NOTES ON HRUSHOVSKI'S ARTICLE "STABLE GROUP THEORY AND APPROXIMATE SUBGROUPS"
VERSION 0.4

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Abstract. Relations between forking and S1 ideals, especially in stable structures.

Contents

1. Introduction

These are the notes for a talk, which is part of a seminar on Hrushovski's article [Hru10] given in Münster. They are intended as a supplement to [vdD09, §1].

$\mathcal{U}$ is a monster model of a first order theory, in a language $\mathcal{L}$. A is a "small" (i.e., small with respect to the monstrosity of $\mathcal{U}$) subset of $\mathcal{U}$, M is a small elementary substructure of $\mathbb{C}$. By "$X(z)$ is a definable family over A" we will mean that there exists an $\mathcal{L}(A)$-formula $\phi(x, y)$, such that, for every tuple b in $\mathcal{U}$, $X(b) = \{c \in \mathcal{U}^n : \mathcal{U} \models \phi(c, b)\}$. By "sequence" we will mean "sequence of tuples in $\mathcal{U}$".

2. Producing indiscernible sequences

Definition 2.1. Let $\bar{c} = \langle c_j : j < \gamma\rangle$ be a sequence. The Ehrenfeucht-Mostowski-type $\mathrm{EM}(\bar{c}/A)$ of $\bar{c}$ over A is the set of $\mathcal{L}(A)$-formulae $\phi(x_1, \dots, x_n)$, such that, for every $j_1 < \ldots < j_n < \gamma$, $\mathcal{U} \models \phi(c_{j_1}, \dots, c_{j_n})$.

Theorem 2.2 (Standard lemma). *Let γ be an infinite ordinal and $\bar{c} = \langle c_j : j < \gamma\rangle$ be a sequence. Then, for every ordinal β, there exists an A-indiscernible sequence $\bar{b} = \langle b_i : i < \beta\rangle$, such that* $\mathrm{EM}(\bar{b}/A) \supseteq \mathrm{EM}(\bar{c}/A)$.

Proof. Ramsey theorem + compactness: see [TZ10, Lemma 15.3]. ☐

Theorem 2.3 (Shelah). *There exists a cardinal κ (depending only on $|A|$ and on $|\mathcal{L}|$) such that: for every sequence $\bar{b} = \langle b_i : i < \kappa\rangle$ there exists an A-indiscernible sequence $\bar{c} = \langle c_j : j < \omega\rangle$, such that for every n there exist $i_0 < \cdots < i_n < \kappa$ with* $\mathrm{tp}(c_0, \dots, c_n/A) = \mathrm{tp}(b_{i_1}, \dots, b_{i_n}/A)$.

Proof. Erdös-Rado theorem + compactness: see [TZ10, Lemma 28.11]. ☐

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3. Forking and S1 ideals

I recall the definitions of S1 ideals and of the forking ideal.

Definition 3.1. Fix a set Z which is $\bigvee$-definable over A. Let $\operatorname{Def}(Z)$ be the collection of definable subsets of Z. By "ideal" I will mean an ideal in the Boolean ring $\operatorname{Def}(Z)$.

A set $X \in \operatorname{Def}(Z)$ divides over A if $X = X(b)$ (for some A-definable family $X(z)$) and there exists an A-indiscernible sequence $\langle b_i : i < \omega \rangle$, such that $b_0 = b$ and $\bigcap_i X(b_i) = \emptyset$. If we can find an A-indiscernible sequence $\langle b_i : i < \omega \rangle$, such that $b_0 = b$ and the sets $X(b_i)$ are pairwise disjoint, then X 2-divides over A.

The forking ideal (in Z over A) is the ideal in $\operatorname{Def}(Z)$ generated by the family of sets in $\operatorname{Def}(Z)$ which divide over A.

An ideal I is S1 over A if it is A-invariant and, for every A-indiscernible sequence $\langle b_i : i < \omega \rangle$ and every A-definable family $X(z)$, if $X(b_i) \cap X(b_{i'}) = \emptyset$ for every $i < i' < \omega$, then $X(b_i) \in I$ for some (equivalently, for every) $i < \omega$.

Lemma 3.2. *If I is an S1 ideal over A, then I contains the forking ideal.*

Proof. See [Hru10, Lemma 2.9] or [vdD09, Lemma 1.18]. ☐

We will now see an application of §2.

Lemma 3.3. *Let κ be as in Theorem 2.3 and I be an A-invariant ideal. T.f.a.e.:*
(1) *I is S1 over A;*
(2) *for every A-definable family $X(z)$ and every sequence $\bar{b} := \langle b_i : i < \kappa \rangle$, if $X(b_i) \cap X(b_{i'}) \in I$ for every $i < i' < \kappa$, then $X(b_i) \in I$ for some $i < \kappa$.*

Notice that in (2) we did not assume that $\bar{b}$ is indiscernible over A.

Proof. ($2 \Rightarrow 1$) Let $\bar{c} = \langle c_j : j < \omega \rangle$ be an A-indiscernible sequence and $X(z)$ be an A-definable family. Assume that $X(c_j) \cap X(c_{j'}) \in I$ for every $j < j' < \omega$. We want to prove that $X(c_0) \in I$. Assume, for contradiction, that $X(c_0) \notin I$. By the Standard Lemma, there exists and A-indiscernible sequence $\bar{b} = \langle b_i : i < \kappa \rangle$, such that $\operatorname{EM}(\bar{b}/A) \supseteq \operatorname{EM}(\bar{c}/A)$. Since $\bar{c}$ is A-indiscernible, this implies that, for every i and j, $c_j \equiv_A b_i$, and therefore c_0 and b_0 are conjugate over A. Since $X(c_0) \notin I$ and I is A-invariant, $X(b_0) \notin I$. Similarly, one proves that $X(b_i) \cap X(b_{i'}) \in I$ for every $i < i' < \kappa$. However, the above contradicts (2).

($1 \Rightarrow 2$) Let $X(z)$ be an A-definable family and $\bar{b} := \langle b_i : i < \kappa \rangle$ be a sequence, such that $X(b_i) \cap X(b_{i'}) \in I$ for every $i < i' < \kappa$. We want to prove that $X(b_i) \in I$ for some $i < \kappa$. Assume, for contradiction, that $X(b_i) \notin I$ for every $i < \kappa$. By Theorem 2.3, there exists an A-indiscernible sequence $\bar{c} = \langle c_j : j < \omega \rangle$, such that, for every $n \in \mathbb{N}$, there exist $i_0 < \cdots < i_n < \kappa$ with $\operatorname{tp}(c_0, \dots, c_n/A) = \operatorname{tp}(b_{i_1}, \dots, b_{i_n}/A)$. Let $j < \omega$; choose $i < \kappa$ such that $c_j \equiv_A b_i$. Since $X(b_i) \notin I$ and I is A-invariant, we have $X(c_i) \notin I$. Similarly, given $i < i' < \omega$, one proves that $X(c_i) \cap X(c_{i'}) \in I$. However, the above contradicts the assumption that I is an S1 ideal. ☐

In (2) in the above lemma we cannot take a sequence $\bar{b}$ of length ω instead of κ; before giving a counterexample we need to produce some S1 ideals.

Example 3.4. Assume that $\mathcal{U}$ is strongly minimal (i.e., every definable subset of $\mathcal{U}$ is either finite or cofinite: for instance, any algebraically closed field is strongly minimal). Let $Z = \mathcal{U}$ and I be the forking ideal over A (in $\operatorname{Def}(\mathcal{U})$). Then, I is an S1 ideal over A.

Later we will give a more general version of the above example.

Proof. Let $X \in \operatorname{Def}(\mathcal{U})$.

Exercise 3.5. X forks over A iff X is finite and $X \setminus \operatorname{acl}(A) \neq \emptyset$

Let $X(z)$ be a definable family over A and $\bar{b} = \langle b_i : i < \omega \rangle$ be an A-indiscernible sequence such that $X(b_i) \cap X(b_j) \in I$ for every $i < j < \omega$; we want to prove that $X(b_i) \in I$ (for every/some $i < \omega$). Let $i < j < \omega$. By the exercise, $X(b_i) \cap X(b_j)$ is finite and not contained in $\operatorname{acl}(A)$. Hence, $X(b_i) \not\subseteq \operatorname{acl}(A)$ for every $i < \omega$. If, for contradiction, $X(b_i) \notin I$, then $X(b_i)$ (and therefore also $X(b_j)$) is infinite, and thus cofinite; therefore, $X(b_i) \cap X(b_j)$ is also infinite, and thus not in I, absurd. □

Here is the promised counterexample.

Example 3.6. Let $\mathcal{U}$ be strongly minimal and A be infinite. Let I be the forking ideal over A in $\operatorname{Def}(\mathcal{U})$. Let $\langle a_i : i < \omega \rangle$ be an infinite tuple of distinct elements in A. Let $X(z)$ be the definable family "$X(z) = \{z\}$". Then, $X(a_i) \cap X(a_j) = \emptyset \in I$ for every $i < j < \omega$, but $X(a_i) \notin I$ for every $i < \omega$.

Exercise 3.7. Let $\mathcal{U}$ be an o-minimal expansion of a field (i.e., $\mathcal{U}$ is an ordered structure, such that every unary definable subset of $\mathcal{U}$ is a finite union of points and intervals with endpoints in $\mathcal{U} \cup \{\pm\infty\}$; e.g., the real field $\langle \mathbb{R}, +, \cdot, < \rangle$ is o-minimal). Let $a < b \in \mathcal{U} \cup \{\pm\infty\}$. Then, the set (a, b) forks over M iff a and b are not in $M \cup \{\pm, \infty\}$ and $(a, b) \cap M = \emptyset$.

Finally, here is the generalization of Example 3.4

Lemma 3.8. *Let $\mathcal{U}$ be stable and Z be any set $\bigvee$-definable over A. Let I be the forking ideal (in $\operatorname{Def}(Z)$ over A). Then, I is an S1 ideal over A.*

Proof. We apply Lemma 3.3. Let κ be large enough. Let $\bar{b} = \langle b_i : i < \kappa \rangle$ be a sequence and $X(z)$ be an A-definable family. Assume, for contradiction, that $X(b_i) \notin I$ and $X(b_i) \cap X(b_{i'} \in I$ for every $i < i' < \kappa$. For every $i < \kappa$, choose a global type $p_i \in S(\mathcal{U})$, such that p_i does not fork over A and $p_i(x)$ extends "$x \in X(b_i)$" (we can do that because $X(b_i) \notin I$). Since $X(b_i) \cap X(b_{i'})$ forks over A for every $i \neq i'$, the p_i are all distinct. Hence, there are at least κ global types which do not fork over A. However, this is impossible, since in a stable theory T there are at most $2^{|T|+A}$ global types which do not fork over A (see [TZ10, Theorem 36.5]). □

The next proposition will be proved later.

Proposition 3.9. *The above lemma (and its proof) generalizes to NIP theories.*

Some more example of S1 ideals.

Definition 3.10. Let X be a subset of Z. Define $I_X = \operatorname{Def}(X)$ to be the family of definable subsets of Z which are contained in X.

Exercise 3.11. Let X be a subset of Z. Then, I_X is an ideal (in $\operatorname{Def}(Z)$). If X is A-invariant (as a set), then I_X is also A-invariant. If X is A-definable and cofinite, then I_X is an S1 ideal.

Exercise 3.12. Let $\langle I_j : j \in J \rangle$ be a family of ideals. Then, $\bigcap_j I_j$ is an ideal. If moreover each I_j is an S1 ideal, then also $\bigcap_j I_j$ is an S1 ideal.

Exercise 3.13. The family of definable subset of Z which do not intersect $\operatorname{acl}(A)$ is an S1 ideal.

Finally, a lemma that will be used in the following talks.

Lemma 3.14. *Let $X(z)$ be an A-definable family and be an infinite ordinal, such that $\lambda > |S(A)|$. Let $\bar{b} = \langle b_i : i < \lambda \rangle$ be a sequence. If the $X(b_i)$ are pairwise disjoint, then, for some $i < \lambda$, $X(b_i)$ 2-divides over A (and, in particular, $X(b_i)$ forks over A).*

Notice that a sequence of length ω is not enough for the conclusion: see Example 3.4.

Proof. Since $\lambda > |S(A)|$, there exists an infinite subsequence $\bar{c}' := \langle c'_i : i' < \omega \rangle$ of $\bar{c}$, such that, for every $i < l < \omega$, c'_i and c'_l have the same type over A. By the Standard Lemma, there exists an A-indiscernible sequence $\bar{b} = \langle b_j : j < \omega \rangle$, such that $\mathrm{EM}(\bar{b}/A) \supseteq \mathrm{EM}(\bar{c}'/A)$. Hence, $X(b_j) \cap X(b_{j'}) = \emptyset$ for every $j < j' < \omega$. Therefore, by definition, $X(b_0)$ 2-divides over A. Since all the c'_i have the same type over A, we have that c'_i and b_0 are conjugate over A; thus, $X(b'_0)$ 2-divides over A, and we are done. ☐

3.1. **Forking in NIP theories.** Here we will examine forking in NIP theories and prove Proposition 3.9. Some of the results and ideas for this section are taken from [Pil08]; for an introduction to NIP theories, see [Adl08].

Definition 3.15. $\mathbb{C}$ has NIP (Not Independence Property) if, for every A, every A-definable family $X(z)$ and every A-indiscernible sequence $\langle b_i : i < \omega \rangle$, and every $c \in \mathbb{C}$, the set of indices such that $c \in X(b_i)$ is either finite or cofinite (see [Adl08, Proposition 4] or [Poi85, Theorem 12.17]). Otherwise, we say that $\mathbb{C}$ has IP.

Examples of NIP theories are stable theories and o-minimal theories.

Definition 3.16. Given b and c, we write $\mathrm{nc}_A(b, c)$ if b and c start an infinite sequence of indiscernibles over A.

Lemma 3.17 ([TZ10, Lemma 29.2])**.** *Assume that $b \equiv_M b'$. Then, there exists c such that $\mathrm{nc}_M(a, c)$ and $\mathrm{nc}_M(c, b)$.*

Remember that a global type forks over A iff it divides over A (see [vdD09, Lemma 1.7]).

Lemma 3.18. *Let $p \in S(\mathbb{C})$. If p is A-invariant, then it does not fork over A. Conversely, if $\mathbb{C}$ has NIP and p does not fork over M, then p is M-invariant.*

Proof. Assume that p is A-invariant. Assume, for contradiction, that p divides over A; let $X \in p$ be a definable set such that X divides over A. We can write $X = X(b)$ for some A-definable family $X(z)$. Since $X(b)$ divides over M, there exists an A-indiscernible sequence $\langle b_i : i < \omega \rangle$, such that $b_0 = b$ and $\bigcap_i X(b_i) = \emptyset$. However, since p is A-invariant, $X_i \in p$ for every $i \in \mathbb{N}$, absurd.

Assume now that $\mathbb{C}$ has NIP and p does not fork over M. Let $X(z)$ be an M-definable family, and assume that $X(b) \in p$. Let $b' \equiv_M b$; we want to show that $X(b') \in p$. Assume, for contradiction, that $X(b') \notin p$; thus, $X(b) \setminus X(b') \in p$. By Lemma 3.17, we can assume that there exists an M-indiscernible sequence $\langle b_i : i < \omega \rangle$, such that $b_0 = b$ and $b_1 = b'$. Notice that the sequence $\langle b_0 b_1, b_2 b_3, b_4 b_5, \dots \rangle$ is also M-invariant and $X(b_0) \setminus X(b_1)$ is in p. Since p does not divide over M, also $X(b_0) \setminus X(b_1)$ does not divide over M; therefore,

$$\bigcap_{i \text{ even}} X(b_i) \setminus X(b_{i+1}) \neq \emptyset.$$

However, the above contradicts our definition of NIP. ☐

Some of the bounds in the following lemmas are not optimal; however, they suffice for our task (since *any* bound would suffice).

Lemma 3.19. *There are at most $2^{2^{|\mathcal{L}|+|A|}}$ global types which are invariant over A.*

Proof. Let $p(x)$ be a global type which is A-invariant. Let $\phi(x, y)$ be an $\mathcal{L}$-formula without parameters. Since p is A-invariant, whether or not $\phi(x, b) \in p$ depends only on $\mathrm{tp}(b/A)$, and the type p is determined by which $\phi(x, b)$ are in p, as $\phi(x, y)$ and

b vary. Since there are at most $2^{|L|+|A|}$-many types over A, and at most $|\mathcal{L}|$-many $\mathcal{L}$-formulae without parameters, we are done. ☐

The bound in the above lemma is optimal if $\mathbb{C}$ has IP (see [Poi85, Theorem 12.28]), but can be lowered to $2^{|\mathcal{L}|+|A|}$ if $\mathbb{C}$ has NIP.

Corollary 3.20. *Assume that $\mathbb{C}$ has NIP. Then, there are at most $2^{2^{|\mathcal{L}|+|A|}}$ global types which do not fork over A.*

Proof. Let $M \supseteq A$ such that $|M| \leq |A| + |\mathcal{L}|$. By lemmas 3.18 and 3.19, there are at most $2^{2^{|\mathcal{L}|+|M|}}$ global types which do not fork over M. If p is a global type which does not fork over A, then it does not fork over M either, proving the conclusion. ☐

The bound in the above Corollary can be lowered to $2^{|\mathcal{L}|+|A|}$.

Proof of Proposition 3.9. Reason as in Lemma 3.8, using the above corollary instead of [TZ10, Theorem 36.5]. ☐

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