

Snowwhite and the ω dwarves

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If $(I, <)$ is a linearly ordered set, and $(d_i)_{i \in I}$ is a sequence of dwarves indexed by I , we say that a property $P(x)$ holds eventually if there exists $i_0 \in I$ such that for all $j > i_0$ we have $P(d_j)$.

Let $(d_i)_{i \in I}$ be a sequence of dwarves, all distinct. The Devil puts a hat on the head of each dwarf. Each dwarf cannot see the colour of his own hat, but receives some additional information (depending on the specific problem). Then, the Devil asks each dwarf to guess the colour of his (the dwarf's) hat; he kills all the dwarves who guess wrongly. Before receiving their hats, the dwarves discuss among themselves a strategy. Their aim is saving as many dwarves as possible; the aim of the devil is killing as many dwarves as possible.

1 Theorem. *Assumptions:*

1. *there are ω dwarves;*
2. *there are 2 possible colours for the hats $\mathcal{H} := \{Red, Blue\}$;*
3. *every dwarf d_n can see the hats of all the dwarves in front of him (i.e., the dwarves d_m with $m > n$);*
4. *every dwarf knows his position in the sequence (i.e., he knows that he is the n th dwarf).*

Then, there exists a strategy for the dwarves that, for every hat distribution, saves all the dwarves eventually.

Proof. Let ${}^\omega\mathcal{H}$ be the set of possible hat distributions. Fix a well-ordering \prec on ${}^\omega\mathcal{H}$. The strategy employed by the dwarves is the following: each dwarf d_n chooses the \prec -first hat distribution $h^n \in {}^\omega\mathcal{H}$ that is compatible with what he sees (that is, if $c(i)$ is the colour of the i th hat, then $c(i) = h^n(i)$ for every $i > n$). Then, he declares that the colour of his hat is $h^n(n)$.

Let $m > n$; note that, since the dwarf d_n can see the colour of d_m , if d_m agrees with d_n , i.e. if $h^m = h^n$, then d_m is saved.

We claim that the sequence $(h^n)_{n \in \mathbb{N}}$ eventually stabilises: with the above observation, this implies that all d_n are eventually saved. In fact, since $({}^\omega \mathcal{H}, \prec)$ is a well-ordering, it suffices to prove that $h^m \preceq h^n$ for every $m \geq n$. However, if $n < m$, the dwarf d_m has less information than d_n ; in particular, the sequence h^n is compatible with what d_m sees, and therefore either d_m chooses h^n , or a sequence before it. \square

With the same proof, we can weaken the hypothesis in Thm. 1.

2 Remark. Assumptions:

1. there are a linearly ordered set I of dwarves;
2. for a fixed cardinal κ , there are κ possible colours for the hats \mathcal{H} (and the dwarves know what κ is);
3. every dwarf d_n can see the hats of all the dwarves in front of him (i.e., the dwarves d_m with $m > n$);
4. every dwarf knows his position in the sequence.

Then, there exists a strategy for the dwarves that, for every hat distribution, saves all the dwarves eventually.

We can weaken the hypothesis in Thm. 1 in a different way.

3 Lemma. *Assumptions:*

1. there are ω of dwarves;
2. for a fixed cardinal κ , there are κ possible colours for the hats \mathcal{H} ;
3. every dwarf d_n can see the hats of all the dwarves in front of him (i.e., the dwarves d_m with $m > n$);
4. every dwarf does not know his position in the sequence.

Then, there exists a strategy for the dwarves that, for every hat distribution, saves all the dwarves eventually.

Proof. The strategy is similar to the one employed in Thm. 1: each dwarf guess the whole hat distribution, and answer accordingly to his global guess.

Let ${}^\omega \mathcal{H}$ be the set of possible hat distributions. Fix a well-ordering \prec on ${}^\omega \mathcal{H}$. The strategy employed by the dwarves is the following. If c is eventually periodic, let $(h(i))_{i \in \mathbb{Z}}$ be the periodic sequence such that $c(i) =$

$h(i)$ eventually; note that all the dwarves know what h is. Then, the dwarf d_n answer $h(n)$: even if d_n is not able to determine his position n , he nevertheless knows what $h(n)$ is, thanks to the periodicity.

Otherwise, each dwarf d chooses the \prec -first hat distribution $h \in {}^\omega\mathcal{H}$ such that there exists a $n \in \mathbb{N}$ such that the guess “ d is the n th dwarf and h is the real sequence of hats c ” is compatible with what d sees; note that, since c (and hence h) is not eventually periodic, n is unique; let $h(n)$ be the answer of d . \square

4 Theorem. *Assumptions:*

1. *there are ω dwarves;*
2. *there are 2 possible colours for the hats $\mathcal{H} := \{\text{Red}, \text{Blue}\}$;*
3. *every dwarf d_n can see the hats of all the dwarves before, and of a finite number of dwarves in front of him (this finite number can be chosen by each dwarf at the running time);*
4. *every dwarf knows his position in the sequence.*

Then, there does not exist a strategy for the dwarves that, for every hat distribution, saves all the dwarves eventually.

Proof. Fix a possible strategy for the dwarves. We will construct a sequence of hat distributions $(c^i)_{i \in \mathbb{N}}$ and a strictly increasing sequence $(p_i)_{i \in \mathbb{N}}$ of natural numbers. We will construct a hat distribution c stepwise, in such a way that the strategy fails at each p_i for c .

c^1 is the constant distribution Red, $p_1 := 1$. Let p_2 be strictly greater than any n , such that the first dwarf d_1 knows the colour of d_n for the distribution c^1 . Let $c(1)$ be different from the answer of d_1 for the sequence c^1 , and c be equal to c_1 on $(1, p_2)$.

Let c^2 be equal to c on $[1, p_2)$, and equal to Red for $i \geq p_2$. Let p_3 be strictly greater than any n such that the dwarf d_{p_2} knows the colour of d_n for the distribution c^2 . Let $c(p_2)$ be different from the answer of d_{p_2} for the sequence c^2 , and c be equal to c^2 on $[1, p_3) \setminus \{p_2\}$.

Proceed in this way. Since c agrees with c^i on $[1, p_{i+1}) \setminus \{p_i\}$, the answer of d_{p_i} to c will be the same as to c^i , and therefore it will be wrong for c . \square

5 Lemma. *In ZF plus “all subsets of 2^ω are measurable” (w.r.t. the canonical measure), Thm. 1 does not hold.*

Proof. Fix a strategy \mathcal{A} that, for every hat distributions, saves almost all dwarves. For each $n \in \mathbb{N}$, let E_n be the set of $c \in \mathcal{H}$, such that the strategy \mathcal{A} saves the dwarf d_n . We claim that, if E_n is measurable, then it has measure $1/2$. In fact, the measure of the complement of E_n is equal to $\mu(E_n)$. If all the E_n are measurable, then $\mu(\bigcup_n \bigcap_{j \geq n} E_j) = 0$. Hence, there exists $c \in {}^\omega \mathcal{H} \setminus \bigcup_n \bigcap_{j \geq n} E_j$, and c will kill infinitely many dwarves. \square

6 Lemma. *Assumptions:*

1. *there are 7 dwarves;*
2. *there are 2 possible colours for the hats $\mathcal{H} := \{\text{Red}, \text{Blue}\}$;*
3. *every dwarf d_n can see the hats of all the dwarves different from him; (i.e., the dwarves d_m with $m \neq n$);*
4. *each dwarf can hear the answer of the first dwarf d_1 .*

Then, there exists a strategy for the dwarves that, for every hat distribution, saves all the dwarves, except maybe d_1 .

Proof. Parity check. \square

With the same proof, we can weaken the hypothesis in Lemma 6, assuming that there are k possible colours, for some fixed $k \in \mathbb{N}$, instead of only 2 (and the dwarves know k).

7 Theorem. *Assumptions:*

1. *there are ω dwarves;*
2. *there are 2 possible colours for the hats $\mathcal{H} := \{\text{Red}, \text{Blue}\}$;*
3. *every dwarf knows his position in the sequence;*
4. *every dwarf d_n can see the hats of all the dwarves different from him; (i.e., the dwarves d_m with $m \neq n$);*
5. *each dwarf can hear the answer of the first dwarf d_1 .*

Then, there exists a strategy for the dwarves that, for every hat distribution, saves all the dwarves, except maybe d_1 .

Proof. Non-standard analysis. Fix V^* be a non-standard universe, and N^* a non-standard natural number. If c is the sequence of hats given by the Devil, let c^* be the corresponding non-standard sequence, and let C be the truncation of c^* to the interval $[1, N^*]$. Note that C is non-standard-finite. Let \mathcal{A} be the (non-standard) strategy for $[1, N^*]$ given by the non-standard version of Lemma 6; use it to determine the answer of each dwarf. \square

Again, with the same proof, we can weaken the hypothesis in Thm 7, assuming that there are k possible colours, for some fixed $k \in \mathbb{N}$, instead of only 2 (and the dwarves know k).

8 Theorem. *Assumptions:*

1. there are 7 dwarves;
2. there are 2 possible colours for the hats $\mathcal{H} := \{\text{Red}, \text{Blue}\}$;
3. every dwarf knows his position in the sequence;
4. every dwarf d_n can see the hats of all the dwarves different him.

Then, all possible strategy will save exactly half of the dwarves on average (where the average is taken on the set ${}^\omega\mathcal{H}$ of hat distribution, with the canonical measure).

Proof. Fix a strategy \mathcal{A} . For each dwarf d_n , let E_n be the set of hat distribution $c \in {}^\omega\mathcal{H}$ such that d_n guesses correctly for c (using \mathcal{A}). Note that E_n has measure exactly $1/2$.

Define $f_n : {}^\omega\mathcal{H} \rightarrow \mathbb{R}$ as $f_n(c) = 1$ if d_n guesses correctly in c , 0 otherwise; by the above observation, the average of f_n is $1/2$. Let $f := f_1 + \dots + f_7 : {}^\omega\mathcal{H} \rightarrow \mathbb{R}$. Since each f_i has average $1/2$, f has average $7/2$, which is equivalent to the conclusion. \square

In the same situation of Thm. 8, there exists a strategy that, for every hat distribution, saves at least $\lfloor 7/2 \rfloor = 3$ dwarves. This strategy is optimal for the worst case analysis (i.e., the max-min).

References

- [1] For a different problem with hats, see the Wikipedia article “Hat puzzle”, http://en.wikipedia.org/wiki/Hat_puzzle
- [2] S. Butler, M.T. Hajiaghayi, R.D. Kleinberg, T. Leighton. Hat Guessing Games. To appear in SIAM Journal on Discrete Mathematics.
- [3] American Mathematical Monthly, Problem 5348 (1965).