O-minimal spectrum

Specialization order and definability of types

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Introduction

On the space of n-type $S_n(M)$ of a first-order structure M there is the Stone topology.
If M is an o-minimal structure, then we can define on $S_n(M)$ the spectral topology too, which is a weaker topology than the Stone one.
In the spectral topology types are *not* closed. Therefore, we can define that a type q specializes a type p if q is in the closure of p.
The set $\mathrm{cl}(p)$ of specialization of p is a finite linearly ordered set.
We show that we can find a type p' which dominated by p, such that from the study of $\mathrm{cl}(p')$ we can understand whether p is rational or not.

Definition of spectrum

Let X be a topological space, with a basis of open sets $\mathcal{U}$. A constructible set is a boolean combination of sets in $\mathcal{U}$. We will assume that singletons are constructible and closed.

$$\begin{aligned} \mathfrak{C} &= \text{Boolean algebra of constructible sets;} \\ \widetilde{X} = \widetilde{(X,\mathcal{U})} &\ \text{(the spectrum of } X) = \text{Stone space of } \mathfrak{C}; \\ \text{Filter} &= \text{subset } p \text{ of } \mathfrak{C} \text{ closed for intersection, and such that} \\ &\quad \text{if } U \in p \text{ and } U \subseteq V, \text{ then } V \in p. \\ \widetilde{X} &= \text{set of ultrafilters } (= \text{maximal filters}) \text{ of } \mathfrak{C}. \end{aligned}$$

Given $C \in \mathfrak{C}$ and $p \in \widetilde{X}$, write $p \in C$ instead of $C \in p$. Let

$$\widetilde{C} := \{\, p \in \widetilde{X} : p \in C \,\}.$$

Stone topology basis: $\{\, \widetilde{U} : U \text{ constructible} \,\}$.
Spectral topology basis: $\{\, \widetilde{U} : U \text{ constructible and open} \,\}$.

The map ι sending $x \in X$ to $\widetilde{\{x\}}$ is a homeomorphism with its image, and $\iota(X)$ is dense in $\widetilde{X}$. We can identify X with $\iota(X)$. Since $\widetilde{X}$ is quasi-compact, $\widetilde{X}$ is a quasi-compactification of X.

Examples

1. The prime spectrum of a ring: the basis is given by the Zariski open sets.
2. Consider on the space $\mathbb{Q}$ the basis $\mathcal{U}$ given by open intervals (a, b), with endpoints in $\mathbb{Q} \cup \{\pm\infty\}$; then, $\widetilde{\mathbb{Q}} = \{ a^{\pm} : a \in \mathbb{Q} \} \cup \mathbb{R} \cup \{\pm\infty\}$.
3. Same as above, but the interval in the basis must also be bounded; then, $\widetilde{\mathbb{Q}} = \{ a^{\pm} : a \in \mathbb{Q} \} \cup \mathbb{R} \cup \{\infty\}$.
4. Let $X = R^n$, where R is a real-closed field, and $\mathcal{U}$ be the set of open constructible subsets of X. $\widetilde{X}$ is the real spectrum of the polynomial ring $R[x_1, \dots, x_n]$.

Spectral space

Lemma

$\widetilde{X}$ is a spectral space.

This means:

1. $\widetilde{X}$ is quasi-compact and it has a basis of quasi-compact sets, closed under finite intersections.
2. Every irreducible closed set is the closure of a unique point.

Normality

Definition

A topological space is normal if every pair of disjoint closed subsets can be separated by open sets.
X is constructibly normal if every pair of disjoint constructible closed subsets can be separated by constructible open sets.

From now on we will assume that X is constructibly normal.

Lemma

$\widetilde{X}$ is normal (but not T_1: not all points are closed).

Closed points

Definition

Let $\mathrm{Max}(\widetilde{X})$ be the set of closed points of $\widetilde{X}$.

Lemma

$\mathrm{Max}(\widetilde{X})$ is compact Hausdorff (and hence normal).

The embedding ι maps X densely into $\mathrm{Max}(\widetilde{X})$, and therefore $\mathrm{Max}(\widetilde{X})$ is a compactification of X.

Remark

If $\mathcal{U}$ is the set of *all* open subsets of X, then $\mathrm{Max}(\widetilde{X})$ is the Stone-Čech compactification of X.

Specialization

Let Y be a topological space, $p, q \in Y$.

Definition

q is a specialization of p, p is a generalization of q, $q \trianglelefteq p$, if q is in the closure of $\{p\}$.

This means that any open set containing q contains also p.
$\trianglelefteq$ is a partial order on Y.

Lemma

Let Y be a spectral space. Y is normal iff every point $p \in Y$ has a unique specialization $\pi(y)$ in $\mathrm{Max}(Y)$.

The map of the above lemma is a continuous retraction $\pi : Y \to \mathrm{Max}(Y)$.

O-minimal structures

Definition

An o-minimal structure (expanding a group) is first-order structure $(M, \leq, +, \dots)$, such that $(M, \leq, +)$ is an ordered (Abelian) group, and every definable subset of M is a finite union of points an intervals with endpoints in $M \sqcup \{\pm\infty\}$.

The order induces a topology on M (and on M^n).

Fact

1. *A definable set is a boolean combination of open definable sets.*
2. *To every definable subset of M^n we can assign a dimension, satisfying the usual conditions.*
3. $\dim(\mathrm{cl}(A) \setminus A) < \dim A$, *if A is definable and non-empty.*
4. *The 1 type of a point x over M is determined by the cut of x over M.*

Example

A real closed field is an o-minimal structure.

O-minimal spectrum

From now on, M will be an o-minimal structure expanding a group, and X will be some definable subset of M^n.
We will take the definable open subsets as the basis for the topology of X.
Note that a set is constructible iff it is definable.

1. X is definably normal, and therefore $\widetilde{X}$ is normal.
2. $\widetilde{X}$ coincides with the set of types on X.
3. The Stone topology of $\widetilde{X}$ is the usual Stone topology for types over some structure.

Every definable subset of $\widetilde{X}$ is normal.
However, $\widetilde{X}$ is *not* completely normal (in general): it contains some (non definable!) subsets which are not normal.
For instance, if M is a ω-saturated expansion of a real closed field, then $\widetilde{M^2}$ is not completely normal.

Dimension

Every C definable subset of X has an (o-minimal) dimension.

Definition

The dimension of a type $p \in \widetilde{X}$ is

$$\{\dim C : p \in C\}.$$

Lemma

$\dim(C) = \max\{\dim p : p \in C\}$.

Specialization

Lemma

1. *The closure of every $p \in \widetilde{X}$ is* linearly *ordered by* $\trianglelefteq$.
2. *If* $q \triangleleft p$, *then* $\dim q < \dim p$.
3. $\dim X$ *is equal to the maximal length (minus 1) of the specialization chains in* $\widetilde{X}$.

Example

If $X = M$, then the generalizations of 0 are 0, 0^+ and 0^-.

$$\mathrm{cl}(0^+) = \{\, 0, 0^+ \,\}; \quad \mathrm{cl}(0^-) = \{\, 0, 0^- \,\}.$$

$\dim X = 1$, and $\left(0 \triangleleft 0^+\right)$ is a specialization chain of maximal length.

Functions on $\widetilde{X}$

Every definable map $f : X \to X'$ induces a map $\widetilde{f} : \widetilde{X} \to \widetilde{X}'$, given by

$$\widetilde{f}(p) = \{\, C \text{ constructible in } X' : p \in f^{-1}(C) \,\}.$$

1. If f is continuous, then $\widetilde{f}$ is also continuous;
2. $\widetilde{f}$ is open iff f is open;
3. $\widetilde{f}$ is closed iff f is definably closed.

If f is not closed, we cannot conclude that $\widetilde{f}$ maps closed points to closed points, even when f is continuous.
If f is continuous, then $\widetilde{f}$ is closed iff $\widetilde{f}$ maps closed points to closed points.

Rudin-Keisler ordering

Let p and q be types over M.

Definition

We say that q is dominated by p, and write $q \underset{RK}{\leq} p$, if there exists a definable map f such that $\widetilde{f}(p) = q$.

Definition

$M(p)$, the extension generated by p, is the smallest elementary extension of M that realizes p.

$M(p)$ is unique up to isomorphism over M.
q is dominated by p iff q is realized in $M(p)$.

$\trianglelefteq$ versus $\underset{RK}{\leq}$

Lemma

1. $\underset{RK}{\leq}$ *is a partial ordering on* $\widetilde{X}$;
2. *if* $q \underset{RK}{<} p$, *then* $\dim q < \dim p$;
3. *if* $q \underset{RK}{\equiv} p$, *then* $\dim q = \dim p$.

Lemma

If $q \trianglelefteq p$, then $q \underset{RK}{\leq} p$.
More precisely, there exist: a cell C containg q, of the same dimension as q; a set $V \subseteq X$ open, definable and containing C; and a definable retraction $r : V \to C$, such that $\widetilde{r}(p) = q$.

Elementary extensions

Aside

Given an elementary extension $N \succcurlyeq M$, let

$$\theta : \widetilde{X(N)} \to \widetilde{X}$$

be the restriction map. θ is:

1. continuous in the Stone and spectral topologies;
2. closed in the Stone topology;
3. not closed nor open in the spectral one;
4. ¿open in the Stone topology?

Lemma

Let $Z \subseteq \widetilde{X}$ be type-definable. If Z is connected, then $\theta^{-1}(Z)$ is connected. If either M expands a field, or Z is definable and definably compact, then θ^{-1} preserves the (Čech) cohomology of Z.

Definably compact sets

From now on, we assume that M **expands a field**.

Definition

X is definably compact (d-compact for short) if it is closed and bounded.

Definition

- A d-compactification is a map $\rho : X \to Y$ such that: Y is d-compact, ρ is a definable homeomorphism with its image, and its image is dense in Y.
- Let $f : X \to X'$ be definable and continuous. A d-compactification $\rho : X \to Y$ is compatible with f if there exists a definable continuous map $g : Y \to X'$ such that $g \circ \rho = f$.

Points near the frontier

Let $C \subseteq X$ be definable, and $p \in \widetilde{X}$.

Definition

C is near p, denoted by $C \trianglelefteq p$, if every (definable) open neighbourhood of C contains p.

Lemma

$C \trianglelefteq p$ iff there exists $q \in C$ such that $q \trianglelefteq p$.

Definition

p is near the frontier of X if, for some d-compactification Y of X, $p \trianglelefteq Y \setminus X$.

Lemma

p is near the frontier of X iff $p \notin C$, for every d-compact $C \subseteq X$.

Compact points

Definition

$p \in \widetilde{X}$ is compact if it is closed and far form the frontier of X.

T.f.a.e.:

1. p is compact;
2. p is closed in *some* d-compactification of X;
3. p is closed in *all* d-compactification of X;
4. p is closed and $p \in C$, for some d-compact $C \subseteq X$.

Lemma

Let $f : X \to X'$ be a definable map (not necessarily continuous), and $p \in \widetilde{X}$. If p is compact (in $\widetilde{X}$), then $\widetilde{f}(p)$ is compact (in $\widetilde{X}'$).

Remember that $\widetilde{f}$ might not map closed points to closed points. However, if X is d-compact, then does $\widetilde{f}$ map closed points to closed points.

Rational types

Definition

A type p (over M) is rational (a.k.a. definable) if for every formula $\phi(x, y)$ there is a formula $\psi(x, y)$ and a tuple $b \in M$ such that

$$\{ x \in M : M(p) \models \phi(x, p) \} = \{ x \in M : M \models \psi(x, b) \}.$$

A tipe is irrational if it is not rational.

Example

The rational 1-type are a, $a^{\pm}$ and $\pm\infty$, where $a \in M$.

Rational extensions

Definition

An elementary extension $N \succcurlyeq M$ is rational (a.k.a. tame) if it realizes only rational types.

Theorem

Let $N \succcurlyeq M$. T.f.a.e.:

1. *N/M is rational;*
2. *N realizes only rational 1-types;*
3. *M is Dedekind complete in N: i.e., every M-bounded element of N has a standard part in M.*

Example

Every type over $\mathbb{R}$ is rational, because $\mathbb{R}$ is Dedekind complete.

Totally irrational types

Remark

Let $q \underset{\text{RK}}{\leq} p$. If p is rational, then q is also rational.

Definition

A type p is totally irrational if for every $q \underset{\text{RK}}{\leq} p$, either q is realized, or q is irrational.

A totally irrational extension is an elementary extension satisfying only totally irrational types.

Lemma

Let $N \succcurlyeq M$. T.f.a.e.:

1. *N/M is irrational;*
2. *every 1-type realized in N/M is irrational;*
3. *M does not realize 0^+;*
4. *M is cofinal in N.*

Totally irrational and compact types

Proposition

p is totally irrational iff it is compact.

Sketch of proof.

If p is totally irrational, then it cannot realize 0^+. If, for contradiction, p were not compact, we could assume that $q \lhd p$ for some q. But then the distance between p and q would be 0^+.

Conversely, if p is rational, there would exists a definable continuous function f such that $\widetilde{f}(p) = 0^+$. Therefore, $f^{-1}(0) \lhd p$, and p would not be compact. □

Specializations and rationality

The ambient space is d-compact.
Let $\mathrm{cl}(p) = (p_0 \lhd p_1 \lhd \ldots \lhd p_n = p)$.
If p is rational, then each p_i is rational, because $p_i \underset{\mathrm{RK}}{\leq} p$. Moreover, p_0 is realized, because it is both rational and totally irrational.
p is totally irrational iff $p = p_0$.
¿If p_0 is realized, is p rational? ¡no! (example: $x_1 = e \cdot x_2$; $x_1 = 0^+$).

Lemma

If $q < p$, q is rational, and $\dim p = \dim q + 1$, then p is rational.

Corollary

If $\dim(p_i) = i$ for all $i \leq n$, then p (and each p_i) is rational.

Example

If p is a 1-type and $a \in M$, with $a \trianglelefteq p$, then then p is rational. In fact, either $p = a$, or $p = a^{\pm}$.

Compactifications fixing a type

Let X be d-compact and $q \lhd p \in \widetilde{X}$.

Definition

$\rho : Y \to Y'$ is a d-compactification fixing p if:

1. $p \in \widetilde{Y}$ and $Y \subseteq X$;
2. ρ is a d-compactification compatible with the inclusion map $\lambda : Y \to X$.

Denote by $\mu : X \to Y'$ an extension of λ to X.

Definition-lemma

If ρ is a d-compactification fixing p, then there exists $q' \in \widetilde{Y}'$ such that $q' \lhd \widetilde{\rho}(p)$ and $\widetilde{\mu}(q') = q$. We call $(q' \lhd \widetilde{\rho}(p))$ is a lifting of $(q \lhd p)$ (compatible with the d-compactification ρ).

Maximal pairs

Let $(q' \lhd p')$ be a lifting of $(q \lhd p)$. Note that $\dim q \leq \dim q'$, because $q \underset{\mathrm{RK}}{\leq} q'$, and that $p \underset{\mathrm{RK}}{\equiv} p'$.

Definition

The pair $(q \lhd p)$ is maximal if, for every lifting $(q' \lhd p')$ of $(q \lhd p)$, we have $\dim q' = \dim q$.

Remark

If $(q \lhd p)$ is maximal, and ρ is a d-compactification fixing p, then there exists a unique q' such that $(q' \lhd \widetilde{\rho}(p))$ is a lifting of $(q \lhd p)$ compatible with ρ.

We can always finding a lifting $(q' \lhd p')$ of $(q \lhd p)$, such that $(q' \lhd p')$ is maximal.

Rational pairs

Let $p \in \widetilde{X}$ and f be a definable function on X.

Definition

(f, p) is rational if for some (eq. for any) $N \succcurlyeq M$ and $d \in N$ such that $\mathrm{tp}(d/M) = p$, we have that $\mathrm{tp}\big(d/M(f(d))\big)$ is rational.

Given $q \lhd p$, there is always a retraction r such that $\widetilde{r}(p) = q$.

Proposition

Let $q \lhd p$, and r be any retraction mapping p to q. If $(q \lhd p)$ is maximal, then (r, p) is rational.

Let p be a type which is not closed, and let $p_0 \trianglelefteq p$ the mimimal type in the closure of p. We can assume that $(p_0 \lhd p)$ is maximal.

Corollary

We can find an intermediate extension $M \preccurlyeq M(p_0) \preccurlyeq M(p)$, with $M(p_0)/M$ totally irrational, and $M(p)/M(p_0)$ rational.

Example

Let M be the field of real algebrac numbers, and $e \in \mathbb{R} \setminus M$. Let $p \in \widetilde{M^2}$ be the following type:

$$p_1 = 0^+,$$
$$p_2 = 0^+,$$
$$p_1/p_2 = e.$$

The closure of p is $\left(0 \lhd p\right)$. However, p is not rational, because e is dominated by it.
Let ρ be the map $\rho(x_1, x_2) = (x_1, x_1/x_2)$. Then, $p' := \widetilde{\rho}(p) = (0^+, e)$; let $q' := (0, e)$. Note that q is totally irrational of dimension 1 and that $(q' \lhd p')$ is a maximal lifting of $(q \lhd p)$. Hence, $p/M(q')$ is rational, and $M(q)/M$ is totally irrational.

Decomposition of extensions

Let $M \preccurlyeq N$.
We can find intermediate extensions P and Q such that

1. P/Q is totally irrational and maximally so (and therefore N/P is rational)
2. Q/N is rational and maximally so (and therefore N/Q is totally irrational).

Theorem

P is unique up to isomorphisms fixing M.

Sketch of proof.

Let P' be another such extension, For every $x \in P$ there exists a unique $f(x) \in P'$, such that $x - f(x)$ is infinitesimal w.r.t. M. The map f is the isomorphism we were looking for. □

In the case $M = N(p)$, we can define $P = M(p_0)$ (if we choose $(p_0 \lhd p)$ maximal).

Amalgalms

Let: M, P, Q, N be o-minimal structures (expaning a field), such that $M \preccurlyeq P \preccurlyeq N$, $M \preccurlyeq Q \preccurlyeq N$, and $N = PQ$.

Proposition

Assume that P/M totally irrational and Q/M rational. Then,

1. *N/P is rational and N/Q is totally irrational;*
2. *M, P, Q, N form both a heir-coheir and a coheir-heir amalgalm;*
3. *P and Q are orthogonal over M.*

Spectrum of the product

Let $\pi_X : X \times Y \to X$ be the projection onto X. For every $q \in \widetilde{X}$, $\widetilde{\pi}_X^{-1}(q)$ is homeomorphic to $Y(\widetilde{M(q)})$.

Lemma

If $Y = [0,1]$, then $\widetilde{\pi}_X^{-1}(q) = \widetilde{[0,1](M(q))}$ has trivial (Čech) cohomology, and therefore $\widetilde{\pi}$ induces an isomorphism in cohomology.

Orthogonal types

Let $\mu : \widetilde{X \times Y} \to \widetilde{X} \times \widetilde{Y}$ be the map induced by the projections π_X and π_Y.

μ is continuous, surjective, but *not* injective. Define $p \times q := \mu^{-1}(p, q)$.

Definition

p and q orthogonal iff $p \times q$ is a singleton.

Remark

There is always $r \in p \times q$ such that $\dim r = \dim p + \dim q$.

T.f.a.e.:

1. p and q are orthogonal;
2. $\dim r = \dim p + \dim q$ for every $r \in p \times q$;
3. p has exactly one extension to $M(q)$;
4. q has exactly one extension to $M(p)$.