Lovely pairs for independence relations

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Introduction

Joint work with G. Boxall.
Lovely pairs of:

Stable and simple structures
Poizat, Ben-Yaacov, Pillay, Vassiliev, ...

O-minimal and geometric structures
Robinson, Macintyre, van den Dries, Berenstein, Boxall, ...

We propose a unified approach.

Notes

We will not give precise attribution to various results, because many of them have been re-proved many times in different situations.

Contents

Elementary pairs

Definition

- An elementary pair is a pair of structures $A \prec B$.
- The language of pairs $\mathcal{L}^2$ is the language $\mathcal{L}$ of B, augmented by a new unary predicate P for the elements of A.
- The theory of elementary pairs (of models of an $\mathcal{L}$-theory T) is the $\mathcal{L}^2$-theory whose models are all the elementary pairs (B, A), with $A \prec B \models T$.

Notes

In the co-density axiom, $\phi(x, \bar{y})$ defines a one-to-finite application $f : B^n \to B$; we require that $U \not\subseteq f(A^n)$.

Dense pairs of geometric structures

Definition

$\mathcal{M}$ monster model. $\mathcal{M}$ is a geometric structure if acl has the exchange property and $\mathcal{M}$ eliminates the quantifier $\exists^\infty$.

Let T be the theory of a geometric structure.

Definition

(B, A) is a dense pair of geometric structures (for T) if:

Elementary pair $A \prec B \models T$;

Density Every infinite T-definable subset of B intersects A;

Co-density For every $\mathcal{L}(B)$-formula $\phi(x, \bar{y})$, if $\phi(B, \bar{b})$ is finite for every $\bar{b} \in B^n$, then there exists $u \in U$ such that, for every $\bar{a} \in A^n$, $B \models \neg\phi(u, \bar{a})$.

Example

Let (B, A) be an elementary pair.

- if B is a geometric expansion of a field, then (B, A) is a dense pair iff it satisfies the density axiom.
- If B is an o-minimal expansion of a group, then (B, A) is a dense pair iff A is topologically dense in B.
- If B is strongly minimal, then (B, A) is a dense pair iff it satisfies the co-density axiom.

Notes

The second result is by van den Dries, and is the reason of the names "density axiom" and "dense pairs".

Axioms of independence relations

$\mathcal{M}$ monster model, $\mathop{\smile\hspace{-0.9em}\mid}$ independence relation on $\mathcal{M}$ (H. Adler).

Invariance $(\forall \sigma \in \mathrm{aut}(\mathcal{M}))\ A \mathop{\smile\hspace{-0.9em}\mid}_B C$ iff $\sigma A \mathop{\smile\hspace{-0.9em}\mid}_{\sigma B} \sigma C$.

Transitivity Assuming $B \subseteq C \subseteq D$, $A \mathop{\smile\hspace{-0.9em}\mid}_B D$ iff $A \mathop{\smile\hspace{-0.9em}\mid}_B C$ and $A \mathop{\smile\hspace{-0.9em}\mid}_C D$.

Normality $A \mathop{\smile\hspace{-0.9em}\mid}_C B$ iff $AC \mathop{\smile\hspace{-0.9em}\mid}_C B$

Extension There is some $A' \equiv_B A$ such $A' \mathop{\smile\hspace{-0.9em}\mid}_B C$.

Finite Character $A \mathop{\smile\hspace{-0.9em}\mid}_B C$ iff $A_0 \mathop{\smile\hspace{-0.9em}\mid}_B C$ for all finite $A_0 \subseteq A$.

Local Character For every A and B there exists $B_0 \subseteq B$ such that $|B_0| \leq |T| + |A|$ and $A \mathop{\smile\hspace{-0.9em}\mid}_{B_0} B$.

Symmetry $A \mathop{\smile\hspace{-0.9em}\mid}_B C$ iff $C \mathop{\smile\hspace{-0.9em}\mid}_B A$.

Note. We do not assume:

Strictness $a \mathop{\smile\hspace{-0.9em}\mid}_B a \Leftrightarrow a \in \mathrm{acl}(B)$;

Boundedness every type has only boundedly many nonforking extensions.

Examples of strict independence relations

$\mathcal{M}$ monster model.

- $\mathcal{M}$ ω-stable/stable/simple, $\mathop{\smile\hspace{-0.9em}\mid}$ = Shelah's forking.
- $\mathcal{M}$ rosy, $\mathop{\smile\hspace{-0.9em}\mid}$ = thorn forking.
- $\mathcal{M}$ pregeometric (i.e., acl has the Exchange Property), $\mathop{\smile\hspace{-0.9em}\mid}$ = algebraic independence; e.g., $\mathcal{M} \succ \mathbb{Q}_p$.

Notes

1. Let $\mathcal{M}$ be pregeometric. $A \mathop{\smile\hspace{-0.9em}\mid}_B C$ iff every subset of A which is algebraically independent over B remains algebraically independent over BC.
2. Later: nonstrict independence relations.

Lovely pairs

Let T be a complete theory, $\mathop{\smile\!\!\!\!\!\!\mid}$ be an independence relation, $\kappa := |T|^+$.

Definition

(B, A) is a $\mathop{\smile\!\!\!\!\!\!\mid}$-lovely pair if:

Elementary pair $A \prec B \models T$;

Density For every $C \subset B$ with $|C| < \kappa$ and $p(x)$ complete $\mathcal{L}$-type over C, there exists $c \models p$ such that $c \mathop{\smile\!\!\!\!\!\!\mid}_C A$;

Co-density For every $C \subset B$ with $|C| < \kappa$ and $p(x)$ complete $\mathcal{L}$-type over C, if p does not fork over $A \cap C$, then p is realized in A.

The above definition was originally given by BPV in the case when T is simple and $\mathop{\smile\!\!\!\!\!\!\mid}$ is Shelah's forking.

Remark

If (B, A) is an $\mathop{\smile\!\!\!\!\!\!\mid}$-lovely pair, then A and B are κ-saturated (as $\mathcal{L}$-structures).

Remark

Assume that B is geometric, $\mathop{\smile\!\!\!\!\!\!\mid}$ is given by geometric independence, and (B, A) is κ-saturated (as an $\mathcal{L}^2$-structure). Then, (B, A) is a dense pair of geometric structures iff (B, A) is a $\mathop{\smile\!\!\!\!\!\!\mid}$-lovely pair.

The same (complete) theory can have many independence relations; each independence relation gives a different class of lovely pairs.

Notes

1. If T is stable and $\mathop{\smile\!\!\!\!\!\!\mid}$ is Shelah's forking, then a $\mathop{\smile\!\!\!\!\!\!\mid}$-lovely pair is (essentially) a “belle paire” in the sense of Poizat.
2. Ben-Yaacov considered lovely pairs for different axioms for $\mathop{\smile\!\!\!\!\!\!\mid}$.

Completeness

From now on: (B, A) is a $\mathop{\smile\!\!\!\!\!\!\mid}$-lovely pair. Let $\bar{b} \in B^n$.

Definition

1. The P-type of $\bar{b}$ is the information of which b_i are in A.
2. $\bar{b}$ is P-independent iff $\bar{b} \mathop{\smile\!\!\!\!\!\!\mid}_{A \cap \bar{b}} A$.

Main Theorem

Let (B, A) and (B', A') be $\mathop{\smile\!\!\!\!\!\!\mid}$-lovely pairs. Let $\bar{b} \in B^n$ and $\bar{b}' \in B'^n$. Assume that $\bar{b}$ and $\bar{b}'$ are both P-independent and have the same P-type and the same $\mathcal{L}$-type. Then, $\bar{b}$ and $\bar{b}'$ have the same $\mathcal{L}^2$-type.

Corollary

Let (B, A) and (B', A') be $\mathop{\smile\!\!\!\!\!\!\mid}$-lovely pairs; then, they are elementarily equivalent.

Notes

Similar results hold for infinite tuples of cardinality $< \kappa$.

Loveliness is first-order

Definition

"Loveliness is first-order" means that there is a theory T^{lovely} such that every sufficiently saturated model of T^{lovely} is a $\mathop{\smile}\limits^{|}$-lovely pair.

Example

1. If $\mathop{\smile}\limits^{|}$ is given by geometric independence, then loveliness is first-order iff T eliminates the quantifier $\exists^{\infty}$.
2. (Poizat) If T is stable and $\mathop{\smile}\limits^{|}$ is Shelah's forking, then loveliness is first-order iff T has non-fcp, i.e. iff T^{eq} eliminates the quantifier $\exists^{\infty}$.

From now on, we will assume that loveliness is first-order, and T^{lovely} is the corresponding theory.

Notes

1. The axioms we gave earlier for dense pairs show that, for geometric theories, loveliness is first-order.
2. When T is simple, the condition when loveliness is first-order is quite complicate.

Inheritance of stability properties

Theorem

Assume that T is stable/simple/superstable/supersimple/ω-stable/NIP. Then T^{lovely} also is.

The above theorem was already known in special cases; the general proof is often an adaption of the proof in the special cases.

Lemma

Assume that T is stable. Then T^{lovely} also is.

Proof.

The proof is by a type-counting argument.

- T is stable iff it is λ-stable for some cardinal λ.
- Choose λ such that $\lambda^\kappa = \lambda$. Let $(B, A) \models T^{\text{lovely}}$ with $|B| = \lambda$ and $(\mathcal{M}, P(\mathcal{M}))$ be a monster model of T^{lovely}.
  Note. $|S^1_\kappa(B)| = \lambda$; we must prove that $|S^2_1(B)| = \lambda$.
- Let $q \in S^2_1(B)$; choose $c \in \mathcal{M}$ satisfying q.
- Local Character: let $\bar{p} \subseteq P(\mathcal{M})$ such that $c \mathop{\smile\!\!\!\!\!|}_{B\bar{p}} P(\mathcal{M})$ and $|\bar{p}| < \kappa$.
- By the Main Theorem, $\text{tp}^2(c/B)$ is determined by $\text{tp}^1(c\bar{p}/B)$ plus the P-type of c; since $|\bar{p}| < \kappa$, we have $|S^2_1(B)| \leq |S^1_\kappa(B)| = \lambda$. □

Notes

We use the superscript 1 for $\mathcal{L}$-notions, and the superscript 2 for $\mathcal{L}^2$-notions.

- The proofs of the superstable/ω-stable claims are minor variations of the above proof.
- The proof of the NIP claim is based on counting coheirs (Boxall).
- The proof of the simple/supersimple claims is based on a proof by BPV for the case when $\mathop{\smile\!\!\!\!\!|}$ is Shelah's forking.

Superior independence relations

Definition

$\mathop{\smile\!\!\!\!\!|}$ is superior if the relationship between types "p is a forking extension of q" is a well-founded partial order. If $\mathop{\smile\!\!\!\!\!|}$ is superior, $U^{\mathop{\smile\!\!\!\!\!|}}$ is the corresponding foundation rank of types.

Example

Let $\mathcal{M}$ be stable/simple, and $\mathop{\smile\!\!\!\!\!|}$ be Shelah's forking. $\mathop{\smile\!\!\!\!\!|}$ is superior iff T is superstable/supersimple; in this case, $U^{\mathop{\smile\!\!\!\!\!|}}$ is Lascar's U-rank.

Similar result holds for T rosy.

Example

If $\mathcal{M}$ is geometric and $\mathop{\smile\!\!\!\!\!|}$ is given by algebraic independence, then $\mathop{\smile\!\!\!\!\!|}$ is superior and $U^{\mathop{\smile\!\!\!\!\!|}}(\mathcal{M}) = 1$.

Coarsening

Assume that $\mathop{\smile\!\!\!\!\!\!\,|}$ is superior; let $U := U^{\mathop{\smile\!\!\!\!\!\!\,|}}$. Let λ be the unique power of ω such that:

- $U^{\mathop{\smile\!\!\!\!\!\!\,|}}(p) \geq \lambda$ for some (finitary $\mathcal{L}$-)type p;
- for every type q there exists $n \in \mathbb{N}$ such that $U^{\mathop{\smile\!\!\!\!\!\!\,|}}(q) = n \cdot \lambda + o(\lambda)$.

Define $\bar{a} \mathop{\smile\!\!\!\!\!\!\,|}^{c}_{C} D$ if $U(\bar{a}/C) = U(\bar{a}/CD) + o(\lambda)$.

Lemma

- $\mathop{\smile\!\!\!\!\!\!\,|}^{c}$ *is a superior independence relation;*
- $\mathop{\smile\!\!\!\!\!\!\,|}$ *refines* $\mathop{\smile\!\!\!\!\!\!\,|}^{c}$*;*
- $U^{\mathop{\smile\!\!\!\!\!\!\,|}^{c}}(q)$ *is finite for every type* q*.*

$\mathop{\smile\!\!\!\!\!\!\,|}^{c}$ is not strict in general.

Notes

1. $\alpha = o(\lambda)$ means $\alpha < \lambda$ (small-o notation).
2. $\mathop{\smile\!\!\!\!\!\!\,|}^{c} = \mathop{\smile\!\!\!\!\!\!\,|}$ if $U(\mathcal{M})$ is finite (i.e., $\lambda = 1$).

Independence relations on lovely pairs

Assume that loveliness is first order. Let $(\mathcal{M}, P(\mathcal{M}))$ be a monster model of T^{lovely}. Define $\mathop{\smile\!\!\!\!\!\!\,|}_{P}$ as $C \mathop{\smile\!\!\!\!\!\!\,|}_{PD} E$ iff $C \mathop{\smile\!\!\!\!\!\!\,|}_{P(\mathcal{M})D} E$.

Lemma

- $\mathop{\smile\!\!\!\!\!\!\,|}_{P}$ *is an independence relation on* $(\mathcal{M}, P(\mathcal{M}))$*.*
- *Assume that* $\mathop{\smile\!\!\!\!\!\!\,|}$ *is superior. Then,* $\mathop{\smile\!\!\!\!\!\!\,|}_{P}$ *is also superior; moreover, for every partial* $\mathcal{L}$*-type* q*,* $U^{\mathop{\smile\!\!\!\!\!\!\,|}}(q) = U^{\mathop{\smile\!\!\!\!\!\!\,|}_{P}}(q)$*.*

$\mathop{\smile\!\!\!\!\!\!\,|}_{P}$ is never strict.

Open problems

Assume that loveliness is first-order.

Conjecture

If T is (super)rosy, then T^{lovely} is too.

Conjecture

$\mathop{\smile\!\!\!\!\!\!\,|}_{P}$-loveliness is first-order.

Open problem

Give some form of elimination of imaginaries for lovely pairs.