O-minimality of the standard part.

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Abstract

Let $\mathfrak{M}$ be an o-minimal structure expanding an ordered field. Let R be the residue field of $\mathfrak{M}$, with the structure generated by the images of definable subsets of $\mathfrak{M}$ under the residue map. Using a theorem by Baisalov and Poizat, we prove that R is weakly o-minimal.

Contents

1 Introduction

Let $\mathfrak{M}$ be a structure expanding a linearly ordered field in a given language $\mathcal{L} := (0, 1, \leq, +, \cdot, \dots)$ with domain the set M. Let v be a convex valuation[1] on $\mathfrak{M}$,

$$\mathcal{O} := \{ x \in M : v(x) \geq 0 \}$$

its valuation ring,

$$\mathcal{M} := \{ x \in M : v(x) > 0 \}$$

the set of infinitesimals,

$$\mathbb{K} := \mathcal{O}/_{\mathcal{M}}$$

its corresponding residue field, and

$$\mathrm{st} : \mathcal{O} \to \mathbb{K}$$

the quotient map.

For every $n \in \mathbb{N}$, st induces a map from $\mathcal{O}^n \to \mathbb{K}^n$, that I will denote with the same symbol.

Given $n \in \mathbb{N}$, let $\mathfrak{A}_n$ be family of all subsets of M^n definable with parameters in $\mathfrak{M}$, and let $\mathfrak{A} := \bigcup_n \mathfrak{A}_n$ be the family of definalbe (with parameters) sets.

[1]This means that $\mathcal{O}$ is a convex subset of $\mathfrak{M}$.

Given $A \subseteq M^n$, $\operatorname{st}(A \cap \mathcal{O}^n)$ is a subset of $\mathbb{K}^n$. Let

$$\mathfrak{E}_n := \{ \operatorname{st}(A \cap \mathcal{O}^n) : A \in \mathfrak{A}_n \}, \quad \mathfrak{E} := \bigcup_n \mathfrak{E}_n.$$

I will denote with $\mathfrak{K} := (\mathbb{K}, \mathfrak{E})$ the structure induced by $\mathfrak{E}$ on $\mathbb{K}$.

I remind that $\mathfrak{M}$ is *o-minimal* iff every subset of M definable (with parameters from M) is a finite union of points and intervals with end-points in M. It is *weakly o-minimal* if the end-points of these intervals are not necessarily in M.

Theorem 1. *With the previous notations, if $\mathfrak{M}$ is o-minimal, then $\mathfrak{K}$ is weakly o-minimal.*

Note that it is not true in general that the complement of a set in $\mathfrak{E}$ is in $\mathfrak{E}$. For instance, if $\mathfrak{M}$ is o-minimal and the valuation v is non-trivial, then every set in $\mathfrak{E}$ is closed (with the interval topology).

Every weakly o-minimal structure on $\mathbb{R}$, the field of real numbers, is o-minimal; therefore, if $\mathbb{K}$ is $\mathbb{R}$, then the previous theorem implies that $\mathfrak{K}$ is an o-minimal structure.

In the proof of Theorem 1, the main ingredient is the following theorem.[2]

Theorem 2 (Baisalov and Poizat). *Let $\mathfrak{M}$ be an o-minimal structure. Let I be a convex subset of M. Then, the structure on M generated by I and $\mathfrak{M}$ is weakly o-minimal.*

Corollary 1.1. *Let $\mathfrak{M}'$ the structure on M generated by $\mathfrak{M}$ and $\mathcal{O}$. Then, $\mathfrak{M}'$ is weakly o-minimal.*

2 Proof

Theorem 1 is be a consequence of the following theorem.

Theorem 3. *With the previous notation, suppose that $\mathfrak{M}$ is weakly o-minimal and that $\mathcal{O}$ is definable (with parameters) in $\mathfrak{M}$. Then, $\mathfrak{K}$ is weakly o-minimal.*

Proof. Note that if $\mathcal{O}$ is definable, then $\mathcal{M}$ is also definable.

Claim 1. Let $E \in \mathfrak{E}_n$ for some $n \in \mathbb{N}$, and

$$A := \operatorname{st}^{-1}(E).$$

Then, $A \in \mathfrak{A}$.

Let $X \in \mathfrak{A}$ such that $\operatorname{st}(X \cap \mathcal{O}) = E$. Then,

$$A = \{ x \in M^n : \exists y \in \mathcal{O}^n \cap X \ (|x - y| \in \mathcal{M}) \},$$

so $A \in \mathfrak{A}$.

Claim 2. $\mathfrak{E}$ is already a structure.

[2] YERZHAN BAISALOV and BRUNO POIZAT, *Paires de structures O-minimales* **J. Sybolic Logic**, vol. 63 (1998), pp. 570–578.

This means that if $E, E' \in \mathfrak{E}_n$, then $E \cap E'$, $E \cup E'$, $E^c := \mathbb{K} \setminus E$, $E \times E'$ and $\pi(E)$ are in $\mathfrak{E}$, where π is the projection to the first $n-1$ coordinates.

In fact, let $A := \mathrm{st}^{-1}(E)$ and $A' := \mathrm{st}^{-1}(E')$. By claim 1, they are both in $\mathfrak{E}$. Then,

$$\begin{aligned} E \cap E' &= \mathrm{st}(A \cap A') \\ E \cup E' &= \mathrm{st}(A \cup A') \\ E \times E' &= \mathrm{st}(A \times A') \\ E^c &= \mathrm{st}(A^c) \\ \pi(E) &= \mathrm{st}(\pi(A)). \end{aligned}$$

Claim 3. If $A \in \mathfrak{A}_1$, then $\pi(A)$ is a finite union of convex sets.

This follows immediately from the fact that $\mathfrak{M}$ is weakly o-minimal.

Let $E \in \mathbb{K}$ be definable in $\mathfrak{K}$. By claim 2, there exists $A \in \mathfrak{A}$ such that $E = \mathrm{st}(A)$. Therefore, by claim 3, A is a finite union of convex sets, so $(\mathbb{K}, \mathfrak{E})$ is weakly o-minimal. □

Proof of Theorem 1. Let $\mathfrak{M}'$ be as in Corollary 1.1. By that corollary, $\mathfrak{M}'$ is weakly o-minimal, therefore it satisfies the hypothesis of Theorem 1, so $\mathfrak{K}'$, the induced structure on $\mathbb{K}$, is weakly o-minimal. But $\mathfrak{K}$ is obviously a restriction of $\mathfrak{K}'$, so a fortiori it is weakly o-minimal. □

Corollary 2.1. *Suppose that $\mathfrak{M}$ is o-minimal and that $\mathbb{K} = \mathbb{R}$. Then, $\mathfrak{K}$ is o-minimal.*

Note that if v is the Archimedean valuation, and $\mathfrak{M}$ is ω-saturated, then $\mathbb{K} = \mathbb{R}$.

Note also that if v is non-trivial and $\mathcal{O}$ is definable in $\mathfrak{M}$, then $\mathfrak{M}$ cannot be o-minimal.

3 Conclusion

I will conclude with some open questions and conjectures.

1. If $\mathfrak{M}$ is o-minimal, is $\mathfrak{K}$ o-minimal too (and not simply weakly o-minimal)?
2. If $\mathfrak{M}$ is weakly o-minimal, but $\mathcal{O}$ is not necessarily definable, is $\mathfrak{K}$ weakly o-minimal too?
3. If $\mathfrak{M}$ is o-minimal and, say, $\mathbb{K} = \mathbb{R}$, what is the relationship between the theory of $\mathfrak{M}$ and that of $\mathfrak{K}$?
4. Can we weaken the hypothesis of Theorem 2 and Corollary 1.1 to $\mathfrak{M}$ weakly o-minimal? This would imply the conjecture 2.

Consider in particular the case where $\mathfrak{M}$ is an ω-saturated o-minimal exponential field, with $\exp(1) = e$, and v is the Archimedean valuation. Then, $\mathfrak{K}$ is the field $\mathbb{R}$ with the usual exponential map. It would be interesting to relate $\mathfrak{K}$ with $\mathfrak{M}$. If it were true that the theory of $(\mathbb{R}, \exp)$ is axiomatised by the functional equation and o-minimality, then $\mathfrak{K}$ would be elementary equivalent to $\mathfrak{M}$.