

O-minimal spectrum

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Abstract

Let X be a definable sub-set of some o-minimal structure. We study the spectrum of X , in relation with the definability of types.

Key words: Spectrum, o-minimality

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1 Introduction

Let M be an o-minimal structure, expanding a group, and $A \subseteq M^n$ be a definable subset. The spectrum of A , denoted by \tilde{A} , as defined by Pillay [8], is the set of complete types over A , with the topology with the following basis of open sets:

$$\{\tilde{U} : U \subseteq A \text{ \& } U \text{ open and definable}\}.$$

Note that \tilde{A} has also the Stone topology, which is finer than the spectral one. When speaking of topological notions about A , we refer to the spectral topology, unless explicitly said otherwise.

\tilde{A} is a T_0 space, but not a T_1 space: namely, points are not closed. In this situation, there is the so-called specialization order: x is a specialization of y (written $x \leq y$) iff $\text{cl}(x) \subseteq \text{cl}(y)$.

[8] proved that \tilde{A} is a spectral space. Namely, \tilde{A} has a basis of quasi-compact open sets stable under finite intersections, and every irreducible closed set is the closure of a unique point. Coste and Carral [2] study spectral spaces in general, and the normal ones in particular (since A is definably normal, \tilde{A} is normal).

We continue the study of the properties of \tilde{A} . The main theme is the relationship between the specialization order \leq , the Rudin-Keisler order \leq_{RK} , and the dichotomy between rational and irrational types (cf. Theorem 3, Lemma 6.19, Corollary 6.29).

In §2, we list some results on o-minimal structures and on topological spaces, which will be used in later sections.

In §3, we collect some basic results on the spectrum of A . In particular, we prove that, for every $x \in \tilde{A}$, $\text{cl}(x)$ is totally ordered by the specialization order (Lemma 3.14).

Any definable function $f : A \rightarrow B$ induces a function $\tilde{f} : \tilde{A} \rightarrow \tilde{B}$, which is continuous if f is. We say that $y \leq_{\text{RK}} x$ iff $y = \tilde{f}(x)$ for some definable function f . In §4, we study the properties of \tilde{f} and of \leq_{RK} . In particular, we prove that, if $y \leq x$, then $x \leq_{\text{RK}} x$ (Theorem 1).

In the remainder of the article, we assume that M expands a field. A type x is called strongly closed iff x is closed in some \tilde{A} , with A definably compact. In §5, we study the definable compactifications of definable sets, and give some results about strongly closed types (Definition 5.15 and Theorem 2).

The main results of the article are in §§6 and 7. In §6, we investigate the relationship between rational and strongly closed types (Theorem 3). We further analyze the relationship between \leq_{RK} and rationality (Theorem 4, Lemma 6.17, Lemma 6.19, and Corollary 6.29).

In §7, we study the amalgam of a rational and a totally irrational extension of M (Theorem 5).

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2 Preliminaries about o-minimal structures

Let M be an o-minimal structure, expanding an ordered group. Let A be a definable sub-set of M^k , for some $k \in \mathbb{N}$.

In the following, if M expands an ordered field, then 1 will denote the neutral element of the multiplication. Otherwise, 1 will be some fixed element of M such that $1 > 0$. Definable will mean “definable with parameters from M ”, unless explicitly stated otherwise.

2.1 Notation. Let X be a topological space. For every $Y \subseteq X$ we shall denote by $\text{cl}_X(Y)$ (or simply by $\text{cl}(Y)$ if X is clear from the context) the topological closure of Y in X . The frontier of Y is

$$\partial_X Y := \text{cl}_X(Y) \setminus Y.$$

2.2 Remark. If $A \subseteq B \subseteq M^k$ are non-empty and definable, then $\dim(\partial_B A) < \dim A$.

2.3 Definition. Let $X \subseteq M^k$. We say that X is definably compact, or **d-compact** for short, iff X is definable, closed and bounded. X is **locally d-compact** iff X is definable, and for every $x \in X$ there exists a d-compact neighborhood of x (in X).

The reader can skip the remainder of this section, and refer back to it when needed.

2.4 Lemma. Let Z be a topological space. Let $C, U \subseteq Z$ such that U is open (in Z). Then,

$$U \cap \text{cl}_Z(C) = U \cap \text{cl}_Z(U \cap C) = \text{cl}_U(U \cap C).$$

Proof. The fact that $U \cap \text{cl}_Z(C) \supseteq U \cap \text{cl}_Z(U \cap C)$ is obvious. For the opposite inclusion, let $x \in U \cap \text{cl}_Z(C)$. Hence, $x \in U$. If, for contradiction, $x \notin \text{cl}_Z(U \cap C)$, then

$$x \in \text{cl}_Z(C \setminus U) \subseteq \text{cl}_Z(Z \setminus U) = Z \setminus U,$$

because U is open, absurd. □

2.5 Corollary. If $C, U \subseteq A$ are definable, and U is open, then

$$\dim(U \cap C) = \dim(U \cap \text{cl}(C)).$$

Proof. $U \cap \text{cl}(C) = \text{cl}_U(U \cap C)$, and $\dim(\text{cl}_U(U \cap C)) = \dim(U \cap C)$. □

2.6 Lemma. Let C_1, C_2 be definable disjoint sub-sets of A , and $m := \max(\dim(C_1), \dim(C_2))$. Then,

$$\dim(\text{cl}(C_1) \cap \text{cl}(C_2)) < m.$$

Proof. In fact,

$$\begin{aligned} \text{cl}(C_1) \cap \text{cl}(C_2) &= ((\text{cl}(C_1) \cap \text{cl}(C_2)) \setminus C_1) \cup ((\text{cl}(C_1) \cap \text{cl}(C_2)) \setminus C_2) \subseteq \\ &\subseteq (\text{cl}(C_1) \setminus C_1) \cup (\text{cl}(C_2) \setminus C_2). \end{aligned}$$

Since $\dim(\text{cl}(C_i) \setminus C_i) < \dim C_i \leq m$ for $i = 1, 2$, we are done. □

2.7 Lemma. Let $C_1, C_2 \subseteq A$ be closed and definable, and $C_0 := C_1 \cap C_2$. Then, there exist $V_i \subseteq A$ open and definable, $i = 1, 2$, such that

1. $C_i \setminus V_i = C_0$, $i = 1, 2$;

2. $V_1 \cap V_2 = \emptyset$;
3. $\text{cl}(V_1) \cap \text{cl}(V_2) \subseteq C_0$.

Proof. Define

$$V_1 := \{ a \in A : d(a, C_1) < \frac{1}{3}d(a, C_2) \},$$

where d is the Euclidean distance (or equivalently the max distance if M has no field structure), and similarly for V_2 .

Alternative proof: let

$$\begin{aligned} U &:= A \setminus C_0, \\ C'_i &:= C_i \setminus C_0, \quad i = 1, 2. \end{aligned}$$

U is definable, and therefore definably normal. Moreover, C'_1 and C'_2 are disjoint open sub-sets of U . Therefore, there exist $V_1, V_2 \subseteq U$ disjoint and open in U (and hence open in X) such that $C'_i \subseteq V_i$. \square

2.8 Definition. Let X be a topological space. X is a T_5 space (also called **completely normal**) iff every sub-space of X is T_4 .¹

Note that any metric space is T_5 . Note moreover that T_4 does not imply T_5 .

2.9 Remark. Let X be a topological space. The following are equivalent:

1. X is T_5 .
2. For every $U \subseteq X$, if U is open, then U is T_4 .
3. Let $C_1, C_2 \subseteq X$ be closed, and $C_0 := C_1 \cap C_2$. Then, there exist $V_1, V_2 \subseteq X$ open, and satisfying conditions 1–3 of Lemma 2.7.
4. For every $D_1, D_2 \subseteq X$, if $D_1 \cap \text{cl}(D_2) = \text{cl}(D_1) \cap D_2 = \emptyset$, then there exist V_1, V_2 disjoint open sub-sets of X , such that $D_i \subseteq V_i$.

2.10 Lemma. Let $C \subseteq U \subseteq D \subseteq M^k$ be definable, such that D is d -compact and U is open in D . Let $f : D \rightarrow M$ be a definable continuous function, such that $C = f^{-1}(0)$. Then, there exists $\varepsilon \in M$ such that $0 < \varepsilon$ and $f^{-1}(\text{]}-\varepsilon, \varepsilon[) \subseteq U$.

Proof. Note that C must be closed. Assume, for contradiction, that for every $t > 0$ there exists $a_t \in D \setminus U$ such that $|f(a_t)| \leq t$. By definable choice, we can find a_t as above that is a definable function of t . Since D is definably compact, there exists $\lim_{t \rightarrow 0^+} a_t =: a$. However, $a \in C \setminus U$, a contradiction. \square

Therefore, if D is d -compact and $f : D \rightarrow M$ is definable and continuous, the family $\{ f^{-1}(\text{]}-t, t[) : t > 0 \}$ is a fundamental system of open neighborhoods of $f^{-1}(0)$. Note that this is not true if D is not d -compact.

A variant of the following lemma is [14, Lemma 1.1].

2.11 Lemma. Let $C \subseteq A \subseteq M^n$ be definable, such that C is a cell. Then, there exists a definable open neighborhood V of C (in A) such that C is a retract of V . Namely, there exists a definable continuous map $\rho : V \rightarrow C$, such that the restriction of ρ to C is the identity.

¹We do not assume that T_4 implies Hausdorff.

Proof. First, we will do the case when $A = M^n$. The proof is by induction on n . If $n = 0$, the conclusion is trivial. Since C is a cell, there exists a cell $D \subseteq M^{n-1}$ such that one of the following cases happens:

1. $C = \Gamma(f) := \{ (x, y) \in M^n : x \in D \text{ \& } f(x) = y \}$, where $f : D \rightarrow M$ is a continuous definable function;
2. $C =]f, g[_D \{ (x, y) \in M^n : x \in D \text{ \& } f(x) < y < g(x) \}$, where $f, g : D \rightarrow M \sqcup \{ \pm\infty \}$ are continuous definable functions such that $f < g$.

By inductive hypothesis, there exist a definable open neighborhood W of D and a retraction $\sigma : W \rightarrow D$. In the first case, let $V := W \times M$, and define

$$\rho(w, t) := (\sigma(w), f(w)).$$

In the second case, define

$$\begin{aligned} F : W \times M &\rightarrow D \times M \\ (w, t) &\mapsto (\sigma(w), t). \end{aligned}$$

Let $V := F^{-1}(C)$, and ρ be the restriction of F to V . Since C is open in $D \times M$, V is an open neighborhood of C .

When $A \neq M^n$, let V' and $\rho' : V' \rightarrow C$ be the open neighborhood of C in M^n and the retraction whose existence we proved above. Define $V := V' \cap A$, and $\rho := \rho' \upharpoonright V$. \square

2.12 Example. Note that in general C is not a retract of all M^n . For instance, let $n = 1$ and $C =]0, 1[$.

2.13 Lemma. *When M expands a real closed field, in Lemma 2.11, we can weaken the hypothesis to C locally closed in A , instead of C cell.*

Proof. It is [12, Proposition 8.3.3]. \square

2.14 Question. What happens if we drop the condition that M expands a real closed field in Lemma 2.13?

2.15 Lemma. *Let X be a topological spaces, and Y, U be sub-spaces of X such that Y is dense and U is open in X . Then, $\text{cl}_X(Y \cap U) = \text{cl}_X(U)$.*

Proof. It is obvious that $\text{cl}(Y \cap U) \subseteq \text{cl}(U)$. For the opposite inclusion, let $b \in \text{cl}(U)$, and V be an open neighborhood of b . Since U is open, $W := V \cap U$ is also open, and since $b \in \text{cl}(U)$, $W \neq \emptyset$. Therefore, $(Y \cap U) \cap V = W \cap Y \neq \emptyset$, because U is dense in X , and thus $b \in \text{cl}(Y \cap U)$. \square

2.16 Lemma. *Let A be definable, closed and locally d -compact, and B be definable, such that $A \subseteq B$ and A is dense in B . Then, A is open in B .*

Proof. Fix $a \in A$. Let K be a d -compact neighborhood of a . Let $U \subseteq K$ be a definable open neighborhood of a (in A). Therefore, there exists $V \subseteq B$ open and definable such that $U := V \cap A$. Therefore,

$$A \supseteq K = \text{cl}_B(K) \supseteq \text{cl}_B(U) = \text{cl}_B(V \cap A).$$

By Lemma 2.15, $\text{cl}_B(V \cap A) = \text{cl}_B(V)$. Therefore, $V \subseteq \text{cl}_B(V) \subseteq A$, and therefore A is a neighborhood of a . \square

2.17 Lemma. *Let $C \subseteq M^{k+h}$ be a cell, and $\pi : M^{k+h} \rightarrow M^k$ be the projection on the first k coordinates. Then, exactly one of the following 2 things happens: either $\pi \upharpoonright C$ is injective, or $\dim \pi(C) < \dim C$.*

2.18 Lemma. *Let $f : M^h \rightarrow M^k$ be definable. Then, there exists a decomposition of M^h into definable sets $\{C_i : i \leq n\}$, such that for every $i \leq n$, either $f \upharpoonright C_i$ is injective, or $\dim f(C_i) < h$.*

*Proof.*² Let

$$S := \{ (f(a), a) \in M^{k+h} : a \in M^h \}.$$

Decompose M^{k+h} into cells, in a way compatible with S . Let $D \subseteq S$ be a cell, $C := f^{-1}(D)$, and $\pi : M^{k+h} \rightarrow M^k$ the projection on the first k coordinates. Note that $f(C) = \pi(D)$, and hence $\dim D \leq h$. Moreover, $f \upharpoonright C$ is injective iff $\pi \upharpoonright D$ is injective. By Lemma 2.17, either $\dim \pi(D) < h$, or $\pi \upharpoonright D$ is injective. \square

2.19 Lemma. *Let X be a topological space, $Z \subseteq X$ be connected, and $U \subseteq X$ be open. If $Z \cap U \neq \emptyset$ and $Z \setminus U \neq \emptyset$, then $Z \cap \partial U \neq \emptyset$.*

Proof. Assume that $Z \cap \partial U = \emptyset$. Then,

$$Z = (Z \cap U) \sqcup (Z \setminus U) = (Z \cap \text{cl}(U)) \sqcup (Z \setminus U),$$

contradicting the fact that Z is connected. \square

3 Specialization

Let M be an o-minimal structure, expanding an ordered group. Let A be a definable sub-set of M^k , for some $k \in \mathbb{N}$, and $X := \widetilde{A}$ be the spectrum of A (namely, the set of complete types over A).

In this section we study the basic properties of X . Most of these results are well-known, at least in the case when M expands a field [2], [7], [8], and [5].

3.1 Definition. Let $C \subseteq X$. C is **definable** iff C is of the form \widetilde{D} , where $D \subseteq A$ is definable. $C \subseteq X$ is **type-definable** iff C is closed in the Stone topology, or, equivalently, C is an intersection of definable sets.

3.2 Definition. The **spectral topology** in X is the topology generated by the sets of the form \widetilde{U} , where $U \subseteq A$ is open and definable; cf. [8].

3.3 Remark. Since the Stone topology is Hausdorff, any finite set is type-definable. Any definable set is type-definable. Since the Stone topology is stronger than the spectral one, any closed set is type-definable. Since the Stone topology is compact, the spectral one is quasi-compact. If $C \subseteq X$ is type-definable, and $f : A \rightarrow B$ is definable, then $\widetilde{f}(C)$ is also type-definable.

3.4 Remark. \sim is an injective morphism of Boolean algebras between the definable subsets of A and the subsets of X . Moreover, if $B \subseteq A$ is definable, then

$$\widetilde{\text{cl}(B)} = \text{cl}(\widetilde{B}).$$

²Thanks to prof. Berarducci for the proof.

3.5 *Example.* However, \sim does *not* preserve infinite unions or infinite intersections. For instance, let $A := [0, 1]$, $U_t :=]-t, t[\subseteq A$. Then, $\bigcap_{0 < t \in M} U_t = \{0\}$, but

$$\bigcap_{0 < t \in M} \tilde{U}_t = \{0^-, 0, 0^+\},$$

where 0^+ is the cut $] -\infty, 0],]0, +\infty[$, and similarly for 0^- .

3.6 Lemma. *Let $C \subseteq X$ be open. Then, C is quasi-compact iff C is definable.*

Proof. The “if” direction is trivial. On the other hand, since C is open, $C = \bigcup_{i \in I} U_i$, where each U_i is open and definable. If C is quasi-compact, then there exists $I_0 \subseteq I$ finite, such that $C = \bigcup_{i \in I_0} U_i$, and thus C is definable. \square

3.7 Definition. Let $x \in X$. The **dimension** of x is

$$\dim x := \min(\dim C),$$

where C varies among the definable sub-sets of X containing x . Given $C \subseteq X$ definable, the **local dimension** of C at x is

$$\text{loc dim}(x; C) := \min(\dim U \cap C),$$

where U varies among the definable *open* sub-sets of X containing x . Define $\text{loc dim}(x; C) = -\infty$ iff there exists an definable open U containing x such $U \cap C = \emptyset$. Note that $\text{loc dim}(x; C) \geq 0$ iff $x \in \text{cl}(C)$, where $\text{cl}(C)$ is the closure of C in the spectral topology.

We shall write $\text{loc dim}(x)$ instead of $\text{loc dim}(x; A)$.

We say that $\text{loc dim}(C)$ is constantly equal to n iff $\text{loc dim}(x; C) = n$ for every $x \in \text{cl}(C)$. Given $D \subseteq A$ definable, we define $\text{loc dim}(x; D) := \text{loc dim}(x; \tilde{D})$.

3.8 Lemma. *$\dim x$ is the dimension of c/M , for any $N \ni c \models x$.*

3.9 *Remark.* For every $x \in X$,

$$\dim x \leq \text{loc dim}(x) \leq \dim A.$$

For every $C \subseteq X$ definable, Corollary 2.5 implies that

$$\text{loc dim}(x; C) = \text{loc dim}(x; \text{cl}(C)).$$

If $D \subseteq A$ is a cell of dimension n , then $\text{loc dim} D$ is constantly equal to n . If $\text{loc dim} C$ is constantly equal to n , then $\text{loc dim}(\text{cl}(C))$ is also constantly equal to n .

3.10 *Remark.* $\text{loc dim}(x)$ is the minimum of the dimensions of the definable open sub-sets containing x .

Intuitively, the local dimension of x is the dimension of the ambient space A in a neighborhood of x . On the other hand, the dimension of x tells us “how big” a definable sub-set containing x must be.

3.11 *Example.* X is not T_1 in general (namely, not every point is closed). For instance, let $A := M$, $x = 0^+$. Then, $\text{cl}(x) = \{0, 0^+\}$.

3.12 Definition. For every $x, y \in X$, we say that x is a **specialization** of y (and y is a **generalization** of x) iff $x \in \text{cl}(y)$, where $\text{cl}(y)$ is the closure of $\{y\}$ in the spectral topology, and we write $x \leq y$ for this. We shall say that x is a closed point iff $\text{cl}(x) = \{x\}$, namely iff x is minimal w.r.t. the order \leq .

3.13 Lemma. 1. \leq is a partial order on X .

2. If $x \leq y$, then $\dim x \leq \dim y \leq \text{loc dim}(x)$.
3. If $x \leq y$ and $D \subseteq X$ is definable, then $\text{loc dim}(x; D) \geq \text{loc dim}(y; D)$.
4. If $x < y$ (namely, $x \leq y$ and $x \neq y$), then $\dim x < \dim y$.

Proof. 1) The transitivity of \leq is obvious. To prove that \leq is a partial order, it is enough to show that if $x \leq y$ and $y \leq x$, then $x = y$. The hypothesis is equivalent to $\text{cl}(x) = \text{cl}(y)$. Assume for contradiction that $x \neq y$. Let $B, C \subseteq X$ be definable subsets such that

$$\begin{aligned} x &\in B \setminus C \text{ and} \\ y &\in C \setminus B. \end{aligned}$$

W.l.o.g., we can assume that $\dim B = \dim x \geq \dim y = \dim C$, and B and C disjoint. Therefore, by Lemma 2.6,

$$\dim(\text{cl}(B) \cap \text{cl}(C)) < \dim B.$$

Thus, $x \notin \text{cl}(B) \cap \text{cl}(C)$, and so $x \notin \text{cl}(C)$. However, $\text{cl}(y) \subseteq \text{cl}(C)$, absurd.

2) Let $U \subseteq X$ be definable and open, such that $x \in U$ and $\dim U = \text{loc dim}(x)$. Moreover, $y \in U$, and therefore $\dim y \leq \dim U = \text{loc dim}(x)$.

Assume, for contradiction, that $x \leq y$, but $m := \dim x > \dim y =: n$. Let $C \subseteq X$ be definable such that $y \in C$ and $\dim C = n$. Moreover, $\dim \text{cl}(C) = \dim C = n$, therefore we can assume that C is closed. Besides, $x \notin C$, because $\dim x > \dim C$. However, C is closed and $y \in C$, and therefore $\text{cl}(y) \subseteq C$, absurd.

3) Let $U \subseteq X$ be a open and definable, such that $\dim(U \cap D) = \text{loc dim}(x; D)$, and $x \in U$. Then, $y \in U$, and we are done.

4) Assume, for contradiction, that $x < y$, but $\dim x = \dim y =: m$. Let $E \subseteq A$ be definable, such that $y \in \widetilde{E}$, and $\dim E = m$. Let $A' := \text{cl}(E)$. Then, $\dim A' = m$, and $x, y \in \widetilde{A}'$. Therefore, w.l.o.g., we can assume that $\dim A = m$. Since $<$ is irreflexive, $y \notin \text{cl}(x)$, namely there exist $U \subseteq X$ open and definable such that $y \in U$, but $x \notin U$. However, $x \in \text{cl}(U) \setminus U$, because $x \in \text{cl}(y)$. Therefore, $\dim x < m$, absurd. \square

3.14 Lemma. For every $x \in X$, $\text{cl}(x)$ is totally ordered by \leq . Moreover, $\#(\text{cl}(x)) \leq 1 + \dim x \leq 1 + \text{loc dim}(x) \leq 1 + \dim A$.

Proof. Assume, for contradiction, that there exist $y_1, y_2 \in \text{cl}(x)$, such that $y_1 \not\leq y_2$ and $y_2 \not\leq y_1$. Let $D_i \subseteq A$ be definable such that

$$\dim D_i = \dim y_i =: m_i, \quad \text{and} \quad y_i \in \widetilde{D}_i \setminus \widetilde{D}_{2-i}, \quad i = 1, 2.$$

W.l.o.g., we can assume that $m_2 \leq m_1$.

Claim 1. $y_2 \leq y_1$.

Note that the claim contradicts $y_2 \not\leq y_1$, which is absurd.

Assume not. Then, there exists $U \subseteq A$ open and definable, such that $y_2 \in \widetilde{U}$, but $y_1 \notin \widetilde{U}$. By substituting D_1 with $D_1 \cap U$, and D_2 with $D_2 \setminus U$, we can assume that $D_2 \subseteq U$, and $D_1 \subseteq A \setminus U$, and in particular that $D_1 \cap D_2 = \emptyset$.

Let $C_i := \text{cl}(D_i)$, $i = 1, 2$, and $C_0 := C_1 \cap C_2$. Note that $C_0 \subseteq \text{cl}(U) \setminus U$. Hence, $y_2 \notin \widetilde{C}_0$. Besides, since $D_1 \cap D_2 = \emptyset$, by Lemma 2.6 we have $\dim C_0 < m_1$, and therefore $y_1 \notin \widetilde{C}_0$. Let V_1 and V_2 be as in Lemma 2.7. Since $V_1 \cap V_2 = \emptyset$, x cannot be both in \widetilde{V}_1 and

in \tilde{V}_2 . Assume that $x \notin \tilde{V}_i$. Since $y_i \notin C_0$, we have $y_i \in \tilde{V}_i$. However, $y_i \in \text{cl}(x) \subseteq X \setminus \tilde{V}_i$, absurd.

Therefore, we have proved that \leq is a total order on $\text{cl}(x)$. Using Lemma 3.13, we conclude that $\#\text{cl}(x) \leq 1 + \dim x$. \square

3.15 Example. Let $x \in A$. The set $\{y \in X : y \geq x\}$ in general is *not* totally ordered by \leq . For instance, let $A = M$, $x = 0$, $y = 0^+$ and $y' = 0^-$. Then, $x < y, y'$, but neither $y \leq y'$ nor $y' \leq y$.

3.16 Example. Both cases are possible: $\#\text{cl}(x) = \dim x$ and $\#\text{cl}(x) < \dim x$. For instance, let $A = M$. Let $x = 0^+$ and $x' = +\infty$. Then, $\dim x = \dim x' = 1$. However, $\text{cl}(x) = \{0, 0^+\}$, while $\text{cl}(x') = \{x'\}$.

3.17 Remark. If $x, y \in X$ are such that neither $x \leq y$ nor $y \leq x$, then there exist disjoint open definable sub-sets $U, V \subseteq X$ such that $x \in U$ and $y \in V$.

3.18 Definition. Let $\iota : A \rightarrow X$ be the natural embedding, sending $c \in A$ to the type $x(a) := "a = c"$.

3.19 Remark. ι is a continuous map.

3.20 Remark. Let $x \in X$. Then, $\dim x = 0$ iff $x = \iota(a)$ for some $a \in A$. In that case, x is a closed point.

3.21 Example. Not all closed point have dimension 0. For instance, if $A = M$, then $+\infty$ is a closed point of dimension 1.

3.22 Remark. Let $x \in X$. Define the following partial types:

$$\begin{aligned}\Phi(a) &:= \{a \in U : U \subseteq A \text{ definable and open, } x \in \tilde{U}\}, \\ \Psi(a) &:= \{a \in C : C \subseteq A \text{ definable and closed, } x \in \tilde{C}\}.\end{aligned}$$

Then,

$$\begin{aligned}\tilde{\Phi} &= \{y \in X : x \leq y\}, \\ \tilde{\Psi} &= \{y \in X : x \geq y\} = \text{cl}(x).\end{aligned}$$

Equivalently,

$$\begin{aligned}\{y \in X : x \leq y\} &= \bigcap \{U : U \subseteq X \text{ definable and open, } x \in U\} \\ \text{cl}(x) &= \bigcap \{C : C \subseteq X \text{ definable and closed, } x \in C\}.\end{aligned}$$

3.23 Remark. More in general, if $D \subseteq X$, then

$$\text{cl}(D) = \bigcap \{C : C \subseteq X \text{ definable and closed, } D \subseteq C\}.$$

Moreover, D is closed iff

$$D = \bigcap \{C : C \subseteq X \text{ definable and closed, } D \subseteq C\}.$$

3.24 Corollary. Let $x \in X$ such that $\dim x = \text{loc dim}(x)$. Then,

$$\{x\} = \bigcap \{U : U \subseteq X \text{ definable and open, } x \in U\}$$

3.25 Lemma. Let $x \in X$. Let $m := \dim x$, and $n := \text{loc dim}(x)$. Then, there exists $y \in X$ such that $x \leq y$ and $\dim y = n$.

Proof. If $m = n$, take $y := x$. If $m < n$, let

$$\Phi(a) := \{a \in U \setminus C : C, U \subseteq A \text{ definable}, x \in \tilde{U}, \dim C < n, U \text{ open}\}.$$

By definition of local dimension, it is easy to see that Φ is consistent set of formulae. Let $y \in \tilde{\Phi}$. By definition of Φ , $\dim y = n$ and $x \leq y$. \square

3.26 Remark. Let $\dim A = n$, and $k \leq n$. The set $X_k := \{x \in X : \dim x \geq k\}$ is of the form $\tilde{\Phi}_k$ (and hence X_k is type-definable), where $\Phi_k(a)$ is the partial type

$$\{a \notin C : C \subset A \text{ definable}, \dim C < k\}.$$

Moreover, both $\iota(A)$ and X_k are dense in X , and, if $k > 0$, they are disjoint. Finally, $\iota(A)$ is open in the Stone topology, because $X \setminus \iota(A) = X_1$.

Proof. $\iota(A)$ is dense, because it is dense in the Stone topology, which is stronger than the spectral one. X_k is dense by Lemma 3.25. \square

3.27 Example. Let $x, y \in X$ with $x < y$ and $m := \dim x$ and $n := \dim y$. It is not true in general that if $m < l < n$, then there exists $z \in X$ such that $x < z < y$ and $\dim z = l$. For instance, let $A' := M^2$, $x = (0, 0)$, $z = (0^+, 0)$ (z is a complete type) and $y \in A'$ such that $\dim y = 2$ and $z < y$ (y exists by Lemma 3.25). Note that $x < z < y$. Let A be $A' := \{(a, 0) : a > 0\}$. Since $z \in X' \setminus X$, and in $\text{cl}(y)$ there is at most one z' such that $\dim z' = 1$, there is no $z' \in X$ such that $z' < y$ and $\dim z' = 1$.

Let A be definably compact. Let $x, y \in X$ with $x < y$ and $m := \dim x$ and $n := \dim y$, and let $l \in \mathbb{N}$ be such that $m < l < n$. Later (examples 6.3 and 6.4) we will show that there might *not* exist $z \in X$ such that $x < z < y$ and $\dim z = l$.

3.28 Lemma. *Let $C \subseteq X$ be type-definable. Then,*

$$\text{cl}(C) = \bigcup \{\text{cl}(x) : x \in C\}.$$

Proof. It is obvious that $\text{cl}(C) \supseteq \bigcup \{\text{cl}(x) : x \in C\}$. For the other inclusion, we have to prove that for every $y \in \text{cl}(C)$ there exists $x \in C$ such that $y \in \text{cl}(x)$. Since C is type-definable, we can write $C = \bigcap_{i \in I} C_i$, where each C_i is definable. Let Φ be the following partial type

$$\Phi(a) = \{a \in U \cap C_i : i \in I, U \subseteq A \text{ open and definable}, y \in \tilde{U}\}.$$

Since $y \in \text{cl}(C)$, Φ is consistent. Any $x \in \tilde{\Phi}$ satisfies $y \leq x$ and $x \in C$. \square

3.29 Corollary. *Let $C \subseteq X$. Then, C is closed iff C is type-definable and*

$$\forall x \in C \quad \text{cl}(x) \subseteq C. \tag{3.1}$$

Proof. The “only if” direction is trivial. For the “if” direction, Lemma 3.28 implies that $\text{cl}(C) = C$. \square

3.30 Example. There are some sub-sets $C \subset X$ that are not closed, but do satisfy (3.1). By the Corollary, any such C cannot be type-definable. For instance, let $C := \iota(A)$. C is not closed (unless A is finite), because $\text{cl}(C) = X$, but C does satisfy (3.1).

3.31 Lemma. *A is definably connected iff X is connected.*

Proof. Easy. □

3.32 Lemma. *Let $M \preceq N$, and*

$$\theta : \widetilde{A(N)} \rightarrow \widetilde{A(M)}$$

be the restriction map. Then, θ is continuous both in the Stone and the spectral topologies.

3.33 Example. θ is neither closed, nor open. For instance, let $A = M$, $x = 0^+$, $c \in N$ such that $c \models 0^+$, $y := \iota(c) \in \widetilde{N}$. Then, y is a closed point, but $\theta(y) = x$ is not. Let $U :=]-c, c[\subseteq \widetilde{N}$. Then, U is open, but $\theta(U) = \{0^-, 0, 0^+\}$ is not open.

3.34 Question. Is θ closed or open in the Stone topology?

3.35 Lemma. *Let A, B be definable, and $\pi : A \times B \rightarrow A$ be the projection onto the first coordinate. Let $N \succeq M$, $c \in A(N)$, and $x := \text{tp}(c/A) \in X = \widetilde{A}$. Let $\tau : B(\widetilde{M(c)}) \rightarrow \widetilde{A} \times B$ defined by*

$$\tau(y(b)) := x(a) \ \& \ \{ \phi(a, b) : \phi(c, b) \in y(b) \}.$$

Then, τ is well-defined, and it is a (surjective) homeomorphism.

The case $B = [0, 1]$ of the above Lemma is in [3].

3.36 Lemma. *Let $Z \subseteq X$ be type-definable and connected. Let θ be as in the Lemma above. Then, $\theta^{-1}(Z)$ is type-definable and connected.³*

Proof. Let $Z = \bigcap_{i \in I} \widetilde{C}_i$. Then,

$$W := \theta^{-1}(Z) = \bigcap_{i \in I} \widetilde{C}_i(N).$$

Assume, for contradiction, that W is disconnected. Namely, there exist T_1 and T_2 open such that

$$\begin{aligned} W &\subseteq T_1 \cup T_2, \\ T_1 \cap T_2 \cap W &= \emptyset, \\ W \cap T_i &\neq \emptyset, \quad i = 1, 2. \end{aligned}$$

Claim 1. We can also assume that the T_i are definable (in N).

In fact,

$$T_i = \bigcup_{j \in J_i} U_{i,j},$$

where the $U_{i,j}$ are open and definable (in N). By compactness of the Stone topology, there exist $J'_i \subseteq J_i$ finite, $i = 1, 2$, such that

$$\begin{aligned} W &\subseteq T'_1 \cup T'_2, \\ W \cap T'_i &\neq \emptyset, \quad i = 1, 2, \end{aligned}$$

where $T'_i := \bigcup_{j \in J'_i} U_{i,j}$. Moreover, $T'_1 \cap T'_2 \cap W \subseteq T_1 \cap T_2 \cap W = \emptyset$.

³Thanks to Berarducci for the proof.

In particular, $\bigcap_{i \in I} \widetilde{C}_i(N) \subseteq T_1 \cup T_2$. Using again the compactness of the Stone topology, we conclude that there exists $I_0 \subseteq I$ finite, such that

$$\begin{aligned} \widetilde{C(N)} &\subseteq T_1 \cup T_2, \\ \widetilde{C(N)} \cap T_1 \cap T_2 &= \emptyset, \end{aligned}$$

where $C := \bigcap_{i \in I_0} C_i$. We have that $Z \subseteq \widetilde{C}$. Since Z is connected, $Z \subseteq \widetilde{D}$ for a (unique) definably connected component D of C . Hence, $W \subseteq \widetilde{D(N)} \subseteq T_1 \cup T_2$. Moreover, $\widetilde{D(N)} \cap T_i \supseteq W \cap T_i \neq \emptyset$. However, $D(N)$ is definably connected, hence $\widetilde{D(N)}$ is connected, a contradiction. \square

3.37 Conjecture. Let $M \preceq N$, $Z \subseteq X$ be type-definable, and $W := \theta^{-1}(Z)$, where θ is as in Lemma 3.32. Then, θ induces an isomorphism between the Čeck cohomology of Z and the one of W .

The conjecture is true if M expands a field. In fact, in that case we know that it holds if Z is definable, and therefore

$$\check{\mathcal{H}}(Z) = \varinjlim_{C \in \mathfrak{D}^A(Z)} \check{\mathcal{H}}(\widetilde{C}) = \varinjlim_{C \in \mathfrak{D}^A(Z)} \check{\mathcal{H}}(\theta^{-1}(\widetilde{C})) = \varinjlim_{D \in \mathfrak{D}^A(N)(\theta^{-1}(Z))} \check{\mathcal{H}}(\widetilde{D}) = \check{\mathcal{H}}(\theta^{-1}(\widetilde{Z})),$$

where

$$\mathfrak{D}^A(Z) := \{ C \subseteq A : Z \subseteq C \text{ \& } C \text{ is definable} \}.$$

The fact that $\check{\mathcal{H}}(Z) = \varinjlim_{C \in \mathfrak{D}^A(Z)} \check{\mathcal{H}}(\widetilde{C})$ will be proved elsewhere. Note that the same proof works when N is an o-minimal expansion of M , instead of an elementary extension.

3.1 Beyond o-minimality

Let M be a first order topological structure (in the sense of Pillay [7]). We shall say that M is definably T_5 iff every definable sub-set of M^k is definably T_4 , for every $k \in \mathbb{N}$.

3.38 Lemma. *The following are equivalent:*

1. M is definably T_5 .
2. For every $U \subseteq M^k$, if U is definable and open, then U is definably T_4 .
3. Lemma 2.7 is true for any definable $A \subseteq M^k$.
4. For every $D_1, D_2 \subseteq X$ definable, if $D_1 \cap \text{cl}(D_2) = \text{cl}(D_1) \cap D_2 = \emptyset$, then there exist V_1, V_2 disjoint definable open sub-sets of X , such that $D_i \subseteq V_i$.
5. For every $A \subseteq M^k$ definable, \widetilde{A} is T_4 .

3.39 Conjecture. M is definably T_5 iff, for every $A \subseteq M^k$ definable, \widetilde{A} is T_5 .

Most of the results in this section apply to the following situation, with the same proofs (in particular, Lemmata 3.13 and 3.14 hold). M is a first order such that for every $k \in \mathbb{N}$ there is a function

$$\dim : \mathfrak{D}^{M^k} \rightarrow [-1, k],$$

where

$$\mathfrak{D}^{M^k} := \{A \subseteq M^k : A \text{ definable}\},$$

satisfying the following conditions:

$$M \text{ is definably } T_5; \quad (3.2)$$

$$\dim A = -1 \text{ iff } A = \emptyset; \quad (3.3)$$

$$\dim(A \cup B) = \max(\dim A, \dim B); \quad (3.4)$$

$$\dim(\partial A) < \dim A \text{ if } A \neq \emptyset; \quad (3.5)$$

$$\dim\{a\} = 0 \forall a \in M^k. \quad (3.6)$$

Note that M must be T_2 . In fact, $\dim(\text{cl}(a) \setminus \{a\}) = -1$ for every $a \in M^k$. Hence, M^k is T_1 , and, since M is also definably T_4 , we have that M is T_2 too.

3.40 Remark. If $A \subseteq M^k$ is definable, then \tilde{A} is a T_4 spectral space.

Proof. The proof of [8, Lemma 1.1] works also in this context. More precisely, the definable open sets form a basis of quasi-compact open sub-sets of \tilde{A} , stable under finite intersections. Therefore, we need only to show that every irreducible closed set is the closure of a unique point. Let $C \subseteq \tilde{A}$ be closed and irreducible. Note that C is type-definable, and hence compact in the Stone topology. Let

$$\mathcal{D} := \{D \subseteq \tilde{A} : D \text{ is definable and closed } \& C \setminus D \neq \emptyset\}.$$

Claim 1. There exists $x \in C \setminus \bigcup \mathcal{D}$.

If, for contradiction, $C \subseteq \bigcup \mathcal{D}$, then \mathcal{D} is a covering of C by definable sets, and therefore, by compactness, there exists $D_1, \dots, D_n \in \mathcal{D}$ such that $C \subseteq \bigcup_{i=1}^n D_i$. Since C is irreducible, $C \subseteq D_i$ for some $i \leq n$, absurd.

It is now easy to see that $C = \text{cl}(x)$. Uniqueness is a consequence of Lemma 3.13.

Since A is definably T_5 , \tilde{A} is T_4 (but we do not know whether it is T_5). \square

3.41 Remark. If $A \subseteq M^k$ is definable, then A is a boolean combination of open definable sets.

Proof. Induction on $\dim A$. Since $\dim(\partial A) < \dim(A)$, we have that ∂A is a boolean combination of open definable sets. The conclusion follows from $A = \text{cl}(A) \setminus \partial A$. \square

Therefore, all the results in [2] about normal spectral spaces are true in this context. Moreover, by Lemma 3.14, the Krull dimension of \tilde{A} [2, ¶1.4] is less or equal to $\dim A$.

3.42 Remark. Let $A \subseteq M^k$ be definable. If A has empty interior, then $\dim A < k$.

Proof. Let $B := M^k \setminus A$. Since A has empty interior, $A = \partial B$. \square

4 Functions

Let $B \subseteq M^h$, for some $h \in \mathbb{N}$, $Y := \tilde{B}$, and $f : A \rightarrow B$ be a definable function.

4.1 Definition. Define

$$\tilde{f} : X \rightarrow Y$$

as

$$\tilde{f}(x)(b) := \{ \phi(b) : \phi(f(a)) \in x(a) \}.$$
⁴

Namely,

$$\tilde{f}(x) = \bigcap \{ \widetilde{f(U)} : U \subseteq A \text{ definable, } x \in \tilde{U} \}.$$

Note that $f(x)$ is indeed a type, since for every $x \in X$, either $\phi(f(a)) \in x(a)$, or $\neg\phi(f(a)) \in x(a)$. Since \sim preserves the composition of maps, we can view \sim as a covariant functor between the category of definable sets with definable maps, and the category of sets.

4.2 Remark. Let $U \subseteq A$ and $V \subseteq B$ be definable. Then

$$\tilde{f}^{-1}(\tilde{V}) = \widetilde{f^{-1}(V)}, \quad \text{and} \quad \tilde{f}(\tilde{U}) = \widetilde{f(U)}.$$

4.3 Remark. \tilde{f} is continuous with the Stone topology.

4.4 Remark. If $Z \subseteq Y$ is type-definable, then $\tilde{f}^{-1}(Z)$ is quasi-compact.

Proof. Since $\tilde{f}^{-1}(Z)$ is type-definable, it is also quasi-compact. □

4.5 Remark. If f is continuous, then \tilde{f} is also continuous.

Proof. Let $U \subseteq B$ be definable and open. It suffices to prove that $\tilde{f}^{-1}(\tilde{U})$ is open in A . Let $V := f^{-1}(U)$. Since f is definable and continuous, V is definable and open. Since $\tilde{f}^{-1}(\tilde{U}) = \tilde{V}$, we are done. □

Therefore, we can also view \sim as a covariant functor between the category of definable sets with definable continuous maps, and the category of topological spaces.

4.6 Remark. Let $x, y \in X$ such that $y \leq x$. If f is continuous, then $\tilde{f}(y) \leq \tilde{f}(x)$.

Proof. Because \tilde{f} is continuous. □

4.7 Remark. For every $x \in X$, $\dim(\tilde{f}(x)) \leq \dim x$.

4.8 Definition (Rudin-Keisler ordering). Let $x \in S_k(M)$ and $y \in S_h(M)$. We will say that y is less or equal to x in for the Rudin-Keisler ordering, and write $x \leq_{\text{RK}} y$, iff $y = \tilde{g}(x)$

for some $g : M^k \rightarrow M^h$ definable. We will say that x and y are RK-equivalent, and write $x \sim_{\text{RK}} y$, iff $x \leq_{\text{RK}} y$ and $y \leq_{\text{RK}} x$.

4.9 Lemma. Let $x \in S_k(M)$ and $y \in S_h(M)$. The following are equivalent:

1. $y \leq_{\text{RK}} x$;
2. for every $N \succeq M$ elementary extension, if N realizes x , then N realizes y ;
3. $M(x)$ realizes y .

Proof. M has definable Skolem functions. □

⁴Here we use a as a mute variable ranging in A , and similarly for b .

Note that \leq_{RK} is a quasi-order, and that the relation \sim_{RK} is an equivalence relation.

4.10 Lemma. *Let $x \in S_k(M)$ and $y \in S_h(M)$ be such that $y \leq_{\text{RK}} x$. Then, $\dim y \leq \dim x$. Moreover, if $\dim y = \dim x$, then $y \sim_{\text{RK}} x$.*

Proof. The fact that $\dim y \leq \dim x$ is trivial. For the second part, let $f : M^k \rightarrow M^h$ be definable such that $\tilde{f}(x) = y$. Let $N \succeq M$ and $c \in N^k \models x$. Therefore, $d := f(c) \models y$, and $d \in M(c)$. Since

$$\dim(c/M) = \dim x = \dim y = \dim(d/M),$$

$\dim(c/M(d)) = 0$, namely $M(c) = M(d)$. \square

4.11 Lemma. *Let $x \in S_k(M)$ and $y \in S_h(M)$ such that $x \sim_{\text{RK}} y$, and $f : M^k \rightarrow M^h$ be definable, such that $\tilde{f}(x) = y$. Then, there exist $A \subseteq M^k$, $B \subseteq M^h$, and $g : B \rightarrow A$ definable such that*

1. $x \in \tilde{A}$ and $y \in \tilde{B}$;
2. $\dim A = \dim x = \dim y = \dim B$;
3. f is continuous on A , and g is continuous on B ;
4. $\tilde{g}(y) = x$;
5. $g \circ f$ is the identity on B ;
6. $f \circ g$ is the identity on A .

Proof. Let $A \subseteq M^k$ be a cell, such that $x \in \tilde{A}$ and $\dim A = \dim x =: n$. W.l.o.g., we can assume that $n = k$ and $A = M^n$. By Lemma 2.18, there exists a cell $C \subseteq M^n$ such that $x \in \tilde{C}$ and either $f \upharpoonright C$ is injective, or $\dim f(C) < n$. If $f \upharpoonright C$ were not injective, then $\dim y < n$, and hence $x \not\sim_{\text{RK}} y$, absurd. Hence, $f \upharpoonright C$ is injective.

Substitute A with C , and let $B := f(A)$. Hence, $f : A \rightarrow B$ is a bijection. Let $g := f^{-1}$; the remainder of the conclusion follows (after restricting A and B if necessary, in order to get the continuity of g). \square

4.12 Lemma. *Let $y \in S_k(M)$. There exists $V \subseteq M^k$ open and definable, and a definable continuous map $\rho : V \rightarrow M^k$ such that, for every $x \in \tilde{V}$, if $x \geq y$, then $\tilde{\rho}(x) = y$.*

Proof. Let $E \subseteq M^n$ be a definable cell, such that $\dim E = \dim y =: n$ and $y \in \tilde{E}$. By Lemma 2.11, there exists V definable open neighborhood of E and a continuous definable retraction $\rho : V \rightarrow E$. Note that V open and $y \in \tilde{E}$ imply that $x \in \tilde{E}$.

Claim 1. For every $x \geq y$, $\tilde{\rho}(x) = y$.

By Remark 4.6 we have $y = \tilde{\rho}(y) \leq \tilde{\rho}(x)$. Moreover, $\tilde{\rho}(x) \in \tilde{E}$. Since y is maximal in \tilde{E} , the claim is true. \square

1 Theorem. *Let $x, y \in S_k(M)$, such that $y \leq x$. Then, $y \leq_{\text{RK}} x$.*

Proof. Let $N \succeq M$ be an elementary extension realizing x .

Let $\rho : V \rightarrow M^k$ be as in Lemma 4.12. Note that if $N^k \ni c \models x$, then $\tilde{\rho}(c) \models y$, and therefore N realizes y . \square

4.1 Closed maps

4.13 Remark. Let $C, D \subseteq X$ be closed and disjoint. Then, there exist $C', D' \subseteq X$ definable, closed and disjoint such that $C \subseteq C'$ and $D \subseteq D'$.

Proof. By compactness. \square

4.14 Definition. $f : A \rightarrow B$ is **definably closed** iff for every $C \subseteq A$ closed and definable, $f(C)$ is also closed.

4.15 Example. Note that a “definably open” map is nothing else than an open map. On the other hand, a (definable) map can be definably closed, without being closed. For instance, let M be countable, $A := [0, 1]^2$, $B := [0, 1]$, $f : A \rightarrow B$ be the projection onto the first coordinate. Since A is d -compact, f is definably closed. Let $\eta \in Y$ be a gap (for instance, if $M = \mathbb{Q}$, we can take $\eta = \frac{\sqrt{2}}{2}$). The intervals $] \eta, 1]$ and $]0, 1]$ are order-isomorphic, because they are both countable, dense, and with no minimum and a maximum. Let $g :]0, 1] \rightarrow] \eta, 1]$ be an order-isomorphism. Define

$$C := \{ (b, g(b)) : 0 < b \leq 1 \}.$$

Then, C is closed (because $\lim_{b \rightarrow 0^+} g(b) = \eta \notin A$), but $f(C) =]0, 1]$ is not closed.

4.16 Lemma. f is an open map iff \tilde{f} is open. f is definably closed iff \tilde{f} is closed.

Proof. The “only if” directions are trivial.

To prove that \tilde{f} is open, it suffices to prove that for every $U \subseteq A$ open and definable, $\tilde{f}(U)$ is also open. However, this is immediate from Remark 4.2.

Also immediate from the remark is the fact that if $C \subseteq A$ is closed and definable, then $\tilde{f}(C)$ is also closed (if f is closed).

It remains to prove that if $C \subseteq X$ is any closed set, then $\tilde{f}(C)$ is closed. Since C is closed, it is type-definable, and therefore $f(C)$ is also type-definable. Thus,

$$\begin{aligned} C &= \bigcap \{ \tilde{D} : D \subseteq A \text{ definable and closed, } C \subseteq \tilde{D} \}, \\ \tilde{f}(C) &= \bigcap \{ \tilde{E} : E \subseteq A \text{ definable, } \tilde{f}(C) \subseteq \tilde{E} \}. \end{aligned}$$

Let $\tilde{f}(C) \subseteq E$, with E definable. Then, $C \subseteq \tilde{f}^{-1}(E)$. By compactness, there exists D closed and definable such that

$$C \subseteq D \subseteq \tilde{f}^{-1}(E).$$

Hence, $\tilde{f}(C) \subseteq \tilde{f}(D) \subseteq E$. However, by what we said above, $\tilde{f}(D)$ is closed. Therefore,

$$\tilde{f}(C) = \bigcap \{ \tilde{f}(D) : D \subseteq A \text{ definable, } C \subseteq \tilde{E} \},$$

and in particular $\tilde{f}(C)$ is intersection of closed sets, and hence closed. \square

4.17 Corollary. If A is d -compact, f is continuous, and $x \in X$ is a closed point, then $f(x)$ is also a closed point.

Proof. The hypothesis imply that f is definably closed. \square

4.18 Lemma. *Assume that f is continuous. f is definably closed iff $\tilde{f}(x)$ is a closed point for every closed point x .*

Proof. The “only if” direction follows from Lemma 4.16.

For the other direction, let $C \subseteq A$ be definable and closed, and $D := f(C)$. We have to prove that D is closed.

If, for contradiction, D is not closed, let $b \in \text{cl}(D) \setminus D$, and $z := \iota(b)$. By Lemma 3.28, there exists $y \in \tilde{D}$ such that $z < y$.

Let y_0 be minimal (w.r.t. the ordering \leq) such that:

$$\begin{aligned} y_0 &\in D \\ \exists b' \in \text{cl}(D) \setminus D \quad \iota(b') &< y. \end{aligned}$$

Choose $b_0 \in \partial D$ such that $z_0 := \iota(b_0) < y$. Let $x \in \tilde{f}^{-1}(y_0) \cap C$ be minimal. If x is closed, $\tilde{f}(x) = y_0$ is also closed, and therefore $z_0 = y_0$, absurd. If x is not closed, then $\text{cl}(x) = \{x_0, \dots, x_n\}$, where $x_0 < x_1 < \dots < x_n = x \in C$, and $n \geq 1$. Since \tilde{f} is continuous, $\tilde{f}(x_i) \in \text{cl}(\tilde{f}(x)) = \text{cl}(y)$ for every $i \leq n$. Since $\tilde{f}(x_i) \in \text{cl}(y_0) \cap f(C)$, $f(x_i) = y_0$ for $i = 0, \dots, n$ by minimality of y_0 . However, this contradicts the minimality of x , since $x_0 < x$. \square

4.19 Example. If f is not closed, we cannot conclude that $\tilde{f}(x)$ is closed for every closed points, even if f is continuous. Here are two examples.

1. Let $A :=]0, 1[$, $B := [0, 1]$, $f : A \rightarrow B$ be the inclusion map, and $x = 0^+$. Then, $\tilde{f}(x) = x$ is *not* closed in Y , because $\text{cl}_Y(x) = \{0, 0^+\}$. However, x is closed in X .
2. Let M expand a real closed field, $A := M^2$, $B := M$, $\pi_i : A \rightarrow B$ $i = 1, 2$ be the projections on the first and second coordinate respectively. Let $x(a_1, a_2) \in X$ be only type satisfying the following conditions:

$$\begin{aligned} \tilde{\pi}_1(x) &= 0^+, \\ \tilde{\pi}_2(x) &= 0^+, \\ a_1 \cdot a_2 &= 1. \end{aligned}$$

x is on the infinite branch of the hyperbola “near infinity”, and it is closed. However, since $\tilde{f}(x) = 0^+$, $\tilde{f}(x)$ is *not* closed.

4.20 Example. Assume that f is not continuous. By Lemma 4.16, if f is definably closed, then $\tilde{f}(x)$ is closed for every closed point $x \in X$. However, the converse is not true. For example, let $A = [-1, 1]$, $B = [-1, 2]$, and $f : A \rightarrow B$ so defined:

$$f(x) = \begin{cases} x & \text{if } x \leq 0 \\ x+1 & \text{if } x > 0. \end{cases}$$

Then, $f(A) = [-1, 0] \sqcup]1, 2]$, and therefore f is *not* definably closed. However, $\tilde{f}(x)$ is closed for every closed point $x \in X$.

4.21 Corollary. *Assume that f is continuous. The following are equivalent:*

1. f is definably closed;

2. \tilde{f} is closed;
3. $f(x)$ is closed for every closed point x .

Proof. $1 \Rightarrow 2$ by Lemma 4.16. $2 \Rightarrow 3$ by definition. $3 \Rightarrow 1$ by Lemma 4.18. \square

4.22 Lemma. *Assume that M expands a real closed field, and let $A \subseteq M^k$ be definable. The following are equivalent:*

1. $A \subseteq M^k$ is d -compact;
2. for every B and every $f : A \rightarrow B$ definable and continuous, f is definably closed;
3. for every B and every $f : A \rightarrow B$ definable and continuous, $f(A)$ is closed in B .

Proof. $1 \Rightarrow 2$ and $2 \Rightarrow 3$ are trivial. It remains to prove that $3 \Rightarrow 1$. Let $g : M^k \rightarrow M^{k'}$ be any definable injective continuous map, such that the image of g is *bounded*. Let $B := M^{k'}$, and f be the restriction of g to A . Since $f(A)$ is closed and bounded, $f(A)$ is d -compact. Since f is invertible, A is also d -compact. \square

In the proof of the above lemma, we used the fact that M expands a field only to construct the map g . However, the existence of such a map is equivalent to the fact that M expands a real closed field [6, Corollary 9.2].

5 Compactification

5.1 Definition. A **d -compactification** of A (also called completion in [12]) is a map $\rho : A \rightarrow C$ such that:

1. $C \subseteq M^h$ is d -compact;
2. ρ is a definable homeomorphism onto its image;
3. $\rho(A)$ is dense in C .

If the map ρ is clear from the context, we will simply say that C is a d -compactification of A .

5.2 Example. In the definition of d -compactification, we cannot weaken (2) to “ ρ is definable, continuous and injective”. For instance, let

$$A := [0, 1) \subset \mathbb{R}, \quad C := \mathbb{S}^1 \subset \mathbb{C}, \quad \rho(t) := e^{i\pi t}.$$

$\rho : A \rightarrow C$ is not a d -compactification.

5.3 Lemma. *Let $\rho : A \rightarrow C$ be a d -compactification of A . Then, A is locally d -compact iff $\rho(A)$ is open in C .*

Proof. For the “only if” direction, use Lemma 2.16. For the other direction, let $a \in A$, and $U \subseteq C$ definable and open such that $\rho(a) \in U$ and $\text{cl}(U) \subseteq \rho(A)$ (U exists because $\rho(A)$ is open and C is normal). Then, $\rho^{-1}(U)$ is a relatively d -compact neighborhood of a . \square

5.4 Definition. Let $f : A \rightarrow B$ be continuous. A d -compactification $\rho : A \rightarrow C$ is **compatible** with f iff there exists a definable continuous map $g : C \rightarrow B$ such that $g \circ \rho = f$.

From now on, we will assume that M **expands a real closed field**.

5.5 Lemma. *Given a definable continuous map $f : A \rightarrow B$, where B is d -compact, there exists a d -compactification of A compatible with f .⁵*

Proof. Assume that $A \subseteq M^k$ is bounded. Let $\Gamma(f) \subseteq A \times B$ be the graph of f , and $C := \text{cl}(\Gamma(f)) \subseteq \text{cl}(A) \times B$ be its closure. Define

$$\begin{aligned} \rho : A &\rightarrow C \\ a &\mapsto (a, f(a)), \end{aligned}$$

and $g : C \rightarrow B$ be the projection on the second coordinate. \square

5.6 Example. Let $A := ([0, 1] \times [-1, 1]) \setminus (0, 0)$, and $B := [0, 1]$. Define $f : A \rightarrow B$ as

$$f(a_1, a_2) = \min\left\{1, \left|\frac{a_2}{a_1}\right|\right\}.$$

The d -compactification of A given in the proof of the lemma is given by the disjoint union of the graph of f (which is homeomorphic to A), and the vertical segment $\{(a_1, a_2, b) \in M^3 : a_1 = a_2 = 0, 0 \leq a_3 \leq 1\}$.

5.7 Remark. If $\dim A = 1$, there exists a **universal d -compactification** of A (namely, one compatible with all the definable functions f with d -compact co-domains).

5.8 Example. If $\dim A > 1$, such universal d -compactification might not exist. Let $A :=]0, 1] \times [-1, 1]$. A universal d -compactification for A does not exist.

5.9 Definition. Let $x \in X$ and $C \subseteq X$ be definable. We shall say that x is **near** C , and write $C \leq x$, iff every definable open neighborhood of C contains x . We shall write $C < x$ iff $C \leq x$ and $x \notin \bar{C}$.

5.10 Lemma. *The following are equivalent:*

1. $C \leq x$;
2. every definable closed set containing x intersects C ;
3. every closed set containing x intersects C ;
4. there exists $y \in C$ such that $y \leq x$ (namely, $\text{cl}(x) \cap C \neq \emptyset$).

Proof. $1 \Leftrightarrow 2$ and $3 \Rightarrow 2$ are trivial.

For $2 \Rightarrow 4$, let

$$\mathcal{C} := \{D \cap C : D \subseteq X \text{ closed and definable, } x \in D\}.$$

\mathcal{C} is a family of definable subsets of C with the F.I.P. Since C is compact (with the Stone topology), $\bigcap \mathcal{C} \neq \emptyset$. Any $y \in \bigcap \mathcal{C}$ is in $\text{cl}(x) \cap C$.

For $4 \Rightarrow 3$, any closed set D containing x must also contain y , and therefore $C \cap D \neq \emptyset$. \square

5.11 Remark.

$$\{x \in X : C \leq x\} = \bigcap \{U : C \subseteq U \subseteq X, U \text{ open and definable}\}.$$

⁵Thanks to M. Mamino for the proof. It is the same proof as in [12].

The above lemma suggests the following extension of the notion $C \leq x$ to the case when C is type-definable.

5.12 Lemma. *Let $Z \subseteq X$ be type-definable. Then,*

$$\begin{aligned} \hat{Z} &:= \bigcap \{ U : Z \subseteq U \subseteq X, U \text{ open and definable} \} = \\ &= \bigcap \{ U : Z \subseteq U \subseteq X, U \text{ open} \} = \{ x \in X : \text{cl}(x) \cap Z \neq \emptyset \}. \end{aligned}$$

We say that x is near Z , and write $Z \leq x$, iff $x \in \hat{Z}$.

Note that \hat{Z} is type-definable.

5.13 Definition. Let $x \in X$. We will say that x is far from the frontier of X iff there exists a definable set $C \subseteq A$ which is d -compact and such that $x \in \tilde{C}$. Otherwise, we will say that x is near the frontier. The **fringe** $\text{Frin}(A)$ of A is the set of points of X near the frontier.

5.14 Lemma. *The following are equivalent:*

1. $x \in \text{Frin}(A)$;
2. for every C d -compactification of A , $\partial A < x$, where ∂A is the frontier of A taken inside C ;
3. for some C d -compactification of A , $\partial A < x$;
4. for every definable $D \subseteq A$, if D is d -compact, then $x \notin \tilde{D}$.

Proof. Assume that $x \in \text{Frin}(A)$. Let U be an open definable neighborhood of ∂A , and $D := C \setminus U$. Since C is d -compact, D is also d -compact and contained in A , and therefore $x \notin \tilde{D}$. Hence, $x \in \tilde{U}$, and thus $\partial A \leq x$. Since $x \in X$, $x \notin \partial A$, and so $\partial A < x$.

Conversely, assume that $\partial A < x$ for some C d -compactification of A . Let $D \subseteq A$ be d -compact, and $U := A \setminus D$. Since D is d -compact, U is open in A . $D \cap \partial A = \emptyset$, because $D \subseteq A$, and so U is an open neighborhood of ∂A . Thus, $x \in \tilde{U}$, and therefore $x \in \text{Frin}(A)$. \square

5.15 Definition. Let $x \in X$. We will say that x is **strongly closed** iff x is closed and x is far from the frontier of A .

5.16 Lemma. *The following are equivalent:*

1. x is strongly closed;
2. x is closed in some d -compactification of A ;
3. x is closed in every d -compactification of A ;
4. $x \in \tilde{D}$, for some $D \subseteq A$ definable and d -compact.

Proof. $1 \Rightarrow 3 \Rightarrow 2$ is trivial. For $2 \Rightarrow 1$, assume that x is closed in some d -compactification C of A . It is then trivially true that x is already closed in A . If, for contradiction, $\partial A < x$, we would have that there exists $y \in \partial A$ such that $y < x$, contradicting the fact that x is closed in \tilde{C} . \square

5.17 Remark. If A is d -compact, then x is strongly closed iff it is closed.

5.18 Lemma. *Let $A \subseteq B$, and $x \in X$. Then, x is strongly closed in A iff it is strongly closed in B .*

Proof. Let $\rho : B \rightarrow D$ be a d -compactification of B . W.l.o.g., we can assume that ρ is the inclusion map. Let $C := \text{cl}_D(A)$. Then, C is a d -compactification of A .

If x is not strongly closed in A , then there exists $y \in \tilde{C}$ such that $y < x$. Since $C \subseteq D$, x is not closed in D either, and therefore x is not strongly closed in B .

If x is not strongly closed in B , then there exists $y \in \tilde{D}$ such that $y < x$. However, C is closed in D , and therefore $y \in \tilde{C}$. Hence, x is not strongly closed in A either. \square

2 Theorem. *Let $f : A \rightarrow B$ be any definable map. If $x \in X$ is strongly closed (in A), then $\tilde{f}(x)$ is also strongly closed (in B).*

Proof. Decompose A into cells C_i such that f is continuous on each cell. Let C be the cell counting x . By Lemma 5.18, x is strongly closed in C . Let E be some d -compactification of B . Consider the map $g : C \rightarrow E$ given by the composition of the immersion of B in E with the restriction of f to C . Let $\rho : C \rightarrow D$ be a d -compactification of C compatible with g . By Corollary 4.17, $\tilde{g}(x) = \tilde{f}(x)$ is closed in E , and therefore $\tilde{f}(x)$ is strongly closed. \square

In a slogan, “being strongly closed” is an intrinsic property of a type x (namely, independent from the ambient space A).

6 Rational and irrational types

6.1 Definition. A type $x \in \tilde{A}$ is **rational** (a.k.a. definable) iff for every formula $\psi(a, u)$ there exists $c \in A$ such that

$$\{b \in M^h : \psi(a, b) \in x(a)\} = \{b \in M^h : M \models \psi(c, b)\}.$$

If x is not rational, then it is **irrational**.

Note that $\iota(a)$ is a rational type for every $a \in A$.

Remember that M expands a field.

6.2 Lemma. *Let N be an elementary extension of M . Let A be a d -compactification of M (e.g., $A = \mathbb{S}^1$). The following are equivalent:*

- x is rational (and not realized);
- either $x = \pm\infty$, or there exists $a \in M$ such that $x = a^\pm$;
- M is not co-final in $M(x)$
- M is Dedekind complete in $M(x)$;
- x is not closed in \tilde{A} ;
- there exists $a \in A$ such that $\iota(a) < x$.

The following are also equivalent:

- x is irrational;
- M is co-final in $M(x)$;

- x is closed in \tilde{A} .

Proof. See [9] (or [1]). Note that the hypothesis imply that $\dim x = 1$. \square

The following example is by Coste.

6.3 Example. Let $M' := (\mathbb{R}, +, \cdot, \exp)$, and $A = [0, 1]^2$. Let $x = x(a_1, a_2) \in \tilde{A}$ be the unique type satisfying the conditions

$$\begin{aligned} a_1 &\models 0^+ \\ a_2 &= \exp(a_1). \end{aligned}$$

Note that $a_2 \models 0^+$. Moreover, $\dim(x) = 1$, and $0 < x$.

Let M be the reduct of M' to the field structure alone, and y be the image of x under the reduct map. Note that $0 < y$, and that $\dim y = 2$, because the germ of \exp near 0 is not definable in M . Moreover, there is no $z \in \tilde{M}$ such that $0 < z < y$, otherwise there would be $z' \in \tilde{M}'$ such that $0 < z' < x$, which is impossible. Note also that y is a rational type, because *all* type over M are rational, since M is Dedekind complete.

6.4 Example. Let M be the field of real algebraic numbers. Let N be the real closure of $M(\varepsilon, e)$, where ε is a positive infinitesimal element (namely, $\text{tp}(\varepsilon/M) = 0^+$), and e be a transcendental real number. Let $A := [0, 1]^2 \subseteq M^2$, and $x(a_1, a_2) \in \tilde{A}$ be given by

$$\begin{aligned} a_1 &\models 0^+ \\ a_2 &= ea_1. \end{aligned}$$

Note that $a_2 \models 0^+$, that $0 < x$, that $\dim x = 2$, but there is no $y \in \tilde{A}$ such that $0 < y < x$. Moreover, x is irrational, because the type $\text{tp}(e/M)$ can be defined using x , and $\text{tp}(e/M)$ is irrational.

Let $M \preceq N$ be an elementary extension.

6.5 Definition. We will say that the extension N/M is **rational** iff, for every $n \in \mathbb{N}$, every n -type over M realized in N is rational.⁶ We will say that N/M is **totally irrational** iff, for every $n \in \mathbb{N}$, and for every $c \in N^n \setminus M^n$, $\text{tp}(c/M)$ is irrational. We will say that a type $x \in S_n(M)$ is totally irrational iff there exists a totally irrational extension N realizing x , or equivalently iff $M(x)/M$ is totally irrational.

6.6 Remark. A totally irrational type is either irrational, or already realized in M .

6.7 Lemma. N/M is rational iff every 1-type realized in N is rational. N/M is totally irrational iff, for every $c \in N \setminus M$, $\text{tp}(c/M)$ is irrational.

Proof. See [9] or [1]. \square

6.8 Remark. N/M is totally irrational iff M is *co-final* in N .

6.9 Remark. If $M \preceq N \preceq P$ and N/M is totally irrational, then \equiv_M and \equiv_N coincide (on P^k).

6.10 Definition. Let $M \preceq N$, and $b, c \in N^k$. We shall say that b is **M -bounded**, (or simply bounded if M is clear from the context) iff there exists $a \in M$ such that $|b| \leq a$. We shall write $b \equiv_M c$ iff for every $a \in M$ such that $a > 0$, we have $|b - c| < a$. Note that \equiv_M is an equivalence relation.

⁶Rational extensions were called *tame* extensions in [13] (in the case of o-minimal theories expanding a real closed field).

6.11 Lemma. N/M is rational iff for every $k \in \mathbb{N}$ and every M -bounded $c \in N^k$ there exists (a necessarily unique) $a \in M^k$ such that $a \equiv_M c$.

We will call a as above the **M -standard part** of c , and write $a = \text{st}_M(c)$ (or simply $a = \text{st}(c)$ if M is clear from the context).

6.12 Remark. $x \in S_n(M)$ is totally irrational iff, for every definable function $f : M^n \rightarrow [0, 1]$, we have $f(x) \neq 0^+$.

6.13 Lemma. Let $C \subseteq A$ be definable. Define $f : A \rightarrow M$ as $f(a) := d(a, C)$, where d is the distance. Let $x \in X$ and $y := \tilde{f}(x) \in \tilde{M}$. Then, $C < x$ iff $y = 0^+$.

Proof. For every $\varepsilon \in M^{>0}$, let $V_\varepsilon := f^{-1}[0, \varepsilon[$.

Assume that $C < x$. Since $x \notin \tilde{C}$, $y \in]0, +\infty[$. Since, $\forall \varepsilon > 0$, V_ε is an open definable neighborhood of C , and $C \leq x$, we have $x \in \tilde{V}_\varepsilon$, and therefore $y \in]0, \varepsilon[$. Thus, $y \in \bigcap_{\varepsilon > 0}]0, \varepsilon[$, and so $y = 0^+$.

Conversely, if $y = 0^+$, let U be a definable open neighborhood of C . By Lemma 2.10, $V_\varepsilon \subseteq U$ for some $\varepsilon > 0$. Since $y = 0^+$, $x \in V_\varepsilon$. Therefore, $C \leq x$. Since $y \neq 0$, $x \notin \tilde{C}$, and therefore $C < x$. \square

3 Theorem. Let $x \in S_n(M)$. Then, x is strongly closed iff it is totally irrational.

Proof. Let us prove the “**if**” direction. Assume that x is not totally irrational. Then, by Remark 6.12, there exists $f : M^n \rightarrow [0, 1]$ definable such that $\tilde{f}(x) = 0^+$. Decompose M^n into cell such that f is continuous on each cell, and let C be the cell containing x . Let $\rho : C \rightarrow D$ be a d-compactification of C compatible with f , and let $g : D \rightarrow [0, 1]$ be the corresponding extension of f . W.l.o.g., ρ is the inclusion map. Moreover, $\tilde{g}(x) = 0^+$. Let $E := g^{-1}(0)$.

Claim 1. $E \leq x$.

Let U be a definable neighborhood of E . By Lemma 2.10, there exists $\varepsilon > 0$ such that $g^{-1}(]0, \varepsilon]) \subseteq U$. Since $\tilde{g}(x) = 0^+$, we have

$$\tilde{x} \in g^{-1}(]0, \varepsilon]) \subseteq U,$$

and the claim is proved.

Since $x \notin E$, we have $E < x$. By Lemma 5.10, there exists $y \in E$ such that $y < x$. Therefore x is not closed in D , and thus x is not strongly closed.

For the “**only if**” direction, assume that x is not strongly closed. W.l.o.g., we can assume that $x \in \tilde{A}$ for some d-compact set A . Let $y < x$, and let $C \subseteq A$ be a closed definable sub-set such that $y \in \tilde{C}$ and $\dim C < \dim x$. By Lemma 5.10, $C \leq x$, and, since $\dim C < \dim x$, $C < x$. Therefore, by Lemma 6.13, $\tilde{f}(x) = 0^+$, where $f(a) := d(a, C)$. Thus, by Remark 6.12, x is not totally irrational. \square

6.14 Lemma. Let A be d-compact. Let $f : A \rightarrow B$ be definable and continuous, $x \in \tilde{A}$ and $y \in \tilde{B}$ such that $y \leq \tilde{f}(x)$. Then, there exists $z \in \tilde{A}$ such that $z \leq x$ and $\tilde{f}(z) = y$. Moreover, if $y < \tilde{f}(x)$, then $z < x$.

Proof. Let $C := \text{cl}(x)$. Since A is d-compact, \tilde{f} is closed, and therefore $\tilde{f}(C)$ is closed. Since $x \in \tilde{f}(C)$ and $y \leq x$, we have $y \in \tilde{f}(C)$. Let $z \in C$ such that $\tilde{f}(z) = y$. \square

6.15 Example. We cannot drop the condition that A is d-compact in Lemma 6.14. For instance, let $B := [0, 1]$, $A := B \setminus \{0\}$, $x := 0^+$, $y := 0$ and $f : A \rightarrow B$ be the inclusion map. A point z as in the conclusion of the Lemma does not exist.

6.16 Remark. Let $y \leq_{\text{RK}} x$. If x is rational, then y is also rational. If x is totally irrational, then y is also totally irrational.

6.17 Lemma. *Let $y < x \in X$. Assume that x is rational and $n := \dim x = 1 + \dim y$. Then, y is rational.*

Proof. W.l.o.g., we can assume that $A \subseteq [0, 1]^k$. We will prove the conclusion by induction on k . The case $k = 0$ is trivial, and the one $k = 1$ is easy. Let $f : M^k \rightarrow [0, 1]$ be definable. We want to prove that $\tilde{f}(x)$ is rational. Decompose M^k into cells in a way compatible with A and with f . Let $C \subseteq M^k$ be the cell containing x .⁷

Let $\rho : C \rightarrow D$ be a d-compactification of C compatible with f and with the inclusion map $\lambda : C \rightarrow [0, 1]^k$, and $g : D \rightarrow [0, 1]$, $\mu : D \rightarrow [0, 1]^k$ be the extensions of f and λ respectively. If we identify C with $\rho(C)$, we can assume that ρ is the inclusion map. By Lemma 6.14, there exists $z < x$ such that $\tilde{\mu}(z) = y$.

Claim 1. $\dim z = \dim y$.⁸

In fact, $z < x$ implies that $\dim z \leq n - 1$. Moreover, since $y = \tilde{\mu}(z)$, $\dim z \geq \dim y$.

Claim 2. z is rational.

In fact, by Lemma 4.10, $z \leq_{\text{RK}} y$, and y is rational.

By Claim 2, $\tilde{g}(z)$ is rational. Moreover, since g is continuous, $\tilde{g}(z) \leq \tilde{g}(x)$. By the case $k = 1$, $\tilde{g}(x)$ is also rational. \square

6.18 Corollary. *Let $x_0, \dots, x_n \in A$ such that $x_0 < x_1 < \dots < x_n$ and $\dim(x_n) = n$. Then, each x_i is rational, for $i = 0, \dots, n$.*

Proof. Note that $\dim(x_i) = i$ for every $i \leq n$. Since $\dim x_0 = 0$, x_0 is rational. By induction on n , x_{n-1} is rational. By Lemma 6.17, x_n is also rational. \square

6.19 Lemma. *Let A be d-compact, and $x \in X$. If x is rational, then there exists a unique $a \in A$ such that $\iota(a) \leq x$.*

Proof. Uniqueness is trivial. Assume, for contradiction, that a does not exist. Let y be the minimum of $\text{cl}(x)$. Note that y is closed in A . Since A is d-compact, y is strongly closed, and therefore, by Theorem 3, y is totally irrational. Since $\dim y > 0$, y is irrational. Since $y \leq x$, by Theorem 1, $y \leq_{\text{RK}} x$, and therefore x is irrational, absurd. \square

6.20 Definition. Let A be d-compact, and $x \in X$. We will say that $\rho : C \rightarrow D$ is a d-compactification fixing x iff

1. $C \subseteq A$ is definable, such $x \in \tilde{C}$;
2. $\rho : C \rightarrow D$ is a d-compactification compatible with the inclusion map $\lambda : C \rightarrow A$.

In this case, we will denote by $\mu : D \rightarrow A$ the extension of λ to D .

6.21 Definition. Let A be d-compact. Let $x, y \in \tilde{A}$ such that $y \leq x$. Let $\rho : C \rightarrow D$ be a d-compactification fixing x . By Lemma 6.14, there exists $z \in D$ such that $z \leq \tilde{\rho}(x)$ and $\tilde{\mu}(z) = y$. We will call the pair $(z, \tilde{\rho}(x))$ a **lifting** of (y, x) compatible with the d-compactification ρ . We will say that the pair (y, x) is **maximal** iff for every lifting (z, x') of (y, x) , $\dim z \leq \dim y$.

⁷Note that C might not contain y .

⁸Here is the point where we use $\dim y = n - 1$.

Note that if (z, x') is a lifting of (y, x) , then $\dim y \leq \dim z$. Therefore, if (y, x) is maximal, we have $\dim y = \dim z$, and hence, by Lemma 4.10, $y \underset{\text{RK}}{\sim} z$.

6.22 Lemma. *Let A be d -compact. Let $y \leq x \in X$, with (y, x) maximal. Let $\rho : C \rightarrow D$ be a d -compactification fixing x , and $x' := \tilde{\rho}(x)$. Then, there exists a unique $z \in \tilde{D}$ such that $z \leq x'$ and $\tilde{\mu}(z) = y$.*

Proof. The existence of z is Lemma 6.14. For the uniqueness, let $z_1 \leq z_2 \leq x'$ such that $\tilde{\mu}(z_1) = \tilde{\mu}(z_2) = y$. However, by maximality of (x, y) , $\dim z_1 = \dim y = \dim z_2$. Thus, $z_1 = z_2$. \square

Therefore, if (y, x) is maximal, and ρ is as in the hypothesis of the lemma, we can speak of *the* lifting of (y, x) compatible with ρ .

6.23 Definition. Let $x \in S_k(M)$, $y \in S_h(M)$, such that $y \underset{\text{RK}}{\leq} x$, and $f : M^k \rightarrow M^h$ be a definable function, such that $\tilde{f}(x) = y$. We will say that (y, x, f) is **rational** iff, for every $N \succeq M$, and every $N^k \ni c \models x$, the type $\text{tp}(c/N(d))$ is rational, where $d := f(c)$.

6.24 Remark. Let x, y, f be as in the above definition. (y, x, f) is rational iff, for *some* $N \succeq M$ and some $N^k \ni c \models x$, the type $\text{tp}(c/N(d))$ is rational, where $d := f(c)$.

6.25 Remark. Note that if $y = \iota(a)$ for some $a \in M^k$, then (y, x, f) is rational (for any $f : M^k \rightarrow M^h$ such that $\tilde{f}(x) = y$) iff x is rational.

6.26 Example. The fact that (y, x, f) is rational does depend on the particular choice of f (such that $\tilde{f}(x) = y$). For instance, let $k := 2$, $h := 1$,

$$x(a_1, a_2) := a_i \models 0^+, a_1 \ll a_2,$$

$y := 0^+$, $f_1(a_1, a_2) := a_1$, $f_2(a_1, a_2) := a_2$. Then, (y, x, f_1) is not rational, while (y, x, f_2) is rational.

6.27 Remark. If $\dim x = 1$, the fact that (y, x, f) is rational does *not* depend on f .

Proof. Let $N \succeq M$, and $N \ni c \models x$, and $d := f(c) \models y$. If $d \in M$, then (y, x, f) is rational iff x is rational. If $d \notin M$, then $M(d) = M(c)$, and therefore (y, x, f) is always rational. \square

The following Lemma is contained in [8, Proposition 2.1].

6.28 Lemma. *Let $N \succeq M$, $d \in N^k$, $N' := M(d)$, $e \in N^h$. Then, there exist U neighborhood of d definable (in M), $h : U \rightarrow N^h$ definable and continuous, such that $h(d) = e$.*

Proof. Since $e \in N^h$, there exists $h' : N^k \rightarrow N^h$ definable (but not necessarily continuous) such that $h'(d) = e$. Let $C \subseteq N^k$ be a cell (definable in M !) such that h' is continuous on C and $d \in C$. By Lemma 2.11, there exists an open neighborhood U of C and retraction $\rho : U \rightarrow C$. Define $h := h' \circ \rho : U \rightarrow N^h$. \square

4 Theorem. *Let $x, y \in S_k(M)$, with $y \leq x$. By Lemma 4.12, there exists $f : U \rightarrow C$ retraction such that $\tilde{f}(x) = y$, where C contains y , and U is a neighborhood of C . If (y, x) is maximal, then (y, x, f) is rational.*

Proof. W.l.o.g., we can assume that $C \subseteq A$, where $A = [0, 1]^k$. Let $N \succeq M$ and $c \in A(N)$ such that $c \models x$, and let $d := f(c) \models y$. Let $N' := M(d)$. Assume, for contradiction, that there exists $g : A(N) \rightarrow N$ definable in N' , such that $z := \text{tp}(g(c)/N')$ is not rational. Note that $g(t) = g'(t, d)$ for some $g' : A \rightarrow M$ definable in M . Hence, $g(c) = g'(c, f(c))$, and therefore w.l.o.g. we can assume that g is definable in M . Moreover, w.l.o.g. we can assume that $g(d) < g(c)$.

Let $E \subseteq U$ be a cell such that g is continuous on E and $x \in \tilde{E}$. Let $\rho : E \rightarrow D$ be a d-compactification of E compatible with f, g , and the embedding $\lambda : E \rightarrow A$. Let (y', x') be the lifting of (y, x) compatible with ρ . By hypothesis, $\dim y' = \dim y$, and therefore $y \underset{\text{RK}}{\sim} y'$. Also, $x \underset{\text{RK}}{\sim} x'$.

Let $\mu : D \rightarrow A$ be the extension of λ . By Lemma 4.11, there exist $F \subseteq E, G \subseteq D$ and $v : G \rightarrow A$ continuous, all definable, such that $y \in \tilde{F}, y' \in \tilde{G}$ and v is the inverse of $\mu \upharpoonright F$. Define $f' := v \circ f \circ \mu$; the domain of f' is $U' := \mu^{-1}(f^{-1}(G)) \subseteq U$: note that $x', y' \in \tilde{U}'$; its co-domain is F . Note that f' is continuous. Moreover,

$$f' \circ f' = v \circ f \circ \mu \circ v \circ f \circ \mu = v \circ f \circ \mu = f'.$$

Let $c' = \rho(c) \models x'$ and $d' := f'(d') \models y'$. It suffices to prove that (y', x', f') is rational. Therefore, w.l.o.g. we can assume that $y \in \tilde{E}$ (and $x = x', y = y', f = f', d = d', c = c'$).

Since z is not rational, there exists $h : E \rightarrow M$ definable, such that

$$g(d) < h(d) < g(c).$$

By Lemma 6.28, we can assume that h is defined and continuous on all E . Let

$$F := \{a \in U \cap E : h(f(a)) \leq g(a)\}$$

Note that $c \in F(N)$, and therefore $x \in \tilde{F}$. Moreover, h, f, g are continuous, hence F is closed in E . Thus, $y \in \tilde{F}$, and therefore

$$h(f(d)) \leq g(d).$$

Since $f(d) = d$, we have $h(d) \leq g(d)$, a contradiction. \square

6.29 Corollary. *Let $a \in A$ and $x \in X$ such that $1a \leq x$. If $(1a, x)$ is maximal, then x is rational.*

6.30 Remark. Let $y \leq x \in X$. Then, there exists a lifting (y', x') of (y, x) , such that (y', x') is maximal.

Lifting the closure of a type. Let A be d-compact, and $x \in X$, with $\dim x = n$. We shall write $\text{cl}(x) = (x_0 < x_1 < \dots < x_m)$ iff $\text{cl}(x) =: \{x_0, x_1, \dots, x_m\}$, with $x_0 < x_1 \dots < x_m = x$.

Let $\text{cl}(x) = (x_0 < \dots < x_m)$. Let $\rho_i : C_i \rightarrow D_i$ be d-compactifications fixing x , $i = 0, 1, 2$, such that

1. $C_0 \subseteq C_1 \cap C_2$;
2. ρ_0 is compatible with ρ_1 and ρ_2 ;

(ρ_0 is a common refinement of ρ_1 and ρ_2). Let $\mu_i : D_i \rightarrow A$ be the extension of the inclusion $\lambda_i : C_i \rightarrow A$, $i = 0, 1, 2$, and $\nu_i : D_0 \rightarrow D_i$ be the extension of ρ_i , $i = 1, 2$. For $i = 0, 1, 2$, let $y_i := \tilde{\rho}_i(x) \in \tilde{D}_i$, and $\text{cl}(y_i) =: \{z_{i,0}, \dots, z_{i,m_i}\}$, with $z_{i,0} < z_{i,1} < \dots < z_{i,m_i} = y_i$. Note that, for each $i \leq 3$ and $j \leq m_i$,

$$\exists! f(i, j) \leq m \quad \tilde{\mu}_i(z_{i,j}) = x_{f(i,j)}.$$

Moreover, $\dim x_{f(i,j)} \leq \dim z_{i,j}$. Similarly, for $i = 1, 2$ and $j \leq m_0$,

$$\exists! g(i, j) \leq m_i \quad \tilde{\nu}_j(z_{0,j}) = z_{i,g(i,j)}, \quad \text{and} \quad \dim z_{i,g(i,j)} \leq \dim z_{0,j}.$$

We will say that $\text{cl}(x)$ is **maximal** iff for every d-compactification $\rho : C \rightarrow D$ fixing x , if we call $x' := \tilde{\rho}(x')$ and $\text{cl}(x') = (x'_0 < \dots < x'_{m'})$, then $m = m'$, $\tilde{\rho}(x'_i) = x_i$, and $\dim(x'_i) = \dim(x_i)$, for $i = 0, \dots, m$.

Therefore, there exist a d-compactification $\rho : C \rightarrow D$ fixing x , such that $\text{cl}(\tilde{\rho}(x))$ is maximal.

Note that x_0 is always strongly closed (because A is d-compact). If moreover $\text{cl}(x)$ is maximal, then (x_i, x_j) is maximal for every $i \leq j \leq m$, and in particular (x_0, x) is maximal.

6.31 Corollary. *Let $N = M(c)$, for some finite tuple $c \in [0, 1]^k(N)$. Let $x := \text{tp}(c/M)$. Let $y \leq x$ such that y is strongly closed and (y, x) is maximal (by the above discussion, we can always assume that y exists). Let $N \ni d \models y$, and $N' := M(d)$. Then, N'/M is totally irrational, and N/N' is rational.*

7 Amalgamation

7.1 Lemma. *Let $M \preceq N$. There exists N' such that $M \preceq N' \preceq N$, N'/M is totally irrational and N/N' is rational. There exists N'' such that $M \preceq N'' \preceq N$, N''/M is rational and N/N'' is totally irrational.*

Proof. See [1, Lemma 3]. □

7.2 Lemma. *Let $M \preceq N \preceq P$.*

1. *(N/M is totally irrational and P/N is totally irrational) iff P/M is totally irrational.*
2. *If N/M is rational and P/N is rational, then P/M is rational.*
3. *If P/M is rational (resp. totally irrational), then N/M is rational (resp. totally irrational).*

7.3 Example. It is not true that P/M rational implies P/N rational. For instance, let $P := M(b, c)$, where $\text{tp}(b/M) = 0^+$ and $\text{tp}(c/M(b)) = 0^+$. Let $N := M(c)$. Then, P/N is not rational (in fact, it is totally irrational).

Another example is in [1], where P is “very saturated” over M .

7.4 Lemma. *Let $N \succeq M$. Let N_i , $i = 1, 2$, such that*

1. $M \preceq N_i \preceq N$;
2. N_i/M is totally irrational;

3. N/N_i is rational.

Then, N_1 and N_2 satisfy the same types over M .

Proof. Let $d \in N_1^k$, and $x := \text{tp}(d/M)$. We have to prove that x is realized in N_2 . W.l.o.g., we can assume that $d \in [0, 1]^k(N)$. By Theorem 3, x is strongly closed.

Let $y := \text{tp}(d/N_2)$, and $z \leq y$ minimum. Note that z is strongly closed, and that, since $d \models y$, we have that z is realized in N . Since, by hypothesis, all types over N_2 realized in N are rational, z is both totally irrational and rational, and therefore z is realized in N_2 . \square

7.5 Proposition. *If M, N, N_i are as in the above lemma, then N_1 and N_2 are isomorphic over M .*

Proof. It was proved in [13, Theorem 2.15]. The proof goes as follows.

Claim 1. For every $b \in N_1$ there exists a unique $c \in N_2$ such that $b \equiv_M c$.

The uniqueness follows from the fact that M is cofinal in N_2 . For the existence, if b did not exist, then $\text{tp}(c/N_2)$ would not be rational, and hence $N_2(c)/N_2$ would be totally irrational, contradicting the maximality of N_2 .

Claim 2. The map sending $b \in N_1$ to the above $c \in N_2$ is an M -isomorphism.

Working by induction, we can assume that $N_1 = M(b)$ and $N_2 = M(c)$. If $b \in M$, then $c = b$, and we are done. If $b \notin M$, then c fills the same gap on M as b , and therefore $M(b)$ is isomorphic to $M(c)$ over M . \square

7.6 Definition. Let $x \in \widetilde{A}$ and $y \in \widetilde{B}$. Define

$$x \times y := \{z \in \widetilde{A \times B} : \widetilde{\pi}_A(z) = x \ \& \ \widetilde{\pi}_B(z) = y\},$$

where $\pi_A : A \times B \rightarrow A$ is the projection onto A , and similarly for π_B . We shall say that x and y are **orthogonal** iff $x \times y = \{z\}$, for some (unique) $z \in \widetilde{A \times B}$.

We shall say that x and y are **independent** iff for every $z \in x \times y$ we have $\dim z = \dim x + \dim y$.

If $M \preceq N$, $a \in A(N)$ and $b \in B(N)$, we will say that b and c are orthogonal over M iff $\text{tp}(a/M)$ and $\text{tp}(b/M)$ are orthogonal. If $M \preceq P$ and $M \preceq Q$, we shall say that P and Q are orthogonal over M iff every $b \in P^k$ and $c \in Q^k$ are orthogonal over M , for every $h, k \in \mathbb{N}$.

What here we called orthogonal types, are called ‘‘almost orthogonal’’ in [10]. We will prove presently that orthogonal and independent types are the same concept.

7.7 Remark. If $x = ta$, for some $a \in A$, then x is orthogonal to any M -type y .

7.8 Lemma. *For every $x \in \widetilde{A}$ and $y \in \widetilde{B}$, there exists at least one $z \in x \times y$ such that $\dim z = \dim x + \dim y$.*

Proof. Assume not. Then, there would exist $C \subseteq A$ and $D \subseteq B$ definable, such that $x \in \widetilde{C}$, $y \in \widetilde{D}$, and for every $z \in \widetilde{C \times D}$, $\dim z < \dim x + \dim y$. However, this would mean that

$$\dim C + \dim D = \dim(C \times D) < \dim x + \dim y,$$

absurd. \square

7.9 Lemma. *Let $x \in \widetilde{A}$ and $y \in \widetilde{B}$. The following are equivalent:*

1. x and y are orthogonal;
2. x has exactly one extension to $M(y)$;
3. y has exactly one extension to $M(x)$.

7.10 Lemma. *Let x and y be orthogonal, with $x' \leq_{RK} x$, and $y' \leq_{RK} y$. Then, x' and y' are orthogonal.*

Proof. Assume, for contradiction, that x and y are orthogonal, but x' and y' are not, where x' and y' are as in the hypothesis. Therefore, there exist $z_1 \neq z_2 \in x' \times y'$. Let $f : M^k \rightarrow M^{k'}$ and $g : M^h \rightarrow M^{h'}$ be definable, such that $\tilde{f}(x) = x'$ and $\tilde{g}(y) = y'$. Define $h := f \times g : M^{k+h} \rightarrow M^{k'+h'}$. Let

$$Z_i := \tilde{h}^{-1}(z_i) \subseteq x \times y, \quad i = 1, 2.$$

Note that Z_1 and Z_2 are disjoint and non-empty. However, $x \times y$ is a singleton, which is absurd. \square

7.11 Lemma. *Let $x \in \tilde{A}$, $y \in \tilde{B}$. Then, x and y are orthogonal iff they are independent.*

Proof. The “only if” direction is immediate from the above Lemma.

For the “if” direction, let $z_1 \neq z_2 \in x \times y$. W.l.o.g., we can assume that $\dim x = \dim A = h$ and $\dim y = \dim B = k$. Let $U \subseteq A \times B$ be definable, such that $z_1 \in \tilde{U}$ and $z_2 \notin \tilde{U}$. Since x and y are independent, $\dim z_1 = \dim z_2 = h + k$. Hence, w.l.o.g., U is open.

Claim 1. $x \times y$ is connected.

Let $N := M(x)$. By Lemma 3.35, $x \times y$ is homeomorphic to $\theta^{-1}(y)$, where $\theta : S_k N \rightarrow S_k M$ is the restriction map. Hence, by Lemma 3.36, $x \times y$ is connected.

Thus, by Lemma 2.19, $(x \times y) \cap \tilde{\partial U} \neq \emptyset$. Let $z \in (x \times y) \cap \tilde{\partial U}$. Then, $\dim z \leq \dim \partial U < \dim U < h + k$, absurd. \square

7.12 Example. If $x \in X \setminus \iota(A)$, then x is *not* orthogonal to itself. In fact, let $z(a_1, a_2) \in x \times x$ such that z also satisfies $a_1 = a_2$. Then, $\dim z = \dim x < \dim x + \dim x$.

7.13 Definition. We shall say that M, P, Q, N are an **amalgam** iff

$$M \preceq P \preceq N \quad \text{and} \quad M \preceq Q \preceq N.$$

In this case, we will denote by PQ the elementary sub-structure of N generated by $P \cup Q$.

M, P, Q, N are a **heir-coheir** amalgam iff for every $\psi(b, c)$ is a formula, and $b \in P^h$, $c \in Q^k$ such that $N \models \psi(b, c)$, there exists $a \in M^k$ such that $N \models \psi(b, a)$ [4].

M, P, Q, N is a **tame-cotame** amalgam iff P/M is totally irrational and Q/M is rational.

7.14 Lemma. *Let M, P, Q, N and M, P', Q', N' be amalgams. Let $\beta : P \rightarrow P'$ and $\gamma : Q \rightarrow Q'$ be M -isomorphisms. If P and Q are orthogonal over M , then there exists a unique isomorphism $\delta : PQ \rightarrow P'Q'$ extending both β and γ .*

Proof. Any element of PQ is of the form $f(b, c)$, for some M -definable $f : N^{h+k} \rightarrow N$, $b \in P^h$, $c \in Q^k$. Define

$$\delta(f(b, c)) := f(\beta(b), \gamma(c)).$$

Since P and Q are orthogonal, δ is well-defined. \square

7.15 Lemma. *Let M, P, Q, N be an amalgam. The following are equivalent:*

1. M, P, Q, N are a heir-coheir amalgam;
2. for every $h, k \in \mathbb{N}$, $b \in P^h$, $c \in Q^k$, we have that $M, M(b), M(c), N$ are a heir-coheir amalgam;
3. for every $h \in \mathbb{N}$ and $b \in P^h$, $\text{tp}(b/Q)$ is a heir of $\text{tp}(b/M)$;
4. for every $k \in \mathbb{N}$ and $c \in Q^k$, $\text{tp}(c/P)$ is a coheir of $\text{tp}(c/M)$;
5. $\text{tp}(P/Q)$ is a heir of $\text{tp}(P/M)$;
6. $\text{tp}(Q/P)$ is a coheir of $\text{tp}(Q/M)$;
7. for every $h, k \in \mathbb{N}$, $b \in P^h$, $c \in Q^k$, $\text{tp}(b/M(c))$ is a heir of $\text{tp}(b/M)$;
8. for every $h, k \in \mathbb{N}$, $b \in P^h$, $c \in Q^k$, $\text{tp}(c/M(b))$ is a coheir of $\text{tp}(c/M)$.

Proof. See [10]. □

7.16 Lemma. *Let M, P, Q, N be an amalgam, with P and Q orthogonal over M . Then, M, P, Q, N is both a heir-coheir and a coheir-heir amalgam.*

Proof. By [10, Theorem 11.01], $\text{tp}(Q/M)$ has at least one heir on P . Since Q and P are orthogonal, $\text{tp}(Q/M)$ has exactly one extension to P , which therefore must be a heir. □

7.17 Lemma. *Let M, P, Q, N be a coheir-heir amalgam, and $N' := PQ$. If Q/M is rational, then N'/P is also rational.*

Therefore, $\forall d \in N^n$, such that d is P -bounded,

$$\exists b_0 \in P^n \ d \equiv_P b_0, \quad \text{and a fortiori } d \equiv_M b_0.$$

Proof. $\text{tp}(Q/P)$ is the (unique) heir of $\text{tp}(Q/M)$. By [10, §11.b], $\text{tp}(Q/P)$ is rational, and therefore N'/P is rational. □

5 Theorem. *Let M, P, Q, N be a tame-cotame amalgam.*

1. Call $N' := PQ$. Then, N'/Q is totally irrational, and N'/P is rational.
2. M, P, Q, N are both a heir-coheir and a coheir-heir amalgam.
3. P and Q are orthogonal over M .

Proof postponed to §7.1.

7.18 Corollary. *Let $x \in \tilde{A}$ be totally irrational, and $y \in \tilde{B}$ be rational. Then, x and y are orthogonal.*

Proof. By the last point in the Theorem. □

Before proving the Theorem, we will prove some additional lemmata.

7.19 Lemma. *Let $A, B \subseteq M^k$ be d -compact, and $f : A \rightarrow B$ be continuous and definable. Let $N \succeq M$, and $c, d \in A(N)$. If $c \equiv_M d$, then $f(c) \equiv_M f(d)$.*

Proof. Same as Lemma [13, 1.13]. □

7.20 Example. In the above lemma, it is essential that A is d-compact, and f is continuous. For instance, if $A =]0, 1[^2$ and $\lim_{a \rightarrow 0} f(a)$ does not exist (where $f : A \rightarrow [0, 1]$ is definable and continuous), then the conclusion does not hold.

7.21 Lemma. *Let N/M be rational, and $f : N^h \rightarrow N^k$ be definable in N . Then, the set*

$$D := \{a \in M^h : f(a) \text{ is } M\text{-bounded}\}$$

and the map

$$\begin{aligned} D &\rightarrow [0, 1]^k(M) \\ a &\mapsto \text{st}_M(f(a)) \end{aligned}$$

are definable in M .

Proof. It is [11, Corollary 1.5]. □

7.22 Lemma. *Let M, P, Q, N be a tame-cotame amalgam. Let $b \in P$ and $c \in C$ such that $b \leq c$. Then, there exists $a \in M$ such that $b \leq a \leq c$. If moreover $b \in P \setminus C$, we can also impose $b < a < c$.*

7.1 Proof of Theorem 5

Note that, by Lemma 7.16, the second point is a consequence of the third one. Moreover, by Lemma 7.17, the fact that N'/P is rational is a consequence of the second point.

First step: it suffices to consider the case when P/M and Q/M are finitely generated.

The only point that needs clarification is that N'/Q is totally irrational. Assume therefore that we have proved it for every finite sub-amalgam M, P', Q', N' such that $Q \preceq Q'$ and $P \preceq P'$. We have to prove that every $d \in N'^n$ is Q -bounded.

Let $f : M^h \times M^k \rightarrow M$ be definable, $b \in P^h$ and $c \in Q^k$ such that $d = f(b, c)$. Since, by assumption, $\text{tp}(d/M(c))$ is totally irrational, d is $M(c)$ -bounded, and *a fortiori* Q -bounded.

Therefore, we can reduce to the case when $P = M(b)$, $Q = M(c)$, and $N = M(b, c)$, for some $b \in P^h$, $c \in Q^k$.

Second step: it suffices to consider the case when $h = 1$, namely $\dim(P/M) = 1$.

We will proceed by induction on h . Let $b := (b_1, b')$, where $b_1 \in P$ and $b' \in P^{h-1}$. Let $P_1 := M(b_1)$, and $Q_1 := Q(b_1)$.

Let us apply the case $h = 1$ to the amalgam M, P_1, Q, Q_1 . Hence, Q_1/P_1 is rational, and Q_1/Q is totally irrational. Moreover, b_1 and c are orthogonal.

Note that P/P_1 is totally irrational. Therefore, P_1, P, Q_1, N are a tame-cotame amalgam, with $\dim(P/P_1) = h - 1$. Hence, by the inductive case $h - 1$, N'/P is rational, and N'/Q_1 is totally irrational. Thus, N'/Q is totally irrational, and the first point is done.

For the last point, let N'' be a sufficiently saturated and homogeneous extension of N , and $e \in N''^h$ and $f \in N''^k$ such that $\text{tp}(e/M) = \text{tp}(b/M)$ and $\text{tp}(f/M) = \text{tp}(c/M)$. We have to prove that $\text{tp}(e, f/M) = \text{tp}(b, c/M)$. It suffices to show that $\text{tp}(e, c/M) = \text{tp}(b, c/M)$, and $\text{tp}(b, f/M) = \text{tp}(b, c/M)$. Namely, that $\text{tp}(e/Q) = \text{tp}(b/Q)$, and $\text{tp}(f/P) = \text{tp}(c/P)$.

By the case $h = 1$, we have that $\text{tp}(e_1/Q) = \text{tp}(b_1/Q)$, where $e =: (e_1, e')$. Let σ be a Q -automorphism of N'' such that $\sigma(b_1) = e_1$, and define $g := \sigma(e) = (b_1, g')$. Thus,

$$\text{tp}(e/Q) = \text{tp}(g/Q) = \text{tp}(b_1, g'/Q).$$

Moreover,

$$\text{tp}(b_1, b'/M) = \text{tp}(b/M) = \text{tp}(e/M) = \text{tp}(g/M) = \text{tp}(b_1, g'/M),$$

and therefore $\text{tp}(b'/P_1) = \text{tp}(g'/P_1)$. Thus, by the case $h - 1$, $\text{tp}(b'/Q_1) = \text{tp}(g'/Q_1)$, namely

$$\text{tp}(b/Q) = \text{tp}(g/Q) = \text{tp}(e/Q).$$

Finally, by the case $h = 1$, we have that $\text{tp}(f/P_1) = \text{tp}(c/P_1)$. Therefore, by the case $h - 1$, we have that $\text{tp}(f/P) = \text{tp}(c/P)$.

Third step: the case $h = 1$.

7.23 Claim. $(b, c)/M$ are independent.

If not, then $(b, c)/M$ would be dependent, and therefore $b/M(c)$ is dependent. Since c/M is independent, this would imply $b \in M(c)$, absurd.

If, for contradiction, N'/Q were not totally irrational, then, since $\dim N'/Q = 1$, it would be rational. Hence, N'/M would be rational, and *a fortiori* P/M would be rational, absurd.

For the third point, let $b' \in N$, $c' \in N^k$ such that $\text{tp}(b'/M) = \text{tp}(b/M)$, and $\text{tp}(c'/M) = \text{tp}(c/M)$. Let $U \subseteq N^{k+1}$ be M -definable. We have to prove that, if $(b, c) \in U$, then $(b', c') \in U$. W.l.o.g., we can assume that U is an open cell. Thus, there exists an open cell $W \subseteq N^k$ and functions $f, g : W \rightarrow N$, all definable (over M) such that $f < g$ and

$$U = \{ (t, u) \in N^{k+1} : u \in W, f(u) < t < g(u) \}.$$

Thus, $f(c) < b < g(c)$. If $f(c) = a_1 \in M$, then $f(c) = f(c') = a_1 < b'$, and similarly for g . Otherwise, there exist $a_1, a_2 \in M$ such that

$$f(c) < a_1 < b < a_2 < g(c).$$

Thus,

$$f(c') < a_1 < b' < a_2 < g(c'),$$

and $(b', c') \in U$. □

7.24 Example. Let M be the field of real algebraic numbers, $b_1, b_2 \in \mathbb{R}$ be algebraically independent, and $\text{tp}(t/\mathbb{R}) = 0^+$. Let $P := M(b_1, b_2)$, $Q := M(t, b_1 + tb_2)$, $N := PQ = M(b_1, b_2, t)$. Then, $P \cap Q = M$, but P and Q are *not* orthogonal over M , because $\dim N/M = 3 < \dim P/M + \dim Q/M$.

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