Hausdorff Measure on O-Minimal Structures

Integral geometry on o-minimal structures

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Introduction

In [Berarducci-Otero, 2004], the authors define an analogue of Lebesgue measure for bounded definable subsets of an o-minimal structure K, which expands a real closed field. Our aim is defining the area of a bounded definable d-dimensional subsets of K^n. Different possible definitions for such area are possible: we will prove that some of them are equivalent.
On the reals, there are many different definitions of the d-area of a subset. However, on smooth sub-manifolds of $\mathbb{R}^n$ all these definitions coincide with the d-dimensional Hausdorff measure $\mathcal{H}^d$. Every definable set in an o-minimal structure on the reals is a finite union of smooth sub-manifolds, therefore the d-area of such a set is $\mathcal{H}^d$.
Many formulae are known for the Hausdorff measure: from Fubini's theorem to the area and coarea formulae. We will generalise some of these formulae to our measure on K.
Elisa Vasquez Rifo obtained independently some of the results exposed here.

1. Berarducci-Otero measure is a *real* number, not an element of K. The same holds for our measure.
2. Elisa is a student of Speissegger.
3. She works expecially in dimension 1 (curves).

Outline

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Outer Measure

The extended real line is $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. Let Y be a set, $\mathcal{Y}$ be a family of subsets of Y, and $\mu : \mathcal{Y} \to \overline{\mathbb{R}}$.

- μ is monotone iff for every $A, B \in \mathcal{Y}$,

$$A \subseteq B \Rightarrow \mu(A) \leq \mu(B).$$

- μ is countably subadditive (or σ-subadditive) iff for every $A, B_i \in \mathcal{Y}$

$$A \subseteq \bigcup_{i\in\mathbb{N}} B_i \Rightarrow \mu(A) \leq \sum_{i\in\mathbb{N}} \mu(B_i).$$

- μ is countably additive (or σ-additive) iff for every $B_i \in \mathcal{Y}$

$$B_i \text{ disjoint and } \bigsqcup_{i\in\mathbb{N}} B_i \in \mathcal{Y} \Rightarrow \mu(\bigcup_{i\in\mathbb{N}} B_i) = \sum_{i\in\mathbb{N}} \mu(B_i).$$

An outer measure on Y is a non-negative, monotone, and countably subadditive function $\mu^\star : \mathcal{P}(Y) \to \overline{\mathbb{R}}$, such that $\mu^\star(\emptyset) = 0$.

Measure

Definition (Boolean rings and σ-algebrae)

A Boolean ring (or simply ring) $\mathcal{R}$ on Y is a non-empty class of subsets of Y closed under finite union and set-difference. $\mathcal{R}$ is a σ-ring iff it is also closed under countable unions. $\mathcal{R}$ is a σ-algebra iff moreover $Y \in \mathcal{R}$.

Definition (Measure)

A measure on a ring $\mathcal{R}$ is a map $\mu : \mathcal{R} \to \overline{\mathbb{R}}$ which is monotone and countably additive, and such that $\mu(\emptyset) = 0$. A measure is σ-finite iff Y is a countable union of sets in $\mathcal{R}$ of finite measure. A measure is complete iff for every $B \in \mathcal{R}$ and $A \subseteq B$ such that $\mu(B) = 0$, we have $A \in \mathcal{R}$ (and consequently $\mu(A) = 0$). A measure *tout court* is a measure on a σ-algebra on Y.

Carathéodory's Construction

Let $\mu^\star$ be an outer measure on Y. A subset $A \subseteq Y$ is $\mu^\star$-measurable iff for every subset $E \subseteq Y$,

$$\mu^\star(E) = \mu^\star(E \cap A) + \mu^\star(E \setminus A).$$

The $\mu^\star$-measurable subsets of Y form a σ-algebra $\bar{S}_\mu$.
We will denote by μ the measure induced by $\mu^\star$, namely the restriction of $\mu^\star$ to $\bar{S}_\mu$. μ is a *complete* measure on $\bar{S}_\mu$.

Hausdorff Measure

n natural number;
d real number, with $0 \leq d \leq n$;
A subset of $\mathbb{R}^n$;
$\mathcal{H}^d(A) = \mathcal{H}^{d,n}(A)$ the d-dimensional Hausdorff measure of A.

$\mathcal{H}^{d,n}$ does not depends on n. It is an outer measure on $\mathbb{R}^n$.
For every $A \subseteq \mathbb{R}^n$ there exists a unique $d_0 \in \mathbb{R}$ such that

$$\forall d > d_0 \quad \mathcal{H}^d(A) = 0 \text{ and}$$
$$\forall d' < d_0 \quad \mathcal{H}^{d'}(A) = +\infty.$$

d_0 is the Hausdorff dimension of A.

Hausdorff Measure on Manifolds

If A is an embedded $\mathcal{C}^1$ manifold of dimension d, then d is equal to the Hausdorff dimension of A, and $0 < \mathcal{H}^d(A)$.
If $f : D \to \mathbb{R}^n$ is a $\mathcal{C}^1$ map, with $D \underset{\text{open}}{\subseteq} \mathbb{R}^d$, $\mathcal{H}^d\big(f(D)\big)$ is computed as the integral of the Jacobian $\mathrm{J}_d(f)$ over D.
The area of a d-dimensional manifold $A \subseteq \mathbb{R}^n$ is defined by calculating it on parametrised portions of A.

Example

- $d = 0$, $\mathcal{H}^0(A) = \#(A)$ cardinality.
- $d = 1$, $\mathcal{H}^1(A)$ length.
- $d = 2$, $\mathcal{H}^2(A)$ area.
- $d = n$, $\mathcal{H}^{n,n}(A) = \mathcal{L}^n(A)$ Lebesgue measure.

Hausdorff Measure and Real O-Minimal Structures

R o-minimal structure on the real line $\mathbb{R}$, expanding the ordered field structure. $A \subseteq R^n$ definable (with parameters) in R. Then,

- The o-minimal dimension and the Hausdorff dimension of A coincide.
- $0 < \mathcal{H}^d(A)$, where $d = \dim(A)$.
- If A is bounded, then $\mathcal{H}^d(A) < +\infty$.

If A_t is a uniformly bounded definable family of subsets of R^n, then $\mathcal{H}^d(A_t)$ is uniformly bounded.

Berarducci-Otero Definition

- K is an $\aleph_1$-saturated o-minimal structure extending a real closed field.
- A subset $A \subseteq K^n$ is bounded iff there exists $m \in \mathbb{N}$ such that $|x| < n$ for every $x \in A$.
- $\mathrm{Fin}(K)$ is the set of bounded elements.
- The family of bounded definable subsets of K^n is a boolean ring.

Berarducci and Otero define an analogue of Lebesgue measure on $\mathrm{Fin}(K)^n$.
More precisely, they define the outer measure $\mathcal{L}_K^{n\star}$ on $\mathrm{Fin}(K)^n$ as

$$\mathcal{L}_K^{n\star}(A) := \inf\Big(\sum_{i=1}^{k} \mathcal{L}_K^{n\star}(Q_i)\Big),$$

for every $A \subseteq \mathrm{Fin}(K)^n$, where each Q_i is a n-dimensional hypercube with rational vertices, such that

$$A \subseteq \bigcup_{i=1}^{k} Q_i,$$

and $\mathcal{L}_K^{n\star}(Q_i) := \ell_i^n$, where ℓ_i is the length of the edge of Q_i.

1. $\mathcal{L}_K^n$ is defined even when K is not $\aleph_1$-saturated.
   However, when K is countable, $\mathcal{L}_K^n$ is not σ-additive, but only additive.

Outer Measure Pullback

X, Y sets;
$\nu^\star$ an outer measure on Y;
$f : X \to Y$ a surjective function.

For every $A \subseteq X$, define

$$\mu^\star(A) := \nu^\star(f(A))$$
$$\widetilde{A} := f^{-1}(f(A)) \cap f^{-1}(f(X \setminus A)).$$

$\mu^\star$ is an outer measure on X.
$A \subseteq X$ is $\mu^\star$-measurable iff

1. $f(A)$ is $\nu^\star$-measurable, and
2. $\mu^\star(\widetilde{A}) = 0$.

Standard Part and Baisalov-Poizat Theorem

K $\aleph_1$-saturated o-minimal structure expanding an ordered field.
$\mathrm{Fin}(K)$ bounded elements of K.
$\sim$ equivalence relation: $x \sim y$ iff $|x - y| < \frac{1}{n}$ for every $n \in \mathbb{N}$.
$\mathrm{Fin}(K)/\sim$ is naturally isomorphic to $\mathbb{R}$.
$\mathrm{st} : \mathrm{Fin}(K) \to \mathbb{R}$ quotient map = standard part map.
$\mathbb{R}_K$ K induces a structure $\mathbb{R}_K$ on the reals via the map st.
$\mathbb{R}_K$ = expansion of $\mathbb{R}$ generated by the subsets of $\mathbb{R}^n$ of the form $\mathrm{st}(A)$, as A varies among the definable subsets of K^n.

As a corollary of a theorem in [Baisalov-Poizat, 1998],

Theorem (Baisalov and Poizat)

$\mathbb{R}_K$ is o-minimal.

Standard Part and O-Minimal Lebesgue Measure

Let $\mathcal{L}^{n\star}_K$ be the pullback of $\mathcal{L}^{n\star}$ via the map st. $\mathcal{L}^{n\star}_K$ is an outer measure on $\mathrm{Fin}(K)^n$. $A \subseteq \mathrm{Fin}(K)^n$ is measurable iff

1. $\mathrm{st}(A)$ is measurable, and
2. $\mathrm{st}(A) \cap \mathrm{st}(\mathrm{Fin}(K)^n \setminus A)$ is negligible.

- Let $A \subseteq K^n$ be bounded and definable.
- By [Baisalov-Poizat, 1998], $\mathrm{st}(A)$ is Borel, and *a fortiori* measurable.
- [Berarducci-Otero, 2004] prove that $\mathrm{st}(A) \cap \mathrm{st}(\mathrm{Fin}(K)^n \setminus A)$ is negligible.
- Therefore, A is $\mathcal{L}^{n\star}_K$-measurable.
- $\mathcal{L}^n_K(A)$ is the measure of A according to [Berarducci-Otero, 2004].

Hausdorff Measure of the Standard Part

When it does not work

The above construction does not work for $\mathcal{H}^{d,n}$ instead of $\mathcal{L}^n$. Let $d < n$, and $\mathcal{H}^d_{st}$ be the outer measure on $\mathrm{Fin}(K)^n$ induced by $\mathcal{H}^{d,n}$ via st.

Example

Let A the d-dimensional unit cube in $K^d \subset K^n$. A is not $\mathcal{H}^d_{st}$-measurable, because $\mathrm{st}(\widetilde{A})$ is not $\mathcal{H}^d$-negligible. For instance, if A' is the translate of A by an infinitesimal vector

$$A' := \left\{ (x_1, \dots, x_d, \varepsilon, 0, \dots, 0 : x_i \in K, 0 \leq x_i \leq 1 \right\},$$

then $\mathcal{H}^d_{st}(A \sqcup A') = 1 < \mathcal{H}^d_{st}(A) + \mathcal{H}^d_{st}(A')$.

Hausdorff Measure of the Standard Part

When it does work

- Let C be a bounded subset of K^n, definable via a semi-algebraic map without parameters, and of dimension d.
- The restriction $\mathcal{H}^d_{st}$ to C is an outer measure.
- Any definable subset of C is $\mathcal{H}^d_{st}$-measurable.
- If $A \subseteq C$ is definable via a semi-algebraic map without parameters, then $\mathcal{H}^d_{st}(A) = \mathcal{H}^d\big(A(\mathbb{R})\big)$.

Proof.

Work in local charts. □

The Grassmannian

Grassmannian $\mathcal{G}(d, n)$ the set of d-dimensional $\mathbb{R}$-linear subspaces of $\mathbb{R}^n$.

$\mathcal{G}(d, n; K)$ the set of d-dimensional K-linear subspaces of K^n.

There is a compact manifold structure on $\mathcal{G}(d, n)$, of dimension $(n-d)d$, such that

$$\mathcal{G}(d, n) \simeq SO(n)/\big(SO(d) \times SO(n-d)\big).$$

Example

$\mathcal{G}(1, n) \simeq \mathcal{G}(n-1, n) \simeq \mathbb{P}^{n-1}(\mathbb{R})$, the projective space.

The Affine Grassmannian

The affine Grassmannian $\mathcal{A}(d, n)$ is the set of $(n-d)$-dimensional $\mathbb{R}$-affine subspaces of $\mathbb{R}^n$, and similarly for $\mathcal{A}(d, n; K)$.

$\mathcal{A}(d, n)$ is a fibred space over $\mathcal{G}(d, n)$, with fibre $\mathbb{R}^d$. To every affine $(n-d)$-space E we associate the pair $\big(E^\perp, x\big)$, where

$E^\perp$ $\in \mathcal{G}(d, n)$ is the d-linear space orthogonal to E;

$\{x\}$ $= E^\perp \cap E$.

$\mathcal{A}(d, n)$ can be also given a manifold structure, of dimension $(n-d)d + d$.

Example

$\mathcal{A}(1, 2)$ is isomorphic to the Möbius band.

Haar Measure on $\mathcal{G}(d, n)$ and $\mathcal{A}(d, n)$

- The group $SO(n)$ of linear isometries on $\mathbb{R}^n$ acts naturally on $\mathcal{G}(d, n)$.
- There is a unique invariant measure $\mu^{\mathcal{G}}$ on $\mathcal{G}(d, n)$.
- The group of isometries of $\mathbb{R}^n$ acts on $\mathcal{A}(d, n)$, and there is a corresponding invariant measure $\mu^{\mathcal{A}}$ on $\mathcal{A}(d, n)$.
- $\mu^{\mathcal{A}}$ is the product of $\mu^{\mathcal{G}}$ with $\mathcal{L}^d$.

Cauchy-Crofton Formula

Let $C \subseteq \mathbb{R}^n$ be a d-dimensional manifold. Then, there exists a constant[1] $\beta = \beta(d, n) \in \mathbb{R}^{>0}$ such that

$$\mathcal{H}^{d,n}(C) = \frac{1}{\beta} \int_{\mathcal{A}(d,n)} \#(C \cap q)\, \mathrm{d}\mu^{\mathcal{A}}(q) = \frac{1}{\beta} \int_{\mathcal{G}(d,n)} f_C \, \mathrm{d}\mu^{\mathcal{G}}, \tag{1}$$

where $f_C(E)$ is the $\mathcal{L}^E$-area of $p_E(C)$, "counted with multiplicity", with p_E the orthogonal projection onto E. In general, the function

$$\mathcal{I}^{d,n}(C) := \frac{1}{\beta} \int_{\mathcal{A}(d,n)} \#(C \cap q)\, \mathrm{d}\mu^{\mathcal{A}}(q)$$

is defined for any Borel set C, and $\mathcal{I}^{d,n}$ is a Borel measure.

[1] $\beta = \Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{n-d+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)^{-1}$.

Example

$d = n$: $\mathcal{A}(n,n) = \mathbb{R}^n$, and (1) becomes a tautology.

$d = 0$: $\mathcal{A}(0,n)$ is a singleton, $\mathcal{H}^0(C) = \#(C)$, and (1) is obvious.

$d = 1,\ n = 2$: For every $\theta \in [0, 2\pi)$, let D_θ be the line forming an angle θ with the x-axis, $N(r,\theta)$ be the number of points on the curve C whose orthogonal projection on D_θ is the point $(r\cos\theta, r\sin\theta)$. Define

$$P(\theta) := \int_{-\infty}^{+\infty} N(r,\theta)\, \mathrm{d}r.$$

Then, the length of C is equal to

$$\mathcal{I}^{1,2}(C) = \frac{1}{4} \int_0^{2\pi} P(\theta)\, \mathrm{d}\theta.$$

Realisation of Grassmannian

There exist a semi-algebraic realisation of $\mathcal{G}(d,n)$ in $\mathbb{R}^m$ (for some m) such that $\mu^{\mathcal{G}}$ coincides with the restriction of $\mathcal{H}^k$ to $\mathcal{G}(d,n)$ (where $k = \dim(\mathcal{G}(d,n))$). Similarly for $\mathcal{A}(d,n)$.
Therefore, the measure $\mu^{\mathcal{G}}_K$ induced by $\mu^{\mathcal{G}}$ via the map st is an outer measure on $\mathcal{G}(d,n;K)$, such that definable subsets of $\mathcal{G}(d,n;K)$ are measurable. Similarly for $\mathcal{A}(d,n)$.

Definition

Let C be a bounded definable subset of K^n. Define

$$\mathcal{I}^{d,n}_K(C) := \frac{1}{\beta} \int_{\mathcal{A}(d,n;K)} \#(C \cap q)\, \mathrm{d}\mu^{\mathcal{A}}_K(q) = \frac{1}{\beta} \int_{\mathcal{G}(d,n;K)} f_C \, \mathrm{d}\mu^{\mathcal{G}}_K,$$

where $f_C(E)$ is the $\mathcal{L}^E$-area of $p_E(C)$, counted with multiplicity.

Basic Properties

$\mathcal{I}_K^{d,n}$ is a measure defined on all bounded definable subsets of K^n. Given $C \subseteq K^n$ definable and bounded, let

$$S := \{ q \in \mathcal{A}(d,n;K) : q \cap C \neq \emptyset \}$$
$$S_\infty := \{ q \in \mathcal{A}(d,n;K) : \#(q \cap C) = \infty \} \subseteq S$$
$$S_0 := S \setminus S_\infty.$$

- S, S_∞ and S_0 are bounded and definable.
- If $\dim(C) \leq d$, then $\dim(S_\infty) < k' := \dim(\mathcal{A}(d,n;K))$.
- If $\dim(C) < d$, then $\dim(S) < k'$.
- Therefore, if $\dim(C) \leq d$, then $\mathcal{I}_K^{d,n}(C) < \infty$, and if $\dim(C) < d$, then $\mathcal{I}_K^{d,n}(C) = 0$.

More Properties

Let $C \subseteq K^n$ be bounded, definable, with $\dim(C) \leq d$, and $(C(t))_{t \in K}$ be a uniformly bounded definable family of subsets of K^n, with $\dim(C(t)) \leq d$.

1. $\mathcal{I}_K^{d,n}$ is invariant under rotations and translations.
2. For every $r \in \mathrm{Fin}(K)^n$, $\mathcal{I}_K^{d,n}(rC) = \mathrm{st}(r)^d \cdot \mathcal{I}_K^{d,n}(C)$.
3. If C is defined by a semi-algebraic formula without parameters, then $\mathcal{I}_K^{d,n}(C) = \mathcal{I}^{d,n}(C(\mathbb{R})) = \mathcal{I}^{d,n}(\mathrm{st}(C))$.
4. $\mathcal{I}_K^{d,n}(C(t))$ is uniformly bounded.
5. If K' is either an elementary extension or an o-minimal expansion of K, then $\mathcal{I}_{K'}^{d,n}(C) = \mathcal{I}_K^{d,n}(C)$.
6. If $d = n$, $\mathcal{I}_K^{n,n}(C) = \mathcal{L}_K^n(C)$.
7. If $C \subseteq E$, where $E \subseteq K^n$ is a d-dimensional plane, then $\mathcal{I}_K^{d,n}(C) = \mathcal{L}^E(C) = \mathcal{H}^{d,n}(\mathrm{st}(C))$, where $\mathcal{L}^E$ is the o-minimal Lebesgue measure on E.

Let $n \leq n'$, and $C \subseteq K^n$ be bounded and definable.

Question

$$¿\ \mathcal{I}_K^{d,n}(C) = \mathcal{I}_K^{d,n'}(C)\ ?$$

Outline

2 Parametric Formula for the Area
- Lebesgue Measure, Integral, and Definable Functions
- Lipschitz and Bi-Lipschitz Functions
- Rectifiable Sets and Hausdorff Measure
- More Properties of Definable Functions: Regular Points, Sard's Lemma, Implicit Function
- Integral Geometry on O-Minimal Structures: Area, Coarea and Cauchy-Crofton Formulae
- Open Problems: Area, General Area-Coarea, General Cauchy-Crofton Formulae

Area and Jacobian

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $a \in \mathbb{R}^n$. The n-Jacobian of f at a is

$$\mathrm{J}_n(f)(a) := \sqrt{\det\left(\mathrm{D}f(a) \cdot \mathrm{D}f(a)^{\mathrm{T}}\right)}.$$

The m-Jacobian of A is

$$\mathrm{J}_m(f)(a) := \sqrt{\det\left(\mathrm{D}f(a)^{\mathrm{T}} \cdot \mathrm{D}f(a)\right)}.$$

Note that $\mathrm{J}_n(f)(a) = 0$ iff $\mathrm{rk}\left(\mathrm{D}f(a)\right) < n$, and similarly for m.
If $n \leq m$ and f is a smooth embedding of $A \subseteq \mathbb{R}^n$ into $\mathbb{R}^m$, then the area of $f(A)$ is given by

$$\mathcal{H}^{n,m}(f(A)) = \int_A \mathrm{J}_n(f)\,\mathrm{d}\mathcal{L}^n.$$

1. For every k it is possible to define $\mathrm{J}_k(f)(a)$ as the square root of the sum of the square of the k minors of $\mathrm{D}f$.
2. $\mathrm{J}_k(f)(a)$ is the maximum k-dimensional volume of the image under $\mathrm{D}f(a)$ of a unit k-dimensional cube.
3. If $k = m$ or n, $\left(\mathrm{J}_k(f)\right)^2$ is the determinant of the $k \times k$ product of $\mathrm{D}f$ with its transpose.

Monads and Definable Functions

Definition

Given $x, y \in K^n$, x and y are in the same monad, $x \sim y$, iff $|x - y|$ is infinitesimal.

Let $f : K^n \to K^m$ be definable.

Definition

Given $c \in K^n$, f preserves the monad of c iff

$$x \sim c \Rightarrow f(x) \sim f(c).$$

For every $x \in K^n$, define

$$\mathcal{S}f : \mathbb{R}^n \to \overline{\mathbb{R}}^m$$

$$\mathrm{st}(x) \mapsto \begin{cases} st\left(f(x)\right) & \text{iff } f \text{ preserves the monad of } x \\ 0 & \text{otherwise.} \end{cases}$$

1. On $\mathrm{Fin}(K)^n$, $x \sim y$ iff $\mathrm{st}(x) = \mathrm{st}(y)$.
2. $\mathcal{S}f$ is definable in $\mathbb{R}_K$.

Lebesgue Measure and Definable Functions

Call $\mathbb{I}_K := [0,1]_K$, and $\mathbb{I}_{\mathbb{R}} := [0,1]_{\mathbb{R}}$. Let $f : \mathbb{I}_K^n \to \mathbb{I}_K^m$ be definable. Then, there exists $C \subseteq \mathbb{I}_{\mathbb{R}}^n$ definable and negligible such that, outside C,

1. f preserves the monads;
2. f is $\mathcal{C}^1$;
3. $\mathcal{S}f$ is $\mathcal{C}^1$;
4. $\mathrm{D}(\mathcal{S}f) = \mathcal{S}(\mathrm{D}f)$.

If moreover f preserves the monads on all $\mathbb{I}_K^n$, then $\mathcal{S}f$ is continuous on all $\mathbb{I}_{\mathbb{R}}^n$.

1. Outside C can mean "outside $st^{-1}(C)$".
2. A proof of 1: let Γ be the graph of f, and $\bar{\Gamma} := \mathrm{st}(\Gamma)$. Since $\dim(\bar{\Gamma}) \leq \dim(\Gamma) = n$, $\bar{\Gamma}$ is, outside a negligible set, the graph of a function.
3. The hypothesis can be weakened to $f : \mathbb{I}_K^n \to K^m$ definable and "almost bounded".
4. We use the saturation property for the continuity of $\mathcal{S}f$.
5. The latter is only a curiosity: we will not use it.

Integral of Definable Functions

Let $f : \mathbb{I}_K^n \to \mathbb{I}_K$ be definable, $g := \mathcal{S}f : \mathbb{I}_{\mathbb{R}}^n \to \mathbb{I}_{\mathbb{R}}$, and $[0,f] := \left\{ (x,y) \in \mathbb{I}_K^n \times \mathbb{I}_K : 0 \leq y \leq f(x) \right\}$ be the sub-graph of f. The following are well-defined and equal:

$$\int_{\mathbb{I}_{\mathbb{R}}^n} g \,\mathrm{d}\mathcal{L}^n$$

$$\int_{\mathbb{I}_K^n} \mathrm{st}(f) \,\mathrm{d}\mathcal{L}_K^n$$

$$(\mathcal{L}_K^n \times \mathcal{L}^1)\big([0, \mathrm{st}(f)]\big)$$

$$\mathcal{L}_K^{n+1}\big([0,f]\big)$$

$$\mathcal{L}^{n+1}\Big(\mathrm{st}\big([0,f]\big)\Big)$$

$$\mathcal{L}^{n+1}\big([0,g]\big).$$

Definition

We define $\displaystyle\int_{\mathbb{I}_K^n} f \,\mathrm{d}\mathcal{L}_K^n = \int_{\mathbb{I}_K^n} f(x)\,\mathrm{d}x := \int_{\mathbb{I}_{\mathbb{R}}^n} g \,\mathrm{d}\mathcal{L}^n.$

1. $\mathrm{st}(f) : \mathbb{I}_K^n \to \mathbb{I}_{\mathbb{R}}$.
2. The hypothesis can be weakened to $f : \mathbb{I}_K^n \to K$ definable and almost bounded.

Lipschitz and Bi-Lipschitz Functions

Let $A \subseteq K^m$ and $f : A \to K^n$ be definable.

Definition

f is Lipschitz iff there exists a rational number $r > 0$ such that

$$\forall x, y \in K^n \; \big|f(x) - f(y)\big| \leq r|x - y|.$$

The infimum of such constants r is $\operatorname{Lip} f$, the Lipschitz constant of f.

Note that if f is and $\mathcal{C}^1$, and $\big|\mathrm{D}f\big| < r$, then $\operatorname{Lip} f < r$.
Conversely, if f is Lipschitz and $\operatorname{Lip} f < r$, then $\big|\mathrm{D}f\big| < r$, where it is defined.

Definition

A definable bijection $f : A \to B$ is bi-lipschitz iff both f and f^{-1} are Lipschitz.

Torricelli's Theorem

Theorem

Let $f : \mathbb{I}_K \to \mathbb{I}_K$ be definable and Lipschitz, and $\dot{f} := \frac{\mathrm{d}f}{\mathrm{d}x}$. Then,

$$\int_0^1 \dot{f}(x)\,\mathrm{d}x = \operatorname{st}\big(f(1) - f(0)\big).$$

Example

Let $0 < \varepsilon \in K$ be infinitesimal. Define

$$\Theta_\varepsilon := \begin{cases} x/\varepsilon & \text{if } 0 \leq x \leq \varepsilon, \\ 1 & \text{if } \varepsilon \leq x \leq 1. \end{cases}$$

Then, $\int_0^1 \dot{\Theta}_\varepsilon(x)\,\mathrm{d}x = 0$, while $\operatorname{st}\big(\Theta_\varepsilon(1) - \Theta_\varepsilon(0)\big) = 1$.

Fubini's Theorem

Theorem

For every $n, m \in \mathbb{N}$, the (completion of the) measure $\mathcal{L}_K^{n+m}$ is equal to the (completion of the) product $\mathcal{L}_K^n \times \mathcal{L}_K^m$.

Theorem

Let $f : \mathbb{I}_K^{n+m} \to \mathbb{I}_K$ be definable. Then,

$$\int_{\mathbb{I}_K^{n+m}} f(x,y)\,\mathrm{d}\mathcal{L}_K^{m+n}(x,y) = \int_{\mathbb{I}_K^m} \int_{\mathbb{I}_K^n} f(x,y)\,\mathrm{d}\mathcal{L}_K^n(x)\,\mathrm{d}\mathcal{L}_K^m(y).$$

Bi-Lipschitz Change of Variable

Theorem

Let $f : \mathbb{I}_K^n \to \mathbb{I}_K^n$ be definable and bi-lipschitz , and $A \subseteq \mathbb{I}_K^n$ be definable. Then,

$$\mathcal{L}_K^n\big(f(A)\big) = \int_A \left|\det(\mathrm{D}f)\right| \mathrm{d}\mathcal{L}_K^n.$$

If moreover $g : \mathbb{I}_K^n \to \mathbb{I}_K$ is definable, then

$$\int_{f(A)} g(x)\,\mathrm{d}x = \int_A g\big(f(y)\big) \cdot \big|\det\big(\mathrm{D}f(y)\big)\big|\,\mathrm{d}y.$$

Rectifiable Sets

Let $d \in \mathbb{N}$.

Definition (Rectifiable sets)

A set $C \subseteq K^n$ is basic d-rectifiable iff there exists $A \subseteq \mathbb{I}_K^d$ definable and open and $f : A \to C$ definable bi-lipschitz bijection.

Theorem (Decomposition into basic rectifiable sets)

Let $C \subseteq \mathbb{I}_K^n$ definable and of dimension d. Then, there exist $C_0, \dots, C_m \subseteq \mathbb{I}_K^m$ such that

1. $C_0, \dots, C_m$ *are disjoint;*
2. $C_0 \sqcup \dots \sqcup C_m = C$;
3. $\dim(C_0) < d$;
4. *each C_i is basic d-rectifiable, $i = 1, \dots, m$.*

1. In particular, C in the definition is basic d-rectifiable.
2. Say what a Keisler measure is.

Hausdorff Measure on Rectifiable Sets

Definition

Let $A \subseteq \mathbb{I}_K^d$ be definable, $f : \mathbb{I}_K^d \to \mathbb{I}_K^n$ be a bi-lipschitz map, and $C := f(A)$. Define

$$\mathcal{H}_K^{d,n}(C) := \int_A \mathrm{J}_d(f) \, \mathrm{d}\mathcal{L}_K^d.$$

The quantity $\mathcal{H}_K^{d,n}(C)$ does not depend on the particular choice of A and of f.

Proof.

By the bi-lipschitz change of variables. □

Lemma

On a d-rectifiable set $C \subseteq \mathrm{Fin}(K)^n$, $\mathcal{H}_K^{d,n}$ is a Keisler measure.

Change of Variables for Hausdorff Measure

Lemma

Let $B \subseteq \mathbb{I}_K^n$ and $C \subseteq \mathbb{I}_K^m$ be basic d-rectifiable, and $f : B \to C$ be a bi-lipschitz bijection. Then,

$$\int_B \mathrm{J}_d(f) \, \mathrm{d}\mathcal{H}_K^{d,n} = \mathcal{H}_K^{d,m}(C).$$

Rectifiable Sets and Hausdorff Measure of the Standard Part

Lemma (*)

Let $d < n$ and $\pi : K^n \to K^d$ be the projection onto the first d coordinates. Let $A \subseteq \mathbb{I}_K^d$ be definable, and $f : A \to \mathbb{I}_K^n$ be definable and Lipschitz, such that

$$\pi \circ f = \mathrm{id}_A .$$

Then,

$$\mathcal{H}_K^{d,n}(f(A)) = \mathcal{H}^{d,n}(\mathrm{st}(f(A))) = \mathcal{H}^{d,n}(\mathcal{S}f(\mathrm{st}(A))).$$

1. Note that f is bi-lipschitz, and therefore $f(A)$ is basic d-rectifiable.
2. It is possible to decompose any d-dimensional definable set C into sets C_i satisfying the condition (namely the existence of a function f_i as in the lemma, after a permutation of variables).

Regular Points

Definition

Let $f : \mathbb{I}_K^n \to \mathbb{I}_K^m$ be a definable function, and $g := \mathcal{S}f : \mathbb{I}_\mathbb{R}^n \to \mathbb{I}_\mathbb{R}^m$. An element $c \in \mathbb{I}_K^n$ is a $\mathcal{S}$-regular point for f iff there exists $\varepsilon \in \mathbb{Q}^{>0}$ such that, such that, if we call B the ball of centre c and radius ε,

1. $B \subseteq (0,1)_\mathbb{R}^n$;
2. f preserves the monads on B;
3. f is $\mathcal{C}^1$ on B;
4. $\mathrm{D}f$ is bounded and preserves the monads on B;
5. g is $\mathcal{C}^1$ on B;
6. $\mathcal{S}(\mathrm{D}f) = \mathrm{D}g$ on B;
7. every $b \in B$ is a regular point for g.

Note that every $x \in \mathrm{st}^{-1}(B)$ is a regular point for f, and that the set of $\mathcal{S}$-regular points is open.

Morse-Sard's Lemma

Let $f : \mathbb{I}_K^n \to \mathbb{I}_K^m$ be definable, with $m \geq n$.

Theorem

Let $C \subseteq \mathbb{I}_K^n$ be the sets of points $c \in \mathbb{I}_K^n$ such that

1. *satisfy the conditions 1–6;*
2. $\mathrm{rk}\big(\mathrm{D}f(c)\big) = n$.

Then, $\mathcal{H}_K^{n,m}\big(f(\mathbb{I}_K \setminus C)\big) = 0$.

In particular, the set of $\mathcal{S}$-singular values for f is $\mathcal{L}_K^n$-negligible.

1. We have no control over the case $m < n$, because the set of non-$\mathcal{C}^1$ points in the domain $\mathbb{I}^n_K$ can have dimension up to $n-1$.
2. The corollary is trivial if $m > n$, because the whole image of f has dimension at most n.

Implicit Function

Theorem

Let $f : \mathbb{I}^n_K \to \mathbb{I}^n_K$ be definable, $g := \mathcal{S}f$, $a \in \mathbb{I}^n_{\mathbb{R}}$ be a $\mathcal{S}$-regular point for f, and $c := g(a)$.
Then, for every $y \in \mathrm{st}^{-1}(c)$ there exists a unique $x \in \mathrm{st}^{-1}(a)$ such that $f(x) = y$.
Moreover, fix $x_0 \in \mathrm{st}^{-1}(a)$. Then, there exist $\rho, r' \in \mathbb{Q}^{>0}$ such that, if

$$B := \mathrm{cl}\big(B_\rho(a)\big) \text{ and}$$
$$B' := \mathrm{cl}\big(B_\rho(x_0)\big) \subseteq \mathrm{st}^{-1}(B),$$

then

1. *$f\restriction_{B'}$ and $g\restriction_B$ are open and injective,*
2. *$B_{r'}(f(x_0)) \subseteq f(B')$,*
3. *for every $a' \in B$ and $y \in \mathrm{st}^{-1}\big(g(a')\big)$ there exists a unique $x \in B$ such that $f(x) = y$.*

Implicit Function

Corollary

In the above theorem, let $c \in \mathbb{I}^n_K$ be a $\mathcal{S}$-regular value, and $\mathrm{st}(y) = c$. Then,

$$\#\big(g^{-1}(c)\big) = \#\big(f^{-1}(y)\big).$$

Sketch of proof of the theorem.

Consider the one of the usual proof of the implicit function theorem from standard calculus, via a contraction $T : B \to B$. Adapt it to the o-minimal situation. Verify that T is definable, and that the various constants can be chosen non-infinitesimal for $\mathcal{S}$-regular points. □

Aside: the Image of a Measurable Set

Theorem

Let $f : \mathbb{I}^n_K \to \mathbb{I}^n_K$ be definable and preserving the monads. Let $X \subseteq \mathbb{I}^n_K$ be $\mathcal{L}^n_K$-measurable. Then,

1. *$f(X)$ is also $\mathcal{L}^n_K$-measurable;*
2. *if X is $\mathcal{L}^n_K$-negligible, then $f(X)$ is also $\mathcal{L}^n_K$-negligible.*

Sketch of proof.

Let $A := \mathrm{st}(X)$, $Y := f(X)$, and $g := \mathcal{S}f$. Then,

1. A is $\mathcal{L}^n_K$-measurable;
2. $g(A) = \mathrm{st}(Y)$;
3. $\mathrm{st}(Y)$ is $\mathcal{L}^n_K$-measurable;
4. if X is negligible, then $g(A)$ is also negligible, and therefore Y is negligible.

□

Counter-Example

Example

Let $0 < \varepsilon \in K$ be infinitesimal. Define

$$\Theta_\varepsilon := \begin{cases} x/\varepsilon & \text{if } 0 \le x \le \varepsilon, \\ 1 & \text{if } \varepsilon \le x \le 1. \end{cases}$$

- Let $Y \subseteq [0, \frac{1}{2}]$ be non $\mathcal{L}^n_K$-measurable, and
- $X := \Theta_\varepsilon^{-1}(Y) \subseteq [0, \varepsilon]$.
- Then, $\mathcal{L}^n_K(X) = 0$, but
- $\Theta_\varepsilon(X) = Y$ is non $\mathcal{L}^n_K$-measurable.

Coarea Formula

Theorem

Let $f : \mathbb{I}^m_K \to \mathbb{I}^n_K$ be a definable Lipschitz function, with $m > n$, and $A \subseteq \mathbb{I}^m_K$ be a definable set. Then,

$$\int_A \mathrm{J}_n f \, \mathrm{d}\mathcal{L}^m_K = \int_{\mathbb{I}^n_K} \mathcal{H}^{m-n,m}_K \big(A \cap f^{-1}(y)\big) \, \mathrm{d}\mathcal{L}^n_K(y).$$

Coarea Formula

Sketch of proof.

1. By Morse-Sard's lemma, we can assume that all points are $\mathcal{S}$-regular.
2. Apply the Coarea formula to $g := \mathcal{S}f$ and $B := \mathrm{st}(A)$, obtaining

$$\int_A \mathrm{J}_n f \, \mathrm{d}\mathcal{L}^m_K = \int_B \mathrm{J}_n g \, \mathrm{d}\mathcal{L}^m = \int_{\mathbb{I}^n_{\mathbb{R}}} \mathcal{H}^{m-n,m}\big(B \cap g^{-1}(z)\big) \, \mathrm{d}\mathcal{L}^n(z).$$

3. By the implicit function theorem, and Lemma (*), for almost every $y \in \mathbb{I}^n_K$, we have

$$\mathcal{H}^{m-n,m}_K\big(A \cap f^{-1}(y)\big) = \mathcal{H}^{m-n,m}\big(B \cap g^{-1}(\mathrm{st}(y))\big).$$

□

Area Formula

Particular case

Theorem

Let $f : \mathbb{I}^n_K \to \mathbb{I}^n_K$ be a definable and Lipschitz function. Then,

$$\int_{\mathbb{I}^n_K} |\det \mathrm{D}f| \, \mathrm{d}\mathcal{L}^n_K = \int_{\mathbb{I}^n_K} \#\big(f^{-1}(y)\big) \, \mathrm{d}\mathcal{L}^n_K(y)$$

If moreover $h : D^n \to D$ is definable, then

$$\int_{\mathbb{I}^n_K} h(x) |\det \mathrm{D}f(x)| \, \mathrm{d}\mathcal{L}^n_K(x) = \int_{\mathbb{I}^n_K} \sum_{x \in f^{-1}(y)} h(x) \, \mathrm{d}\mathcal{L}^n_K(y).$$

Area Formula

Sketch of proof.

- By Morse-Sard's lemma, we can assume that all points are $\mathcal{S}$-regular.
- Apply the corollary to the implicit function theorem:

$$\forall c\ \forall y \in \mathrm{st}^{-1}(c) \quad \#(f^{-1}(y)) = \#(g^{-1}(c)).$$

- Apply the area formula to g. □

1. $\mathcal{L}^E$ coincides with the $\mathcal{H}^{d,n}_K$-measure restricted to E.
2. To prove the lemma, apply the implicit function theorem and Morse-Sard's lemma to the orthogonal projection onto E, compose with a bi-lipschitz bijection

$$f : \mathbb{I}^d_K \to A.$$

Cauchy-Crofton Formula

Lemma

Let $A \subseteq \mathbb{I}^n_K$ be a basic d-rectifiable set, $E \in \mathcal{G}(d,n;K)$, and $\mathcal{L}^E$ be the Lebesgue measure on E
Then, for $\mathcal{L}^E$-almost every $q \in E$, we have

$$\#(q \cap A) = \#(\mathrm{st}(q) \cap \mathrm{st}(A)).$$

Theorem (Cauchy-Crofton)

Let $f : \mathbb{I}^m_K \to \mathbb{I}^n_K$ be a definable Lipschitz function, with $m \le n$, and $C \subseteq \mathbb{I}^m_K$ be a definable set. Then,

$$\mathcal{H}^{d,n}_K(C) = \mathcal{I}^{d,n}_K(C) = \frac{1}{\beta}\int_{\mathcal{A}(d,n;K)} \#(q \cap C)\,\mathrm{d}\mu^{\mathcal{A}}_K = \frac{1}{\beta}\int_{\mathcal{G}(d,n;K)} f_C\,\mathrm{d}\mu^{\mathcal{G}}_K.$$

Cauchy-Crofton Formula

Sketch of proof.

Call $B := \mathrm{st}(C)$. We can assume that C is basic d-rectifiable and that $\mathcal{H}^{d,n}(B) = \mathcal{H}^{d,n}_K(C)$. By the Lemma,

$$\int_{E\in\mathcal{G}(d,n;K)} \int_{q\in E} \#(q\cap C)\,\mathrm{d}\mathcal{L}^E(q)\,\mathrm{d}\mu^{\mathcal{G}}_K(E) =$$
$$\int_{E\in\mathcal{G}(d,n;K)} \int_{p\in \mathrm{st}(E)} \#(p\cap B)\,\mathrm{d}\mathcal{L}^{\mathrm{st}(E)}(p)\,\mathrm{d}\mu^{\mathcal{G}}_K(E) =$$
$$\int_{F\in\mathcal{G}(d,n)} \int_{p\in F} \#(p\cap B)\,\mathrm{d}\mathcal{L}^{F}(p)\,\mathrm{d}\mu^{\mathcal{G}}(F) = \mathcal{H}^{d,n}\mathrm{st}(B),$$

where we used the Cauchy-Crofton formula for the last equality. □

More Properties

Let $C \subseteq K^n$ be bounded, definable, with $\dim(C) \leq d$, and $\big(C(t)\big)_{t \in K}$ be a uniformly bounded definable family of subsets of K^n, with $\dim\big(C(t)\big) \leq d$.

1. $\mathcal{I}^{d,n}_K$ is invariant under rotations and translations.
2. For every $r \in \mathrm{Fin}(K)^n$, $\mathcal{I}^{d,n}_K(rC) = \mathrm{st}(r)^d \cdot \mathcal{I}^{d,n}_K(C)$.
3. If C is defined by a semi-algebraic formula without parameters, then $\mathcal{I}^{d,n}_K(C) = \mathcal{I}^{d,n}\big(C(\mathbb{R})\big) = \mathcal{I}^{d,n}\big(\mathrm{st}(C)\big)$.
4. $\mathcal{I}^{d,n}_K\big(C(t)\big)$ is uniformly bounded.
5. If K' is either an elementary extension or an o-minimal expansion of K, then $\mathcal{I}^{d,n}_{K'}(C) = \mathcal{I}^{d,n}_K(C)$.
6. If $d = n$, $\mathcal{I}^{n,n}_K(C) = \mathcal{L}^n_K(C)$.
7. If $C \subseteq E$, where $E \subseteq K^n$ is a d-dimensional plane, then $\mathcal{I}^{d,n}_K(C) = \mathcal{L}^E(C) = \mathcal{H}^{d,n}\big(\mathrm{st}(C)\big)$, where $\mathcal{L}^E$ is the o-minimal Lebesgue measure on E.

8. Let $n \leq n'$, and $C \subseteq K^n$ be bounded and definable. Then,

$$\mathcal{H}^{d,n}_K(C) = \mathcal{H}^{d,n'}_K(C).$$

Open Problems

Conjecture (Area formula)

Let $f : \mathbb{I}^m_K \to \mathbb{I}^n_K$ be a definable Lipschitz function, with $m \leq n$.

1. *If $A \subseteq \mathbb{I}^m_K$ is a $\mathcal{L}^m$-measurable set, then*

$$\int_A \mathrm{J}_m(f)\,\mathrm{d}\mathcal{L}^m_K = \int_{\mathbb{I}^n_K} \#\big(f^{-1}(y) \cap A\big)\,\mathrm{d}\mathcal{H}^m_K(y).$$

2. *If $g : \mathbb{I}^m_K \to \mathbb{I}_K$ is a definable function, then*

$$\int_{\mathbb{I}^m_K} g \cdot \mathrm{J}_m(f)\,\mathrm{d}\mathcal{L}^m_K = \int_{\mathbb{I}^n_K} \sum_{x \in f^{-1}(y)} g(x)\,\mathrm{d}\mathcal{H}^m_K(y).$$

Conjecture (General Area-Coarea formula)

Let W be a subset definable of $\mathbb{I}^n_K$ of dimension m, and Z a definable subset of $\mathbb{I}^\nu_K$ of dimension μ, with $m \geq \mu \geq 1$.
Let $f : W \to Z$ be a definable Lipschitz function.

1. *If $A \subseteq \mathbb{I}^m_K$ is definable, then*

$$\int_A \mathrm{J}_\mu(f)\,\mathrm{d}\mathcal{H}^m_K = \int_Z \mathcal{H}^{m-\mu}_K\big(f^{-1}(z)\big)\,\mathrm{d}\mathcal{H}^\mu_K(z).$$

2. *If $g : W \to \mathbb{I}_K$ is a definable function, then*

$$\int_W g \cdot \mathrm{J}_\mu(f)\,\mathrm{d}\mathcal{H}^m_K = \int_Z \int_{f^{-1}(z)} g\,\mathrm{d}\mathcal{H}^{m-\mu}_K\,\mathrm{d}\mathcal{H}^\mu_K(z).$$

Conjecture (General Cauchy-Crofton formula)

Let $0 \leq k \leq m \leq n$, and $d := m - k$. Let $A \subseteq \mathbb{I}^n_K$ be definable and of dimension m. Then,

$$\mathcal{H}^m_K(A) = \frac{1}{\beta} \int_{q \in \mathcal{A}(n-d,n;K)} \mathcal{H}^k_K(A \cap q)\, \mathrm{d}\mathcal{H}^{n-d}_K(q),$$

where $\beta = \beta(k, m, n)$ is a constant not depending on K or A.

Summary

- We can define the Hausdorff measure on o-minimal structures.
- This measure behaves as expeted, with respect to definable Lipschitz maps. In particular, it satisfies the formulae of Cauchy-Crofton, change of variables and coarea.
- Some other properties are only conjectured.

Bibliography

P. R. Halmos.
Measure Theory.
D. Van Nostrand Company, Inc., New York, N. Y., 1950.

F. Morgan.
Geometric Measure Theory.
Academic Press, 1988
An introduction to Federer's book by the same title.

Y. Baisalov and B. Poizat.
Paires de structures o-minimales.
J. Symbolic Logic, 63(2):570–578, 1998.
O-minimality of the standard part.

A. Berarducci and M. Otero.
An additive measure in o-minimal expansions of fields.
Q. J. Math., 55(4):411–419, 2004.
Lebesgue measure on o-minimal structures.