

Introduction

Joint work with G. Boxall.

Lovely pairs of:

Stable and simple structures

Poizat, Ben-Yaacov, Pillay, Vassiliev, ...

O-minimal and geometric structures

Robinson, Macintyre, van den Dries, Berenstein, Boxall, ...

We propose a unified approach.

NOTES

We will not give precise attribution to various results, because many of them have been re-proved many times in different situations.

Contents

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- 2 Independence relations
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Elementary pairs

Definition

- An **elementary pair** is a pair of structures $A < B$.
- The language of pairs \mathcal{L}^2 is the language \mathcal{L} of B , augmented by a new unary predicate P for the elements of A .
- The theory of elementary pairs (of models of an \mathcal{L} -theory T) is the \mathcal{L}^2 -theory whose models are all the elementary pairs (B, A) , with $A < B \models T$.

NOTES

In the co-density axiom, $\phi(x, \bar{y})$ defines a one-to-finite application $f : B^n \rightarrow B$; we require that $U \not\subseteq f(A^n)$.

Dense pairs of geometric structures

Definition

\mathcal{M} monster model. \mathcal{M} is a **geometric structure** if acl has the exchange property and \mathcal{M} eliminates the quantifier \exists^∞ .

Let T be the theory of a geometric structure.

Definition

(B, A) is a dense pair of geometric structures (for T) if:

Elementary pair $A < B \models T$;

Density Every infinite T -definable subset of B intersects A ;

Co-density For every $\mathcal{L}(B)$ -formula $\phi(x, \bar{y})$, if $\phi(B, \bar{b})$ is finite for every $\bar{b} \in B^n$, then there exists $u \in U$ such that, for every $\bar{a} \in A^n$, $B \models \neg\phi(u, \bar{a})$.

Example

Let (B, A) be an elementary pair.

- if B is a geometric expansion of a field, then (B, A) is a dense pair iff it satisfies the density axiom.
- If B is an o-minimal expansion of a group, then (B, A) is a dense pair iff A is topologically dense in B .
- If B is strongly minimal, then (B, A) is a dense pair iff it satisfies the co-density axiom.

NOTES

The second result is by van den Dries, and is the reason of the names “density axiom” and “dense pairs”.

Axioms of independence relations

\mathcal{M} monster model, \perp independence relation on \mathcal{M} (H. Adler).

Invariance $(\forall \sigma \in \text{aut}(\mathcal{M})) A \perp_B C \text{ iff } \sigma A \perp_{\sigma B} \sigma C.$

Transitivity Assuming $B \subseteq C \subseteq D$, $A \perp_B D$ iff $A \perp_B C$ and $A \perp_C D.$

Normality $A \perp_C B$ iff $AC \perp_C B$

Extension There is some $A' \equiv_B A$ such $A' \perp_B C.$

Finite Character $A \perp_B C$ iff $A_0 \perp_B C$ for all finite $A_0 \subseteq A.$

Local Character For every A and B there exists $B_0 \subseteq B$ such that $|B_0| \leq |T| + |A|$ and $A \perp_{B_0} B.$

Symmetry $A \perp_B C$ iff $C \perp_B A.$

Note. We do not assume:

Strictness $a \perp_B a \Leftrightarrow a \in \text{acl}(B);$

Boundedness every type has only boundedly many nonforking extensions.

Examples of strict independence relations

\mathcal{M} monster model.

- \mathcal{M} ω -stable/stable/simple, \perp = Shelah's forking.
- \mathcal{M} rosy, \perp = thorn forking.
- \mathcal{M} pregeometric (i.e., acl has the Exchange Property), \perp = algebraic independence; e.g., $\mathcal{M} \succ \mathbb{Q}_p.$

NOTES

1. Let \mathcal{M} be pregeometric. $A \perp_B C$ iff every subset of A which is algebraically independent over B remains algebraically independent over $BC.$
2. Later: nonstrict independence relations.

Lovely pairs

Let T be a complete theory, \perp be an independence relation, $\kappa := |T|^+$.

Definition

(B, A) is a \perp -lovely pair if:

Elementary pair $A < B \models T$;

Density For every $C \subset B$ with $|C| < \kappa$ and $p(x)$ complete \mathcal{L} -type over C , there exists $c \models p$ such that $c \perp_C A$;

Co-density For every $C \subset B$ with $|C| < \kappa$ and $p(x)$ complete \mathcal{L} -type over C , if p does not fork over $A \cap C$, then p is realized in A .

The above definition was originally given by BPV in the case when T is simple and \perp is Shelah's forking.

NOTES

1. If T is stable and \perp is Shelah's forking, then a \perp -lovely pair is (essentially) a "belle paire" in the sense of Poizat.
2. Ben-Yaacov considered lovely pairs for different axioms for \perp .

Remark

If (B, A) is an \perp -lovely pair, then A and B are κ -saturated (as \mathcal{L} -structures).

Remark

Assume that B is geometric, \perp is given by geometric independence, and (B, A) is κ -saturated (as an \mathcal{L}^2 -structure). Then, (B, A) is a dense pair of geometric structures iff (B, A) is a \perp -lovely pair.

The same (complete) theory can have many independence relations; each independence relation gives a different class of lovely pairs.

Completeness

From now on: (B, A) is a \perp -lovely pair. Let $\bar{b} \in B^n$.

Definition

- 1 The P -type of \bar{b} is the information of which b_i are in A .
- 2 \bar{b} is P -independent iff $\bar{b} \perp_{A \cap \bar{b}} A$.

Main Theorem

Let (B, A) and (B', A') be \perp -lovely pairs. Let $\bar{b} \in B^n$ and $\bar{b}' \in B'^n$. Assume that \bar{b} and \bar{b}' are both P -independent and have the same P -type and the same \mathcal{L} -type. Then, \bar{b} and \bar{b}' have the same \mathcal{L}^2 -type.

Corollary

Let (B, A) and (B', A') be \perp -lovely pairs; then, they are elementarily equivalent.

NOTES

Similar results hold for infinite tuples of cardinality $< \kappa$.

Loveliness is first-order

Definition

“Loveliness is first-order” means that there is a theory T^{lovely} such that every sufficiently saturated model of T^{lovely} is a \perp -lovely pair.

Example

- ① If \perp is given by geometric independence, then loveliness is first-order iff T eliminates the quantifier \exists^∞ .
- ② (Poizat) If T is stable and \perp is Shelah’s forking, then loveliness is first-order iff T has non-fcp, i.e. iff T^{eq} eliminates the quantifier \exists^∞ .

From now on, we will assume that loveliness is first-order, and T^{lovely} is the corresponding theory.

NOTES

1. The axioms we gave earlier for dense pairs show that, for geometric theories, loveliness is first-order.
2. When T is simple, the condition when loveliness is first-order is quite complicate.

Inheritance of stability properties

Theorem

*Assume that T is stable/simple/superstable/supersimple/ ω -stable/NIP.
Then T^{lovely} also is.*

The above theorem was already known in special cases; the general proof is often an adaption of the proof in the special cases.

Lemma

Assume that T is stable. Then T^{lovely} also is.

Proof.

The proof is by a type-counting argument.

- T is stable iff it is λ -stable for some cardinal λ .
- Choose λ such that $\lambda^\kappa = \lambda$. Let $(B, A) \models T^{\text{lovely}}$ with $|B| = \lambda$ and $(\mathcal{M}, P(\mathcal{M}))$ be a monster model of T^{lovely} .
Note. $|S_\kappa^1(B)| = \lambda$; we must prove that $|S_1^2(B)| = \lambda$.
- Let $q \in S_1^2(B)$; choose $c \in \mathcal{M}$ satisfying q .
- Local Character: let $\bar{p} \subseteq P(\mathcal{M})$ such that $c \perp_{B\bar{p}} P(\mathcal{M})$ and $|\bar{p}| < \kappa$.
- By the Main Theorem, $\text{tp}^2(c/B)$ is determined by $\text{tp}^1(c\bar{p}/B)$ plus the P -type of c ; since $|\bar{p}| < \kappa$, we have $|S_1^2(B)| \leq |S_\kappa^1(B)| = \lambda$. □

- The proofs of the superstable/ ω -stable claims are minor variations of the above proof.
- The proof of the NIP claim is based on counting coheirs (Boxall).
- The proof of the simple/supersimple claims is based on a proof by BPV for the case when \perp is Shelah's forking.

NOTES

We use the superscript 1 for \mathcal{L} -notions, and the superscript 2 for \mathcal{L}^2 -notions.

Superior independence relations

Definition

\perp is **superior** if the relationship between types “ p is a forking extension of q ” is a **well-founded** partial order. If \perp is superior, U^\perp is the corresponding foundation rank of types.

Example

Let \mathcal{M} be stable/simple, and \perp be Shelah's forking. \perp is superior iff T is superstable/supersimple; in this case, U^\perp is Lascar's U -rank.

Similar result holds for T rosy.

Example

If \mathcal{M} is geometric and \perp is given by algebraic independence, then \perp is superior and $U^\perp(\mathcal{M}) = 1$.

Coarsening

Assume that \perp is superior; let $U := U^\perp$. Let λ be the unique power of ω such that:

- $U^\perp(p) \geq \lambda$ for some (finitary \mathcal{L} -)type p ;
- for every type q there exists $n \in \mathbb{N}$ such that $U^\perp(q) = n \cdot \lambda + o(\lambda)$.

Define $\bar{a} \perp_C^\circ D$ if $U(\bar{a}/C) = U(\bar{a}/CD) + o(\lambda)$.

Lemma

- \perp° is a superior independence relation;
- \perp refines \perp° ;
- $U^\circ(q)$ is finite for every type q .

\perp° is not strict in general.

Independence relations on lovely pairs

Assume that loveliness is first order. Let $(\mathcal{M}, P(\mathcal{M}))$ be a monster model of T^{lovely} . Define \perp_P as $C \perp_{PD} E$ iff $C \perp_{P(\mathcal{M})D} E$.

Lemma

- \perp_P is an independence relation on $(\mathcal{M}, P(\mathcal{M}))$.
- Assume that \perp is superior. Then, \perp_P is also superior; moreover, for every partial \mathcal{L} -type q , $U^\perp(q) = U^{\perp_P}(q)$.

\perp_P is never strict.

NOTES

1. $\alpha = o(\lambda)$ means $\alpha < \lambda$ (small-o notation).
2. $\perp^\circ = \perp$ if $U(\mathcal{M})$ is finite (i.e., $\lambda = 1$).

Open problems

Assume that loveliness is first-order.

Conjecture

If T is (super)rosy, then T^{lovely} is too.

Conjecture

\perp_P -loveliness is first-order.

Open problem

Give some form of elimination of imaginaries for lovely pairs.