

**NOTES ON HIERONYMI'S THEOREM: EVERY DEFINABLY
COMPLETE STRUCTURE IS DEFINABLY BAIRE
VERSION 1.6**

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ABSTRACT. We give an exposition and strengthening of P. Hieronymi's theorem: if C is a nonempty closed set definable in a definably complete expansion of an ordered field, then C satisfies an analogue of Baire's category theorem.

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1. INTRODUCTION

Let \mathbb{K} be a Definably Complete structure expanding a field (that is, every definable subset of \mathbb{K} has a least upper bound in $\mathbb{K} \cup \{\pm\infty\}$: see [Mil01]).

Tamara Servi and I conjectured in [FS10] that \mathbb{K} is definably Baire (see Definition 1.1). In [Hie11], Philipp Hieronymi proved our conjecture. The aim of this note is to give (a sketch of) an alternative proof of Hieronymi Theorem, together with a generalization of Hieronymi Theorem and of Kuratowski-Ulam Theorem to definable closed subsets of \mathbb{K}^n .

We recall the relevant definitions.

Definition 1.1 ([FS10]). Let $A \subseteq B \subseteq \mathbb{K}^n$ be definable sets. A is nowhere dense in B if the closure of A has interior (inside B); otherwise, A is somewhere dense in B . A is meager in B if there exists a definable increasing family $(Y_t : t \in \mathbb{K})$ of nowhere dense subsets of B , such that $A \subseteq \bigcup_t Y_t$. A is residual in B if $B \setminus A$ is meager in B .

B is definably Baire if every nonempty open definable subset of B is not definably meager in B (or, equivalently, in itself).

A is an \mathcal{F}_σ subset of B if there exists a definable increasing family $(Y_t : t \in \mathbb{K})$ of closed subsets of B , such that $A = \bigcup_t Y_t$; $A \subseteq B$ is a \mathcal{G}_δ subset of B if $B \setminus A$ is an \mathcal{F}_σ subset of B ; if we don't specify the ambient space B , we mean that $B = \mathbb{K}^n$.

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Fact 1.2 ([FS10, §3]). *A finite Boolean combination of closed definable subsets of \mathbb{K}^n is \mathcal{F}_σ in \mathbb{K}^n . Moreover, the family of \mathcal{F}_σ subsets of \mathbb{K}^n , for some n , is closed under finite union and intersections, and under images under definable continuous functions.*

Theorem 1.3 (Baire Category). *Let $C \subseteq \mathbb{K}^n$ be a nonempty \mathcal{G}_δ subset of \mathbb{K}^n . Then, C is definably Baire.*

Notice that the case when $n = 1$ and $C = \mathbb{K}$ in Theorem 1.3 is exactly Hieronymi's Theorem (see [Hie11]). We will prove Theorem 1.3 in §5.

Theorem 1.4 (Kuratowski-Ulam). *Let $\pi := \Pi_m^{m+n}$. Let $C \subseteq \mathbb{K}^{m+n}$ be definable and $E := \pi(C)$. Let $F \subseteq C$ be a definable set. Assume that either F or C are \mathcal{F}_σ in \mathbb{K}^{m+n} . Define $T := T_C^m(F) := \{\bar{x} \in E : F_{\bar{x}} \text{ is meager in } C_{\bar{x}}\}$. If F is meager in C , then T is residual in E .*

Notice that the case when $C = \mathbb{K}^{m+n}$ in Theorem 1.4 is [FS10, Theorem 4.1]. We will prove Theorem 1.4 in §4.

As an application, we prove the following results.

Corollary 1.5. *Let C be a nonempty, d -compact subset of \mathbb{K}^{m+n} (see §2). Let $\pi : \mathbb{K}^{n+m} \rightarrow \mathbb{K}^m$ be the projection onto the first n coordinates, let $A := \pi(C)$, and define $f : A \rightarrow \mathbb{K}^n$, $f(\bar{x}) := \text{lex min}(C_{\bar{x}})$. Let E be the set of $\bar{x} \in A$ such that either \bar{x} is an isolated point of A , or f is continuous at \bar{x} . Then, E is residual in A , and therefore it is dense in A .*

Proof. The fact that E is residual in A is Fact 2.9 (see [DMS10, 1.9]). By Theorem 1.3, A is definably Baire, and every residual subset E of a definably Baire set A is dense in A . \square

Corollary 1.6. ⁽¹⁾ *Let $F \subseteq C \subseteq \mathbb{K}^{m+n}$ be nonempty definable, closed subsets of \mathbb{K}^{m+n} . Let $\pi : \mathbb{K}^{m+n} \rightarrow \mathbb{K}^m$ be the projection onto the first m coordinates, and $E := \pi(C)$. Assume that E is closed inside \mathbb{K}^m , and that the set*

$$T' := T_C^m(F) := \{\bar{x} \in E : F_{\bar{x}} \text{ has no interior inside } C_{\bar{x}}\}$$

is not dense in E . Then, F has interior inside C .

Proof. By Theorem 1.3 (applied to each fiber $C_{\bar{x}}$), $T' = T_C^m(F)$. By Theorem 1.3 again (applied to the set E), T' is not residual inside E (cf. Lemma 4.1). Thus, by Theorem 1.4, F is not meager inside C ; therefore F is somewhere dense inside C , and thus it has interior inside C . \square

Question 1.7. What is the most general form of Theorem 1.4? For instance, can we drop the assumption that either F or C are \mathcal{F}_σ subsets of \mathbb{K}^{m+n} ?

Definition 1.8 ([For11, §4]). A pseudo- \mathbb{N} set is a set $\mathcal{N} \subset \mathbb{K}_{\geq 0}$, such that \mathcal{N} is definable, closed, discrete, and unbounded.

A quasi-order $\langle D, \trianglelefteq \rangle$ is a forest if, for every $a \in D$, the set $\{c \in D : c \trianglelefteq a\}$ is totally ordered by \triangleleft .

The following lemma is at the core of the proof: we hope it may be of independent interest.

Lemma 1.9 (Leftmost Branch). *Let \mathcal{N} be a pseudo- \mathbb{N} set. Let \trianglelefteq be a definable quasi-order of \mathcal{N} (i.e., \trianglelefteq is a reflexive and transitive binary relation on \mathcal{N} , whose graph is definable). Assume that \trianglelefteq is a forest. Then, there exists a definable set $E_0 \subseteq \mathcal{N}$, such that:*

⁽¹⁾ [Mil01, §4] asked a related question: notice however that Miller's original question has a misprint.

- (a) the minimum of \mathcal{N} is in E_0 ;
(b) for every $d \in E_0$, the successor of d in E_0 (if it exists) is

$$n(d) := \min\{e \in \mathcal{N} : d < e \ \& \ d \triangleleft e\};$$

conversely, if $n(d)$ exists, then it is the successor of d in E_0 .

Besides, E_0 is unique, satisfying the above conditions. Besides, \leq and \trianglelefteq coincide on E_0 (and, in particular, E_0 is linearly ordered by \trianglelefteq).

If moreover we have

- (*) For every $d \in \mathcal{N}$ there exists $e \in \mathcal{N}$ such that $d \triangleleft e$,
then E_0 is unbounded (and hence cofinal in \mathcal{N}).

We call the set E_0 defined in the above lemma the leftmost branch of \trianglelefteq (inside \mathcal{N}); notice \mathcal{N} has a minimum, and every element of \mathcal{N} has a successor in \mathcal{N} (see [For11, §4]).

The proof of a particular case of the above lemma is given in [Hie11, Def. 12, Lemma 14, Def. 15, Lemma 16, Lemma 17], where $d \triangleleft e$ if “ $f(e)$ extends $f(d)$ ” (in [Hie11] terminology). We will give a sketch of the proof in §3. P. Hieronymi pointed out a mistake in a previous version of these notes, when we did not require the condition that \trianglelefteq is a forest in Lemma 1.9 (see §3.1).

Let \mathcal{N} be a pseudo- \mathbb{N} set. While it is quite clear how to prove statements in \mathcal{N} by (a kind of) induction (see [For11, Remark 4.15]), *a priori* it is not clear how to construct (definable) sets by recursion: Lemma 1.9 gives a way to produce a definable set E_0 whose definition is recursive; this will allow us to prove that \mathbb{K} is definably Baire (see §5.1). However, to prove that a \mathcal{G}_δ set C is definably Baire we need to use a different method (since, for technical reason, our proof in §5.1 requires the assumption that C contains a dense pseudo-enumerable set, and we do not know if the assumption holds for C), that relies on Theorem 1.4, used inductively (see §5.2).

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2. PRELIMINARIES

Fact 2.1 ([FS10, Proposition 2.11]). *Let $U \subseteq \mathbb{K}^n$ be open and definable. U is meager in \mathbb{K}^n iff U is meager in itself.*

Definition 2.2. A d -compact set is a definable, closed, and bounded subset of \mathbb{K}^n (for some n).

The following fact will be used many times without mentioning it explicitly.

Fact 2.3 ([Mil01]). (1) *Let $X \subseteq \mathbb{K}^n$ be a d -compact set, and $f : X \rightarrow \mathbb{K}^m$ be a definable and continuous function. Then, $f(X)$ is d -compact.*

(2) *Let $(X_t : t \in \mathbb{K})$ be a definable decreasing family of nonempty d -compact subsets of \mathbb{K}^n . Then, $\bigcap_t X_t$ is nonempty.*

Definition 2.4. Let $a \in \mathbb{K}^n$, $X \subseteq \mathbb{K}^n$, and $r > 0$. Define

$$B(a; r) := \{x \in \mathbb{K}^n : |x - a| < r\};$$

$$\overline{B}(a; r) := \{x \in \mathbb{K}^n : |x - a| \leq r\};$$

$$B_X(a; r) := X \cap B(a; r);$$

$$\overline{B}_X(a; r) := X \cap \overline{B}(a; r).$$

Given $Y \subseteq X$, denote by $\text{cl}_X(Y)$ (resp., $\text{int}_X(Y)$) the topological closure (resp., the interior) of Y inside X , and denote $\text{cl}(Y) := \text{cl}_{\mathbb{K}^n}(Y)$.

Remark 2.5. Let X be a topological space and $A \subseteq X$. A is somewhere dense in X iff there exists $V \neq \emptyset$ an open subset of X , such that, for every $W \neq \emptyset$ open subset of V , $W \cap A \neq \emptyset$.

Lemma 2.6. (1) Let X be a topological space, U be a dense subset of X , and $A \subseteq X$ be any subset. T.f.a.e.:

- i) A is nowhere dense in X ;
- ii) $A \cap U$ is nowhere dense in X ;
- iii) $A \cap U$ is nowhere dense in U .

(2) Let $X \subseteq \mathbb{K}^n$ be a definable, U be a dense open definable subset of X , and $A \subseteq X$ be any definable subset. T.f.a.e.:

- i) A is meager in X ;
- ii) $A \cap U$ is meager in X ;
- iii) $A \cap U$ is meager in U .

(3) Let $X \subseteq \mathbb{K}^n$ be a definable set and $U \subseteq X$ be a definable dense open subset of X . Then, X is definably Baire iff U is definably Baire.

Proof. (1) follows from Remark 2.5.

(2) follows from (1).

(3) follows from (2). □

Corollary 2.7. Let $A \subseteq X \subseteq \mathbb{K}^n$ be definable nonempty sets. Assume that A is a dense subset of X .

(1) If A is definably Baire, then X is also definably Baire.

(2) If X is definably Baire and A is a \mathcal{G}_δ subset of X , then A is also definably Baire.

Proof. (1) is clear from Lemma 2.6(1).

(2) Assume, for a contradiction, that A is not definably Baire. We can easily reduce to the case when A is meager in itself. Let $F := X \setminus A$. By our assumption on A , F is an \mathcal{F}_σ subset of X with empty interior; thus, F is meager in X . Since X is Baire and $X = A \cup F$, A is not meager in X . Since A is meager in itself, $A = \bigcup_t C_t$, for some definable increasing family $(C_t : t \in \mathbb{K})$ of nowhere dense subsets of A . By Lemma 2.6(1), each C_t is also nowhere dense in X , contradicting the fact that A is not meager in X . □

Definition 2.8. Let $E \subseteq \mathbb{K}^m$ and $f : E \rightarrow \mathbb{K}^n$ be definable. Given $\varepsilon > 0$, define

$$\mathcal{D}(f; \varepsilon) := \{ \bar{a} \in E : \forall \delta > 0 \ f(B_E(\bar{a}; \delta)) \not\subseteq B(f(\bar{a}); \varepsilon) \}.$$

Fact 2.9 ([DMS10, 1.9]). Let $C \subset \mathbb{K}^{m+n}$ be a nonempty d -compact set and $E := \Pi_m^{m+n}(C) \subset \mathbb{K}^m$. Define $f : E \rightarrow \mathbb{K}^n$, $f(x) := \text{lex min}(C_{\bar{x}})$. Then, for every $\varepsilon > 0$, $\mathcal{D}(f; \varepsilon)$ is nowhere dense in E .

Conjecture 2.10. Let X_1, X_2 be definable subsets of \mathbb{K}^n , and $X := X_1 \cup X_2$. If both X_1 and X_2 are definably Baire, then X is also definably Baire.

3. PROOF OF LEMMA 1.9

We will proceed by various reductions. Define

$$E := \{ d \in \mathcal{N} : (\forall e \in \mathcal{N}) \ e < d \rightarrow d \not\leq e \}.$$

Let $e_0 := \min(\mathcal{N})$. Define

$$E_1 := \{ d \in E : e_0 \leq d \}.$$

Define E_2 as the set of elements $d \in E_1$, such that d is the minimum of the set $\{ d' \in E_2 : d' \leq d \ \& \ d \leq d' \}$. Notice that E_2 satisfies the following conditions, for all $d, d' \in E_2$:

- (i) $e_0 \in E_2$;
- (ii) $e_0 \leq d$;
- (iii) $n(d) \in E_2$;
- (iv) $d \trianglelefteq d' \rightarrow d \leq d'$;
- (v) \trianglelefteq is a partial order on E_2 ;
- (vi) $\langle E_2, \trianglelefteq \rangle$ is a forest.

For every $a, b \in E_2$, define $a \perp b$ if $a \not\trianglelefteq b$ and $b \not\trianglelefteq a$. Given $a, b \in E_2$ such that $a \perp b$, define $c(a, b) := \min\{a' \in E_2 : a' \trianglelefteq a \text{ \& } a' \perp b\}$ (where the minimum is taken w.r.t. \leq). Notice that if $a \perp b$, $a' \trianglelefteq a$, and $a' \perp b$, then $c(a, b) \leq a'$ and, by (iv), either $c(a, b) \trianglelefteq a'$, or $c(a, b) \perp a'$; moreover, $c(a, b) \leq a$, $c(a, b) \trianglelefteq a$, and $c(a, b) \perp b$.

Finally, define E_0 as the set of $a \in E_2$, such that, for every $b \in E_2$, if $b \perp a$, then $c(a, b) < b$.

We have to show that E_2 is the leftmost branch of \trianglelefteq inside \mathcal{N} . W.l.o.g., we can assume that $\mathcal{N} = E_2$.

Claim 1. For every $a \in E_0$, $n(a) \in E_0$.

Assume not. Let $a \in E_0$ be such that $b := n(a) \notin E_0$. Thus, by definition, there exists $d \in \mathcal{N}$ such that $c := c(b, d) \geq d$. If $d \trianglelefteq a$, then $d \trianglelefteq b$, absurd. If $a \trianglelefteq d$, then $d \geq b$ because $b = n(a)$, also absurd. If $d \perp a$, then, since $a \in \mathcal{N}$, we have that $c' := c(a, d) < d$. Moreover, $c' \trianglelefteq a \triangleleft b$ and $c' \perp d$; thus, by definition, $c \leq c'$, and therefore $c' > d$, absurd.

The next claim is the only place where we use the fact that \mathcal{N} is a forest.

Claim 2. Let $b \in E_0$ and $a \in \mathcal{N}$ with $a \trianglelefteq b$. Then, $a \in E_0$.

Assume not. Let $d \in \mathcal{N}$ such that $d \perp a$ and $c := c(a, d) \geq d$. Since $c \leq a$ and $a \neq d$, we have $d < a$, and therefore $d < b$. If $d \trianglelefteq b$, then, since $\langle \mathcal{N}, \trianglelefteq \rangle$ is a forest, we have $d \trianglelefteq a$, absurd. Thus, we have $d \perp b$. Since $b \in E_0$, we have $c' := c(b, d) < d$. Moreover, since $c \perp d$ and $c \trianglelefteq b$, the definition of c' implies $c' \leq c$, and therefore, since $\langle \mathcal{N}, \trianglelefteq \rangle$ is a forest, $c' \trianglelefteq c$. Conversely, since $c' \trianglelefteq a$ and $c' \perp d$, the definition of c implies $c \leq c'$, and therefore $c = c' < d$, absurd.

Claim 3. \leq and \trianglelefteq coincide on E_0 .

Assume not. By (iv), there exist $a, b \in E_0$, such that $a \perp b$; let $a \in E_0$ be minimal such there exists $b \in E_0$ with $a \perp b$. Let $c := c(b, a)$. By Claim 2, since $c \trianglelefteq b$, we have $c \in E_0$. Since $b \in E_0$, we have $c < a$, contradicting the minimality of a . \square

3.1. A counterexample: the forest hypothesis is necessary. The following counterexample is due to P. Hieronymi.⁽²⁾ We show that the conclusion of Lemma 1.9 may fail if we remove the assumption that $\langle \mathcal{N}, \trianglelefteq \rangle$ is a forest, even under the assumption that \mathbb{K} expands the reals and satisfies strong ‘‘tameness’’ condition (i.e., d-minimality).

Let $P := \{2^{2^n} : n \in \mathbb{N}\}$ and let $\mathbb{K} := \langle \mathbb{R}, P \rangle$ be the expansion of the real field by a predicate for P ; by [MT06], \mathbb{K} is d-minimal. It is quite clear that P is a pseudo- \mathbb{N} set. We now define a partial ordering on P . For every $a \in P$, denote by $s(a)$ the successor of a in P , i.e. $s(a) = a^2$. Let $a, b \in P$; define $a \trianglelefteq b$ iff either $a = b$ or $b > s(a)$ (that is, if $b \geq a^4$). It is clear that $\langle P, \trianglelefteq \rangle$ is a partially ordered set, satisfying (*), and that the leftmost branch Q of $\langle P, \trianglelefteq \rangle$ is the set of elements with even index, i.e. $Q := \{2^{4^n} : n \in \mathbb{N}\}$. However, the set Q is not definable in \mathbb{K} : see [For11, Lemma 2.2] and [MT06] for the details.

⁽²⁾ Private communication.

4. PROOF OF THEOREM 1.4

Lemma 4.1. *Let $m, n \in \mathbb{N}_{\geq 1}$. Let $\pi := \Pi_m^{m+n}$. Let $C \subseteq \mathbb{K}^{m+n}$ be definable and $E := \pi(C)$. Let $F \subseteq C$ be a d -compact definable set. Define*

$$T' := T'_C(F) := \{ \bar{x} \in E : \text{int}_{C_{\bar{x}}}(F_{\bar{x}}) = \emptyset \}.$$

If $\text{int}_C(F) = \emptyset$, then T' is residual in E .

Proof. The proof of the Lemma is similar to [FS10, §4, Case 1]. Fix $\varepsilon > 0$; define

$$F(\varepsilon) := \{ \langle \bar{x}, \bar{y} \rangle \in F : B_C(\bar{y}; \varepsilon) \subseteq F_{\bar{x}} \};$$

$$X(\varepsilon) := \text{cl}_C(F(\varepsilon)) = \text{cl}_F(F(\varepsilon));$$

$$Y(\varepsilon) := \pi(X(\varepsilon)) \subseteq E.$$

Claim 4. $Y(\varepsilon)$ is nowhere dense in E .

Since F is d -compact and $X(\varepsilon)$ is closed in F , $X(\varepsilon)$ is d -compact. Since $Y(\varepsilon) = \pi(X(\varepsilon))$, $Y(\varepsilon)$ is also d -compact, and therefore it is closed in E . Assume, for a contradiction, that $Y(\varepsilon)$ is somewhere dense in A : thus, $U := \text{int}_E(Y(\varepsilon)) \neq \emptyset$. Define $f : U \rightarrow \mathbb{K}^n$, $\bar{x} \mapsto \text{lex min}(X(\varepsilon)_{\bar{x}})$; notice that $\Gamma(f) \subseteq X(\varepsilon)$. By Fact 2.9, $\mathcal{D}(f; \varepsilon/4)$ is nowhere dense in U . Thus, there exist $\bar{a} \in U$ and $\delta > 0$ such that $B_E(\bar{a}, \delta) \subseteq U \setminus \text{cl}_E(\mathcal{D}(f; \varepsilon/4))$, and $\delta < \varepsilon/4$. Let $\bar{b} := f(\bar{a})$; thus, $\langle \bar{a}, \bar{b} \rangle \in \Gamma(f) \subseteq X(\varepsilon) \subseteq F$. The following Claim 4 contradicts the fact that F is nowhere dense in C , and therefore Claim 4 will follow.

Claim 5. $B_C(\langle \bar{a}, \bar{b} \rangle; \delta_1) \subseteq F$, for some $\delta_1 > 0$.

Choose $\delta_1 > 0$ such that $\delta_1 < \delta$ and $f(B_E(\bar{a}; \delta_1)) \subseteq B(\bar{b}; \delta)$ (δ_1 exists because $\bar{a} \notin \mathcal{D}(f; \varepsilon/4)$). Let $\langle \bar{x}, \bar{y} \rangle \in B_C(\langle \bar{a}, \bar{b} \rangle; \delta_1)$. Thus, $\bar{x} \in E$, $|\bar{x} - \bar{a}| < \delta_1$, $\bar{y} \in C_{\bar{x}}$, and $|\bar{y} - \bar{b}| < \delta_1$. Therefore, $\bar{x} \in B_E(\bar{a}; \delta) \subseteq U \setminus \text{cl}_A(\mathcal{D}(f; \varepsilon/4))$. Thus,

$$|\bar{y} - f(\bar{x})| \leq |\bar{y} - \bar{b}| + |\bar{b} - f(\bar{x})| \leq \delta_1 + \delta < 2\delta < \varepsilon,$$

and therefore $\bar{y} \in B_{C_{\bar{x}}}(f(\bar{x}); \varepsilon)$. Since $\langle \bar{x}, f(\bar{x}) \rangle \in X(\varepsilon)$, we have $B_{C_{\bar{x}}}(f(\bar{x}); \varepsilon) \subseteq F_{\bar{x}}$; thus, Claim 5 is proven.

Claim 6. $E \setminus T' \subseteq \bigcup_{\varepsilon_0} Y(\varepsilon)$.

Let $\bar{x} \in E \setminus T'$. Since $F_{\bar{x}}$ is d -compact, we have, $\text{int}_{C_{\bar{x}}}(F_{\bar{x}}) \neq \emptyset$. Let $\bar{y} \in \text{int}_{C_{\bar{x}}}(F_{\bar{x}})$ and $\varepsilon > 0$ such that $B_{C_{\bar{x}}}(\bar{y}; \varepsilon) \subseteq F_{\bar{x}}$. Thus, $\langle \bar{x}, \bar{y} \rangle \in F(\varepsilon) \subseteq X(\varepsilon)$, and $\bar{x} \in Y(\varepsilon)$.

Thus, by claims 4 and 6, T' is residual in E . \square

Proof of Theorem 1.4. The proof of the Lemma is as in [FS10, §4, Case 2].

Case 1: F is \mathcal{F}_σ in \mathbb{K}^{m+n} . Then, $F = \bigcup_{s>0} F(s)$, for some definable increasing family $(F(s) : s \in \mathbb{K})$ of d -compact sets. By the proof of Lemma 4.1, for each $t \in \mathbb{K}$, $E \setminus T^m(F(s)) \subseteq \bigcup_{\varepsilon>0} Y(t, \varepsilon)$, where $Y(t, \varepsilon)$ is a definable family of nowhere dense subsets of E , which is increasing in t and decreasing in ε . Thus, $T^m(F)$ is residual in E .

Case 2: C is \mathcal{F}_σ in \mathbb{K}^{m+n} . By definition, there exists $F' \subseteq C$, such that F' is a meager \mathcal{F}_σ subset of C , and $F \subseteq F'$; thus, by replacing F with F' , w.l.o.g. we can assume that F is \mathcal{F}_σ in C . Then, since C is an \mathcal{F}_σ set, F is \mathcal{F}_σ also in \mathbb{K}^{m+n} , and we can apply Case 1. \square

5. PROOF OF THEOREM 1.3

Lemma 5.1. *Let $m \in \mathbb{N}$ and $C \subseteq \mathbb{K}^n$ be a definable nonempty set. Assume that, for every $a \in C$, there exists $U \subseteq C$, such that U is a definable neighbourhood (in C) of a which is definably Baire. Then, C is definably Baire.*

Proof. Let $V \subseteq C$ be a definable open nonempty subset of C . Assume, for a contradiction, that V is meager in itself. Let $a \in V$ and let U be a definable neighbourhood (in C) of a which is definably Baire. Let $W := \text{int}_C(U \cap V)$. Notice that W is a nonempty open subset of C . Since V is meager in itself and W is an open subset of V , W is also meager in itself. Since W is open in U , W is meager in U . Since W is a nonempty open subset of U and U is definably Baire, W is not meager in U , absurd. \square

Lemma 5.2. *Let $C \subseteq \mathbb{K}^n$ be definable, closed (in \mathbb{K}^n), and nonempty. If C is not definably Baire, then there exists $E \subseteq C$, such that E is definable, nonempty, d-compact, and meager in itself.*

Proof. By Lemma 5.1, there exists a d-compact B such that $C' := B \cap C$ is not definably Baire; thus, by replacing C with C' , w.l.o.g. we can assume that C is d-compact. Let $U \subseteq C$ be definable, nonempty, and open in C , such that U is meager in itself. Let $E := \text{cl}(U) = \text{cl}_C(U)$. By assumption, U is an open and dense subset of E ; thus, by Lemma 2.6(2) (applied to $A = X = E$), E is meager in itself. \square

5.1. The case $n = 1$.

Definition 5.3 ([For11, §4]). Let $C \subseteq \mathbb{K}^n$ be a definable set. C is at most pseudo-enumerable if there exists a pseudo- \mathbb{N} set \mathcal{N} and a definable surjective function $f : \mathcal{N} \rightarrow C$. C is pseudo-enumerable if it is at most pseudo-enumerable, and it is not closed, discrete, and bounded.

Lemma 5.4. *Let $C \subseteq \mathbb{K}$ be definable, nonempty, and closed. If C is meager in itself, then there exists a pseudo-enumerable set $P \subseteq C$, such that P is dense in C .*

Proof. Assume that C is meager in itself. Let $U := \text{int}_{\mathbb{K}}(C)$ and $E := C \setminus U$. Notice that E is definable, closed, and nowhere dense in \mathbb{K} . If U is nonempty, then, since U is open in C and meager in itself, then, by [For11, Proposition 6.4], there is a pseudo-enumerable set $P_0 \subseteq U$, such that P_0 is dense in U . Let P_1 be the set of endpoints of $\mathbb{K} \setminus E$; notice that P_1 is at most pseudo-enumerable. Since E is nowhere dense in \mathbb{K} , P_1 is dense in E . Define $P := P_0 \cup P_1$. P is pseudo-enumerable (because it is the union of finitely many at most pseudo-enumerable sets) and dense in C . \square

The first step in the proof of Theorem 1.3 is the case when $m = 1$ and C is closed. Thus, we have to prove the following lemma.

Lemma 5.5. *Let $C \subseteq \mathbb{K}$ be definable, nonempty, and closed. Then, C is definably Baire.*

The remainder of this subsection is the proof of the above lemma.

Lemma 5.6. *Let $C \subseteq \mathbb{K}^n$ be definable, nonempty, and meager in itself. Then, C has no isolated points.*

Let $U \subseteq C$ be a nonempty definable open subset of C . Then, U is not pseudo-finite: that is, there is no discrete and d-compact subset D of \mathbb{K} , such that there is a definable surjective function $f : D \rightarrow U$.

Proof. Clear. \square

Definition 5.7. Let $C \subseteq \mathbb{K}^n$ be a nonempty definable set. Let $\mathcal{A} := (A_i : i \in I)$ be a definable family of subsets of C . We say that \mathcal{A} is a **weak basis** for C if:

- (1) for every $i \in I$, $\text{int}_C(A_i) \neq \emptyset$;
- (2) if $U \subseteq C$ is a nonempty open subset of C , then there exists $i \in I$ such that $A_i \subseteq U$.

We say that \mathcal{A} is pseudo-enumerable if its index set I is pseudo-enumerable.

Lemma 5.8. *Let $C \subseteq \mathbb{K}$ be definable, nonempty, and closed. Assume that C is meager in itself. Then, C has a pseudo-enumerable weak basis of d -compact sets.*

Proof. By Lemma 5.4, there exists $P \subseteq C$ which is pseudo-enumerable and dense in C ; thus, we can write $P := \{p_i : i \in \mathcal{N}\}$, for some pseudo- \mathbb{N} set $\mathcal{N} \subset \mathbb{K}_{\geq 1}$ and some definable function $i \mapsto p_i$. For every $i \in \mathcal{N}$, define $A_i := \overline{B}_C(p_i; 1/i)$. Let $\mathcal{A} := (A_i : i \in \mathcal{N})$. The lemma follows from the following claim.

Claim 7. Let $U \subseteq C$ be a definable open nonempty subset of C . Then, there exists $i \in \mathcal{N}$ such that $C_i \subseteq U$.

Choose $i_0 \in \mathcal{N}$ such that $q := p_{i_0} \in U$. Choose $r > 0$ such that $B_C(q; 3r) \subseteq U$.

Claim 8. There exists $i \in \mathcal{N}$ such that $p_i \in B_C(q; r)$ and $i > 1/r$

We know that C has no isolated points. Let $F := \{q\} \cup \{j \in \mathcal{N} : j \leq r\}$; notice that F is d -compact and discrete; thus, $V := B_C(q; r) \setminus F$ is open in C , and, since C is not pseudo-finite, V is nonempty. Thus, by density, there exists $i \in \mathcal{N}$ such that $p_i \in V$, proving the claim.

Then,

$$\overline{B}_C(p_i; 1/i) \subseteq B_C(p_i; 2r) \subseteq B_C(q; 3r) \subseteq U. \quad \square$$

We now turn to the proof of Lemma 5.5 proper. Assume, for a contradiction, that $C \subseteq \mathbb{K}$ is nonempty, definable, closed, but it is not definably Baire. Let $E \subseteq C$ be as in Lemma 5.2. By replacing C with E , w.l.o.g. we can also assume that C is also d -compact and meager in itself.

By Lemma 5.8, there exists a pseudo- \mathbb{N} set $\mathcal{N} \subset \mathbb{K}_{\geq 1}$ and a definable family $\mathcal{A} := (A_i : i \in \mathcal{N})$, such that \mathcal{A} is a weak basis for C of d -compact sets. Moreover, since C is meager in itself, there exists a definable decreasing family $(U_j : j \in \mathcal{N})$, such that each U_j is a dense open subset of C , and $\bigcap_j U_j = \emptyset$.

Since each U_d is open and dense, and U_d has nonempty interiors, $A_d \cap U_d$ has nonempty interior. Since \mathcal{A} is a weak basis, there exists $e \in \mathcal{N}$ such that $A_e \subseteq U_d \cap A_d$; since moreover C has no isolated points, we can find e as above such that $e \geq d$. For every $d \in \mathcal{N}$, let $g(d)$ be the minimum element of \mathcal{N} , such that $g(d) \geq d$ and $A_{g(d)} \subseteq U_d \cap A_d$. Notice that $\mathcal{A}' := (A_{g(d)} : d \in \mathcal{N})$ is also a weak basis; thus, by replacing \mathcal{A} with \mathcal{A}' (and each A_d with $A_{g(d)}$), we can assume that $A_d \subseteq U_d$ for every $d \in \mathcal{N}$.

For every $d \in \mathcal{N}$, notice that $E_d := \bigcup_{e \leq d, e \in \mathcal{N}} \text{bd}(A_e)$ is a pseudo-finite union of closed nowhere dense subsets of C ; thus, $A_d \setminus E_d$ is non-empty and d -compact, and therefore we can find $f(d) \in A_d$; moreover, there is a definable function $f : \mathcal{N} \rightarrow C$ such that $f(d) \in A_d$ for every $d \in \mathcal{N}$ (e.g., we can take $f(d) := \text{lex min } A_d$).

For every $a \in \mathcal{N}$, define

$$T(a) := \{d \in \mathcal{N} : d \leq a \text{ \& } f(d) \in A_d\}.$$

Notice that each $T(a)$ is a pseudo-finite set, and that $a = \max(T(a))$.

We now define the following partial order on \mathcal{N} : $a \trianglelefteq b$ if $T(a)$ is an initial segment of $T(b)$, that is:

$$\forall c \leq a \quad c \in T(a) \leftrightarrow c \in T(b).$$

Lemma 5.9. *$(\mathcal{N}, \trianglelefteq)$ is a partially ordered set, which is a forest and satisfies condition (*) in Lemma 1.9.*

Proof. $a \trianglelefteq a$ by definition.

Notice that $a \trianglelefteq b$ implies $a \leq b$, by definition. Moreover, if $a \trianglelefteq b$, then $a \in T(b)$, since $a \in T(a)$.

Claim 9. If $a \trianglelefteq b$ and $b \trianglelefteq c$, then $a \trianglelefteq c$.

In fact, let $d \leq a$. Then, $d \in T(a)$ iff $d \in T(b)$ iff $d \in T(c)$.

Thus, \trianglelefteq is a partial order.

The fact that $\langle \mathcal{N}, \trianglelefteq \rangle$ is a forest is clear.

We now prove that \trianglelefteq satisfies (*). Let $a \in \mathcal{N}$ and $b \leq a$. For every $a \in \mathcal{N}$, let

$$J(a) := \bigcap_{b \in T(a)} \text{int } A_b \setminus \bigcap_{b \leq a \text{ \& } b \notin T(a)} A_b.$$

Since each A_b is closed, and the set $\{b \in \mathcal{N} : b \leq a \text{ \& } b \notin T(a)\}$ is pseudo-finite, we have that $J(a)$ is an open set. We claim that $J(a)$ is nonempty. It suffices to prove the following claim.

Claim 10. $f(a) \in J(a)$.

In fact, by our choice of f , we have that, for every $b \leq a$, $b \in T(a)$ iff $f(a) \in A_b$ iff $f(a) \in \text{int } A_b$; the claim is then obvious from the definition of $J(a)$.

Since C has no isolated points, the set $\{f(b) : b \leq a\}$ is pseudo-finite, and $J(a)$ is open and nonempty, the set $J'(a) := J(a) \setminus \{f(b) : b \leq a\}$ is also open and nonempty; thus, there exists $b \in \mathcal{N}$ such that $A_b \subseteq J'(a)$. The lemma then follows from the following claim.

Claim 11. $a \triangleleft b$.

The fact that $b > a$ is clear from the fact that $f(c) \notin A_b$ for every $c \leq a$.

Let $c \leq a$. We have to show that $c \in T(b)$ iff $c \in T(a)$. If $c \in T(a)$, then $J(a) \subseteq \text{int } A_c$, therefore $f(b) \in \text{int}(A_c)$, and thus $c \in T(b)$. Conversely, if $c \in T(b)$, then $f(b) \in \text{int } A_c \cap J(a)$; thus, $J(a) \cap \text{int } A_c \neq \emptyset$, and therefore, by definition of $J(a)$, we have $c \in T(a)$. □

We now continue the proof of Lemma 5.5. By Lemma 5.9, we can apply Lemma 1.9 to the partial order \trianglelefteq : denote by E_0 the leftmost branch of \trianglelefteq inside \mathcal{N} .

Therefore, by Fact 2.3

$$\emptyset \neq \bigcap_{d \in E_0} A_d \subseteq \bigcap_{d \in E_0} U_d = \bigcap_{d \in \mathcal{N}} U_d = \emptyset,$$

absurd. □

5.2. The inductive step.

Lemma 5.10. *Let $m \in \mathbb{N}$. Let $C \subseteq \mathbb{K}^m$ be nonempty and d -compact. Then, C is definably Baire.*

Proof. We will prove the lemma by induction on m .

Let $1 \leq m \in \mathbb{N}$. We denote by $(5.10)_m$ the instantiation of Lemma 5.10 at m . Notice that $(5.10)_1$ follows from Lemma 5.5. Thus, we assume that we have already proven $(5.10)_m$ and $(5.10)_1$; we need to prove $(5.10)_{m+1}$.

Let $C \subseteq \mathbb{K}^{m+1}$ be d -compact and nonempty. We have to show that C is definably Baire. Assume not. Let F be a definable nonempty open subset of C , such that F is meager in C . Define $\pi := \Pi_m^{m+1}$, and $E := \pi(C)$. By Theorem 1.4, the set $S := \{\bar{x} \in E : F_{\bar{x}} \text{ is not meager in } C_{\bar{x}}\}$ is meager in E . Since F is open in C , $F_{\bar{x}}$ is open in $C_{\bar{x}}$ for every $\bar{x} \in E$; thus, by $(5.10)_1$, $S = \pi(F)$. Notice that E is also d -compact and nonempty; thus, by $(5.11)_m$, E is definably Baire. Since moreover S is open in E , the fact that S is meager in E imply that S is empty, contradicting the fact that F is nonempty. □

Lemma 5.11. *Let $m \in \mathbb{N}$. Let $C \subseteq \mathbb{K}^m$ be closed, nonempty, and definable. Then, C is definably Baire.*

Proof. We want to apply Lemma 5.1; thus, given $a \in C$; it suffices to find $A \subseteq C$, such that A is a definable neighbourhood of a inside C , and A is definably Baire. Fix $r > 0$ (e.g., $r = 1$); Let $A := \overline{B}_C(a; r)$. It is clear that A is a definable neighbourhood of a inside C . Moreover, A is d-compact; thus, by Lemma 5.10, A is definably Baire. \square

Proof of Theorem 1.3. Let $Y := \text{cl}(C)$. By Lemma 5.11, Y is definably Baire. Since C is dense in Y , the conclusion follows from Corollary 2.7. \square

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