bar{x}, y_1, \dots, y_s]$ such that

$$\begin{aligned} \frac{\partial f_i}{\partial x_j}(\bar{x}) &= p_{ij}(\bar{x}, f_1(\bar{x}), \dots, f_i(\bar{x})) & i \leq s, j \leq n \\ F_k(\bar{x}) &= q_k(\bar{x}, f_1(\bar{x}), \dots, f_s(\bar{x})) & k \leq m \end{aligned}$$

The *complexity* of F is the sequence of integers $(n, m, s, \deg q_k, \deg p_{ij} : i \leq s, j \leq n, k \leq m)$.

We prove the following.

Theorem 8.2. *Let \mathbb{K} be a definably complete Baire structure. Let \mathcal{F} be a family of \mathbb{K} -definable Pfaffian chains and let $\mathbb{K}_{\mathcal{F}}$ be the reduct of \mathbb{K} generated by $+$, \cdot and \mathcal{F} . Then $\mathbb{K}_{\mathcal{F}}$ is o-minimal.*

Corollary 8.3. *Let \mathbb{R}_{\exp} be the real ordered field with the exponential function. Then the following statements axiomatize a recursive subtheory of $Th(\mathbb{R}_{\exp})$ which is o-minimal.*

- *Axioms of ordered field.*
- *Axioms ensuring that the models are definably complete and Baire.*
- $\forall x \exp'(x) = \exp(x) \ \& \ \exp(0) = 1.$

Analogous statements hold, for example, for the structures $\langle \mathbb{R}; +, \cdot, 0, 1, \exp, \sin \upharpoonright [0, 1] \rangle$ and $\langle \mathbb{R}; +, \cdot, 0, 1, x^\alpha \rangle$ (α a real number).

To obtain Theorem 8.2 it is enough to show that $\mathbb{K}_{\mathcal{F}}$ satisfies the hypotheses of Theorem 7.7. Hence, it suffices to prove the following version of Khovanskii's Theorems (see [Khov91, Theorems 1 and 2]):

Theorem 8.4.

1. *Suppose $F : \mathbb{K}^n \rightarrow \mathbb{K}^n$ is Pfaffian. Then the number of regular zeroes of F is finite and can be bounded by a function of the complexity of F .*
2. *Suppose $F : \mathbb{K}^n \rightarrow \mathbb{K}^m$ is Pfaffian. Then the number of definably connected components of $F^{-1}(0)$ is finite and can be bounded by a function of complexity of F .*

The fact that the bounds in the above theorem depend only on the complexity imply, in particular, that they do not depend on the coefficients of the polynomials in the Pfaffian chain, or on other parameters in the definition of F . Moreover, the reader can verify that the explicit bounds given in [Khov91] continue to work in this context.

Before proceeding with the proof, we need to develop a version of Sard's Lemma holding true in this context.

8.1 The Sard property and Noetherian Differential Rings

In Subsection 7.2 we have seen that a strong version of Sard's Lemma holds true for functions definable in an o-minimal weak structure (see [Wilkie99, Theorems 2.7 and 2.8]). In this subsection we will show a version of Sard's Lemma for functions belonging to a Noetherian differential ring (Theorem 8.9). The proof of the two mentioned statements are completely different from one another; in particular, the proof of 8.9 is a quite simple modification of the classical argument for Sard's Lemma.

Notation 8.5. Fix $n \in \mathbb{N} \setminus \{0\}$ and a definably connected definable open set $U \subseteq \mathbb{K}^n$. Let $\mathcal{C}^\infty(U, \mathbb{K})$ be the ring of definable \mathcal{C}^∞ functions from U to \mathbb{K} .

Definition 8.6. A ring M with the following properties

- $M \subseteq \mathcal{C}^\infty(U, \mathbb{K})$;
- M is Noetherian;
- M is closed under partial differentiation;
- $M \supseteq \mathbb{K}[x_1, \dots, x_n]$.

is called a *Noetherian differential ring*.

If $G := (g_1, \dots, g_k) \in M^k$, we denote by $V(G)$ the set of zeroes of G , and by $V^{\text{reg}}(G)$ the set of *regular* zeroes of G .

Generalities on Noetherian differential rings of functions over definably complete structures can be found in [Servi07]. In particular, we will need the following result, which states that in a Noetherian differential ring there are no flat functions.

Proposition 8.7. *Let $M \subseteq C^\infty(U, \mathbb{K})$ be a Noetherian differential ring and let $0 \neq g \in M$. Then for every $x \in U$ such that $g(x) = 0$, there exist $k \in \mathbb{N}$ and a derivative θ of order k such that $\theta g(x) \neq 0$.*

Fix a Noetherian differential ring $M \subseteq C^\infty(U, \mathbb{K})$.

Remark 8.8. For $g_1, \dots, g_k \in M$, the set $V := V^{\text{reg}}(g_1, \dots, g_k)$ is in \mathcal{F}_σ ; in fact consider the following closed definable subset of $U \times \mathbb{K}$:

$$C := \bigcup_{E(x)} \{(x, y) \in U \times \mathbb{K} : \bigwedge_{i=1}^k g_i(x) = 0 \wedge \det(E(x))y - 1 = 0\},$$

where $E(x)$ ranges over all maximal rank minors of the Jacobian matrix of (g_1, \dots, g_k) in x . Now, $V = \Pi_n^{n+1}(C)$; since C is an \mathcal{F}_σ of \mathbb{K}^{n+1} and Π_n^{n+1} is continuous, V is also an \mathcal{F}_σ .

In this subsection we prove the following version of Sard's Lemma:

Theorem 8.9. *Fix $k, m \in \mathbb{N}$, $k \leq n$. Let*

- $H = (h_1, \dots, h_{n-k}) \in M^{n-k}$ and $V := V^{\text{reg}}(H) \neq \emptyset$;
- $F = (F_1, \dots, F_m) \in M^m$ and $f := F \upharpoonright V : V \rightarrow \mathbb{K}^m$;
- $\Delta_f \subseteq V$ be the set of singular points of f , and $\Sigma_f := f(\Delta_f)$ be the set of singular values of f .

Then, $f : V \rightarrow \mathbb{K}^m$ has the Sard property, i.e. Σ_f is a meager set (in \mathbb{K}^m).

Proof. We proceed by induction on $\dim V$ and m . If $m = 0$, there are no singular points. If $\dim V = 0$, then V is discrete. In particular, for every $a \in \Delta_f$ there exists U_a neighbourhood of a such that $\Delta_f \cap U_a = \{a\}$. Hence we can apply Corollary 3.7 and we are done.

Consider now the general case.

Claim 1. We can restrict to the case $V = \mathbb{K}^k$.

By Corollary 3.7, it suffice to prove that for every $a \in \Delta_f$ there exists a neighbourhood U_a of a such that $f(U_a \cap \Delta_f)$ is meager. Fix $a \in \Delta_f$. Using the Implicit Function Theorem, it is easy to check that there is a neighbourhood U_a of a and a definable diffeomorphism $\Phi : \mathbb{K}^k \rightarrow V \cap U_a$ such that $H \circ \Phi \equiv 0$ and each $F_i \circ \Phi$ belong to a Noetherian differential ring $M' \subseteq \mathcal{C}^\infty(\mathbb{K}^k, \mathbb{K})$ (see [Servi07] for the details). Hence Claim 1 is proved and we may assume that $f : \mathbb{K}^k \rightarrow \mathbb{K}^m$, and $f \in M \subseteq \mathcal{C}^\infty(\mathbb{K}^k, \mathbb{K})$.

Let $X_0 := \{a \in \Delta_f : Df(a) \neq 0\}$, where Df is the Jacobian matrix of f . We first prove that $f(X_0)$ is meager.

Again by Corollary 3.7, it suffice to prove that for every $a \in X_0$ there exists a neighbourhood U_a of a such that $f(U_a \cap X_0)$ is meager.

Fix $a \in X_0$.

Claim 2. We may assume that $f(x) = (x_1, f_2(x), \dots, f_m(x))$.

In fact, since $Df(a) \neq 0$, w.l.o.g. we can assume that $\partial f_1(a)/\partial x_1 \neq 0$ and $a = 0$.

Consider definable neighbourhoods O and $\tilde{O} \subset \mathbb{K}^k$ of 0, where the following map is a diffeomorphism:

$$\begin{aligned} G : O &\rightarrow \tilde{O} \\ x &\mapsto (f_1(x), x_2, \dots, x_k). \end{aligned}$$

Let Δ be the determinant of the Jacobian of G and let $\hat{M} := \{g \circ G^{-1} \mid g \in M\} \subset \mathcal{C}^\infty(\tilde{O}, \mathbb{K})$; then the ring $\tilde{M} := \hat{M}[\Delta^{-1}]$ is clearly Noetherian and differentially closed; define $\tilde{f} := f \circ G^{-1} \in \tilde{M}$. Since G is a diffeomorphism, it is enough to prove the statement for \tilde{M} and \tilde{f} , and Claim 2 is proved.

For every $t \in \mathbb{K}$, consider the Noetherian differential ring

$$N_t := \{g_t := g(t, x_2, \dots, x_k) \mid g \in M\} \subset \mathcal{C}^\infty(\tilde{O} \cap \mathbb{K}^{k-1}, \mathbb{K}).$$

Let $f_t : \mathbb{K}^{k-1} \rightarrow \mathbb{K}^{m-1}$ be the map $((f_2)_t, \dots, (f_m)_t)$. By inductive hypothesis, the set Σ_{f_t} is meager in \mathbb{K}^{m-1} . Moreover, $f(X_0 \cap \tilde{O}) \cap (\{t\} \times \mathbb{K}^{m-1}) \subseteq \{t\} \times \Sigma_{f_t}$. Hence $f(X_0 \cap \tilde{O}) \subseteq D := \{(t, y) \in \mathbb{K} \times \mathbb{K}^{k-1} \mid y \in \Sigma_{f_t}\}$. By what we have just observed, $T(D) := \{t \in \mathbb{K} : D_t \text{ is meager}\}$ is residual, because $D_t = \Sigma_{f_t}$, hence by Lemma 5.6, D is meager. It follows by Corollary 3.7 that $f(X_0)$ is meager.

Now, let $a \in \Delta_f$ such that $Df(a) = 0$, and let P be the least natural number such that there exists $i \leq m$ and a derivative θ of order P such that, if $g_\theta := \theta f_i$, then $g_\theta(a) = 0$ and $Dg_\theta(a) \neq 0$. Such a P exists by Proposition 8.7. Let $W_\theta := V^{\text{reg}}(g_\theta) \subset \mathbb{K}^k$ (notice that the inclusion is

strict, hence $\dim W_\theta < k$). Then there is a definable open neighbourhood O of a such that

$$\Delta_f \cap O \subseteq \bigcup_{\text{ord}(\theta) \leq P} W_\theta.$$

Hence it is enough to prove that $f(\Delta_f \cap W_\theta)$ is meager. Let $h_\theta := f \upharpoonright W_\theta$. By inductive hypothesis, Σ_{h_θ} is meager. Note that if $x \in W_\theta$ is a singular point for f , then x is also a singular point for h_θ ; that is, $\Delta_f \cap W_\theta \subseteq \Delta_{h_\theta}$, and we are done. \square

Corollary 8.10. *Let $F \in M^k$ and $G \in M$. Define $X := V^{\text{reg}}(F) \subseteq U$, and, for every $\bar{a} \in \mathbb{K}^n$, $g_{\bar{a}} : X \rightarrow \mathbb{K}$ as $g_{\bar{a}}(\bar{x}) := G(\bar{x}) + \sum a_i x_i$. Then, the set $A = \{(a_1, \dots, a_n) \in \mathbb{K}^n : g_{\bar{a}} \text{ is not a Morse function}^{(5)} \text{ on } X\}$ is meager.*

Proof. We proceed as in [GP74].

Claim 1. The lemma is true if $k = 0$, i.e. if $X = U$.

In fact, $\bar{a} \in A$ iff $-\bar{a}$ is critical values of ∇G , and we can apply Theorem 8.9.

By the Implicit Function Theorem, around every point $p \in X$ there exists an open definable neighbourhood U_p , such that the restriction of some $n - k$ of the coordinate functions on \mathbb{K}^n (w.l.o.g., the first $n - k$), constitute a coordinate system in U_p ; let $V_p := \Pi_{n-k}^n(U_p)$ and $\phi_p : V_p \rightarrow U_p$ be the inverse map of $\Pi_{n-k}^n \upharpoonright U_p$. Let \tilde{M} be the ring of functions on V_p of the form $h \circ \phi$, where $h \in M$: notice that \tilde{M} is contained in some Noetherian differential ring M_p (see [Servi07]). Let $A_p = \{(a_1, \dots, a_n) \in \mathbb{K}^n : g_{\bar{a}} \text{ is not a Morse function on } V_p\}$. Proceeding as in [GP74], using Lemma 5.6 instead of Fubini's theorem, and Claim 1 applied to functions in the ring M_p , we see that A_p is meager for every $p \in X$. Since $A = \bigcup_{p \in X} A_p$, Corollary 3.7 implies that A is meager. \square

Remark 8.11. Note that if (f_1, \dots, f_s) is a Pfaffian chain, then the ring $\mathbb{K}[\bar{x}, f_1, \dots, f_s]$ is a Noetherian differential ring. In particular, Theorem 8.9 holds for functions in this ring.

8.2 Proof of Theorem 8.4

We will follow the outline of [Marker97].

⁽⁵⁾A definable \mathcal{C}^2 function f , from a \mathcal{C}^2 \mathbb{K} -manifold to \mathbb{K} , is a Morse function if, as in the classical definition, every singular point of f is nondegenerate.

We argue by induction on the length s of the Pfaffian chain. If $s = 0$ then F is a polynomial map and the bound is given by [BCR98, Proposition 11.5.4].

Let $s > 0$.

Inductive Hypothesis. *We suppose that, for all Pfaffian chains of length $\leq s - 1$, the two statements of Theorem 8.4 hold true.*

We first prove the first statement of Theorem 8.4. Let $F : \mathbb{K}^n \rightarrow \mathbb{K}^n$ be Pfaffian with respect to a Pfaffian chain (f_1, \dots, f_s) , with $F = (F_1, \dots, F_n)$ and $F_i(\bar{x}) = q_i(\bar{x}, f_1(\bar{x}), \dots, f_s(\bar{x}))$.

Lemma 8.12. *There are Pfaffian maps $H : \mathbb{K}^{n+1} \rightarrow \mathbb{K}^n$ and $G : \mathbb{K}^{n+1} \rightarrow \mathbb{K}$ such that*

1. H has length $s - 1$ and G has length s .
2. $V(G) = V^{\text{reg}}(G)$.
3. If $\bar{a} \in V^{\text{reg}}(F) \subseteq \mathbb{K}^n$, then $\exists b \in \mathbb{K}$ such that $(\bar{a}, b) \in V^{\text{reg}}(H, G) \subseteq \mathbb{K}^{n+1}$.

Proof. Define $H_i(\bar{x}, y) := q_i(\bar{x}, f_1(\bar{x}), \dots, f_{s-1}(\bar{x}), y)$ ($i = 1, \dots, n$) and $G(\bar{x}, y) := y - f_s(\bar{x})$. \square

Hence it is enough to bound the cardinality of $V^{\text{reg}}(H, G)$.

Definition 8.13. A definable continuous function $f : \mathbb{K}^d \rightarrow \mathbb{K}^{d'}$ is *proper* if the pre-image of every d -compact set is d -compact.

Remark 8.14. A definable continuous function $f : \mathbb{K}^d \rightarrow \mathbb{K}^{d'}$ is proper iff $\lim_{|x| \rightarrow \infty} |f(x)| = +\infty$.

Lemma 8.15. *We may assume that H is proper.*

Proof. Suppose H is not proper. For all $r \in \mathbb{K}$, we define a proper Pfaffian map $Q^r : \mathbb{K}^{n+2} \rightarrow \mathbb{K}^{n+1}$ such that:

1. the length of Q^r is $s - 1$ and its complexity does not depend on r ;
2. for all $(\bar{a}, b) \in V^{\text{reg}}(H, G)$, there exist $r \in \mathbb{K}$ and $c \in \mathbb{K}$ such that $(\bar{a}, b, c) \in V^{\text{reg}}(Q^r, G)$.

It follows that, if $\forall r \ |V^{\text{reg}}(Q^r, G)| < N$, then $|V^{\text{reg}}(H, G)| < N$. The components of Q^r as defined as follows: $Q_0^r(\bar{x}, y, z) = \sum_{i=1}^n x_i^2 + y^2 + z^2 - r^2$; $Q_i^r(\bar{x}, y, z) = H_i(\bar{x}, y)$ for $i = 1, \dots, n$. \square

Lemma 8.16. *We may assume that $V(H) = V^{\text{reg}}(H)$ and that $V(H, G) = V^{\text{reg}}(H, G)$.*

Proof. Suppose this is not the case. For every $\bar{b} \in \mathbb{K}^n$, we consider the Pfaffian proper map $H_{\bar{b}} := H - \bar{b}$. Let B be the set of all $\bar{b} \in \mathbb{K}^n$ such that $V(H_{\bar{b}}) = V^{\text{reg}}(H_{\bar{b}})$ and $V(H_{\bar{b}}, G) = V^{\text{reg}}(H_{\bar{b}}, G)$. By Theorem 8.9, B is a co-meager subset of \mathbb{K}^n . Note that $H_{\bar{b}}$ has length $s - 1$ and same complexity as H . Suppose we did prove that, for every $\bar{b} \in B$, $|V^{\text{reg}}(H_{\bar{b}}, G)| \leq N$. The set of all $\bar{a} \in \mathbb{K}^n$ such that $|V^{\text{reg}}(H_{\bar{a}}, G)| \geq N + 1$ is open (by the Implicit Function Theorem, applied to H restricted to the manifold $V(G) = V^{\text{reg}}(G)$) and disjoint from B , and therefore empty. \square

We have thus reduced our problem to the following situation: $\Gamma := V^{\text{reg}}(H) = V(H) \subseteq \mathbb{K}^{n+1}$ is a smooth d-compact Pfaffian curve of length $s - 1$ and $G : \mathbb{K}^{n+1} \rightarrow \mathbb{K}$ is a Pfaffian map of length s such that $G \upharpoonright \Gamma$ has only regular zeroes. We need to bound the number of such zeroes.

Definition 8.17. An *arc* of a non singular curve Γ is the image of a differentiable function $\phi : I \rightarrow \Gamma$ such that $I \subseteq \mathbb{K}$ is an interval and $\phi'(t)$ is nonzero for all $t \in I$. The function ϕ is called a *parametrization* of the arc. When no confusion is possible we use the word “arc” both for ϕ and its image.

Lemma 8.18. Γ is the union of finitely many arcs.

Proof. By the inductive hypothesis and Theorem 7.7. \square

Definition 8.19. Given a \mathcal{C}^1 function $f : \mathbb{K}^d \rightarrow \mathbb{K}^d$, let $J(f) : \mathbb{K}^d \rightarrow \mathbb{K}$ be the determinant of the Jacobian matrix of f .

Definition 8.20. Let ξ_H be the unique vector field on \mathbb{K}^{n+1} such that for every smooth definable function $g : \mathbb{K}^{n+1} \rightarrow \mathbb{K}$ we have $\xi_H(\bar{x}) \cdot \nabla g(\bar{x}) = J(H, g)(\bar{x})$. Note that ξ_H is tangent to Γ and is never zero on Γ . We say that the arc $\phi : I \rightarrow \Gamma$ is *orientation preserving* if $\phi'(t) \cdot \xi_H(\phi(t)) > 0$ for every $t \in I$. Note that if $\phi : (a, b) \rightarrow \Gamma$ is not orientation preserving, then its reverse arc $-\phi(t) = \phi(b - t + a)$ is orientation preserving.

Definition 8.21. We say that two points $\bar{x}, \bar{y} \in V(H, G)$ are *consecutive* if there are an orientation preserving arc $\phi : I \rightarrow \Gamma = V(H)$ and $t_1 < t_2$ in I such that $\bar{x} = \phi(t_1), \bar{y} = \phi(t_2)$ and $\phi(t) \notin V(G)$ for every $t \in (t_1, t_2)$.

Lemma 8.22. *Let \bar{x}, \bar{y} be consecutive points in $V(H, G)$. Then $J(H, G)$ assumes opposite signs at \bar{x}, \bar{y} . So in particular $\bar{x} \neq \bar{y}$.*

Proof. We are going to use the elementary fact that if a function $h: I \rightarrow \mathbb{K}$ defined on an interval $I \subseteq \mathbb{K}$ has two consecutive zeros $t_1 < t_2$ in I , and has nonzero derivative at these points, then $h'(t_1)$ and $h'(t_2)$ have opposite signs.

To reduce to this situation consider an orientation preserving arc $\phi: I \rightarrow \Gamma$ with $\bar{x} = \phi(t_1) \in V(G)$, $\bar{y} = \phi(t_2) \in V(G)$ and $\phi(t) \notin V(G)$ for every $t \in (t_1, t_2)$. The derivative $\frac{d(G \circ \phi)(t)}{dt}$ equals $\phi'(t) \cdot \nabla G(\phi(t))$, which has the same sign as $\xi_H(\phi(t)) \cdot \nabla G(\phi(t)) = J(H, G)(\phi(t))$ (this is nonzero since $J(H, G) \neq 0$ on $V(G)$). So if $J(H, G)$ assumes the same sign at \bar{x}, \bar{y} , then $G \circ \phi: I \rightarrow \mathbb{K}$ would contradict the elementary fact stated above. \square

Lemma 8.23. *For each $\bar{x} \in V(H, G)$, there is $\bar{y} \in V(H, G)$ such that \bar{x}, \bar{y} are consecutive.*

Proof. Let Γ be the union of the orientation preserving arcs ϕ_0, \dots, ϕ_k . We can assume that this family of arcs is essential, i.e. no arc ϕ_i is contained in the union of the remaining arcs. Suppose ϕ_0 contains \bar{x} . If this arc does not contain a consecutive point to \bar{x} , then it cannot contain any points of $V(G)$ coming after \bar{x} . Let ϕ_1 be the arc such that $\lim_{t \rightarrow \sup I} \phi_0(t) \in \phi_1$ and $\lim_{t \rightarrow \inf I} \phi_1(t) \in \phi_0$. There is only one such arc, because Γ is a smooth curve and otherwise the Implicit Function Theorem would be violated. We prolong the arc ϕ_0 by attaching ϕ_1 to it. Suppose that the arc ϕ_1 contains no consecutive points to \bar{x} . If $\lim_{t \rightarrow \sup I} \phi_1(t) \in \phi_0$ and $\lim_{t \rightarrow \inf I} \phi_0(t) \in \phi_1$, then the orientation reversing arc $-\phi_0$ must contain a consecutive point to \bar{x} , or else \bar{x} would be consecutive to itself, contradicting Lemma 8.22. Otherwise, let ϕ_2 be the unique arc which contains $\lim_{t \rightarrow \sup I} \phi_1(t)$. Notice that, again by the Implicit Function Theorem, it is not possible that $\lim_{t \rightarrow \sup I} \phi_2(t) \in \phi_1$. We carry on attaching arcs with this procedure, until we either find a consecutive point to \bar{x} or we find an arc ϕ_i such that $\lim_{t \rightarrow \sup I} \phi_i(t) \in \phi_0$. In this case, by the argument above, the arc ϕ_0 must contain a consecutive point to \bar{x} . \square

Lemma 8.24. *There is a Pfaffian function $\hat{J}: \mathbb{K}^{n+1} \rightarrow \mathbb{K}$ of length $s - 1$ which coincides with $J(H, G)$ on $V(G)$.*

Proof. Let $\hat{J}(\bar{x}, y)$ be such that $\hat{J}(\bar{x}, f_s(\bar{x})) = J(H, G)(\bar{x})$. \square

Define \hat{j} to be the restriction of \hat{J} to Γ . Note that \hat{j} assumes opposite signs at two consecutive points \bar{x}, \bar{y} of $V(H, G)$.

Lemma 8.25. *$V(H, G)$ is finite, and we can compute a bound N on its cardinality in terms of the complexity of H, G .*

Proof. Let $\varepsilon > 0$ be the minimum of the absolute value of \hat{j} on the closed and bounded set $V(H, G)$. Then \hat{j} assumes every value between $-\varepsilon$ and $+\varepsilon$ between any two consecutive points \bar{x}, \bar{y} of $V(H, G)$. By Theorem 8.9 \hat{j} has a regular value $t \in (-\varepsilon, +\varepsilon)$. Since \hat{J} has length $\leq s - 1$, using the inductive hypothesis we can compute a finite bound on the cardinality of $\hat{j}^{-1}(t)$. This is also a bound on $V(H, G)$ since we can associate injectively to each $\bar{x} \in V(H, G)$ a point of $\hat{j}^{-1}(t)$ lying in the arc between \bar{x} and the point consecutive to \bar{x} (which exists by Lemma 8.23). \square

Combining all the lemmas, we obtain a proof of the first statement of Theorem 8.4. We now prove the second statement.

Let $F : \mathbb{K}^n \rightarrow \mathbb{K}^m$ be Pfaffian with respect to a Pfaffian chain (f_1, \dots, f_s) .

We need some preliminary results.

Definition 8.26. Let $\text{Cofin}(\mathbb{K})$ be the cofinality of \mathbb{K} . A *sequence* is a map $x : \text{Cofin}(\mathbb{K}) \rightarrow \mathbb{K}^m$. If $(x_k)_{k < \text{Cofin}(\mathbb{K})}$ is a sequence, we say that $x_k \rightarrow l$ if for every neighbourhood V of l there exists $\mu < \text{Cofin}(\mathbb{K})$ such that $x_k \in V$ for every $k > \mu$. We call $(x_k)_{k < \text{Cofin}(\mathbb{K})}$ *infinitesimal* if $x_k \rightarrow 0$.

Lemma 8.27. *Let $F : \mathbb{K}^n \rightarrow \mathbb{K}$ be definable, continuous, proper and nonnegative, and $M \in \mathbb{N}$. Suppose there is an infinitesimal nonnegative sequence $(\varepsilon_k)_{k < \text{Cofin}(\mathbb{K})}$ such that for every $k < \text{Cofin}(\mathbb{K})$, $F^{-1}(\varepsilon_k)$ has less than M def-connected components. Then $F^{-1}(0)$ has less than M def-connected components.*

Proof. Since F is proper, $F^{-1}(0)$ and $F^{-1}(\varepsilon_k)$ are d-compact. Assume, for contradiction, that there exists a partition $\{C_0, \dots, C_M\}$ of $F^{-1}(0)$ into non-empty definable clopen subsets. Let

$$\begin{aligned} \delta &:= \frac{1}{3} \min_{i \neq j} d(C_i, C_j), \\ W &:= \{\bar{x} \in \mathbb{K}^n : d(\bar{x}, F^{-1}(0)) < \delta\}, \\ B_i &:= \{\bar{x} \in \mathbb{K}^n : d(\bar{x}, C_i) < \delta\}, \\ J_i &:= F(B_i). \end{aligned}$$

Note that $\delta > 0$, that the B_i s are open (in K^n) and disjoint, that $B_i \cap F^{-1}(0) = C_i$, and that $W = \bigsqcup_i B_i$. We note that each J_i is def-connected: consider w.l.o.g. $i = 0$. Let $\varepsilon \in J_0$, and let $\bar{y} \in B_0$ such that $F(\bar{y}) = \varepsilon$. Let $\bar{x} \in C_0$ such that $d(\bar{x}, \bar{y}) < \delta$. Note that the segment $[\bar{x}, \bar{y}]$ is contained

in B_0 . Since $[\bar{x}, \bar{y}]$ is def-connected, $[0, \varepsilon] = F([\bar{x}, \bar{y}])$ is also def-connected, and therefore J_0 is def-connected.

Let

$$\begin{aligned}\theta_i &:= \sup J_i \\ \eta_1 &:= \min_i \theta_i.\end{aligned}$$

We claim that there exists $\eta_2 > 0$ such that $F^{-1}([0, \eta_2]) \subseteq W$. Let $D := F^{-1}([0, 1]) \setminus W$. Note that D is d-compact, because F is proper. If $D = \emptyset$, we can define $\eta_2 = 1$. Otherwise, $F(D)$ is d-compact and non-empty. Let $\eta_2 := \min D$. Since $F^{-1}(0) \cap D = \emptyset$, we have that $\eta_2 > 0$. Let $F(\bar{x}) < \eta_2$. Then, $\bar{x} \notin D$, and therefore $\bar{x} \in W$.

Define $\eta = \min(\eta_1, \eta_2)$. Therefore, for every $\varepsilon < \eta$, we have

$$\begin{aligned}F^{-1}(\varepsilon) &\subseteq \bigsqcup_i B_i \\ \varepsilon &\in \bigcap_i F(B_i).\end{aligned}$$

Let $k < \text{Cofin}(\mathbb{K})$ such that $\varepsilon_k < \eta$. Since $F^{-1}(\varepsilon_k)$ has at most M def-connected components, we deduce that $F^{-1}(\varepsilon_k) \cap B_i = \emptyset$ for at least one i . However, this contradicts $\varepsilon_k \in \bigcap_i F(B_i)$. \square

We turn to the proof of the second statement of Theorem 8.4.

Claim 1. We may assume that F is proper (the preimage of a d-compact is d-compact).

Proof of Claim 1. For every $r \in \mathbb{K}_+$, define the proper map

$$G_r(x_1, \dots, x_{n+1}) = (F(x_1, \dots, x_n, x_1^2 + \dots + x_{n+1}^2 - r^2)).$$

If we find a bound for the number of connected components of $G_r^{-1}(0)$ not depending on the parameter r , then the same number will be a bound valid for F . \square

Claim 2. We may assume $m = 1$ and $F \geq 0$.

Proof of Claim 2. We may replace F by $\sum F_i^2$. \square

Claim 3. We may assume that 0 is a regular value for F .

Proof of Claim 3. Consider the function $F_\varepsilon := F - \varepsilon$, for $\varepsilon \in \mathbb{K}_+$. It follows from Theorem 8.9 that the set of critical values of F is meager, hence we can find an infinitesimal sequence $(\varepsilon_n)_{n < \text{Cofin}(\mathbb{K})}$ such that ε_n is a regular value for F . If we find a bound which works for F_{ε_n} , then by Lemma 8.27, the same bound will work for F . \square

Claim 4. We may assume that x_n is a Morse function of $V(F)$.

Proof of Claim 4. By Corollary 8.10, we can choose $(a_1, \dots, a_n) \in \mathbb{K}^n$ such that $a_n \neq 0$ and $\sum a_i x_i$ is a Morse function on $V(F)$. Define $G(x_1, \dots, x_n) = F(x_1, \dots, x_{n-1}, x_n - \frac{1}{a_n} \sum_{i=1}^{n-1} a_i x_i)$. Then G is proper, 0 is a regular value of G , x_n is a Morse function on $V(G)$ and a bound on the number of connected components of $V(G)$ will also work for $V(F)$. \square

Once these four claims are established, note that every non-empty clopen definable subset C of $V(F)$ is d-compact, and hence the function x_n has at least one critical point on C ⁽⁶⁾; it follows by a standard argument that the number of def-connected components of $V(F)$ finite and is bounded by the number of critical points of x_n on $V(F)$, if the latter is also finite.

We can then proceed as in [Marker97]: a calculation shows that the critical points of x_n on $V(F)$ are regular zeroes of the map $(F, \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_{n-1}})$, a bound on whose number is given by the first statement in Theorem 8.4. This concludes the proof.

9 Relative Pfaffian closure

A consequence of Wilkie’s Theorem of the Complement [Wilkie99, Theorem 1.9] is that the structure generated by the real ordered field together with *all* Pfaffian chains is o-minimal.

Here we prove that the Pfaffian closure of an o-minimal structure *inside a definably complete Baire structure* is o-minimal. This will be obtained by proving that such a structure satisfies the hypotheses of Theorem 7.35, and this is the reason why we proved such a general statement (of which [Wilkie99, Theorem 1.8] and [KM99, Theorem 1] are special cases, even if in 7.35 one considers only expansions of the real field).

9.1 Preliminary results on o-minimal structures

We will need the following results about o-minimal structures.

Let \mathbb{F} be an o-minimal structure expanding a (real closed) field. In this subsection, by “definable” we will mean “definable with parameters from \mathbb{F} ”, and, by “cell”, “cell definable in \mathbb{F} ”.

⁽⁶⁾If $\dim V(F) > 0$, then x_n has actually at least two critical points on C .

Proposition 9.1. *For every $N \geq 1$ and every $Y \subseteq \mathbb{F}^n$ closed and definable there exists $h : \mathbb{F}^n \rightarrow [0, 1]$ definable and \mathcal{C}^N , such that $Y = V(h)$. In particular, \mathbb{F} is generated by its \mathcal{C}^N definable functions. Moreover, if Z is a closed definable subset of \mathbb{F}^n disjoint from Y , then we can also require that $Z = V(1 - h)$.*

Proof. We can use [DM96, Corollary C.12], since the proof works also for o-minimal structures expanding any real closed field, not just \mathbb{R} . \square

Lemma 9.2. *For every $X \subseteq \mathbb{F}^n$ definable there exists $Y \subseteq \mathbb{F}^{n+1}$, also definable, such that Y is closed and $X = \pi(Y)$. If moreover X is a \mathcal{C}^N cell, then Y can be also chosen to be a (closed) \mathcal{C}^N cell of the same dimension as X .*

Proof. Since X is a finite union of \mathcal{C}^N cells, and projection commutes with topological closure, it suffices to deal with the case when X is a \mathcal{C}^N cell. If X is closed, define $Y := X \times \{0\}$. Otherwise, ∂X is a closed non-empty set; let $h : \mathbb{F}^n \rightarrow [0, 1]$ be definable and \mathcal{C}^N such that $\partial X = V(h)$. Define

$$Y := \pi^{-1}(\overline{X}) \cap \{(\bar{x}, z) \in \mathbb{F}^{n+1} : z \cdot h(\bar{x}) = 1\}.$$

It is easy to see that Y is a cell satisfying the conclusion. \square

Lemma 9.3. *Let $Y \subseteq \mathbb{F}^n$ be a closed \mathcal{C}^N cell. Then, there exists a definable \mathcal{C}^N retraction $r : \mathbb{F}^n \rightarrow Y$.*

Proof. After a permutation of variables, w.l.o.g. $Y = \Gamma(f)$, for some definable \mathcal{C}^N function $f : W \rightarrow \mathbb{F}^{n-d}$, where W is an open cell in \mathbb{F}^d . Let $U := W \times \mathbb{F}^{n-d}$ and define $r_0 : U \rightarrow Y$, $r_0(\bar{z}, \bar{y}) := (\bar{z}, f(\bar{z}))$. Notice that U is an open neighbourhood of Y and r_0 is a retraction. Let V be an open definable subset of \mathbb{F}^n , such that $Y \subseteq V$ and $\overline{V} \subseteq U$. By Proposition 9.1 there exists $h : \mathbb{F}^n \rightarrow [0, 1]$ definable and \mathcal{C}^N such that $Y = h^{-1}(1)$ and $\mathbb{F}^n \setminus V = h^{-1}(0)$.

Since Y is a \mathcal{C}^N cell, there exists $\phi : Y \rightarrow \mathbb{F}^d$ definable \mathcal{C}^N diffeomorphism, with $d := \dim Y$. W.l.o.g., we can assume that $\phi(0) = 0$. For every $t \in \mathbb{F}$ and $\bar{x} \in Y$, define

$$t * \bar{x} := \phi^{-1}(t \cdot \phi(\bar{x})) \in Y.$$

Define

$$r(\bar{x}) := \begin{cases} 0 & \text{if } \bar{x} \notin U; \\ h(\bar{x}) * r_0(\bar{x}) & \text{if } \bar{x} \in U. \end{cases} \quad \square$$

9.2 Expansions of o-minimal structures by total smooth functions

In this subsection we generalize Theorem 7.7 to the situation where $\mathbb{K}_{\mathcal{F}}$ expands an o-minimal structure. More precisely, let \mathbb{K} be a definably complete Baire structure, \mathbb{K}_0 be an o-minimal reduct of \mathbb{K} , expanding the field structure, and \mathcal{F} be a family of total C^∞ functions definable in \mathbb{K} . We assume that \mathcal{F} is closed under permutation of variables, contains the coordinate functions $(x_1, \dots, x_n) \mapsto x_i$, and that if $f \in \mathcal{F}$, then $(\bar{x}, y) \mapsto f(\bar{x})$ is also in \mathcal{F} . Let $\mathbb{K}_0(\mathcal{F})$ be the reduct of \mathbb{K} generated by \mathbb{K}_0 and \mathcal{F} . We give necessary and sufficient conditions for $\mathbb{K}_0(\mathcal{F})$ to be an o-minimal structure.

Definition 9.4. Let \mathcal{G}_0 be the set of all total continuous functions definable in \mathbb{K}_0 , and \mathcal{G} be the set of functions of the form $h \circ f$, for some $f : \mathbb{K}^n \rightarrow \mathbb{K}^m$ in \mathcal{F}^m and some $h : \mathbb{K}^m \rightarrow \mathbb{K}$ in \mathcal{G}_0 (notice that $\mathcal{G}_0 \subseteq \mathcal{G}$).

For every $n \in \mathbb{N}$, let \mathcal{S}_n be the family of subsets of \mathbb{K}^n of the form $V(g)$, for some $g : \mathbb{K}^n \rightarrow \mathbb{K}$ in \mathcal{G} , and let $\mathcal{S} := (\mathcal{S}_n)_{n \in \mathbb{N}}$.

Theorem 9.5. $\mathbb{K}_0(\mathcal{F})$ is o-minimal iff, for every X in \mathcal{S} , $\gamma(X) < \infty$.

Proof. Notice that \mathcal{S} is a closed weak structure. It is obvious that every set in \mathcal{S} is definable in $\mathbb{K}_0(\mathcal{F})$. Conversely, since \mathbb{K}_0 is o-minimal, Prop. 9.1 and the fact that $\mathcal{G}_0 \subseteq \mathcal{G}$ imply that the structure generated by \mathcal{S} expands \mathbb{K}_0 ; since moreover $\mathcal{F} \subseteq \mathcal{G}$, \mathcal{S} generates $\mathbb{K}_0(\mathcal{F})$.

Hence, by Theorem 7.35, it suffices to show that \mathcal{S} satisfies DAC^N for all N . That is, let $n \in \mathbb{N}$ and fix $A \in \mathcal{S}_n$. It is enough to prove the following:

(*) There exists $m \geq n$, such that, for every $N \in \mathbb{N}$, A is of the form $\pi(V(g_N))$ for some $g_N : \mathbb{K}^m \rightarrow \mathbb{K}$ in \mathcal{G} and \mathcal{C}^N .

Let $g \in \mathcal{G}$ such that $A = V(g)$. Hence, $g = h \circ f$, for some $f : \mathbb{K}^n \rightarrow \mathbb{K}^m$ in \mathcal{F}^m and some $h : \mathbb{K}^m \rightarrow \mathbb{K}$ in \mathcal{G}_0 . Let $h_N : \mathbb{K}^m \rightarrow \mathbb{K}$ be \mathcal{C}^N and definable in \mathbb{K}_0 , such that $V(h) = V(h_N)$ (the existence of h_N is given by Prop. 9.1), and define $g_N := h_N \circ f : \mathbb{K}^n \rightarrow \mathbb{K}$. Note that g_N is \mathcal{C}^N and in \mathcal{G} . Note moreover that

$$A = V(g) = f^{-1}(V(h)) = f^{-1}(V(h_N)) = V(g_N),$$

and we are done (in fact, we see that we can take $m = n$ in (*)). \square

9.3 Speissegger's theorem

We proceed to define a notion of relative Pfaffian closure. We recall that Speissegger's results in [Speiss99] concern expansions of the real

field. Let \mathbb{R}_0 be an o-minimal expansion of the real field. Let $U \subseteq \mathbb{R}^n$ be an open subset definable in \mathbb{R}_0 , and ω be an \mathbb{R}_0 -definable \mathcal{C}^1 -form on U which is never 0. A **leaf** with data (U, ω) is a closed connected real submanifold of U of dimension $n - 1$ that is orthogonal to ω at every point. A **Rolle leaf** (RL) is a leaf L which moreover satisfies the condition: if $\gamma : [0, 1] \rightarrow U$ is a \mathcal{C}^1 curve with end-points in L , then γ is orthogonal to ω in at least one point. Speissegger proved that if we add to \mathbb{R}_0 all Rolle leaves with data definable in \mathbb{R}_0 , then we still get an o-minimal structure.

We now generalize Speissegger's theorem to o-minimal structure outside the real line. We remark that the first results in this direction are due to Fratarcangeli in [Fratarc08]. However his definitions and methods are substantially different from ours (he follows [Speiss99] whereas we follow [KM99]) and the results he obtains are a special case of the main theorem in this section. We will use a definition of "Rolle leaves" (which we call Virtual Rolle Leaves) which is more complicated than the one in [Fratarc06], but which will allow us to give in Section 10 effective bounds on a series of topological invariants of sets definable in the Pfaffian closure of an o-minimal expansion of the real field, thus answering a question of Fratarcangeli [Fratarc06, p.6].

Proviso. Let \mathbb{K}_0 be an o-minimal structure (expanding a field), and \mathbb{K} be an expansion of \mathbb{K}_0 that is definably complete and Baire. For the rest of this section, by "connected" we will mean "definably connected" (in \mathbb{K}), by "connected component" we will mean "definably connected component", and by "cell" we will mean "cell definable (with parameters) in \mathbb{K}_0 ".

\mathbb{K} -manifolds (which we will simply call manifolds) have already been defined (Def. 1.17).

Definition 9.6. Let $\omega = a_1 dx_1 + \cdots + a_n dx_n$ be a definable \mathcal{C}^1 differential form, defined on some definable open subset $U \subseteq \mathbb{K}^n$, such that $\omega \neq 0$ on all U . A **multi-leaf** with data (U, ω) is a \mathcal{C}^1 manifold M contained in U and closed in U , of dimension $n - 1$, such that M is orthogonal to ω at all of its points (i.e., $T_a M = \ker(\omega(a))$, for every $a \in M$).

Compare the above with the definition of \mathbb{K}_0 -leaf in [Fratarc06, 5.2], where he asks that M is connected.

We must now face the problem of generalizing the notion of Rolle leaf to the context of definably complete Baire structures.

We let an **arc** be a definable \mathcal{C}^1 map $\gamma : [0, 1] \rightarrow \mathbb{K}^n$, such that γ' is always non-zero.

The most natural notion of generalized Rolle leaf would be the following (cf. [Fratarc06, Remark p. 33]):

Definition 9.7. An **alternate Rolle leaf** (ARL) is a connected multi-leaf L with data (U, ω) which moreover satisfies the condition: if $\gamma : [0, 1] \rightarrow U$ is an arc in L , then γ is orthogonal to ω in at least one point.

Unfortunately it is not clear whether in definably complete Baire structures definable C^1 connected manifolds of dimension one are parametraizable as a finite union of arcs. This fact creates an impediment, as will be clear later (see Subsection 9.5), and forces us to modify this definition.

One could think of replacing the use of arcs with the use of connected manifolds of dimension 1 (see the definition of Rolle leaf according to Fratarcangeli [Fratarc06, 1.5]).

The drawback of this choice is that it is not possible to express with a first-order formula the fact that a set is definably connected. However, for the application we have in mind (see Section 10) we need the definition to be first-order (in a sense which will be made precise later).

Hence we will introduce the notion of **Virtual Rolle Leaf** (VRL, see Definition 9.19), which has the advantage of being first order (as ARL is) and at the same time of involving the notion of manifold of dimension one, rather than that of arc.

We are now ready to define the notion of relative Pfaffian closure:

Definition 9.8. Inductive definition: for every $n \in \mathbb{N}$, let \mathbb{K}_{n+1} be the expansion of \mathbb{K}_n to a language L_{n+1} with a new predicate for every VRL with \mathbb{K}_n -definable data. Let $L^* = \bigcup_n L_n$ and define the **relative Virtual Pfaffian closure of \mathbb{K}_0 inside \mathbb{K}** , denoted by $\mathcal{VP}(\mathbb{K}_0, \mathbb{K})$, as the L^* -expansion of \mathbb{K}_0 where every predicate is interpreted as the corresponding Rolle leaf.

Our aim is to prove the following version of Speissegger's Theorem:

Theorem 9.9. *Let \mathbb{K} be a definably complete Baire structure and \mathbb{K}_0 be an o-minimal reduct of \mathbb{K} . Then $\mathcal{VP}(\mathbb{K}_0, \mathbb{K})$ is o-minimal.*

9.4 Virtual Rolle Leaves

We will now give the precise definition of Virtual Rolle leaf. The idea is the following: unlike the definition of ARL, where we considered all arcs X , in the definition of VRL we we consider closed manifolds X of

dimension 1 (not necessarily connected) such that X does not have compact connected components. We want to find a first order condition on X that implies a bound on the number of connected components of X : every component has two “end-points at infinity” (see Definition 9.14 below); hence, if we ask that X has at most $2k$ end-points at infinity, we obtain that X has at most k connected components. It remains to express the requirement that X have no compact components in a first order way: this is done by asking the existence of a definable \mathcal{C}^1 function without critical points on X .

Finally, the Rolle condition for a leaf L is expressed by asking that for any X as above that intersects L in a number of points which is greater than the number of connected components of X , there is a point where X is orthogonal to the 1-form defining L .

Definition 9.10. A **weak cell** of dimension d is a \mathbb{K}_0 -definable set $U \subseteq \mathbb{K}^n$ which is diffeomorphic, via a \mathbb{K}_0 -definable map ϕ_U , to \mathbb{K}^d . For every $0 < t \in \mathbb{K}$, we define $U_t := \phi_U^{-1}(\{x \in \mathbb{K}^d : \|x\| = t\})$.

We consider the diffeomorphism ϕ_U as part of a weak cell: the same subset U of \mathbb{K}^n two different choices of diffeomorphisms should be considered two different weak cells. Notice that, for $0 < t \in \mathbb{K}$, U_t is a compact manifold of dimension $d - 1$.

Definition 9.11. Let $U \subseteq \mathbb{K}^n$ be a weak cell. We say that $X \subseteq U$ is a **twine** in U if X is a 1-dimensional \mathcal{C}^1 manifold, such that X is closed in U . We say that $X \subseteq U$ is a **good twine** in U if X is a twine in U and moreover there exists a definable \mathcal{C}^1 function $\rho : X \rightarrow \mathbb{K}$ without critical points.

Remark 9.12. Let $X \subseteq \mathbb{K}^n$ be definable. We denote by $\mathfrak{B}(X)$ the Boolean algebra of definable clopen subsets of X . $\mathfrak{B}(X)$ is finite iff $\text{cc}(X)$ (the number of connected components of X) is finite, and in that case each connected component of X is definable and an atom of $\mathfrak{B}(X)$, and moreover $|\mathfrak{B}(X)| = 2^{\text{cc}(X)}$.

Moreover, for every $n \in \mathbb{N}$, the following are equivalent:

1. $\mathfrak{B}(X) \leq 2^n$;
2. $\text{cc}(X) \leq n$;
3. if Y_1, \dots, Y_{n+1} are disjoint element of $\mathfrak{B}(X)$, then at least one of them is empty.

Remark 9.13. Let U be a weak cell and X be a twine in U . Let $\emptyset \neq Y \in \mathfrak{B}(X)$. Then, Y is also a twine. If moreover X is good, then Y is also good and not compact. In particular, if X is a good twine and $\text{cc}(X) < \infty$, then no connected component of X is compact.

Definition 9.14. Let U be a weak cell and X be twine in U . For each $0 < t \in \mathbb{K}$, let $X_t := \{x \in X \cap U_t : X \text{ is transversal to } U_t \text{ at } x\}$. We denote by

$$\text{vb}_U(X) := \limsup_{t \rightarrow +\infty} |X_t| \in \mathbb{N} \cup \{\infty\},$$

the **virtual boundary** of X .

Notice that X_t is a 0-dimensional manifold, and hence $|X_t| = \text{cc}(X_t)$. Notice also that, unlike the number of connected components, $\text{vb}_U(X)$ can be defined with a first order formula.

Lemma 9.15. *Let U be a weak cell and $X \subseteq U$ be a good twine in U . Assume that $X = X_1 \sqcup X_2$, where $\emptyset \neq X_i \in \mathfrak{B}(X)$, $i = 1, 2$. Then $\text{vb}_U(X) = \text{vb}_U(X_1) + \text{vb}_U(X_2)$.*

Lemma 9.16. *Assume that \mathbb{K} is o-minimal. Let U be a weak cell and $X \subseteq U$ be a good twine in U . If X is a connected, then $\text{vb}_U(X) = 2$. More generally, $\text{vb}_U(X) = 2 \cdot \text{cc}(X)$.*

Proof. It suffices to do the case when X is connected. Since \mathbb{K} is o-minimal, X is then the image of some definable \mathcal{C}^1 function $\gamma : (0, 1) \rightarrow \mathbb{K}$. The conclusion follows from the o-minimality of \mathbb{K} . \square

Lemma 9.17. *Let U be a weak cell and X be a good twine in U . If X is non-empty, then $\text{vb}_U(X) \geq 1$, and if moreover \mathbb{K} is an expansion of \mathbb{R} , then $\text{vb}_U(X) \geq 2$.*

Moreover, if $\text{vb}_U(X)$ is finite, then $\text{cc}(X) \leq \text{vb}_U(X)$, and in particular X has a finite number of connected component, and each component of X is not compact.

Proof. By Remark 9.12, it suffices to show that if X is non-empty, then $\text{vb}_U(X) \geq 1$; the remainder follows from Remark 9.13. Assume, for contradiction, that $\text{vb}_U(X) = 0$. Since $\text{vb}_U(X) = 0$, there exists $R > 0$ such that, for every $t > R$, X meets U_t only non-transversally. Define $r : U \rightarrow \mathbb{K}$, $r(x) := |\phi_U(x)|$, let $U_{>R} := r^{-1}(R, +\infty) = \{x \in U : |\phi_U(x)| > R\}$, $Y := X \cap U_{>R}$ and $s := r \upharpoonright Y$; notice that Y is open in X . Notice also that s has only critical points (on Y), and therefore s is locally constant. Hence, for every $t > R$, $s^{-1}(t)$ is clopen in Y , and therefore it is open in X . By Remark 9.13, X is not compact; thus, there exists $t_0 > R$

such that $Z := X \cap U_t = Y \cap U_t$ is non-empty, and, by what we have said before, it is clopen in X . Hence, by Remark 9.13, Z is not compact. However, Z is closed and bounded in \mathbb{K}^n , contradiction.

The case \mathbb{K} expanding \mathbb{R} follows from the fact that each connected component of X (not necessarily definable!) is the image of some \mathcal{C}^1 function $f : (0, 1) \rightarrow \mathbb{R}$, and some standard analysis. \square

Remark 9.18. Let $X \subset \mathbb{K}^n$ be a weak cell of dimension 1. Then, X is a good twine in itself, and $\text{vb}_X(X) = 2$.

Definition 9.19. A **virtual Rolle leaf** (VRL) is a multi-leaf L with data (U, ω) which satisfies the following condition: for every $n \in \mathbb{N}$, for every $V \subseteq U \times \mathbb{K}^n$ weak cell and every X good twine in V , if $|X \cap (L \times \mathbb{K}^n)| > \text{vb}_V(X)$, then X is orthogonal to $\pi^*(\omega)$ in at least one point, where $\pi^*(\omega)$ is the 1-form on $U \times \mathbb{K}^n$ induced by ω via the projection $\pi : U \times \mathbb{K}^n \rightarrow U$.

With the notation of the above definition, if $\text{vb}_V(X)$ is infinite, then the premise is false, and therefore the condition is automatically satisfied (for the given X). Therefore, to verify whether L is a VRL, we need to check only the good twines X such that $\text{vb}_V(X)$ is finite. Moreover, by Lemma 9.17, such a good twine X satisfies $\text{cc}(X) \leq \text{vb}_V(X)$. Therefore, if X_1, \dots, X_m are the components of X , and $|X \cap L| > \text{vb}_V(X)$, then for at least one i we have $|X_i \cap L| > 1$.

9.5 O-minimality of $\mathcal{VP}(\mathbb{K}_0, \mathbb{K})$

In this subsection we prove Theorem 9.9. For this subsection, a Rolle leaf will be a virtual Rolle leaf.

Definition 9.20. Let $\text{Rolle}(\mathbb{K}_0, \mathbb{K}) = \{(\text{Rolle}(\mathbb{K}_0, \mathbb{K}))_n \mid n \in \mathbb{N}\}$ be such that $(\text{Rolle}(\mathbb{K}_0, \mathbb{K}))_n$ consists of all the finite unions of sets $A \cap L_1 \cap \dots \cap L_k$, which we call *basic Rolle sets*, where $A \subseteq \mathbb{K}^n$ is \mathbb{K}_0 -definable, and each L_i is a Rolle leaf with data (U_i, ω_i) in \mathbb{K}_0 .

We will show:

Proposition 9.21. $\text{Rolle}(\mathbb{K}_0, \mathbb{K})$ is a semi-closed o-minimal weak structure, satisfying DAC^N for all N .

Since $\text{Rolle}(\mathbb{K}_0, \mathbb{K})$ generates \mathbb{K}_1 in Def. 9.8, this, together with Theorem 7.35, shows that \mathbb{K}_1 is o-minimal; by applying inductively the same result to each \mathbb{K}_n , we obtain a proof of 9.9.

We will prove Proposition 9.21 via a series of lemmas.

Lemma 9.22. $\text{Rolle}(\mathbb{K}_0, \mathbb{K})$ is a weak structure.

Proof. As in [KM99, Lemma 3]. Notice that $\text{Rolle}(\mathbb{K}_0, \mathbb{K})$ is closed under cartesian products by definition of VRL. \square

Remark 9.23. Every basic Rolle set is the projection of another basic Rolle set, such that all the open sets U_i in the data are the same open set U .

Proof. As in [KM99, ¶3.4]. \square

Proposition 9.24. Let $\Omega = (\omega_1, \dots, \omega_q)$ be a tuple of \mathbb{K}_0 -definable non-singular 1-forms defined on some common open subset U of \mathbb{K}^n , and let A be a \mathbb{K}_0 -definable subset of U . Then, there is a natural number N such that, whenever L_i is a VRL of $\omega_i = 0$ for each $i = 1, \dots, q$, then $A \cap L_1 \cdots \cap L_q$ is the union of fewer than N connected manifolds. Moreover, N does not depend on the parameters used in defining Ω , U , and A (and on the choices of the leaves L_i).

The proof of this proposition is in Subsection 9.6. Notice the similarity with [Fratarc06, Theorem 1.7].

Proposition 9.25. Let U be a \mathbb{K}_0 -definable open subset of \mathbb{K}^n , and ω be a \mathbb{K}_0 -definable 1-form on U , such that $\omega \neq 0$ on all U . Let L be a multi-leaf with data (U, ω) . Let C be a definable connected \mathcal{C}^1 manifold of dimension at most $n - 1$ contained in U , such that C is orthogonal to ω at all of its points. Then, either C is contained in L , or C is disjoint from L .

Proof. [Fratarc06, Lemma 5.4]. \square

Lemma 9.26. $\text{Rolle}(\mathbb{K}_0, \mathbb{K})$ is semi-closed.

Proof. We use:

1. union commutes with projection;
2. the class of projections (from various \mathbb{K}^n) of closed sets in $\text{Rolle}(\mathbb{K}_0, \mathbb{K})$ is closed under intersections.

It suffices to prove that any Rolle leaf $L \subseteq \mathbb{K}^n$ is the projection of a closed set in $\text{Rolle}(\mathbb{K}_0, \mathbb{K})$. Let (U, ω) be the data (definable in \mathbb{K}_0) of L . Do a \mathcal{C}^1 cell decomposition of U . It suffices to prove that, for each cell E_i in the decomposition, $L \cap E_i$ is the projection of a closed set in $\text{Rolle}(\mathbb{K}_0, \mathbb{K})$.

Let U_i be an open cell of the decomposition, and $L_i := L \cap U_i$. Consider a \mathcal{C}^1 closed cell $D_i \subseteq \mathbb{K}^{n+1}$ such that $U_i = \pi(D_i)$, and D_i is of dimension n (D_i exists by Lemma 9.2). Let $\tilde{\omega} := \pi^*(\omega)$, the 1-form on $U \times \mathbb{K}$ induced by ω , and $\tilde{L} := \pi^{-1}(L)$; note that \tilde{L} is a Rolle leaf, with data $(U \times \mathbb{K}, \tilde{\omega})$. Define $C_i := D_i \cap \tilde{L}$; C_i is a basic Rolle set, closed in \mathbb{K}^{n+1} , and $\pi(C_i) = L_i$.

If instead E_i is a \mathcal{C}^1 cell in the decomposition of dimension less than n , consider the \mathbb{K}_0 -definable set of points in E_i whose tangent space (w.r.t. the manifold E_i) is contained in $\ker(\omega)$. Decompose this again into \mathbb{K}_0 -definable connected submanifolds. By Prop. 9.25, any of these is either disjoint or contained in L . Hence, $L \cap E_i$ is a finite union of sets definable in \mathbb{K}_0 , and hence is itself definable in \mathbb{K}_0 , and thus projection of a closed set (in \mathbb{K}_0). \square

Lemma 9.27. *Rolle(\mathbb{K}_0, \mathbb{K}) is an o-minimal weak structure.*

Proof. The conclusion can be easily obtained from Proposition 9.24, reasoning as in [Speiss99, Corollary 2.7]. \square

Hence, we can conclude that Rolle(\mathbb{K}_0, \mathbb{K}) is a semi-closed o-minimal weak structure. The last step is proving that Rolle(\mathbb{K}_0, \mathbb{K}) satisfies DAC^N for all N , and hence is an o-minimal *structure*. Notice that the following lemma does not imply Lemma 9.26, because the DAC^N condition does not imply that a weak structure is semi-closed.

Lemma 9.28. *Rolle(\mathbb{K}_0, \mathbb{K}) satisfies DAC^N for all N .*

Proof. Proceed as in the proof of the preceding lemma. Using Lemma 7.36, we are reduced to prove:

(*) If $U \subseteq \mathbb{K}^n$ is open and definable in \mathbb{K}_0 , ω is a \mathcal{C}^1 form, also definable in \mathbb{K}_0 , and L is a Rolle leaf with data (U, ω) , then there exists a natural number r , such that, for every $N \geq 1$, there is a set $S \subseteq \mathbb{K}^{n+1}$, such that S is a finite union of sets, each of whose is an intersection of at most r sets of the form $V(f_{N,i})$, where each $f_{N,i} : \mathbb{K}^{n+1} \rightsquigarrow \mathbb{K}$ is a \mathcal{C}^N admissible correspondence in $\widetilde{\text{Rolle}(\mathbb{K}_0, \mathbb{K})}$, $i = 1, \dots, l$, and $L = \pi(S)$, where $\pi := \Pi_n^{n+1}$.

Note that the above claim is the DAC^N hypothesis for L , with $m = n + 1$. By inspecting the following proof, the reader can easily verify that r indeed does not depend on N .

Fix N . Do a decomposition of U into \mathcal{C}^N cells E_i , such that on each open cell ω is a \mathcal{C}^N form. It suffices to prove (*) for each $L \cap E_i$.

CASE 1. If E_i is a cell of dimension less than n , then, as in the proof of the previous lemma, $E_i \cap L$ is definable in \mathbb{K}_0 , and hence is the projection of $V(f_N)$, for some \mathcal{C}^N function $f_N : \mathbb{K}^{n+1} \rightarrow \mathbb{K}$ definable in \mathbb{K}_0 .

CASE 2. If E_i is an open cell, by Proposition 9.24, $L \cap E_i$ has a finite number of connected components L_1, \dots, L_k ; moreover, since L is a manifold of dimension $n - 1$ and E_i is open, each L_j is also a manifold of dimension $n - 1$, and moreover it is a Rolle leaf with data (E_i, ω) ; hence, by substituting U with E_i , w.l.o.g. we can assume that U is an open cell.

Let $\omega := a_1 dx_1 + \dots + a_n dx_n$, and $V_j := \{\bar{x} \in U : a_j(\bar{x}) \neq 0\}$, $j = 1, \dots, n$. Note that V_j is open and definable in \mathbb{K}_0 . Decompose again U into \mathcal{C}^N cells, in a way compatible with each V_j . For the non-open cells, proceed as in Case 1. For the open ones, do the same trick as before, and reduce to the case $a_n(\bar{x})$ never 0 on U , and therefore we can assume that a_n is the constant function 1.

Hence, L is a closed subset of U , and satisfies all conditions for being the graph of an admissible \mathcal{C}^N correspondence $l : \mathbb{K}^{n-1} \rightsquigarrow \mathbb{K}$, except that L might not be closed in \mathbb{K}^n . If $U = \mathbb{K}^n$, we can easily conclude as in [KM99, Lemma 6]. Otherwise, we have more work to do.

Let $\theta := \Pi_{n-1}^n$, $U' := \theta(U)$ be the basis of the cell U , $\phi' : \mathbb{K}^{n-1} \xrightarrow{\sim} U'$ and $\phi : \mathbb{K}^n \xrightarrow{\sim} U$ be \mathbb{K}_0 -definable \mathcal{C}^N diffeomorphisms, such that $\phi' \circ \theta = \theta \circ \phi$. Let $\tilde{L} := \phi^{-1}(L)$, and $\tilde{\omega} := \phi^*(\omega)$. Then, \tilde{L} is a Rolle leaf, with data $(\mathbb{K}^n, \tilde{\omega})$. Moreover, \tilde{L} is the graph of a \mathcal{C}^N admissible correspondence $\tilde{l} : \mathbb{K}^{n-1} \rightsquigarrow \mathbb{K}$ (in fact, \tilde{L} is closed in \mathbb{K}^n).

Define $\tilde{g}(x_1, \dots, x_n) := \tilde{l}(x_1, \dots, x_{n-1}) - x_n$, $\tilde{g} : \mathbb{K}^n \rightsquigarrow \mathbb{K}$. By Lemma 7.16, \tilde{g} is admissible; notice that $\tilde{L} = V(\tilde{g})$.

We would like to pullback \tilde{g} via ϕ ; the problem is that $\tilde{g} \circ \phi^{-1}$ is not defined on all \mathbb{K}^n .

Let $D \subseteq \mathbb{K}^{n+1}$ be a closed \mathcal{C}^N cell, such that $\pi(D) = U$, $f_{N,1} : \mathbb{K}^{n+1} \rightarrow \mathbb{K}$ be a \mathcal{C}^N and \mathbb{K}_0 -definable function, such that $D = V(f_{N,1})$, and $r : \mathbb{K}^{n+1} \rightarrow D$ be a \mathbb{K}_0 -definable \mathcal{C}^N retraction (D , $f_{N,1}$ and r exists by Lemmas 9.2 and 9.3). Let $f_{N,2} := \tilde{g} \circ \phi^{-1} \circ \pi \circ r$. Notice that $\phi^{-1} \circ \pi \circ r$ is a total \mathcal{C}^N function, and therefore, by Lemmas 7.15 and 9.27, $f_{N,2}$ is admissible; $f_{N,1}$ is also obviously admissible. It is also clear that $L = \pi(V(f_{N,1}) \cap V(f_{N,2}))$. \square

Remark 9.29. In the proof of [KM99, Lemma 6] there is a gap, in that $f_{N,2}$ might not be a total function: this is *the* reason why we had to work with admissible correspondences instead of total functions. It is still true that the proof contained in [KM99] implies the o-minimality of the closure under *total* \mathcal{C}^∞ R -Pfaffian functions of an o-minimal expansion

R of the real field (cf. Theorem 9.5).

However, the correspondences under consideration are single valued, due to the Rolle condition.

Lemma 9.30. *Let $\omega := a_1 dx_1 + \cdots + a_n dx_n$ be a C^N 1-form on \mathbb{K}^n , such that $a_n \equiv 1$, and let F be a Rolle leaf for ω . Then, F is the graph of a C^{N+1} partial function $f : U \rightarrow \mathbb{K}$, with open domain $U \subseteq \mathbb{K}^{n-1}$.*

Proof. The fact that f is an admissible C^{N+1} correspondence is clear. It remains to prove that f is single-valued. If not, there exist $\bar{x} \in \mathbb{K}^{n-1}$ and $y_1 < y_2 \in \mathbb{K}$, such that, for $i = 1, 2$, $p_i := (\bar{x}, y_i) \in F$. Let J be the “vertical” segment with endpoints p_1 and p_2 . By the Rolle condition, there exists $q \in J$ such that J is orthogonal to ω at q . Since J is vertical, this means that $\omega(q)$ is “horizontal”, contradicting the fact that $a_n \equiv 1$. \square

However, as we said before, the partial function $f_{N,2}$ might not be total, as the following example shows.

Example 9.31. Let $f : \mathbb{R} \hookrightarrow \mathbb{R}$ be the partial function $f(x) := 1/x$, defined on \mathbb{R}_+ , and $F \subset \mathbb{R}^2$ be the graph of f . Let $\omega(x, y) := y^2 dx + dy$ be a 1-form defined on \mathbb{R}^2 . Then, F is a C^∞ Rolle leaf of $\omega = 0$. In fact, f solves the differential equation $f' = -f^2$, and therefore we can apply [Speiss99, Example 1.3].

9.6 Proof of Proposition 9.24

We will assume familiarity with [Fratarc06]. Some important but easy observations are the following ones:

- [Fratarc06, Lemma 5.9] does not require that the manifolds L_i are connected, and therefore can be applied to L_i multi-leaves.
- [Fratarc06, Prop. 5.10] does not use neither the conditions that the L_i are connected nor the Rolle condition, and remains true for L_i multi-leaves.
- [Fratarc06, Prop. 5.7] can be used in the following form:

Proposition 9.32. *Let U and V be definable open subsets of \mathbb{K}^n , and let $\sigma : V \rightarrow U$ be a definable diffeomorphism. Let ω be a definable 1-form on U , and L be a multi-leaf with data (U, ω) . Then, $\sigma^{-1}(L)$ is a multi-leaf with data $(V, \sigma^*(\omega))$.*

If L is a VRL and σ is \mathbb{K}_0 -definable, then $\sigma^{-1}(L)$ is a VRL.

Hence, the Rolle condition is used directly only at the end of the proof, on [Fratarc06, p. 39]. We will show how to use the Virtual Rolle condition.

The proof will proceed by induction on q . If $q = 0$, the conclusion follows from o-minimality of \mathbb{K}_0 ; hence, we can assume $q \geq 1$.

For the inductive step, we assume that we have already proved the conclusion for $q - 1$: that is, we assume that we have proved the result for every $(q - 1)$ -tuple Ω' of \mathbb{K}_0 -definable nonsingular 1-forms defined on some open set U' of $\mathbb{K}^{n'}$, for every \mathbb{K}_0 -definable set $A' \subseteq U'$, and for every corresponding $(q - 1)$ -tuple of VRL with data (U', Ω') .

Fix $U, \Omega, L_1, \dots, L_q$, and A as in the assumption of the theorem. Let $d := \dim(A)$. We prove the conclusion by a further induction on d .

As in the proof of [Fratarc06, Theorem 1.7], we can reduce to the case when A is a \mathbb{K}_0 -definable \mathcal{C}^1 -cell of dimension $d \geq q$, contained in U , and Ω is transverse to A ; that is, for every $a \in A$, the projections of (the vector fields associated to) $\omega_1, \dots, \omega_q$ on $T_a(A)$ are linearly independent. Notice that “ Ω transverse to A ” is equivalent to “the projection on $T(A)$ of the q -form $\omega_1 \wedge \dots \wedge \omega_q$ is never null”.

If $d > q$, we can conclude by induction on d as in [Fratarc06, p. 39, “CASE $d > q$ ”]; as we noticed before, the Rolle condition is not used in [Fratarc06, 5.10], and therefore we can use it in our situation.

Hence, it remains to treat the case $d = q$.

If $d = q$, we treat first as a way of exemplification the case $d = q = 1$. Then, A is a good twine in itself, thus, by the Rolle condition, and the fact that ω_1 is transverse to A , $|A \cap L_1| \leq \text{vb}_A(A) = 2$, and we are done.

In general, if $d = q$, define $L' := A \cap L_1 \dots L_{q-1}$ (or $L' := A$ if $q = 1$). Notice that L' is a twine in A . Let $\omega' := \omega'_1 \wedge \dots \wedge \omega'_{q-1}$, where each ω'_i is the projection of ω_i onto (the tangent space of) A . Notice that ω' is a non-singular $(q - 1)$ -form on A . If we identify ω' with the corresponding vector field on A , then ω' is always tangent to L' . Notice also that $A \cap L_1 \cap \dots \cap L_q$ is a 0-dimensional manifold, and therefore $\text{cc}(A \cap L_1 \cap \dots \cap L_q) = |L' \cap L_q|$.

We have to further decompose A in order to transform L' into a *good* twine. Fix a map $p : A \rightarrow \mathbb{K}$, such that p is \mathbb{K}_0 -definable, is \mathcal{C}^1 , and has no critical points on A . For every $x \in A$, let $c(x)$ be the gradient vector of p at x (by definition, $c(x)$ is tangent to A).

Define A_{crit} to be the set of points in A such that ω' is orthogonal to c , and $A_{\text{reg}} := A \setminus A_{\text{crit}}$. After a further cell decomposition, w.l.o.g. we can assume that either $A = A_{\text{reg}}$, or $A = A_{\text{crit}}$.

If $A = A_{\text{reg}}$, let ρ be the restriction of p to L' . Notice that, by definition of A_{reg} , ρ is a definable \mathcal{C}^1 function without critical points, and

hence L' is a good twine in A . Fix a \mathbb{K}_0 -definable diffeomorphism ϕ_A between A and \mathbb{K}^d , and define A_t accordingly. By induction on q , there is $N \in \mathbb{N}$ such that $L' \cap A_t \cap \{x \in A : \omega' \text{ is not orthogonal to } A_t\}$ has at most N connected components, where N does not depend on t . Hence, by definition, $\text{vb}_A(L') \leq N$. Thus, since L_q is a VRL and Ω is transverse to A , $|L' \cap L_q| \leq N$, and we are done.

If instead $A = A_{\text{crit}}$, for every $t \in \mathbb{K}$ let $B(t) := \{x \in A : p(x) = t\}$: each $B(t)$ is a \mathbb{K}_0 -definable set of dimension $d - 1$. By induction on q , L' has a uniformly bounded number of connected components M_1, \dots, M_r . Moreover, p is constant on each M_i , and therefore for each $i \leq r$ there exists $t_i \in \mathbb{K}$ such that $M_i \subseteq B(t_i)$. Thus, $M_i \cap L_q \subseteq L_1 \cap \dots \cap L_q \cap B(t_i)$, and therefore

$$A \cap L_1 \cap \dots \cap L_q \subseteq \bigcup_{i=1}^r B(t_i) \cap L_1 \cap \dots \cap L_q.$$

By induction on d , there exists a uniform (independent from t) bound r' for $\text{cc}(B(t_i) \cap L_1 \cap \dots \cap L_q)$, and therefore $\text{cc}(A \cap L_1 \cap \dots \cap L_q) \leq rr'$.

9.7 Variants of the Rolle Property

In this subsection we compare different notions of Rolle leaves: the original definition of Rolle leaf (RL), due to Speissegger, which makes sense only for expansions of the real field was given at the beginning of Section 9.3. Alternate Rolle leaves (ARL) and Virtual Rolle leaves (VRL) were defined in Definitions 9.7 and 9.19 respectively.

Definition 9.33. A Rolle leaf according to Fratarcangeli (FRL) is a connected multi-leaf L with data (U, ω) , which moreover satisfies the condition: for every $m \in \mathbb{N}$, if $X \subset U \times \mathbb{K}^m$ is a definable connected \mathcal{C}^1 submanifold of $U \times \mathbb{K}^m$ of dimension 1, and X intersects L in at least two points, then X is orthogonal to ω in at least one point (compare with [Fratarc06, 1.5]).

Proposition 9.34. *Let \mathbb{K} be an expansion of the real field. Then every RL is a VRL.*

In particular, we recover Speissegger's theorem is a special case of ours.

Proof. Let $L \subset \mathbb{R}^n$ be a RL with data (U, ω) . Let $V \subseteq U$ be a weak cell and X be a good twine in V . Assume that $|X \cap L| > \text{vb}_V(X) =: m$ (the case when $V \subseteq U \times \mathbb{R}^k$ can be treated similarly). We must show that

X is orthogonal to ω in at least one point; assume, for contradiction, that this is not the case. Let X_i be a connected component of X (notice that X_i is not necessarily definable). Since X is a good twine, X_i is not compact; moreover, X has at most m connected components. Hence, X_i intersects L in at least two points, for some connected component X_i . Thus, since L is a RL and X_i is arc-connected, X_i is orthogonal to ω in at least one point, contradiction. \square

Proposition 9.35. *Let \mathbb{K} be definably complete. Then every FRL is a VRL.*

In particular, we recover Fratarcangeli's theorem is a special case of ours.

Proof. Let L be a FRL with data (U, ω) . Let $V \subseteq U$ be a weak cell and $X \subseteq V$ be a good twine in V , such that $|X \cap L| > \text{vb}_V(X) =: m$ (for simplicity, we are dealing with the case $n = 0$ in Definition 9.19). By Lemma 9.17, X has at most m connected components; therefore, there exists Y component of X such that $|Y \cap L| \geq 2$. Thus, since L is a FRL, Y it orthogonal to ω at some point. \square

There is the following question left. Let \mathbb{K} be definably complete and Baire. Let $F : \mathbb{K}^n \rightarrow \mathbb{K}$ be a Pfaffian function (e.g., F is a definable C^∞ function satisfies $dF/dx_i = g_i(x, F(x))$, for some C^∞ \mathbb{K}_0 -definable functions $g_i : \mathbb{K}^n \rightarrow \mathbb{K}$). Let $\mathbb{K}_0(F)$ be the expansion of \mathbb{K}_0 by F . Is $\mathbb{K}_0(F)$ o-minimal? Let C be the graph of F , and ω be the 1-form on $U := \mathbb{K}^{n+1} g_1 dx_1 + \dots g_n dx_n - dy$. Notice that C is a connected Leaf with data (U, ω) . The question has positive answer if C is either a FRL or a VRL. We don't know it either is true, but, since being a VRL is a first-order condition, we can add either the condition “ C is VRL” to the axioms of $\mathbb{K}_0(F)$, or we can add the condition “every graph of a Pfaffian function is a VRL” to the axioms of \mathbb{K} . In both ways, we obtain an axiomatization of $\mathbb{K}_0(F)$ that ensures o-minimality.

10 Effective bounds

In this section we apply our results to derive uniform and effective bounds on some topological invariants (e.g. the number of connected components) of sets definable in the Pfaffian closure of an o-minimal expansion of the real field.

Let T_0 be a recursively axiomatized (not necessarily complete) o-minimal theory (if T_0 is not recursively axiomatized, then the effective

results of these section are still valid with respect to an oracle for T_0). Let \mathbb{R}_0 be an o-minimal expansion of the real field, which is a model of T_0 and let $\mathcal{P}(\mathbb{R}_0)$ be the Pfaffian closure of \mathbb{R}_0 (in the sense of [Speiss99]).

Definition 10.1. Let $X \subseteq \mathbb{R}^n$ be definable in $\mathcal{P}(\mathbb{R}_0)$. We call the **topological complexity** of X (denoted by $\text{t.c.}(X)$) the least $N \in \mathbb{N}$, such that there exist:

1. a simplicial complex Z composed by less than N simplexes, each of dimension less than N ;
2. and a $\mathcal{P}(\mathbb{R}_0)$ -definable homeomorphism to $f : X \approx |Z|$.

Note that, since $\mathcal{P}(\mathbb{R}_0)$ is o-minimal, the topological complexity is a well defined natural number.

Let X be defined by a formula φ , where some of the variables are evaluated as a suitable tuple of parameters. This definition will involve a finite number of Rolle leaves L_1, \dots, L_k . As one can see from the inductive definition of Pfaffian closure, every leaf L_i will have data (U_i, ω_i) definable (by a formula ϕ_i , where some of the variables are evaluated as a suitable tuple of parameters) in terms of a finite number of Rolle leaves $L_{i,1}, \dots, L_{i,n_i}$ of lower complexity (i.e. appearing at some earlier stage of the inductive construction). Hence, to the set X (or better, to its definition φ) we can associate a finite sequence $F_1 = L_1, \dots, F_k = L_k, F_{k+1} = L_{1,1}, \dots, F_{k+1+n_1} = L_{1,n_1}, \dots, F_m$ of Rolle leaves, which are involved in its definition. The aim of the following definition is to code the set X by this sequence of leaves (cf. [GV04, Fratarc06]).

Definition 10.2. Let \mathcal{L}_P be the language of \mathbb{R}_0 to which we adjoin a countable set of new predicates $\{P_1, \dots, P_n, \dots\}$. A **format** of a definable set X is the following finite sequence of \mathcal{L}_P -formulae (without parameters): $(\varphi, \mathbf{P}, \Phi)$, where

- for a suitable choice of parameters \bar{a} , the set X is defined by $\varphi(\cdot, \bar{a})$;
- $\mathbf{P} = (P_1, \dots, P_m)$ and every P_i represents a Rolle leaf F_i involved in this definition of X ;
- $\Phi = (\phi_1, \dots, \phi_m)$ and, for a suitable choice of parameters \bar{a}_i , the formula $\phi_i(\cdot, \bar{a}_i)$ defines the graph of ω_i on U_i , where (U_i, ω_i) is the data of the leaf F_i .

We did not allow the parameters in the definition. In particular, every other Rolle leaf with the same data (U, ω) has the same format.

Example 10.3. Let $X = L_1 \cup L_2$, where L_i are Rolle leaves with data (U_i, ω_i) . Let L_3 be a Rolle leaf with \mathbb{R}_0 -definable data (U_3, ω_3) . Suppose (U_1, ω_1) are $\langle \mathbb{R}_0, L_3 \rangle$ -definable and (U_2, ω_2) are \mathbb{R}_0 -definable. Let the graphs of $\omega_1, \omega_2, \omega_3$ be defined by formulas $\phi_1(\bar{a}_1, \bar{x}, \bar{y}), \phi_2(\bar{a}_2, \bar{x}, \bar{y}), \phi_3(\bar{a}_3, \bar{x}, \bar{y})$ respectively, where $\bar{a}_1, \bar{a}_2, \bar{a}_3$ are tuples of parameters. Then a format for X is given by the $L_0 \cup \{P_1, P_2, P_3\}$ -formulas $(\varphi, \mathbf{P}, \Phi)$, where $\varphi = P_1 \vee P_2$; $\mathbf{P} = (P_1, P_2, P_3)$; $\Phi = (\phi_1, \phi_2, \phi_3)$.

The next definition requires the notion of Rolle leaf to be first order. This is the reason why we introduced Virtual Rolle Leaves: the property of being a VRL is type-definable, i.e. it can be expressed by a countable (recursive) conjunction of first order formulae.

Definition 10.4. Let X be a definable set and $\theta = (\varphi, \mathbf{P}, \Phi)$ be a format for X . Let T_θ be the first order theory (in the language of \mathbb{R}_0 adjoined with the predicates P_1, \dots, P_m) with the following *recursive* (but not necessarily complete) axiomatization:

- Axioms of T_0 ;
- Axioms of Definably Complete Baire Structure;
- ϕ_i defines the graph of a non-singular C^1 1-form ω_i on some definable open subset U_i ;
- P_i is a VRL with data (U_i, ω_i) .

We now show the existence of a bound on the topological complexity of X , which depends (recursively) only on a format for X .

Theorem 10.5. *There is a recursive function η which, given a set X definable in $\mathcal{P}(\mathbb{R}_0)$ and a format θ for X , returns a natural number $\eta(\theta)$ which is an upper bound on the topological complexity of X .*

Proof. Let X be a definable set and $\theta = (\varphi, \mathbf{P}, \Phi)$ be a format for X . Note that, by Theorem 9.9, the theory T_θ is o-minimal. In particular, there is a natural number N such that $\text{t.c.}(X) < N$. Moreover, T_θ is recursively enumerable hence we can recursively enumerate all the formulas which (for every choice of the parameters) are provable in this theory. Take the first formula in this enumeration which defines a homeomorphism between the set defined by φ and some simplicial complex Z . Define $\eta(\theta)$ as the number of complexes which form Z . \square

Corollary 10.6. *There are recursive bounds on the following topological invariants of sets definable in $\mathcal{P}(\mathbb{R}_0)$: number of connected components, sum of the Betti numbers, number of generators of the fundamental group.*

Proof. Let X be a set definable in $\mathcal{P}(\mathbb{R}_0)$ and N be the recursive bound on $\text{t.c.}(X)$ given by the above theorem. Let F be a simplicial complex with at most N simplexes, each of them of dimension at most N , such that $|F|$ is homeomorphic to X (F exists by definition of $\text{t.c.}(X)$). Clearly, the number of connected components of $|F|$ (and hence of X) is at most N .

If F were a *closed* complex, by classical algebraic topology theory, N would be also a bound for the other mentioned topological invariants. Otherwise, let F' be the barycentric subdivision of F . By [EW08, Lemma7.1], there exists a closed simplicial complex C which is also a sub-complex of F' , such that $|F'|$ (and hence X) is homotopic to $|C|$. Since C is a closed complex, the number m of simplexes of C gives an upper bound to the sum of the Betti numbers of $|C|$ (and hence of X), and to the number of generators of $\pi_1(X, x_0)$ (for any $x_0 \in X$), and m is bounded by a recursive function of N . \square

We can also obtain bounds on the Hausdorff measure of definable sets.

Lemma 10.7. *Let $(C_i)_{i \in I}$ be a collection (not necessarily definable) of sets definable in $\mathcal{P}(\mathbb{R}_0)$, all with the same format Φ , such that each is contained in $B(0; 1)$, and of dimension at most d . Then there exists a uniform bound on their d -dimensional Hausdorff measure $\mathcal{H}^d(C_i)$.*

Proof. By a compactness argument, [Speiss99] implies that there exists a uniform bound on $\gamma(C_i)$ (where γ is as in [Wilkie99]). We then conclude using the Cauchy-Crofton formula: see [Dries03] for details. \square

11 Conclusion

We conclude with some open problems.

Open problem 11.1. Let T be an o-minimal theory (expanding RCF). Let exp be a new unary function symbol, and $T(\text{exp})$ be the following expansion of T :

- $T(\text{exp})$ is definably complete and Baire;

- $\exp(0) = 1$;
- $\exp' = \exp$;
- the graph of \exp is a VRL.

Is $T(\exp)$ consistent? Notice that, by Theorem 9.9, if consistent, $T(\exp)$ is o-minimal. Moreover, any o-minimal structure is either power bounded, or already defines an exponential function [Miller96] (and therefore in the latter case it is already a model of $T(\exp)$). Whether $\text{RCF}(\exp)$ is complete or not is not known, but it is surely consistent, since if T has an Archimedean model, then $T(\exp)$ is consistent. Notice also that there are real closed fields which do not have expansions to models of $\text{RCF}(\exp)$ [KKS97]; however, any real closed field has an elementary extension which admits such an expansion.

Open problem 11.2. Is $\text{RCF}(\exp)$ complete? Assume that Schanuel's Conjecture holds. Then we can combine our results with those in [MW96] and obtain the following result: if \mathbb{K} is a model of $\text{RCF}(\exp)$, such that every unary function definable via $\exp \upharpoonright (0, 1)$ has rational exponent⁽⁷⁾, then \mathbb{K} is elementarily equivalent to $\mathbb{R}(\exp)$. An analogous result has been obtained in [JS08] for the expansion of the real field with a power function x^α ($\alpha \in \mathbb{R}$) and, for α sufficiently generic, it is not necessary to assume Schanuel's or any other unproven conjecture.

Open problem 11.3. Let \mathbb{F} be an o-minimal structure. For every definable (with parameters) continuous function $g : \mathbb{F} \rightarrow \mathbb{F}$, let G be a new unary function symbol, and T' be the following expansion of the elementary diagram of \mathbb{F} with the new symbols:

- T' is definably complete and Baire;
- G is a \mathcal{C}^1 function, and $G' = g$;
- the graph of G is a VRL.

Is T' consistent? Again, notice that, by Theorem 9.9, if consistent, T' is o-minimal. Moreover, if \mathbb{F} expands the real line, then T' is consistent. A positive answer to the above question would allow to define an integral for functions defined in o-minimal structures outside the real line (at the price of enlarging the structure).

⁽⁷⁾If \mathbb{K} is an expansion of an ordered field, we say that a definable function $g : \mathbb{K} \rightarrow \mathbb{K}$ has rational exponent if there exists $q \in \mathbb{Q}$ such that $\lim_{x \rightarrow +\infty} g(x)x^q$ is finite and nonzero.

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