

Dedekind cuts of ordered Abelian groups

Antongiulio Fornasiero
antongiulio.fornasiero@googlemail.com
joint work with Marcello Mamino

University of Freiburg

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Introduction

Let G be a linearly ordered set. A **Dedekind cut** of G is a partition of G

$$\Lambda := \langle \Lambda^L \mid \Lambda^R \rangle$$

such that $\Lambda^L < \Lambda^R$ (that is, $x < y$ for all $x \in \Lambda^L$ and $y \in \Lambda^R$).

G^C will denote the set of (Dedekind) cuts of G .

G^C has a natural ordering, induced by the ordering of G .

If moreover G is an ordered Abelian group, the operations $+$ and $-$ on G induce analogous operations on G^C . We will study the resulting first order structures, and we will give axioms for the universal part of their theory.

Basic definitions

Ordered Abelian groups

An ordered Abelian group is a structure $\langle G, +, 0, \leq \rangle$, where

- $\langle G, +, 0 \rangle$ is an Abelian group, with 0 as neutral element,
- \leq is a linear ordering on G ,
- \leq is compatible with the group operations:

$$x \leq x' \Rightarrow x + y \leq x' + y.$$

From now on, all groups will be Abelian, all orders linear, and G will be an ordered group.

Basic definitions

Order and relations on cuts

Let G^C be the set of cuts of G . For every $X \subseteq G$, let $X^+ \in G^C$ be the **upper edge** of X , and similarly let X^- be the **lower edge** of X . **Define:**

order $\Lambda \leq \Gamma$ if $\Lambda^L \subseteq \Gamma^L$, or, equivalently, if $\Gamma^R \subseteq \Lambda^R$;

left sum $\Lambda + \Gamma = (\Lambda^L + \Gamma^L)^+$;

right sum $\Lambda +^R \Gamma = (\Lambda^R + \Gamma^R)^-$;

unary minus $-\Lambda = \langle -\Lambda^R \mid -\Lambda^L \rangle$;

binary minus $\Lambda - \Gamma = (\Lambda^R - \Gamma^L)^-$.

Moreover, for every $g \in G$,

$$g^+ = \langle (-\infty, g] \mid (g, +\infty) \rangle;$$

$$g^- = \langle (-\infty, g) \mid [g, +\infty) \rangle = -((-g)^+);$$

$$\mathbf{0} = 0^+.$$

Examples

- Let G be a densely ordered group. For every $g, h \in G$,

$$g^+ + h^+ = (g + h)^+ = g^+ +^R h^+;$$

$$g^- + h^- = (g + h)^- = g^- +^R h^-;$$

$$g^+ + h^- = (g + h)^-;$$

$$g^+ +^R h^- = (g + h)^+ > g^+ + h^-.$$

- $\mathbb{Q}^C = \{q^\pm : q \in \mathbb{Q}\} \sqcup \{r : r \in \mathbb{R} \setminus \mathbb{Q}\} \sqcup \{\pm\infty\}$.

- Cuts on $G = \mathbb{Q} \times \mathbb{Q}$ (ordered lexicographically): given $q \in \mathbb{Q}$, $\Lambda \in \mathbb{Q}^C \setminus \{\pm\infty\}$, and $\Gamma \in \mathbb{Q}^C$, define

$$\textcircled{1} \quad q \times \Lambda := (q \times \Lambda^L)^+ = (q \times \Lambda^R)^-;$$

$$\textcircled{2} \quad \Gamma \times \mathbb{Q} := \{(r, s) \in G : r \in \Gamma^L\}^+ = \{(r, s) \in G : r \in \Gamma^R\}^-.$$

$$(q \times \Lambda) + (q' \times \Lambda') = (q + q') \times (\Lambda + \Lambda');$$

$$(\Gamma \times \mathbb{Q}) + (\Gamma' \times \mathbb{Q}) = (\Gamma + \Gamma') \times \mathbb{Q};$$

$$(q \times \Lambda) + (\Gamma \times \mathbb{Q}) = (q + \Gamma) \times \mathbb{Q}.$$

Basic facts

(G^C, \leq) is a complete linear order: every cut on G^C has a least upper bound. Equivalently, G^C is compact in the order topology.

$(G^C, +, \mathbf{0}, \leq)$ is an ordered commutative monoid:

- $+$ is associative and commutative, with $\mathbf{0}$ as neutral element;
- the ordering \leq is compatible with the operation $+$.

$$\Lambda + \Gamma = -((- \Lambda) +^R (- \Gamma));$$

$$\Lambda - \Gamma = \Lambda +^R (- \Gamma);$$

$$\Lambda + \Gamma \leq \Lambda +^R \Gamma.$$

The group G acts on G^C in a way compatible with the structure of G^C :

$$g + \Gamma = \langle g + \Gamma^L \mid g + \Gamma^R \rangle.$$

Invariance group

G acts on G^C ; for every $\Lambda \in G^C$, the invariance group of Λ is

$$\mathfrak{I}(\Lambda) := \{g \in G : g + \Lambda = \Lambda\}.$$

$\mathfrak{I}(\Lambda)$ is always a convex subgroup of G . The **width** of Λ is

$$\widehat{\Lambda} := (\mathfrak{I}(\Lambda))^+.$$

Lemma

$$\widehat{\Lambda} = \Lambda - \Lambda = \{g'' - g' : g' < \Lambda < g''\}^- = (\Lambda^R - \Lambda^L)^-.$$

Moreover, $\mathbf{0} \leq \widehat{\Lambda} = \widehat{\Lambda} + \widehat{\Lambda}$, and $\widehat{\widehat{\Lambda}} = \widehat{\Lambda}$.

Conversely, if Λ satisfies $\mathbf{0} \leq \Lambda = \Lambda + \Lambda$, then $\Lambda = \widehat{\Lambda}$.

The set of widths of G^C is $\mathcal{W}(G^C) := \{\widehat{\Lambda} : \Lambda \in G^C\}$. There is a bijection between the convex subgroups of G and $\mathcal{W}(G^C)$, mapping H to H^+ , and conversely $\widehat{\Lambda}$ to $\mathfrak{I}(\Lambda)$.

Examples

Cuts on $G = \mathbb{Q} \times \mathbb{Q}$: given $q \in \mathbb{Q}$, $\Lambda \in \mathbb{Q}^C \setminus \{\pm\infty\}$, and $\Gamma \in \mathbb{Q}^C$,

$$\textcircled{1} \quad q \times \Lambda := (q \times \Lambda^L)^+ = (q \times \Lambda^R)^-;$$

$$\textcircled{2} \quad \Gamma \times \mathbb{Q} := \{(r, s) \in G : r \in \Gamma^L\}^+ = \{(r, s) \in G : r \in \Gamma^R\}^-.$$

Let $S := G^C$ and $\omega := (\mathbf{0} \times \mathbb{Q})^+$.

$$\mathcal{G}(q \times \Lambda) = (0); \quad \mathcal{G}(\Gamma \times \mathbb{Q}) = \mathcal{G}(\Gamma) \times \mathbb{Q} = \begin{cases} \{0\} \times \mathbb{Q} & \text{if } \Gamma \neq \pm\infty, \\ G & \text{otherwise;} \end{cases}$$

$$\mathcal{W}(S) = \{\mathbf{0}, \omega, +\infty\}.$$

- $(-\omega, \omega)$ is isomorphic to $\mathbb{Q}^C \setminus \{\pm\infty\}$;
- $S^{(0)} \cong \mathbb{Q} \times (-\omega, \omega)$; $S^{(\geq \omega)} \cong \mathbb{Q}^C$; $S^{(\infty)} = \{\pm\infty\}$;
- $[-\omega, \omega]$ is also a scroll, isomorphic to \mathbb{Q}^C .

Examples

- G and G^C are scrolls.
- Let $\tilde{G} := G \sqcup G^C$, with the ordering $g^- < g < g^+$. \tilde{G} is a scroll.

$$\begin{aligned} \mathbb{Z}^C &= \{n^+ : n \in \mathbb{Z}\} \sqcup \{\pm\infty\} \\ x^+ + y^+ &= (x + y)^+ \\ -(0^+) &= 0^- = (-1)^+ \\ -(x^+) &= (-x)^- = (-x - 1)^+. \end{aligned}$$

$\langle \mathbb{Z}, \leq, +, 0 \rangle$ is isomorphic to $\langle \mathbb{Z}^C \setminus \{\pm\infty\}, \leq, +, \mathbf{0} \rangle$ (via the map $n \mapsto n^+$), but the isomorphism does not preserve the $-$.

Scrolls

Definition

A **prescroll** is a tuple $\langle S, \leq, +, \mathbf{0}, - \rangle$, where S is a set, \leq is a binary relation (on S), $\mathbf{0}$ is an element, $+$ is a binary operation and $-$ is a unary operation, satisfying:

- $\langle S, +, \mathbf{0}, \leq \rangle$ is an ordered commutative monoid;
- $-$ is an order-reversing map, such that $-(-x) = x$.

Define:

- $|x| = \max\{x, -x\}$;
- $x +^R y = -((-x) + (-y))$;
- $x - y = x +^R (-y) = -((-x) + y)$.

A **scroll** is a prescroll satisfying the following axioms:

$$\text{SA} \quad -\mathbf{0} \leq \mathbf{0};$$

$$\text{SB} \quad |x| \geq \mathbf{0};$$

$$\text{SC} \quad x \geq y \text{ iff } x - y \geq \mathbf{0}.$$

Duality

If $\langle S, \leq, +, \mathbf{0}, - \rangle$ is a prescroll, then $S^{\text{dual}} = \langle S, \geq, +^R, -\mathbf{0}, - \rangle$ is also a prescroll, the **dual** of S . Moreover, if S is a scroll, the S^{dual} is also a scroll. Therefore, every theorem about (pre)scrolls has a dual theorem.

For instance, the dual of the formula (true for scrolls)

$$\forall x \forall y (x + y) - y \geq x$$

is equivalent to

$$\forall x \forall y (x - y) + y \leq x.$$

Classification

Every scroll S satisfies exactly one of the following conditions:

Scroll of group type: $\mathbf{0} = -\mathbf{0}$;

Scroll of dense type: $\mathbf{0} > -\mathbf{0}$ and $\mathbf{0} +^R \mathbf{0} = \mathbf{0}$;

Scroll of discrete type: $\mathbf{0} > -\mathbf{0}$ and $\mathbf{0} +^R \mathbf{0} > \mathbf{0}$.

If G is a group, then G itself and \tilde{G} are scrolls of group type;
 G^C is of discrete type if G is discrete, it is of dense type if G is dense.

Trivial scrolls

A scroll is **trivial** if

$$x + y = \begin{cases} x & \text{if } |x| > |y|; \\ y & \text{if } |x| < |y|; \\ \min\{x, y\} & \text{if } |x| = |y|. \end{cases}$$

A scroll S is trivial iff for every $x \in S$ we have $\widehat{x} = |x|$.

Given an ordered set S , with a distinguished point $\mathbf{0}$ and an order-reversing involution $-$, such that either $-\mathbf{0} = \mathbf{0}$, or $-\mathbf{0}$ is the immediate predecessor of $\mathbf{0}$, we can define an unique structure of trivial scroll on S using the formula above.

Lemma

For every $n \in \mathbb{N}^*$ there exists a unique (up to isomorphism) scroll \mathbf{n} with n elements, and this scroll is trivial. Moreover, if we fix a linear ordering on $\{1, \dots, n\}$, there exists a unique way to extend the ordering to a scroll.

Width of an element

The **width** of $x \in S$ is $\widehat{x} = x - x$.

Lemma

$$\begin{aligned} \widehat{x} &\geq \mathbf{0}; & -\widehat{x} &= \widehat{-x}; \\ x + \widehat{x} &= x; & \widehat{\widehat{x}} &= \widehat{x}; \\ \widehat{x + y} &= \max\{\widehat{x}, \widehat{y}\}. \end{aligned}$$

- The width measures how much the equality $x + y = x +^R y$ fails;
- $x + y \leq x +^R y \leq (x + y) +^R \widehat{x}$;
- if $\widehat{x} \neq \widehat{y}$, then $x +^R y = x + y$.

Levels of a scroll

- Let $\mathcal{W}(S) := \{\widehat{x} : x \in S\}$. For every $V \subseteq \mathcal{W}(S)$, define

$$S^{(V)} := \{x \in S : \widehat{x} \in V\}.$$

- If V has a minimum θ , then $S^{(V)}$ is a scroll, with neutral element θ . Moreover, if $\theta = \mathbf{0}$, then $S^{(V)}$ is a subscroll of S .
- For every $w \in \mathcal{W}(S)$, the following sets are subscrolls of S :
 $S^{(\mathbf{0})} := \{x \in M : \widehat{x} = \mathbf{0}\}$; $S^{(<w)} := \{x \in M : \widehat{x} < w\}$ (if $w > \mathbf{0}$);
 $S^{(\leq w)} := \{x \in M : \widehat{x} \leq w\}$.
- The following are scrolls, with neutral element w :
 $S^{(w)} := \{x \in M : \widehat{x} = w\}$; $S^{(\geq w)} := \{x \in M : \widehat{x} \geq w\}$.
- If $w > \mathbf{0}$, then $(-w, w) = \{x \in S : -w < x < w\}$ is a subscroll of S .

Examples

Cuts on $G = \mathbb{Q} \times \mathbb{Q}$: given $q \in \mathbb{Q}$, $\Lambda \in \mathbb{Q}^C \setminus \{\pm\infty\}$, and $\Gamma \in \mathbb{Q}^C$,

$$\textcircled{1} \quad q \times \Lambda := (q \times \Lambda^L)^+ = (q \times \Lambda^R)^-;$$

$$\textcircled{2} \quad \Gamma \times \mathbb{Q} := \{(r, s) \in G : r \in \Gamma^L\}^+ = \{(r, s) \in G : r \in \Gamma^R\}^-.$$

Let $S := G^C$ and $\omega := (\mathbf{0} \times \mathbb{Q})^+$.

$$\mathfrak{G}(q \times \Lambda) = (0); \quad \mathfrak{G}(\Gamma \times \mathbb{Q}) = \mathfrak{G}(\Gamma) \times \mathbb{Q} = \begin{cases} \{0\} \times \mathbb{Q} & \text{if } \Gamma \neq \pm\infty, \\ G & \text{otherwise;} \end{cases}$$

$$\mathcal{W}(S) = \{\mathbf{0}, \omega, +\infty\}.$$

- $(-\omega, \omega)$ is isomorphic to $\mathbb{Q}^C \setminus \{\pm\infty\}$;
- $S^{(0)} \cong \mathbb{Q} \times (-\omega, \omega)$; $S^{(\geq \omega)} \cong \mathbb{Q}^C$; $S^{(\infty)} = \{\pm\infty\}$;
- $[-\omega, \omega]$ is also a scroll, isomorphic to \mathbb{Q}^C .

Level 0

- **Define** $G(S) := S^{(0)}/\sim_0$.
 $G(S)$ is a group (the group associated to S).
 If S is of group type or of discrete type, then \sim_0 is trivial, and $G(S) = S^{(0)}$.
 However, if S is of discrete type, the minus operation on $S^{(0)}$ is not the group minus of $G(S)$.
- If S is of dense type, every $g \in G(S)$ has either 1 or 2 preimages in S .
Define $H(S)$ to be the set of $g \in G(S)$ with 2 preimages.
Fact: $H(S)$ is a subgroup of $G(S)$.
 Otherwise, we define $H(S) := G(S)$.
- When $S = H^C$, we have $H(S) = H$ and $G(S)$ is equal to the **Cauchy completion** of H ; the elements of $S^{(0)}$ are called **regular cuts**.
 If H is discrete, then $H = G$; if H is dense, then we have $h^- \sim_0 h^+$.

Equivalence relations

An equivalence relation \sim on S is a **scroll-equivalence relation** if it preserve the structure, i.e., for every $x \sim x'$,

$$\begin{aligned} -x &\sim -x'; \\ x + y &\sim x' + y; \\ x \leq y \leq x' &\rightarrow x \sim y. \end{aligned}$$

The quotient S/\sim is a scroll;

if \sim is non-trivial, then S/\sim is of group type (i.e. $-\mathbf{0} \sim \mathbf{0}$).

Given $w \in \mathcal{W}(S)$, define $x \sim_w x'$ if $x +^R w = x' +^R w$.

\sim_w is a scroll-equivalence relation.

The quotient S/\sim_w is a scroll, with neutral element $[w]$.

If we restrict ourselves to $S^{(\leq w)}$, \sim_w can also be defined by $x \sim_w x'$ iff $|x - x'| \leq w$.

Main theorem

Note that the axioms of scrolls are **universal**;
 therefore, every substructure of a scroll is also a scroll.

Theorem

Every scroll of dense type (of discrete type) can be embedded in G^C , for some densely ordered (discretely ordered) group G .

Equivalent formulation:

Corollary

Let \mathcal{Dense} be the theory of scrolls of dense type (i.e., the axioms of scrolls, plus $-\mathbf{0} < \mathbf{0} = \mathbf{0} +^R \mathbf{0}$ ()). Then, \mathcal{Dense} is the universal part of the theory of cuts on densely ordered groups.*

That is, a universal formula is true in every G^C (for G dense) iff it can be proved from the axioms of scrolls and ().*

Similarly for discrete instead of dense.

Example

Lemma

Let $x, x', y \in S, j, j', k, m, d \in \mathbb{N}^*$ such that $j, j', k < m$ and $j + j' = m + d$.

Then

$$(x - jy) + (x' - j'y) + (my - ky) \leq (x + x') - (d - k)y.$$

Note that, if S is a group, we have equality.

Open problems

Let $Cuts$ be the theory of cuts on ordered groups.

- ① Give axioms for $Cuts$.
- ② Is there an algorithm to decide whether a **universal** formula follows from $Cuts$?

Scrolls are undecidable

An n -chain of ordered (Abelian) groups is a tuple $\langle G_1, \dots, G_n \rangle$ such that $G_i \subseteq G_{i+1}$, where the G_i are ordered groups.

Theorem

Let T_n be the theory of n -chains of ordered Abelian groups. For n large enough, T_n is undecidable.

The proof follows immediately from a theorem of W. Baur.

Corollary

$Dense$ interprets T_n for every n . Therefore, $Dense$ is undecidable. Similarly for scrolls of discrete and of group types.

Proof.

Let $\langle G_1, \dots, G_n \rangle$ be a chain of ordered groups.

Define $S_n := G_n \times \mathbf{2}$, and $S_i := S_{i+1} \sqcup (G_i \times \mathbf{2})$.

The scroll S_1 interprets the chain. □

Cuts of scrolls

Let S be a scroll, and S^C be the set of Dedekind cuts of S .

If S is of group type, and we define

$$\Lambda + \Gamma := (\Lambda^L +^R \Gamma^L)^+;$$

$$\mathbf{0} := \mathbf{0}_S^+;$$

then S^C is also scroll (either of dense, or of discrete type).

The other possible definitions of $+$ do not give a scroll.

Valuations on scrolls

Definition

A **valued scroll** is a triple $\langle S, v, C \rangle$, where S is a scroll, C is an ordered set with a minimum $-\infty$, and $v : S \rightarrow C$ is a surjective map, satisfying

- ① $v(\mathbf{0}) = -\infty$;
- ② $v(-x) = v(x)$;
- ③ $v(x + y) \leq \max(v(x), v(y))$.

Example

- ① If S is actually a group, a valuation on S as a group is a valuation on S as a scroll, satisfying $v(x) = -\infty$ iff $x = 0$.
- ② The map sending x to \widehat{x} is a valuation, satisfying equality in 3.
- ③ **Define** $x \leq y$ iff there exists $n \in \mathbb{N}$ such that $|x| \leq |y| +^R \dots +^R |y|$.
Define $x \sim_{\mathbb{N}} y$ iff $x \leq y$ and $y \leq x$. The quotient map $S \rightarrow S/\sim_{\mathbb{N}}$ is a valuation.

Proof of main theorem

Proper scrolls

Definition

A scroll S is **proper** if for every $x < y \in S$, if $\widehat{y} > \mathbf{0}$, then there exists $z \in S^{(\mathbf{0})}$ such that $x < z < y$.

Example

G, G^C and \tilde{G} are proper scrolls.
If $\mathcal{W}(S) = \{\mathbf{0}\}$, then S is proper.

Example

Give chain of ordered groups $\langle G_1, \dots, G_n \rangle$, the scroll S_1 defined before is not proper. (*)

$(\mathbb{Q} \times \mathbf{2}) \sqcup \mathbb{Z}$ is not proper.

Trivial scrolls are not proper. (*)

(*) Except in trivial cases.

Idea of proof

For simplicity, we will treat only the case of scrolls of dense type (the other cases are similar, or easier). We want to prove the following:

Main Theorem

Every scroll (of dense type) can be embedded in some G^C .

The proof splits into two steps.

Main Lemma

Every proper scroll can be embedded in some G^C .

Exercise

Prove the Main Lemma for the scroll $\mathbb{Q} \times_{\mathbb{Z}} \mathbf{2}$.

Proper Lemma

Every scroll can be embedded in a proper scroll.

Embedding in proper scrolls

Proper Lemma

Every scroll can be embedded in a proper scroll.

Lemma

Let S be a scroll. If $\mathcal{W}(S)$ is finite, then S can be embedded in a proper scroll.

Proof.

If $\#(\mathcal{W}(S)) = 1$, then S is already proper.

Proceed by induction on $\#(\mathcal{W}(S))$. □

Note that “being a proper scroll” is a first-order property.

Fact

Let T and U be theories in the same language, such that U is universal and $T \vdash U$. The following are equivalent:

- ① $U = T_V$;
- ② Every model of U can be embedded in a model of T ;
- ③ Every **finitely generated** model of U can be embedded in a model of T .

Proof of Proper Lemma.

Observe that if S is finitely generated, then $\mathcal{W}(S)$ is finite.

Let $\mathcal{P}roper$ be the theory of proper scrolls. Apply the above fact to $U = \mathcal{P}roper$ and $T = \mathcal{S}croll$. □

Claim: For every $g \in G \setminus H$, $\widehat{g} = \mathbf{0}$.

This means that, for every $n \in \mathbb{N}$ there exist $x_1, x_2 \in K$ such that $x_1 < g < x_2$ and $x_2 - x_1 < \varepsilon^n$. In fact, assuming $g > 0$, we can take $x_1 = g(1 - \varepsilon^{2n+1})$ and $x_2 = g(1 + \varepsilon^{2n})$.

Define $\phi : S \rightarrow K^C$

$$\phi(s) = \begin{cases} \text{either } [s]^- \text{ or } [s]^+ & \text{if } \widehat{s} = \mathbf{0} \text{ \& } [s] \in H; \\ [s] & \text{if } \widehat{s} = \mathbf{0} \text{ \& } [s] \in G \setminus H; \\ \langle x \in K : x_0 < [s] \text{ in } S/\sim_0 \mid x \in K : x_0 > [s] \text{ in } S/\sim_0 \rangle & \text{if } \widehat{s} > \mathbf{0}. \end{cases}$$

Since S is proper, ϕ is a well-defined embedding of scrolls. □

Proof of Main Lemma**Main Lemma**

Every proper scroll S can be embedded in K^C , for some ordered group K .

For simplicity, we treat only the case when S is of dense type. Let

$$G := G(S) = S^{(0)}/\sim_0,$$

$$H := H(S) = \{g \in G : g \text{ has 2 preimages in } S\},$$

$$G[\varepsilon] := \{g_0 + g_1\varepsilon + \cdots + g_n\varepsilon^n : n \in \mathbb{N}, g_i \in G\}, \text{ with } 0 < \varepsilon \text{ infinitesimal.}$$

Define $K := \{h + g_1(1 - \varepsilon) + g_2(1 + \varepsilon^2) + \cdots + g_n(1 + (-\varepsilon)^n) : n \in \mathbb{N}, g_i \in G, h \in H\} \subseteq G[\varepsilon]$.

Identify each $g \in G$ with $g \cdot \varepsilon^0 \in G[\varepsilon]$.

Claim: $G \cap K = H$.

Hence, each $g \in G \setminus H$ determines 1 cut on K (which we also denote by g), while each $h \in H$ determines 2 cuts, h^+ and h^- .