Pairs of fields

A. Fornasiero

Talk on 9 June 2008, Freiburg, v. 1.2

1 Introduction

Definition. Given a language L, let L^2 be the expansion of L with a new unary predicate U. Given an L-theory T, let T^2 be the L^2-expansion of T, whose models are given by pairs (B, A), where $A \prec B \models T$, $A \neq B$, and U is interpreted by A.

Let B be a field, and A be an elementary substructure of B (in the language of rings $L_{Ring} := (0, 1, +, \cdot)$). What can we say about the theory of the pair (B, A) (i.e., what can we say about T^2_{Field})?

We will focus on 2 cases: $T = T_{ACF}$, and $T = T_{RCF}$.

Let T_{ACFp} be the theory of algebraically closed fields of characteristic p.

1.1 Theorem (Robinson '59). *T^2_{ACFp} is complete (and decidable), for every $p \geq 0$.*

T^2_{RCF} is not complete: for instance, A could be dense in B, or not.

Definition. Given a theory T expanding T_{RCF}, let T^{dense} be the extension of T^2 by adding the condition that A is dense in B.

1.2 Theorem (Robinson '59, van den Dries '98). *If T is o-minimal and complete, then T^{dense} is complete. In particular, T^{dense}_{RCF} is complete.*

1.3 Theorem (Macintyre '69). *Given a prime p, let T_p be the theory of p-adically closed fields. Then, T^{dense}_p is complete.*

1.4 Theorem (Macintyre '69). *If (B_i, A_i) are pairs of Henselian valued fields of residue characteristic 0, such that $A_1 \equiv A_2 \equiv B_1 \equiv B_2$, and A_i are dense proper subfields of B_i, then $(B_1, A_1) \equiv (B_2, A_2)$.*

2 Proof of Thm. 1.1

(Modified from Keisler '64).

Given a pair of fields (B, A), let $\dim(B/A)$ be the transcendence degree of B over A.

2.1 Lemma. *Let κ be an uncountable cardinal, and (B, A) be a κ-saturated model of T^2_{ACF}. Then, $\dim B/A \geq \kappa$.*

Proof. Let E be a basis of B/A. If, for contradiction, $|E| < \kappa$, let $q(x)$ be the partial unary L^2-type over E, saying for every $P \in \mathbb{Z}[x, \bar{y}, \bar{z}]$ polynomial and every $\bar{e}$ tuple of E,

$$\forall \bar{y} \, \big(U(\bar{y}) \wedge P(\cdot, \bar{y}, \bar{e}) \not\equiv 0 \rightarrow P(x, \bar{y}, \bar{e}) \neq 0\big).$$

The type q is consistent; therefore, it has a realization $b \in B$. However, $b \notin \mathrm{acl}(A \cup E)$, absurd. □

2.2 Lemma. *Let κ be an uncountable cardinal, $p \geq 0$, and (B, A) and (B', A') be saturated models of T^2_{ACFp} of cardinality κ. Then, (B, A) and (B', A') are isomorphic (as L^2_{Ring}-structures).*

Proof. It is clear that A and A' are isomorphic (as fields), because T_{ACFp} is a complete theory; let $\phi : A \cong A'$ be such an isomorphism.

By the previous Lemma, $\dim(B/A) = \dim(B'/A') = \kappa$. Thus, there exists an isomorphism of fields $\psi : B \cong B'$ extending ϕ (by well known facts about algebraically closed fields). Therefore, (B, A) and (B', A') are isomorphic. □

Proof of Thm. 1.1. Assume that GCH, the Generalized Continuum Hypothesis, is true. Then, for every 2 models (B, A) and (D, C) of T^2_{ACFp}, there exists saturated models (B^*, A^*) and (D^*, C^*) of the same cardinality, such that $(B, A) \preceq (B^*, A^*)$, and $(D, C) \preceq (D^*, C^*)$. By the previous Lemma, $(B^*, A^*) \cong (D^*, C^*)$, and therefore $(B, A) \equiv (D, C)$, proving completeness.

If GCH is not true, notice that the statement $\Omega :=$ "T_{ACFp} is complete" is an arithmetical one (i.e., it is an assertion about natural numbers, involving only quantifiers over $\mathbb{N}$). Since the constructible universe L satisfies GCH, Ω is true in L. By absoluteness, Ω is true also in the "true" universe V (that is, if ψ is a sentence in L_{Ring}, then, by Gödel completeness theorem, in L we can prove either ψ or $\neg\psi$ from T^2_{ACFp}; such a proof is also a proof in V). □

3 Proof of Thm. 1.2

(From van den Dries '98)

Let T be a complete o-minimal theory, expanding a field, and having universal axioms. Given (B, A) a pair of models of T, let $\dim(B/A)$ be the o-minimal dimension of B over A.[1]

3.1 Proposition. *Let $\kappa > |T|$ be an infinite cardinal, and (B, A) be a κ-saturated model of T^{dense}. Then,* $\dim(B/A) \geq \kappa$.

Proof. If $T = T_{RCF}$, the proposition follows from Lemma 2.1. The general case is postponed. □

Definition. Let $(B, A) \subseteq (D, C) \models T^2$. We say that B and C are **free** over A if every set $Y \subseteq B$ that is independent over C remains independent over A.[2]

3.2 Remark. If B and C are free over A, then

1. $C \cap B = A$;
2. C and B are free over A (symmetry);
3. if $Z \subseteq C$, then $(B, A) \subseteq (B\langle Z\rangle, A\langle Z\rangle) \subseteq (D, C)$, and $B\langle Z\rangle$ and C are free over $A\langle Z\rangle$.

3.3 Remark. If $(B, A) \preceq (B^*, A^*) \models T^2$, then B and A^* are free over A.

3.4 Remark. Let $p(x)$ be a 1-type over $A \models T$. If p is not realized in A, then p is "open": that is, for every $L(A)$-formula contained in p and for every $b \models p$ ($b \in B \succ A$)), there exists an open interval I with end-points in A, such that $b \in I \subset \phi(B)$.

Let $\kappa > |T|$ be a cardinal number, and (B^*, A^*) and (D^*, C^*) be κ-saturated models of T^{dense}. Define Γ be the set of all isomorphisms $i : (B, A) \cong (D, C)$ between substructures of (B^*, A^*) and (D^*, C^*) respectively, such that $|B| = |D| < \kappa$, B and A^* are free over A, and D and C^* are free over C.

3.5 Lemma. Γ *has the back-and-forth property.*

Theorem 1.2 follows easily from the above Lemma.

[1] Remember that T has definable Skolem functions, and that the definable closure dcl has the Exchange Property. Thus, dcl is a pre-geometry, and dim is the dimension induced by that pre-geometry.

[2] Free models are also called linearly disjoint in the case $T = T_{RCF}$.

Proof. Let $i : (B, A) \cong (D, C)$ be in Γ. Given $b^* \in B^* \setminus B$, we want to extend i to $j \in \Gamma$, such that b^* is in the domain of j.

Case 1: $b^* := a^* \in A^* \setminus B$. Define $B' := B\langle a^* \rangle$, $A' := A\langle a^* \rangle$. Notice that B' and A^* are free over A' (and, in particular, $B' \cap A^* = A'$). Let $p(x)$ be the L-type of a^* over B, and $q(x)$ be the image of p via i. Notice that $q(x)$ is consistent, and therefore, by saturation, it has a realization $d^* \in D^* \setminus D$. Since A^* is dense in D^*, $q(x)$ has also a realization $c^* \in C^* \setminus D$. Define $D' := D\langle c^* \rangle$, $C' := C\langle c^* \rangle$. Notice that D' and C^* are free over C'. Hence, there exists a unique isomorphisms $j : (B', A') \cong (D', C')$ in Γ extending i and mapping a^* to c^*.

Let $K := A^*\langle B \rangle = B\langle A^* \rangle$, and $K' := C^*\langle D \rangle = D\langle C^* \rangle$.

Case 2: $b^* \in K$. Then, $b \in B\langle a_1^*, \dots, a_n^* \rangle$, for some $n \in \mathbb{N}$ and $a_1^*, \dots a_n^* \in A^*$. Apply the Case 1 construction n times to extend i.

Case 3: $b^* \in B^* \setminus K$. Define $B' := B\langle b^* \rangle$, $A' := A$. Notice that B' and A^* are free over A'. Let $p(x)$ be the L-type of b^* over B, and $q(x)$ be the image of p via i. By Proposition 3.1, $K' \neq D^*$. Hence, by saturation, q has a realization $d^* \in D^* \setminus K'$. Define $D' := D\langle d^* \rangle$, $C' := C$. Notice that D' and C^* are free over C'. Hence, there exists a unique isomorphisms $j : (B', A') \cong (D', C')$ in Γ extending i and mapping b^* to d^*. □

3.6 Corollary (of the proof). *Let $i \in \Gamma$, and $\bar{b} := (b_1, \dots, b_n)$ be in the domain of i. Then, $\bar{b}$ realizes the same L^2-type over $\emptyset$ as $i(\bar{b})$.*

Call a L^2-formula **special** if it is of the form $\exists \bar{y} \big(\psi(\bar{x}, \bar{y}) \wedge U(\bar{y}) \big)$, where ψ is an L-formula.

3.7 Corollary. *Every L^2-formula $\phi(\bar{x})$ is equivalent, modulo T^{dense}, to a Boolean combination of special formulae. I.e., for every L^2-formula $\phi(\bar{x})$ there exists a Boolean combination of special formulae $\theta(\bar{x})$, such that $T^{dense} \vdash \forall \bar{x} \big(\phi(\bar{x}) \leftrightarrow \theta(\bar{x}) \big)$.*

Proof. By a compactness argument, it suffices to show that, if $\bar{b} \in (B^*)^n$ and $\bar{d} \in (D^*)^n$ satisfy the same special formulae, then $\bar{b}$ and $\bar{d}$ realize the same L^2-types over $\emptyset$.

W.l.o.g., we can assume that $b_1, \dots, b_r$ are independent over A^*, where that $r := \dim(\bar{b}/A^*) = r$.

1 Claim. $\dim(\bar{d}/C^*) = r$, and $d_1, \dots, d_r$ are independent over C^*.

In fact, if for instance, $d_r \in A^*\langle d_1, \dots, d_{r-1} \rangle$, then $d_r = f(d_1, \dots, d_{r-1}, \bar{c})$ for some $\emptyset$-definable function f and some $\bar{c} \in (C^*)^l$. By hypothesis, there exists $\bar{a} \in (A^*)^l$ such that $b_r = f(b_1, \dots, b_{r-1}, \bar{a})$, absurd.

Let $\bar{a} \in A^m$ be such that $\dim(\bar{b}/A) = r$, where A is the substructure of A^* generated by $\bar{a}$. Notice that $A\langle\bar{b}\rangle$ and A^* are free over A. Let $p(\bar{x}, \bar{y})$ be the L-type of $(\bar{a}, \bar{b})$ over $\emptyset$, $p'(\bar{x}, \bar{y})$ be the partial L^2-type extending p with "$U(\bar{x})$", and $q(\bar{x}) := p'(\bar{x}, \bar{d})$. By hypothesis on $\bar{d}$, q is consistent; let $\bar{c} \in (C^*)^m$ be a realization of q, and C be the definable closure of $\bar{c}$.

Notice that $\dim(\bar{d}/C) = r$, and $C\langle\bar{d}\rangle$ and C^* are free over C. Thus, using the Case 3 construction r times, we obtain an isomorphism $j : (A\langle\bar{b}\rangle, A) \cong (C\langle\bar{d}\rangle, C)$ in Γ. Thus, $(\bar{b}, \bar{a})$ and $(\bar{d}, \bar{c})$ realize the same L^2-type over $\emptyset$. □

3.1 Sketch of proof of Prop. 3.1

Call a subset $X \subseteq B$ **small** if it is of the form $f(A^n)$, for some $T(B)$-definable function f.

3.8 Lemma. *Let $(B, A) \models T^2$, $f : B^{n+1} \to B$ be A-definable in B, and $b \in B \setminus A$. Then, there exist $a_0, \dots, a_n \in A$ such that*

$$a_0 + a_1 b + \dots + a_n b^n \notin f\big(A^n \times \{b\}\big).$$

In particular, $f\big(A^n \times \{b\}\big) \neq B$.

Proof. See [1, Lemma 1.1]. □

3.9 Lemma. *Let $(B, A) \models T^{dense}$. Then, each small subset of B is a proper subset of B.*

Proof. Let $g : B^n \to B$ be T-definable in B. We have to prove that $g(A^n) \neq B$. After increasing A, we can assume that $B = A\langle b\rangle$, for some $b \in B \setminus A$. Hence, $g(xv)$ is of the form $f(\bar{x}, b)$, for some A-definable function f. The conclusion follows from the previous Lemma. □

We can now prove Prop. 3.1 in the same way as Lemma 2.1. In fact, let E be a basis of B/A, and assume, for contradiction, that $|E| < \kappa$. Let $p(x)$ be the partial unary L^2-type over E, saying, for every function $f(\bar{y}, \bar{z})$ T-definable over $\emptyset$, and every $\bar{e}$ tuple of E,

$$\forall \bar{y}\ \big(U(\bar{y}) \to x \neq f(\bar{y}, \bar{e})\big).$$

By the previous Lemma, p is consistent (because $f(A^n, \bar{e})$ is small, and a finite union of small sets is small); therefore, by saturation, it has a realization $b \in B$. However, $b \notin A\langle E\rangle$, absurd.

References

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