

# Hausdorff Measure on O-Minimal Structures

## Integral geometry on o-minimal structures

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## Introduction

In [Berarducci-Otero, 2004], the authors define an analogue of Lebesgue measure for bounded definable subsets of an o-minimal structure  $K$ , which expands a real closed field. Our aim is defining the area of a bounded definable  $d$ -dimensional subsets of  $K^n$ . Different possible definitions for such area are possible: we will prove that some of them are equivalent. On the reals, there are many different definitions of the  $d$ -area of a subset. However, **on smooth sub-manifolds of  $\mathbb{R}^n$  all these definitions coincide with the  $d$ -dimensional Hausdorff measure  $\mathcal{H}^d$** . Every definable set in an o-minimal structure on the reals is a finite union of smooth sub-manifolds, therefore the  $d$ -area of such a set is  $\mathcal{H}^d$ .

Many formulae are known for the Hausdorff measure: from Fubini's theorem to the area and coarea formulae. We will generalise some of these formulae to our measure on  $K$ .

Elisa Vasquez Rifo obtained independently some of the results exposed here.

1. Berarducci-Otero measure is a *real* number, not an element of  $K$ .  
The same holds for our measure.
2. Elisa is a student of Speissegger.
3. She works especially in dimension 1 (curves).

## Outline

- 1 Cauchy-Crofton Formula and Integral-Geometric Measure
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  - Hausdorff Measure on the Reals
  - O-Minimal Lebesgue Measure
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  - Cauchy-Crofton Formula and Integral-Geometric Measure
  - Integral-Geometric Measure on O-Minimal Structures

## Outer Measure

The extended real line is  $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ . Let  $Y$  be a set,  $\mathcal{Y}$  be a family of subsets of  $Y$ , and  $\mu : \mathcal{Y} \rightarrow \overline{\mathbb{R}}$ .

- $\mu$  is **monotone** iff for every  $A, B \in \mathcal{Y}$ ,

$$A \subseteq B \Rightarrow \mu(A) \leq \mu(B).$$

- $\mu$  is **countably subadditive** (or  **$\sigma$ -subadditive**) iff for every  $A, B_i \in \mathcal{Y}$

$$A \subseteq \bigcup_{i \in \mathbb{N}} B_i \Rightarrow \mu(A) \leq \sum_{i \in \mathbb{N}} \mu(B_i).$$

- $\mu$  is **countably additive** (or  **$\sigma$ -additive**) iff for every  $B_i \in \mathcal{Y}$

$$B_i \text{ disjoint and } \bigsqcup_{i \in \mathbb{N}} B_i \in \mathcal{Y} \Rightarrow \mu\left(\bigsqcup_{i \in \mathbb{N}} B_i\right) = \sum_{i \in \mathbb{N}} \mu(B_i).$$

An **outer measure** on  $Y$  is a non-negative, monotone, and countably subadditive function  $\mu^* : \mathcal{P}(Y) \rightarrow \overline{\mathbb{R}}$ , such that  $\mu^*(\emptyset) = 0$ .

## Carathéodory's Construction

Let  $\mu^*$  be an outer measure on  $Y$ . A subset  $A \subseteq Y$  is  **$\mu^*$ -measurable** iff for every subset  $E \subseteq Y$ ,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A).$$

The  $\mu^*$ -measurable subsets of  $Y$  form a  $\sigma$ -algebra  $\tilde{\mathcal{S}}_{\mu}$ .

We will denote by  $\mu$  the measure induced by  $\mu^*$ , namely the restriction of  $\mu^*$  to  $\tilde{\mathcal{S}}_{\mu}$ .  $\mu$  is a **complete** measure on  $\tilde{\mathcal{S}}_{\mu}$ .

## Measure

### Definition (Boolean rings and $\sigma$ -algebras)

A **Boolean ring** (or simply ring)  $\mathcal{R}$  on  $Y$  is a non-empty class of subsets of  $Y$  closed under finite union and set-difference.  $\mathcal{R}$  is a  **$\sigma$ -ring** iff it is also closed under countable unions.  $\mathcal{R}$  is a  **$\sigma$ -algebra** iff moreover  $Y \in \mathcal{R}$ .

### Definition (Measure)

A **measure** on a ring  $\mathcal{R}$  is a map  $\mu : \mathcal{R} \rightarrow \overline{\mathbb{R}}$  which is monotone and countably additive, and such that  $\mu(\emptyset) = 0$ . A measure is  **$\sigma$ -finite** iff  $Y$  is a countable union of sets in  $\mathcal{R}$  of finite measure. A measure is **complete** iff for every  $B \in \mathcal{R}$  and  $A \subseteq B$  such that  $\mu(B) = 0$ , we have  $A \in \mathcal{R}$  (and consequently  $\mu(A) = 0$ ). A measure *tout court* is a measure on a  $\sigma$ -algebra on  $Y$ .

## Hausdorff Measure

$n$  natural number;

$d$  real number, with  $0 \leq d \leq n$ ;

$A$  subset of  $\mathbb{R}^n$ ;

$\mathcal{H}^d(A) = \mathcal{H}^{d,n}(A)$  the  $d$ -dimensional Hausdorff measure of  $A$ .

$\mathcal{H}^{d,n}$  does not depend on  $n$ . It is an outer measure on  $\mathbb{R}^n$ .

For every  $A \subseteq \mathbb{R}^n$  there exists a unique  $d_0 \in \mathbb{R}$  such that

$$\forall d > d_0 \quad \mathcal{H}^d(A) = 0 \text{ and}$$

$$\forall d' < d_0 \quad \mathcal{H}^{d'}(A) = +\infty.$$

$d_0$  is the **Hausdorff dimension** of  $A$ .

## Hausdorff Measure on Manifolds

If  $A$  is an embedded  $\mathcal{C}^1$  manifold of dimension  $d$ , then  $d$  is equal to the Hausdorff dimension of  $A$ , and  $0 < \mathcal{H}^d(A)$ .

If  $f : D \rightarrow \mathbb{R}^n$  is a  $\mathcal{C}^1$  map, with  $D \subseteq \mathbb{R}^d$ ,  $\mathcal{H}^d(f(D))$  is computed as the integral of the Jacobian  $J_d(f)$  over  $D$ .

The area of a  $d$ -dimensional manifold  $A \subseteq \mathbb{R}^n$  is defined by calculating it on parametrised portions of  $A$ .

### Example

- $d = 0$ ,  $\mathcal{H}^0(A) = \#(A)$  cardinality.
- $d = 1$ ,  $\mathcal{H}^1(A)$  length.
- $d = 2$ ,  $\mathcal{H}^2(A)$  area.
- $d = n$ ,  $\mathcal{H}^{n,n}(A) = \mathcal{L}^n(A)$  Lebesgue measure.

## Hausdorff Measure and Real O-Minimal Structures

$R$  o-minimal structure on the real line  $\mathbb{R}$ , expanding the ordered field structure.  $A \subseteq R^n$  definable (with parameters) in  $R$ . Then,

- The o-minimal dimension and the Hausdorff dimension of  $A$  coincide.
- $0 < \mathcal{H}^d(A)$ , where  $d = \dim(A)$ .
- If  $A$  is bounded, then  $\mathcal{H}^d(A) < +\infty$ .

If  $A_t$  is a uniformly bounded definable family of subsets of  $R^n$ , then  $\mathcal{H}^d(A_t)$  is uniformly bounded.

## Berarducci-Otero Definition

- $K$  is an  $\aleph_1$ -saturated o-minimal structure extending a real closed field.
- A subset  $A \subseteq K^n$  is **bounded** iff there exists  $m \in \mathbb{N}$  such that  $|x| < m$  for every  $x \in A$ .
- $\text{Fin}(K)$  is the set of bounded elements.
- The family of bounded definable subsets of  $K^n$  is a **boolean ring**.

Berarducci and Otero define an analogue of Lebesgue measure on  $\text{Fin}(K)^n$ . More precisely, they define the outer measure  $\mathcal{L}_K^{n,*}$  on  $\text{Fin}(K)^n$  as

$$\mathcal{L}_K^{n,*}(A) := \inf \left( \sum_{i=1}^k \mathcal{L}_K^{n,*}(Q_i) \right),$$

for every  $A \subseteq \text{Fin}(K)^n$ , where each  $Q_i$  is a  $n$ -dimensional hypercube with rational vertices, such that

$$A \subseteq \bigcup_{i=1}^k Q_i,$$

and  $\mathcal{L}_K^{n,*}(Q_i) := \ell_i^n$ , where  $\ell_i$  is the length of the edge of  $Q_i$ .

1.  $\mathcal{L}_K^n$  is defined even when  $K$  is not  $\aleph_1$ -saturated. However, when  $K$  is countable,  $\mathcal{L}_K^n$  is not  $\sigma$ -additive, but only additive.

## Outer Measure Pullback

$X, Y$  sets;

$\nu^*$  an outer measure on  $Y$ ;

$f : X \rightarrow Y$  a surjective function.

For every  $A \subseteq X$ , define

$$\begin{aligned}\mu^*(A) &:= \nu^*(f(A)) \\ \tilde{A} &:= f^{-1}(f(A)) \cap f^{-1}(f(X \setminus A)).\end{aligned}$$

$\mu^*$  is an outer measure on  $X$ .

$A \subseteq X$  is  $\mu^*$ -measurable iff

- ①  $f(A)$  is  $\nu^*$ -measurable, and
- ②  $\mu^*(\tilde{A}) = 0$ .

## Standard Part and O-Minimal Lebesgue Measure

Let  $\mathcal{L}_K^{n*}$  be the pullback of  $\mathcal{L}^{n*}$  via the map  $\text{st}$ .  $\mathcal{L}_K^{n*}$  is an outer measure on  $\text{Fin}(K)^n$ .  $A \subseteq \text{Fin}(K)^n$  is measurable iff

- ①  $\text{st}(A)$  is measurable, and
- ②  $\text{st}(A) \cap \text{st}(\text{Fin}(K)^n \setminus A)$  is negligible.
  - Let  $A \subseteq K^n$  be bounded and definable.
  - By [Baisalov-Poizat, 1998],  $\text{st}(A)$  is Borel, and *a fortiori* measurable.
  - [Berarducci-Otero, 2004] prove that  $\text{st}(A) \cap \text{st}(\text{Fin}(K)^n \setminus A)$  is negligible.
  - Therefore,  $A$  is  $\mathcal{L}_K^{n*}$ -measurable.
  - $\mathcal{L}_K^n(A)$  is the measure of  $A$  according to [Berarducci-Otero, 2004].

## Standard Part and Baisalov-Poizat Theorem

$K$   $\aleph_1$ -saturated o-minimal structure expanding an ordered field.

$\text{Fin}(K)$  bounded elements of  $K$ .

$\sim$  equivalence relation:  $x \sim y$  iff  $|x - y| < \frac{1}{n}$  for every  $n \in \mathbb{N}$ .

$\text{Fin}(K)/\sim$  is naturally isomorphic to  $\mathbb{R}$ .

$\text{st} : \text{Fin}(K) \rightarrow \mathbb{R}$  quotient map = standard part map.

$\mathbb{R}_K$   $K$  induces a structure  $\mathbb{R}_K$  on the reals via the map  $\text{st}$ .

$\mathbb{R}_K =$  expansion of  $\mathbb{R}$  generated by the subsets of  $\mathbb{R}^n$  of the form  $\text{st}(A)$ , as  $A$  varies among the definable subsets of  $K^n$ .

As a corollary of a theorem in [Baisalov-Poizat, 1998],

## Theorem (Baisalov and Poizat)

$\mathbb{R}_K$  is o-minimal.

## Hausdorff Measure of the Standard Part

When it does not work

The above construction does not work for  $\mathcal{H}^{d,n}$  instead of  $\mathcal{L}^n$ . Let  $d < n$ , and  $\mathcal{H}_{\text{st}}^d$  be the outer measure on  $\text{Fin}(K)^n$  induced by  $\mathcal{H}^{d,n}$  via  $\text{st}$ .

## Example

Let  $A$  the  $d$ -dimensional unit cube in  $K^d \subset K^n$ .  $A$  is not  $\mathcal{H}_{\text{st}}^d$ -measurable, because  $\text{st}(\tilde{A})$  is not  $\mathcal{H}^d$ -negligible. For instance, if  $A'$  is the translate of  $A$  by an infinitesimal vector

$$A' := \{ (x_1, \dots, x_d, \varepsilon, 0, \dots, 0 : x_i \in K, 0 \leq x_i \leq 1 \},$$

then  $\mathcal{H}_{\text{st}}^d(A \sqcup A') = 1 < \mathcal{H}_{\text{st}}^d(A) + \mathcal{H}_{\text{st}}^d(A')$ .

## Hausdorff Measure of the Standard Part

When it does work

- Let  $C$  be a bounded subset of  $K^n$ , definable via a semi-algebraic map without parameters, and of dimension  $d$ .
- The restriction  $\mathcal{H}_{st}^d$  to  $C$  is an outer measure.
- Any definable subset of  $C$  is  $\mathcal{H}_{st}^d$ -measurable.
- If  $A \subseteq C$  is definable via a semi-algebraic map without parameters, then  $\mathcal{H}_{st}^d(A) = \mathcal{H}^d(A(\mathbb{R}))$ .

**Proof.**

Work in local charts. □

## The Affine Grassmannian

The **affine Grassmannian**  $\mathcal{A}(d, n)$  is the set of  $(n - d)$ -dimensional  $\mathbb{R}$ -**affine** subspaces of  $\mathbb{R}^n$ , and similarly for  $\mathcal{A}(d, n; K)$ .

$\mathcal{A}(d, n)$  is a **fibred space** over  $\mathcal{G}(d, n)$ , with fibre  $\mathbb{R}^d$ . To every affine  $(n - d)$ -space  $E$  we associate the pair  $(E^\perp, x)$ , where

$E^\perp \in \mathcal{G}(d, n)$  is the  $d$ -linear space orthogonal to  $E$ ;

$\{x\} = E^\perp \cap E$ .

$\mathcal{A}(d, n)$  can be also given a manifold structure, of dimension  $(n - d)d + d$ .

**Example**

$\mathcal{A}(1, 2)$  is isomorphic to the Möbius band.

## The Grassmannian

**Grassmannian**  $\mathcal{G}(d, n)$  the set of  $d$ -dimensional  $\mathbb{R}$ -linear subspaces of  $\mathbb{R}^n$ .

$\mathcal{G}(d, n; K)$  the set of  $d$ -dimensional  $K$ -linear subspaces of  $K^n$ .

There is a **compact manifold** structure on  $\mathcal{G}(d, n)$ , of dimension  $(n - d)d$ , such that

$$\mathcal{G}(d, n) \simeq \text{SO}(n)/(\text{SO}(d) \times \text{SO}(n - d)).$$

**Example**

$\mathcal{G}(1, n) \simeq \mathcal{G}(n - 1, n) \simeq \mathbb{P}^{n-1}(\mathbb{R})$ , the projective space.

## Haar Measure on $\mathcal{G}(d, n)$ and $\mathcal{A}(d, n)$

- The group  $\text{SO}(n)$  of linear isometries on  $\mathbb{R}^n$  acts naturally on  $\mathcal{G}(d, n)$ .
- There is a unique invariant measure  $\mu^{\mathcal{G}}$  on  $\mathcal{G}(d, n)$ .
- The group of isometries of  $\mathbb{R}^n$  acts on  $\mathcal{A}(d, n)$ , and there is a corresponding invariant measure  $\mu^{\mathcal{A}}$  on  $\mathcal{A}(d, n)$ .
- $\mu^{\mathcal{A}}$  is the product of  $\mu^{\mathcal{G}}$  with  $\mathcal{L}^d$ .

## Cauchy-Crofton Formula

Let  $C \subseteq \mathbb{R}^n$  be a  $d$ -dimensional manifold. Then, there exists a constant<sup>1</sup>  $\beta = \beta(d, n) \in \mathbb{R}^{>0}$  such that

$$\begin{aligned} \mathcal{H}^{d,n}(C) &= \frac{1}{\beta} \int_{\mathcal{A}(d,n)} \#(C \cap q) d\mu^{\mathcal{A}}(q) = \\ &= \frac{1}{\beta} \int_{\mathcal{G}(d,n)} f_C d\mu^{\mathcal{G}}, \end{aligned} \quad (1)$$

where  $f_C(E)$  is the  $\mathcal{L}^E$ -area of  $p_E(C)$ , “counted with multiplicity”, with  $p_E$  the orthogonal projection onto  $E$ . In general, the function

$$\mathcal{J}^{d,n}(C) := \frac{1}{\beta} \int_{\mathcal{A}(d,n)} \#(C \cap q) d\mu^{\mathcal{A}}(q)$$

is defined for any Borel set  $C$ , and  $\mathcal{J}^{d,n}$  is a Borel measure.

$${}^1\beta = \Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{n-d+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)^{-1}.$$

## Realisation of Grassmannian

There exist a semi-algebraic realisation of  $\mathcal{G}(d, n)$  in  $\mathbb{R}^m$  (for some  $m$ ) such that  $\mu^{\mathcal{G}}$  coincides with the restriction of  $\mathcal{H}^k$  to  $\mathcal{G}(d, n)$  (where  $k = \dim(\mathcal{G}(d, n))$ ). Similarly for  $\mathcal{A}(d, n)$ .

Therefore, the measure  $\mu_K^{\mathcal{G}}$  induced by  $\mu^{\mathcal{G}}$  via the map  $st$  is an outer measure on  $\mathcal{G}(d, n; K)$ , such that definable subsets of  $\mathcal{G}(d, n; K)$  are measurable. Similarly for  $\mathcal{A}(d, n)$ .

## Example

$d = n$  :  $\mathcal{A}(n, n) = \mathbb{R}^n$ , and (1) becomes a tautology.

$d = 0$  :  $\mathcal{A}(0, n)$  is a singleton,  $\mathcal{H}^0(C) = \#(C)$ , and (1) is obvious.

$d = 1, n = 2$  : For every  $\theta \in [0, 2\pi)$ , let  $D_\theta$  be the line forming an angle  $\theta$  with the  $x$ -axis,  $N(r, \theta)$  be the number of points on the curve  $C$  whose orthogonal projection on  $D_\theta$  is the point  $(r \cos \theta, r \sin \theta)$ . Define

$$P(\theta) := \int_{-\infty}^{+\infty} N(r, \theta) dr.$$

Then, the length of  $C$  is equal to

$$\mathcal{J}^{1,2}(C) = \frac{1}{4} \int_0^{2\pi} P(\theta) d\theta.$$

## Definition

Let  $C$  be a bounded definable subset of  $K^n$ . Define

$$\begin{aligned} \mathcal{J}_K^{d,n}(C) &:= \frac{1}{\beta} \int_{\mathcal{A}(d,n;K)} \#(C \cap q) d\mu_K^{\mathcal{A}}(q) = \\ &= \frac{1}{\beta} \int_{\mathcal{G}(d,n;K)} f_C d\mu_K^{\mathcal{G}}, \end{aligned}$$

where  $f_C(E)$  is the  $\mathcal{L}^E$ -area of  $p_E(C)$ , counted with multiplicity.

## Basic Properties

$\mathcal{J}_K^{d,n}$  is a measure defined on all bounded definable subsets of  $K^n$ .  
Given  $C \subseteq K^n$  definable and bounded, let

$$\begin{aligned} S &:= \{q \in \mathcal{A}(d, n; K) : q \cap C \neq \emptyset\} \\ S_\infty &:= \{q \in \mathcal{A}(d, n; K) : \#(q \cap C) = \infty\} \subseteq S \\ S_0 &:= S \setminus S_\infty. \end{aligned}$$

- $S$ ,  $S_\infty$  and  $S_0$  are bounded and definable.
- If  $\dim(C) \leq d$ , then  $\dim(S_\infty) < k' := \dim(\mathcal{A}(d, n; K))$ .
- If  $\dim(C) < d$ , then  $\dim(S) < k'$ .
- Therefore, if  $\dim(C) \leq d$ , then  $\mathcal{J}_K^{d,n}(C) < \infty$ ,  
and if  $\dim(C) < d$ , then  $\mathcal{J}_K^{d,n}(C) = 0$ .

Let  $n \leq n'$ , and  $C \subseteq K^n$  be bounded and definable.

### Question

$$i \mathcal{J}_K^{d,n}(C) = \mathcal{J}_K^{d,n'}(C) ?$$

## More Properties

Let  $C \subseteq K^n$  be bounded, definable, with  $\dim(C) \leq d$ , and  $(C(t))_{t \in K}$  be a uniformly bounded definable family of subsets of  $K^n$ , with  $\dim(C(t)) \leq d$ .

- 1  $\mathcal{J}_K^{d,n}$  is invariant under rotations and translations.
- 2 For every  $r \in \text{Fin}(K)^n$ ,  $\mathcal{J}_K^{d,n}(rC) = \text{st}(r)^d \cdot \mathcal{J}_K^{d,n}(C)$ .
- 3 If  $C$  is defined by a semi-algebraic formula without parameters, then  $\mathcal{J}_K^{d,n}(C) = \mathcal{J}^{d,n}(C(\mathbb{R})) = \mathcal{J}^{d,n}(\text{st}(C))$ .
- 4  $\mathcal{J}_K^{d,n}(C(t))$  is uniformly bounded.
- 5 If  $K'$  is either an elementary extension or an o-minimal expansion of  $K$ , then  $\mathcal{J}_{K'}^{d,n}(C) = \mathcal{J}_K^{d,n}(C)$ .
- 6 If  $d = n$ ,  $\mathcal{J}_K^{n,n}(C) = \mathcal{L}_K^n(C)$ .
- 7 If  $C \subseteq E$ , where  $E \subseteq K^n$  is a  $d$ -dimensional plane, then  $\mathcal{J}_K^{d,n}(C) = \mathcal{L}^E(C) = \mathcal{H}^{d,n}(\text{st}(C))$ , where  $\mathcal{L}^E$  is the o-minimal Lebesgue measure on  $E$ .

## Outline

- 2 Parametric Formula for the Area
  - Lebesgue Measure, Integral, and Definable Functions
  - Lipschitz and Bi-Lipschitz Functions
  - Rectifiable Sets and Hausdorff Measure
  - More Properties of Definable Functions: Regular Points, Sard's Lemma, Implicit Function
  - Integral Geometry on O-Minimal Structures: Area, Coarea and Cauchy-Crofton Formulae
  - Open Problems: Area, General Area-Coarea, General Cauchy-Crofton Formulae

## Area and Jacobian

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be differentiable at  $a \in \mathbb{R}^n$ . The  $n$ -Jacobian of  $f$  at  $a$  is

$$J_n(f)(a) := \sqrt{\det(Df(a) \cdot Df(a)^T)}.$$

The  $m$ -Jacobian of  $A$  is

$$J_m(f)(a) := \sqrt{\det(Df(a)^T \cdot Df(a))}.$$

Note that  $J_n(f)(a) = 0$  iff  $\text{rk}(Df(a)) < n$ , and similarly for  $m$ .

If  $n \leq m$  and  $f$  is a smooth embedding of  $A \subseteq \mathbb{R}^n$  into  $\mathbb{R}^m$ , then the area of  $f(A)$  is given by

$$\mathcal{H}^{n,m}(f(A)) = \int_A J_n(f) d\mathcal{L}^n.$$

1. For every  $k$  it is possible to define  $J_k(f)(a)$  as the square root of the sum of the square of the  $k$  minors of  $Df$ .
2.  $J_k(f)(a)$  is the maximum  $k$ -dimensional volume of the image under  $Df(a)$  of a unit  $k$ -dimensional cube.
3. If  $k = m$  or  $n$ ,  $(J_k(f))^2$  is the determinant of the  $k \times k$  product of  $Df$  with its transpose.

## Monads and Definable Functions

### Definition

Given  $x, y \in K^n$ ,  $x$  and  $y$  are in the same **monad**,  $x \sim y$ , iff  $|x - y|$  is infinitesimal.

Let  $f : K^n \rightarrow K^m$  be definable.

### Definition

Given  $c \in K^n$ ,  $f$  **preserves the monad** of  $c$  iff

$$x \sim c \Rightarrow f(x) \sim f(c).$$

For every  $x \in K^n$ , define

$$\mathcal{S}f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}^m$$

$$\text{st}(x) \mapsto \begin{cases} \text{st}(f(x)) & \text{iff } f \text{ preserves the monad of } x \\ 0 & \text{otherwise.} \end{cases}$$

1. On  $\text{Fin}(K)^n$ ,  $x \sim y$  iff  $\text{st}(x) = \text{st}(y)$ .
2.  $\mathcal{S}f$  is definable in  $\mathbb{R}_K$ .

## Lebesgue Measure and Definable Functions

Call  $\mathbb{I}_K := [0, 1]_K$ , and  $\mathbb{I}_{\mathbb{R}} := [0, 1]_{\mathbb{R}}$ . Let  $f : \mathbb{I}_K^n \rightarrow \mathbb{I}_K^m$  be definable. Then, there exists  $C \subseteq \mathbb{I}_{\mathbb{R}}^n$  **definable and negligible** such that, outside  $C$ ,

- 1  $f$  preserves the monads;
- 2  $f$  is  $\mathcal{C}^1$ ;
- 3  $\mathcal{S}f$  is  $\mathcal{C}^1$ ;
- 4  $D(\mathcal{S}f) = \mathcal{S}(Df)$ .

If moreover  $f$  preserves the monads on all  $\mathbb{I}_K^n$ , then  $\mathcal{S}f$  is continuous on all  $\mathbb{I}_{\mathbb{R}}^n$ .

1. Outside  $C$  can mean “outside  $st^{-1}(C)$ ”.
2. A proof of 1: let  $\Gamma$  be the graph of  $f$ , and  $\bar{\Gamma} := st(\Gamma)$ . Since  $\dim(\bar{\Gamma}) \leq \dim(\Gamma) = n$ ,  $\bar{\Gamma}$  is, outside a negligible set, the graph of a function.
3. The hypothesis can be weakened to  $f : \mathbb{I}_K^n \rightarrow K^m$  definable and “almost bounded”.
4. We use the saturation property for the continuity of  $\mathcal{S}f$ .
5. The latter is only a curiosity: we will not use it.

## Integral of Definable Functions

Let  $f : \mathbb{I}_K^n \rightarrow \mathbb{I}_K$  be definable,  $g := \mathcal{S}f : \mathbb{I}_{\mathbb{R}}^n \rightarrow \mathbb{I}_{\mathbb{R}}$ , and  $[0, f] := \{ (x, y) \in \mathbb{I}_K^n \times \mathbb{I}_K : 0 \leq y \leq f(x) \}$  be the sub-graph of  $f$ . The following are well-defined and equal:

$$\begin{aligned} & \int_{\mathbb{I}_{\mathbb{R}}^n} g \, d\mathcal{L}^n \\ & \int_{\mathbb{I}_K^n} st(f) \, d\mathcal{L}_K^n \\ & (\mathcal{L}_K^n \times \mathcal{L}^1)([0, st(f)]) \\ & \mathcal{L}_K^{n+1}([0, f]) \\ & \mathcal{L}^{n+1}(st([0, f])) \\ & \mathcal{L}^{n+1}([0, g]). \end{aligned}$$

### Definition

We define  $\int_{\mathbb{I}_K^n} f \, d\mathcal{L}_K^n = \int_{\mathbb{I}_K^n} f(x) \, dx := \int_{\mathbb{I}_{\mathbb{R}}^n} g \, d\mathcal{L}^n$ .

1.  $st(f) : \mathbb{I}_K^n \rightarrow \mathbb{I}_{\mathbb{R}}$ .
2. The hypothesis can be weakened to  $f : \mathbb{I}_K^n \rightarrow K$  definable and almost bounded.

## Lipschitz and Bi-Lipschitz Functions

Let  $A \subseteq K^m$  and  $f : A \rightarrow K^n$  be definable.

### Definition

$f$  is **Lipschitz** iff there exists a **rational** number  $r > 0$  such that

$$\forall x, y \in K^n \quad |f(x) - f(y)| \leq r|x - y|.$$

The infimum of such constants  $r$  is  $\text{Lip } f$ , the Lipschitz constant of  $f$ .

Note that if  $f$  is and  $\mathcal{C}^1$ , and  $|Df| < r$ , then  $\text{Lip } f < r$ .

Conversely, if  $f$  is Lipschitz and  $\text{Lip } f < r$ , then  $|Df| < r$ , where it is defined.

### Definition

A definable bijection  $f : A \rightarrow B$  is **bi-lipschitz** iff both  $f$  and  $f^{-1}$  are Lipschitz.

## Fubini's Theorem

### Theorem

For every  $n, m \in \mathbb{N}$ , the (completion of the) measure  $\mathcal{L}_K^{n+m}$  is equal to the (completion of the) **product**  $\mathcal{L}_K^n \times \mathcal{L}_K^m$ .

### Theorem

Let  $f : \mathbb{I}_K^{n+m} \rightarrow \mathbb{I}_K$  be definable. Then,

$$\int_{\mathbb{I}_K^{n+m}} f(x, y) d\mathcal{L}_K^{m+n}(x, y) = \int_{\mathbb{I}_K^m} \int_{\mathbb{I}_K^n} f(x, y) d\mathcal{L}_K^n(x) d\mathcal{L}_K^m(y).$$

## Torricelli's Theorem

### Theorem

Let  $f : \mathbb{I}_K \rightarrow \mathbb{I}_K$  be definable and **Lipschitz**, and  $\dot{f} := \frac{df}{dx}$ . Then,

$$\int_0^1 \dot{f}(x) dx = \text{st}(f(1) - f(0)).$$

### Example

Let  $0 < \varepsilon \in K$  be infinitesimal. Define

$$\Theta_\varepsilon := \begin{cases} x/\varepsilon & \text{if } 0 \leq x \leq \varepsilon, \\ 1 & \text{if } \varepsilon \leq x \leq 1. \end{cases}$$

Then,  $\int_0^1 \dot{\Theta}_\varepsilon(x) dx = 0$ , while  $\text{st}(\Theta_\varepsilon(1) - \Theta_\varepsilon(0)) = 1$ .

## Bi-Lipschitz Change of Variable

### Theorem

Let  $f : \mathbb{I}_K^n \rightarrow \mathbb{I}_K^n$  be definable and **bi-lipschitz**, and  $A \subseteq \mathbb{I}_K^n$  be definable. Then,

$$\mathcal{L}_K^n(f(A)) = \int_A |\det(Df)| d\mathcal{L}_K^n.$$

If moreover  $g : \mathbb{I}_K^n \rightarrow \mathbb{I}_K$  is definable, then

$$\int_{f(A)} g(x) dx = \int_A g(f(y)) \cdot |\det(Df(y))| dy.$$

## Rectifiable Sets

Let  $d \in \mathbb{N}$ .

### Definition (Rectifiable sets)

A set  $C \subseteq K^n$  is **basic  $d$ -rectifiable** iff there exists  $A \subseteq \mathbb{I}_K^d$  definable and open and  $f : A \rightarrow C$  definable bi-lipschitz bijection.

### Theorem (Decomposition into basic rectifiable sets)

Let  $C \subseteq \mathbb{I}_K^n$  definable and of dimension  $d$ . Then, there exist  $C_0, \dots, C_m \subseteq \mathbb{I}_K^m$  such that

- ①  $C_0, \dots, C_m$  are disjoint;
- ②  $C_0 \sqcup \dots \sqcup C_m = C$ ;
- ③  $\dim(C_0) < d$ ;
- ④ each  $C_i$  is basic  $d$ -rectifiable,  $i = 1, \dots, m$ .

1. In particular,  $C$  in the definition is basic  $d$ -rectifiable.
2. Say what a Keisler measure is.

## Hausdorff Measure on Rectifiable Sets

### Definition

Let  $A \subseteq \mathbb{I}_K^d$  be definable,  $f : \mathbb{I}_K^d \rightarrow \mathbb{I}_K^n$  be a bi-lipschitz map, and  $C := f(A)$ . Define

$$\mathcal{H}_K^{d,n}(C) := \int_A J_d(f) d\mathcal{L}_K^d.$$

The quantity  $\mathcal{H}_K^{d,n}(C)$  does not depend on the particular choice of  $A$  and of  $f$ .

### Proof.

By the bi-lipschitz change of variables. □

### Lemma

On a  $d$ -rectifiable set  $C \subseteq \text{Fin}(K)^n$ ,  $\mathcal{H}_K^{d,n}$  is a **Keisler measure**.

## Change of Variables for Hausdorff Measure

### Lemma

Let  $B \subseteq \mathbb{I}_K^n$  and  $C \subseteq \mathbb{I}_K^m$  be basic  $d$ -rectifiable, and  $f : B \rightarrow C$  be a bi-lipschitz bijection. Then,

$$\int_B J_d(f) d\mathcal{H}_K^{d,n} = \mathcal{H}_K^{d,m}(C).$$

## Rectifiable Sets and Hausdorff Measure of the Standard Part

### Lemma (\*)

Let  $d < n$  and  $\pi : K^n \rightarrow K^d$  be the projection onto the first  $d$  coordinates. Let  $A \subseteq \mathbb{I}_K^d$  be definable, and  $f : A \rightarrow \mathbb{I}_K^n$  be definable and Lipschitz, such that

$$\pi \circ f = \text{id}_A.$$

Then,

$$\mathcal{H}_K^{d,n}(f(A)) = \mathcal{H}^{d,n}(\text{st}(f(A))) = \mathcal{H}^{d,n}(\mathcal{S}f(\text{st}(A))).$$

1. Note that  $f$  is bi-lipschitz, and therefore  $f(A)$  is basic  $d$ -rectifiable.
2. It is possible to decompose any  $d$ -dimensional definable set  $C$  into sets  $C_i$  satisfying the condition (namely the existence of a function  $f_i$  as in the lemma, after a permutation of variables).

## Regular Points

### Definition

Let  $f : \mathbb{I}_K^n \rightarrow \mathbb{I}_K^m$  be a definable function, and  $g := \mathcal{S}f : \mathbb{I}_K^n \rightarrow \mathbb{I}_K^m$ . An element  $c \in \mathbb{I}_K^n$  is a **S-regular point** for  $f$  iff there exists  $\varepsilon \in \mathbb{Q}^{>0}$  such that, such that, if we call  $B$  the ball of centre  $c$  and radius  $\varepsilon$ ,

- 1  $B \subseteq (0, 1)_{\mathbb{R}}^n$ ;
- 2  $f$  preserves the monads on  $B$ ;
- 3  $f$  is  $\mathcal{C}^1$  on  $B$ ;
- 4  $Df$  is bounded and preserves the monads on  $B$ ;
- 5  $g$  is  $\mathcal{C}^1$  on  $B$ ;
- 6  $\mathcal{S}(Df) = Dg$  on  $B$ ;
- 7 every  $b \in B$  is a **regular point** for  $g$ .

Note that every  $x \in \text{st}^{-1}(B)$  is a regular point for  $f$ , and that the set of S-regular points is open.

## Morse-Sard's Lemma

Let  $f : \mathbb{I}_K^n \rightarrow \mathbb{I}_K^m$  be definable, with  $m \geq n$ .

### Theorem

Let  $C \subseteq \mathbb{I}_K^n$  be the sets of points  $c \in \mathbb{I}_K^n$  such that

- 1 satisfy the conditions 1–6;
- 2  $\text{rk}(Df(c)) = n$ .

Then,  $\mathcal{H}_K^{n,m}(f(\mathbb{I}_K \setminus C)) = 0$ .

In particular, the set of S-singular values for  $f$  is  $\mathcal{L}_K^n$ -negligible.

1. We have no control over the case  $m < n$ , because the set of non- $\mathcal{C}^1$  points in the domain  $\mathbb{I}_K^n$  can have dimension up to  $n - 1$ .
2. The corollary is trivial if  $m > n$ , because the whole image of  $f$  has dimension at most  $n$ .

## Implicit Function

### Theorem

Let  $f : \mathbb{I}_K^n \rightarrow \mathbb{I}_K^n$  be definable,  $g := \mathcal{S}f$ ,  $a \in \mathbb{I}_{\mathbb{R}}^n$  be a  $\mathcal{S}$ -regular point for  $f$ , and  $c := g(a)$ .

Then, for every  $y \in \text{st}^{-1}(c)$  there exists a unique  $x \in \text{st}^{-1}(a)$  such that  $f(x) = y$ .

Moreover, fix  $x_0 \in \text{st}^{-1}(a)$ . Then, there exist  $\rho, r' \in \mathbb{Q}^{>0}$  such that, if

$$B := \text{cl}(B_\rho(a)) \text{ and} \\ B' := \text{cl}(B_{r'}(x_0)) \subseteq \text{st}^{-1}(B),$$

then

- 1  $f \upharpoonright_{B'}$  and  $g \upharpoonright_B$  are **open and injective**,
- 2  $B_{r'}(f(x_0)) \subseteq f(B')$ ,
- 3 for every  $a' \in B$  and  $y \in \text{st}^{-1}(g(a'))$  there exists a unique  $x \in B$  such that  $f(x) = y$ .

## Implicit Function

### Corollary

In the above theorem, let  $c \in \mathbb{I}_K^n$  be a  $\mathcal{S}$ -regular value, and  $\text{st}(y) = c$ .

Then,

$$\#(g^{-1}(c)) = \#(f^{-1}(y)).$$

### Sketch of proof of the theorem.

Consider the one of the usual proof of the implicit function theorem from standard calculus, via a contraction  $T : B \rightarrow B$ . Adapt it to the o-minimal situation. Verify that  $T$  is definable, and that the various constants can be chosen non-infinitesimal for  $\mathcal{S}$ -regular points.  $\square$

## Aside: the Image of a Measurable Set

### Theorem

Let  $f : \mathbb{I}_K^n \rightarrow \mathbb{I}_K^n$  be definable and **preserving the monads**. Let  $X \subseteq \mathbb{I}_K^n$  be  $\mathcal{L}_K^n$ -measurable. Then,

- 1  $f(X)$  is also  $\mathcal{L}_K^n$ -measurable;
- 2 if  $X$  is  $\mathcal{L}_K^n$ -negligible, then  $f(X)$  is also  $\mathcal{L}_K^n$ -negligible.

### Sketch of proof.

Let  $A := \text{st}(X)$ ,  $Y := f(X)$ , and  $g := \mathcal{S}f$ . Then,

- 1  $A$  is  $\mathcal{L}_K^n$ -measurable;
- 2  $g(A) = \text{st}(Y)$ ;
- 3  $\text{st}(Y)$  is  $\mathcal{L}_K^n$ -measurable;
- 4 if  $X$  is negligible, then  $g(A)$  is also negligible, and therefore  $Y$  is negligible.  $\square$

## Counter-Example

### Example

Let  $0 < \varepsilon \in K$  be infinitesimal. Define

$$\Theta_\varepsilon := \begin{cases} x/\varepsilon & \text{if } 0 \leq x \leq \varepsilon, \\ 1 & \text{if } \varepsilon \leq x \leq 1. \end{cases}$$

- Let  $Y \subseteq [0, \frac{1}{2}]$  be non  $\mathcal{L}_K^n$ -measurable, and
- $X := \Theta_\varepsilon^{-1}(Y) \subseteq [0, \varepsilon]$ .
- Then,  $\mathcal{L}_K^n(X) = 0$ , but
- $\Theta_\varepsilon(X) = Y$  is non  $\mathcal{L}_K^n$ -measurable.

## Coarea Formula

### Sketch of proof.

- 1 By Morse-Sard's lemma, we can assume that all points are  $\mathcal{S}$ -regular.
- 2 Apply the Coarea formula to  $g := \mathcal{S}f$  and  $B := \text{st}(A)$ , obtaining

$$\int_A J_n f \, d\mathcal{L}_K^m = \int_B J_n g \, d\mathcal{L}^m = \int_{\mathbb{I}_K^n} \mathcal{H}^{m-n,m}(B \cap g^{-1}(z)) \, d\mathcal{L}^n(z).$$

- 3 By the implicit function theorem, and Lemma (\*), for almost every  $y \in \mathbb{I}_K^n$ , we have

$$\mathcal{H}_K^{m-n,m}(A \cap f^{-1}(y)) = \mathcal{H}^{m-n,m}(B \cap g^{-1}(\text{st}(y))). \quad \square$$

## Coarea Formula

### Theorem

Let  $f : \mathbb{I}_K^m \rightarrow \mathbb{I}_K^n$  be a definable Lipschitz function, with  $m > n$ , and  $A \subseteq \mathbb{I}_K^m$  be a definable set. Then,

$$\int_A J_n f \, d\mathcal{L}_K^m = \int_{\mathbb{I}_K^n} \mathcal{H}_K^{m-n,m}(A \cap f^{-1}(y)) \, d\mathcal{L}_K^n(y).$$

## Area Formula

Particular case

### Theorem

Let  $f : \mathbb{I}_K^n \rightarrow \mathbb{I}_K^n$  be a definable and Lipschitz function. Then,

$$\int_{\mathbb{I}_K^n} |\det Df| \, d\mathcal{L}_K^n = \int_{\mathbb{I}_K^n} \#(f^{-1}(y)) \, d\mathcal{L}_K^n(y)$$

If moreover  $h : D^n \rightarrow D$  is definable, then

$$\int_{\mathbb{I}_K^n} h(x) |\det Df(x)| \, d\mathcal{L}_K^n(x) = \int_{\mathbb{I}_K^n} \sum_{x \in f^{-1}(y)} h(x) \, d\mathcal{L}_K^n(y).$$

## Area Formula

### Sketch of proof.

- By Morse-Sard's lemma, we can assume that all points are  $\mathcal{S}$ -regular.
- Apply the corollary to the implicit function theorem:

$$\forall c \forall y \in \text{st}^{-1}(c) \quad \#(f^{-1}(y)) = \#(g^{-1}(c)).$$

- Apply the area formula to  $g$ . □

1.  $\mathcal{L}^E$  coincides with the  $\mathcal{H}_K^{d,n}$ -measure restricted to  $E$ .
2. To prove the lemma, apply the implicit function theorem and Morse-Sard's lemma to the orthogonal projection onto  $E$ , compose with a bi-lipschitz bijection

$$f : \mathbb{I}_K^d \rightarrow A.$$

## Cauchy-Crofton Formula

### Lemma

Let  $A \subseteq \mathbb{I}_K^n$  be a basic  $d$ -rectifiable set,  $E \in \mathcal{G}(d, n; K)$ , and  $\mathcal{L}^E$  be the Lebesgue measure on  $E$

Then, for  $\mathcal{L}^E$ -almost every  $q \in E$ , we have

$$\#(q \cap A) = \#(\text{st}(q) \cap \text{st}(A)).$$

### Theorem (Cauchy-Crofton)

Let  $f : \mathbb{I}_K^m \rightarrow \mathbb{I}_K^n$  be a definable Lipschitz function, with  $m \leq n$ , and  $C \subseteq \mathbb{I}_K^m$  be a definable set. Then,

$$\mathcal{H}_K^{d,n}(C) = \mathcal{J}_K^{d,n}(C) = \frac{1}{\beta} \int_{\mathcal{A}(d,n;K)} \#(q \cap C) d\mu_K^A = \frac{1}{\beta} \int_{\mathcal{G}(d,n;K)} f_C d\mu_K^{\mathcal{G}}.$$

## Cauchy-Crofton Formula

### Sketch of proof.

Call  $B := \text{st}(C)$ . We can assume that  $C$  is basic  $d$ -rectifiable and that  $\mathcal{H}^{d,n}(B) = \mathcal{H}_K^{d,n}(C)$ . By the Lemma,

$$\begin{aligned} \int_{E \in \mathcal{G}(d,n;K)} \int_{q \in E} \#(q \cap C) d\mathcal{L}^E(q) d\mu_K^{\mathcal{G}}(E) &= \\ \int_{E \in \mathcal{G}(d,n;K)} \int_{p \in \text{st}(E)} \#(p \cap B) d\mathcal{L}^{\text{st}(E)}(p) d\mu_K^{\mathcal{G}}(E) &= \\ \int_{F \in \mathcal{G}(d,n)} \int_{p \in F} \#(p \cap B) d\mathcal{L}^F(p) d\mu^{\mathcal{G}}(F) &= \mathcal{H}^{d,n} \text{st}(B), \end{aligned}$$

where we used the Cauchy-Crofton formula for the last equality. □

## More Properties

Let  $C \subseteq K^n$  be bounded, definable, with  $\dim(C) \leq d$ , and  $(C(t))_{t \in K}$  be a uniformly bounded definable family of subsets of  $K^n$ , with  $\dim(C(t)) \leq d$ .

- ①  $\mathcal{J}_K^{d,n}$  is invariant under rotations and translations.
- ② For every  $r \in \text{Fin}(K)^n$ ,  $\mathcal{J}_K^{d,n}(rC) = \text{st}(r)^d \cdot \mathcal{J}_K^{d,n}(C)$ .
- ③ If  $C$  is defined by a semi-algebraic formula without parameters, then  $\mathcal{J}_K^{d,n}(C) = \mathcal{J}^{d,n}(C(\mathbb{R})) = \mathcal{J}^{d,n}(\text{st}(C))$ .
- ④  $\mathcal{J}_K^{d,n}(C(t))$  is uniformly bounded.
- ⑤ If  $K'$  is either an elementary extension or an o-minimal expansion of  $K$ , then  $\mathcal{J}_{K'}^{d,n}(C) = \mathcal{J}_K^{d,n}(C)$ .
- ⑥ If  $d = n$ ,  $\mathcal{J}_K^{n,n}(C) = \mathcal{L}_K^n(C)$ .
- ⑦ If  $C \subseteq E$ , where  $E \subseteq K^n$  is a  $d$ -dimensional plane, then  $\mathcal{J}_K^{d,n}(C) = \mathcal{L}^E(C) = \mathcal{H}^{d,n}(\text{st}(C))$ , where  $\mathcal{L}^E$  is the o-minimal Lebesgue measure on  $E$ .

## Open Problems

### Conjecture (Area formula)

Let  $f : \mathbb{I}_K^m \rightarrow \mathbb{I}_K^n$  be a definable Lipschitz function, with  $m \leq n$ .

- ① If  $A \subseteq \mathbb{I}_K^m$  is a  $\mathcal{L}^m$ -measurable set, then

$$\int_A J_m(f) d\mathcal{L}_K^m = \int_{\mathbb{I}_K^n} \#(f^{-1}(y) \cap A) d\mathcal{H}_K^m(y).$$

- ② If  $g : \mathbb{I}_K^m \rightarrow \mathbb{I}_K$  is a definable function, then

$$\int_{\mathbb{I}_K^m} g \cdot J_m(f) d\mathcal{L}_K^m = \int_{\mathbb{I}_K^n} \sum_{x \in f^{-1}(y)} g(x) d\mathcal{H}_K^m(y).$$

- ⑧ Let  $n \leq n'$ , and  $C \subseteq K^n$  be bounded and definable. Then,

$$\mathcal{H}_K^{d,n}(C) = \mathcal{H}_K^{d,n'}(C).$$

### Conjecture (General Area-Coarea formula)

Let  $W$  be a subset definable of  $\mathbb{I}_K^n$  of dimension  $m$ , and  $Z$  a definable subset of  $\mathbb{I}_K^{\nu}$  of dimension  $\mu$ , with  $m \geq \mu \geq 1$ .

Let  $f : W \rightarrow Z$  be a definable Lipschitz function.

- ① If  $A \subseteq \mathbb{I}_K^m$  is definable, then

$$\int_A J_\mu(f) d\mathcal{H}_K^m = \int_Z \mathcal{H}_K^{m-\mu}(f^{-1}(z)) d\mathcal{H}_K^\mu(z).$$

- ② If  $g : W \rightarrow \mathbb{I}_K$  is a definable function, then

$$\int_W g \cdot J_\mu(f) d\mathcal{H}_K^m = \int_Z \int_{f^{-1}(z)} g d\mathcal{H}_K^{m-\mu} d\mathcal{H}_K^\mu(z).$$

## Conjecture (General Cauchy-Crofton formula)

Let  $0 \leq k \leq m \leq n$ , and  $d := m - k$ . Let  $A \subseteq \mathbb{I}_K^n$  be definable and of dimension  $m$ . Then,





$$\mathcal{H}_K^m(A) = \frac{1}{\beta} \int_{q \in \mathcal{A}(n-d, n; K)} \mathcal{H}_K^k(A \cap q) d\mathcal{H}_K^{n-d}(q),$$

where  $\beta = \beta(k, m, n)$  is a constant not depending on  $K$  or  $A$ .

## Summary

- We can define the **Hausdorff measure** on o-minimal structures.
- This measure behaves as expected, with respect to definable **Lipschitz** maps. In particular, it satisfies the formulae of **Cauchy-Crofton**, **change of variables** and **coarea**.
- Some other properties are only conjectured.

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