

DEFINABLY CONNECTED NONCONNECTED SETS
VERSION 0.4

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ABSTRACT. We give an example of a structure \mathbb{K} on the real line, and a manifold M definable in \mathbb{K} , such that M is definably connected but is not connected.

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1. INTRODUCTION

Let \mathbb{K} be an expansion of the real field $\bar{\mathbb{R}} := \langle \mathbb{R}, +, \cdot, < \rangle$. Let M be a definable subset of \mathbb{K} . We remind that M is called **definably connected** if there is no clopen definable subset Y of M such that $\emptyset \neq Y \neq M$.

Main Theorem. *There exists a structure \mathbb{K} expanding $\bar{\mathbb{R}}$ and set M definable in \mathbb{K} , such that:*

- (1) M is a 1-dimensional embedded C^1 submanifold of \mathbb{R}^3 ,
- (2) M has 2 connected components,
- (3) M is definably connected.

We know that such \mathbb{K} cannot be o-minimal, because in an o-minimal expansion of \mathbb{R} every definable and definably connected set is arc-connected (and hence connected). However, we can find \mathbb{K} as above which is also d-minimal.

The main ingredient is the following result, which follows easily from the proof of [MT06, Theorem 1];⁽¹⁾ we will give some details of the proof in §3.

Lemma 1.1. *There exists a sequence $P = \langle c_n : n \in \mathbb{N} \rangle$ of real numbers, such that*

- (1) P is strictly increasing and unbounded;
- (2) the set $Q := \{c_n : n \text{ even}\}$ (as a set) is not definable in \mathbb{K} , the expansion of $\bar{\mathbb{R}}$ with a new predicate for P (where P is also regarded as a set). Moreover, we can find P as above such that \mathbb{K} is also d-minimal.

We will show that \mathbb{K} as in the above lemma satisfies the conclusion of the Main Theorem.

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⁽¹⁾ Thanks to P. Hieronymi for pointing this out.

1.1. Application to Pfaffian functions. Let \mathbb{K} and M be as in the proof of the Main Theorem. In [Fra06], S. Fratarcangeli introduced the relative Pfaffian closure of $\bar{\mathbb{R}}$ inside \mathbb{K} . Consider the following 1-form on \mathbb{R}^3 $\omega(x, y, z) := dz$. Let L be the xy -coordinate plane. Notice that L is a Rolle Leaf with data $\langle \mathbb{R}^2, \omega \rangle$, according to the definition in [Spe99]. Let X_0 be the translate of M along the z -axis, such that the endpoints of M are on L , X_1 be its mirror image along the xy -plane, and $X := X_0 \cup X_1$. Then, X is a 1-dimensional \mathcal{C}^1 manifold which is definable in \mathbb{K} and definably connected, which intersects L in 2 points, but which is never orthogonal to ω . Thus, L is not a \mathbb{K} -Rolle Leaf, according to [Fra06, Definition 5.2]: therefore, it is not always the case that a Rolle Leaf (à la Speissegger) definable in \mathbb{K} and with data definable in $\bar{\mathbb{R}}$ is a \mathbb{K} -Rolle Leaf à la Fratarcangeli.

2. PROOF OF THE MAIN THEOREM

\mathbb{K} is the structure introduced in Lemma 1.1. We have to produce the manifold M . Let C_0 be the double helix in \mathbb{R}^3 obtained as union of the helices

$$\begin{cases} x = \sin(\theta) \\ y = \cos(\theta) \\ z = \theta/\pi, \end{cases} \quad \text{and} \quad \begin{cases} x = -\sin(\theta) \\ y = -\cos(\theta) \\ z = \theta/\pi, \end{cases}$$

with $0 \leq \theta \leq 1$. Let C be an embedded 1-dimensional manifold definable in $\bar{\mathbb{R}}$, of the “same shape” as C_0 , such that the endpoints of C and C_0 coincide. What we mean is the following:

- (1) C is the union of 2 disjoint \mathcal{C}^1 arcs, C_1 and C_2 , where C_1 is the graph of a \mathcal{C}^1 function $g_1 : [0, 1] \rightarrow [-1, 1]^2$ (with domain the z -axis), and the same for C_2 ;
- (2) $g_1(0) = \langle 0, 1 \rangle$, $g_1(1) = \langle 0, -1 \rangle$, $g_2(0) = \langle 0, -1 \rangle$, $g_2(1) = \langle 0, 1 \rangle$;
- (3) $g_i(t) \in (-1, 1)^2$ for $i = 1, 2$ and $t \in (0, 1)$;
- (4) We also ask that $\langle 0, 0 \rangle = g'_1(0) = g'_1(1) = g'_2(0) = g'_2(1)$ (notice that this latter condition is not satisfied by C_0).

For every $n \in \mathbb{N}$, let D_n be the double helix obtained from C by translation and dilation along the z -axis, such that the lower endpoints of D_n are $\langle 0, \pm 1, c_n \rangle$ and the upper endpoints are $\langle 0, \pm 1, c_{n+1} \rangle$. Finally, let $M := \bigcup_n D_n$. We claim that M satisfies the conclusion of the Main Theorem. The fact that M is definable in \mathbb{K} and has 2 connected components M_1 and M_2 is clear (where M_1 is the component containing the point $\langle 0, 1, d_0 \rangle$). The fact that M is \mathcal{C}^1 is clear from (4). It remains to show that M is definably connected. If not, then M_1 would be definable in \mathbb{K} . However, $\langle 0, 1, z \rangle \in M_1$ iff $z \in Q$: hence, Q would be definable in \mathbb{K} , contradiction.

3. PROOF OF LEMMA 1.1

In this section we sketch the proof of Lemma 1.1. Let $1 < c \in \mathbb{R}$ and P be the sequence $\{c^{2^n}; n \in \mathbb{N}\}$, and $\mathbb{K} := \bar{\mathbb{R}}(P)$. By [MT06, Corollary 7], \mathbb{K} is d-minimal. In the proof of [MT06, Theorem 1], Miller and Tyne establish the following result.

Fact 3.1. *Let $\bar{\mathcal{L}} := \langle 0, 1, +, \cdot, < \rangle$ be the language of ordered fields, and $\bar{\mathcal{L}}(P)$ be its extension by the new unary predicate P (thus, \mathbb{K} is naturally an $\bar{\mathcal{L}}(P)$ -structure). Let $\langle \mathbb{F}, P^* \rangle$ be an elementary extension of $\langle \mathbb{R}, P \rangle$. Let b and b' be in P^* , such that b and b' be in Q such that satisfy the same $\bar{\mathcal{L}}$ -type over \mathbb{R} . Then, b and b' satisfy the same $\bar{\mathcal{L}}(P)$ -type over \mathbb{R} .*

Assume now, by contradiction, that Q , the subset of P given by the elements with even index, is definable in \mathbb{K} , by an $\bar{\mathcal{L}}(P)$ -formula (with parameters from \mathbb{R}) $\phi(x)$. Let $\langle \mathbb{F}, P^* \rangle$ be an ω -saturated elementary extension of \mathbb{K} . Let b be in P^* , such that $b > \mathbb{R}$ and $\phi(b)$ holds. Let b' be the successor of b in P^* . Then, $\neg\phi(b')$ holds. However, b and b' satisfy the same $\bar{\mathcal{L}}$ -type over \mathbb{R} , contradicting Fact 3.1.

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