

Recursive definitions on surreal numbers

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Abstract

Let \mathbf{No} be Conway's class of surreal numbers. I will make explicit the notion of a function f on \mathbf{No} recursively defined over some family of functions. Under some 'tameness' and uniformity condition, f must satisfy some interesting properties; in particular, the supremum of the class

$$\{x \in \mathbf{No} : f(x) \geq 0\}$$

is actually an element of \mathbf{No} . As an application, I will prove that concatenation function $x : y$ cannot be defined recursively in a uniform way over polynomial functions.

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1 Introduction

The class of surreal numbers \mathbf{No} was introduced by Conway in [4]. I will present some results on regarding the properties of functions defined recursively on \mathbf{No} . As a particular case, I will give a different proof of the fact that \mathbf{On} is a real closed field.

I will assume on the part of the reader some familiarity with the theory of surreal numbers, as exposed in [4, 8, 1]; however, I will repeat some of its fundamental properties along the way.

1.1 Basic definitions and properties

I will recall some of the basic properties of the class of surreal numbers. The definitions and theorems of this section can be found in [4] and [8].

I will work in the set theory NBG of von Neumann, Bernays and Gödel with global choice. A well formed formula of NBG is a formula with set and class variables, without quantifications over classes.

I remind that \mathbf{On} is the class of all ordinals.

1.1 Definition (Surreal numbers). Following Gonshor, I define a surreal number x as a function with domain an ordinal α and codomain the set $\{+, -\}$. The ordinal α is called the *length* of the x , in symbol $\ell(x)$. The collection of all surreal numbers is the proper class \mathbf{No} .

1.2 Definition (Linear order). On \mathbf{No} there is a linear order, defined according to the rule $x \leq y$ (x is less or equal to y) iff

$$\begin{array}{l} x = y \quad \text{or} \\ x(\gamma) \text{ undefined and } y(\gamma) = + \quad \text{or} \\ x(\gamma) = - \quad \text{and } y(\gamma) = + \quad \text{or} \\ x(\gamma) = - \quad \text{and } y(\gamma) \text{ undefined,} \end{array}$$

where

$$\gamma := \min \{ \beta \in \mathbf{On} : x(\beta) \neq y(\beta) \}.$$

1.3 Definition (Simpler). There is also a partial order $x \prec y$ (x is strictly *simpler* than y , or x is a *canonical option* of y), iff x is the restriction of y to an ordinal strictly smaller than $\ell(y)$.

1.4 Definition (Convex). A subclass $A \subseteq \mathbf{No}$ is *convex* iff

$$\forall x, y \in A \forall z \in \mathbf{No} \quad x < z < y \rightarrow z \in A.$$

Given $L, R \subseteq \mathbf{No}$ and $z \in \mathbf{No}$, $L < z$ means that

$$\forall x \in L \quad x < z;$$

similar definition for $x < R$.

$L < R$ means that

$$\forall x \in L \forall y \in R \quad x < y.$$

The fundamental properties of \prec and of $<$ are given by the following theorem.

Theorem 1. • (\mathbf{No}, \preceq) is a well founded partial order.

- (\mathbf{No}, \leq) is a dense linear order.
- $\forall a \in \mathbf{No}, \mathcal{S}(a) := \{ x \in \mathbf{No} : a \preceq x \}$ is a convex subclass of \mathbf{No} .
- If $A \subseteq \mathbf{No}$ is convex and non-empty, then there is a unique simplest element $a \in A$, i.e.

$$\forall x \in A \quad a \preceq x.$$

- If $L < R$ are subsets of \mathbf{No} , then the cut

$$(L | R) := \{x \in \mathbf{No} : L < x < R\}$$

is convex and non-empty. Its simplest element is called $\langle L | R \rangle$.

- $\langle L | R \rangle = \langle L' | R' \rangle$ iff for every $x^L \in L, x^R \in R, x^{L'} \in L', x^{R'} \in R'$,

$$\begin{aligned} x^L &< \langle L' | R' \rangle < x^R \\ x^{L'} &< \langle L | R \rangle < x^{R'}. \end{aligned}$$

1.5 Definition (Canonical representation). Given $x \in \mathbf{No}$, let

$$\begin{aligned} L^x &:= \{y \in \mathbf{No} : y < x \ \& \ y \prec x\} \\ R^x &:= \{y \in \mathbf{No} : y > x \ \& \ y \prec x\}. \end{aligned}$$

Then,

$$x = \langle L^x | R^x \rangle.$$

$\langle L^x | R^x \rangle$ is called the *canonical representation* of x .

Theorem 2 (Inverse cofinality theorem). Let $x, z \in \mathbf{No}$, $z \prec x$. Let $x = \langle L | R \rangle$ be any representation of x . Then:

- If $z < x$, there exists $y \in L$ such that $z \leq y < x$.
- If $z > x$, there exists $y \in R$ such that $z \geq y > x$.

2 Recursive definitions

2.1 Functions of one variable

2.1 Definition (Recursive functions). Let $f : \mathbf{No} \rightarrow \mathbf{No}$ be a function, L, R be two sets of functions. I write

$$f = \langle L | R \rangle$$

iff for all $x \in \mathbf{No}$

$$f(x) = \left\langle f^L(x, x^L, x^R, f(x^L), f(x^R)) \mid f^R(x, x^L, x^R, f(x^L), f(x^R)) \right\rangle, \quad (2.1)$$

where x^L, x^R vary in L^x and R^x respectively, and f^L, f^R vary in L and R respectively.

The formula (2.1) gives a recursive definition of f ; in fact, if $f(x^\circ)$ has already been defined for every x° canonical option of x , it defines uniquely $f(x)$ as the simplest element in the cut

$$\left(f^L(x, x^L, x^R, f(x^L), f(x^R)) \mid f^R(x, x^L, x^R, f(x^L), f(x^R)) \right)_{x^L \in L^x, x^R \in R^x},$$

if it is non-empty.

The elements f° of $L \cup R$ are called *options* of f ; they are functions with codomain \mathbf{No} and domain classes containing $A \times B$, with

$$A = \{ (x, y, z) \in \mathbf{No}^3 : y < x < z \}$$

$$B = \{ (f(y), f(z)) : y < z \in \mathbf{No} \}.$$

I will often use the notations

$$f = \langle f^L \mid f^R \rangle$$

instead of $f = \langle L \mid R \rangle$, and

$$f^L(x, x^L, x^R)$$

instead of $f^L(x, x^L, x^R, f(x^L), f(x^R))$, and similarly for f^R .

2.2 Definition (Uniform definitions). The recursive definition $f = \langle f^L \mid f^R \rangle$ is *uniform* iff the value of $f(x)$ does not depend on the chosen representation of x . This means that:

- $\forall x, y, z \in \mathbf{No}$ such that $y < x < z$

$$f^L(x, y, z, f(y), f(z)) < f(x) < f^R(x, y, z, f(y), f(z)).$$

- If $x \in \mathbf{No}$ and $x = \langle x^L \mid x^R \rangle$ is any representation of x , then

$$f(x) = \langle f^L(x, x^L, x^R) \mid f^R(x, x^L, x^R) \rangle.$$

If \mathfrak{A} is a family of functions, and $f : \mathbf{No} \rightarrow \mathbf{No}$, then f is (*uniformly*) *recursive* over \mathfrak{A} iff there exist two subsets L, R of \mathfrak{A} such that $f = \langle L \mid R \rangle$ is a (uniform) recursive definition of f .

Analogous definitions can be given for f a function having as domain a convex subclass of \mathbf{No} of the form $\langle L' \mid R' \rangle$, with $L' < R'$ subsets of \mathbf{No} .

If $a = \langle L \mid R \rangle$ then the variable a° , an option of a , will range in $L \cup R$, a^L will be an element of L , a^R of R . If do not specify otherwise, $\langle L \mid R \rangle$ will be the canonical representation of a , unless I am defining a , i.e. I am constructing L and R .

Similarly, if $f = \langle L \mid R \rangle$, f° will be ranging in $L \cup R$, f^L in L , f^R in R .

2.2 Functions of many variables

In this subsection, $n > 0$ is a fixed natural number. K is the set

$$K := \{ +, -, 0 \}^n \setminus \{ (0, \dots, 0) \}$$

and $k := |K| = 3^n - 1$.

2.3 Definition. The partial order \preceq on \mathbf{No} induces a well-founded partial order on \mathbf{No}^n given by

$$(x_1, \dots, x_n) \preceq (y_1, \dots, y_n) \quad \text{iff} \quad \forall i = 1, \dots, n \ x_i \preceq y_i.$$

Given $\sigma \in K$ and $\vec{x}, \vec{y} \in \mathbf{No}^n$, \vec{y} is a σ -option of \vec{x} iff $\vec{y} \prec \vec{x}$ and for all $i = 1, \dots, n$

$$y_i < x_i \quad \text{iff} \quad \sigma_i = -$$

$$y_i > x_i \quad \text{iff} \quad \sigma_i = +$$

$$y_i = x_i \quad \text{iff} \quad \sigma_i = 0.$$

2.4 Definition. Let $f : \mathbf{No}^n \rightarrow \mathbf{No}$, L, R be sets of functions. I write $f = \langle L \mid R \rangle$ iff for all $\vec{x} \in \mathbf{No}^n$

$$f(x) = \langle f^L(\vec{x}, \vec{x}^{\sigma(1)}, \dots, \vec{x}^{\sigma(k)}, f(\vec{x}^{\sigma(1)}), \dots, f(\vec{x}^{\sigma(k)})) \mid f^R(\vec{x}, \vec{x}^{\sigma(1)}, \dots, \vec{x}^{\sigma(k)}, f(\vec{x}^{\sigma(1)}), \dots, f(\vec{x}^{\sigma(k)})) \rangle,$$

where f^L, f^R vary in L and R respectively, σ is a fixed enumeration of K , and the $\vec{x}^{\sigma(i)}$ vary among the $\sigma(i)$ -options of \vec{x} .

For shorthand, I will write $f^L(x, x^\circ, f(x^\circ))$ instead of

$$f^L(\vec{x}, \vec{x}^{\sigma(1)}, \dots, \vec{x}^{\sigma(k)}, f(\vec{x}^{\sigma(1)}), \dots, f(\vec{x}^{\sigma(k)})).$$

Again, the definition of f is uniform iff $f(\vec{x})$ does not depend on the chosen representations of the components of \vec{x} .

2.3 Examples

2.5 Example. Let $f(x) := x + 1 = \langle x^L + 1, x \mid x^R + 1 \rangle$. Then,

$$f = \langle f_1^L, f_2^L \mid f^R \rangle \quad \text{where}$$

$$f_1^L(f(x^L)) := f(x^L) \qquad f_2^L(x) := x$$

$$f^R(f(x^R)) := f(x^R).$$

As we can see in the previous example, it might be that some option of f does not depend on some of the variables; for instance, f_1^L does not depend on x^R nor on $f(x^R)$. In this case, I have to impose that $f(x) > f_1^L(f(x^L))$, even for those x that do not have canonical right options.

2.6 Example. In general, the recursive definition of $f(x, y) := x + y$ is

$$f(x, y) = \langle f(x^L, y), f(x, y^L) \mid f(x^R, y), f(x, y^R) \rangle.$$

2.7 Example. Let

$$f(x, y) := xy = \langle x^L y + xy^L - x^L y^L, x^R y + xy^R - x^R y^R \mid x^L y + xy^R - x^L y^R, x^R y + xy^L - x^R y^L \rangle.$$

Then, among the left options of f there is

$$f_1^L(f(x^L, y), f(x, y^L), f(x^L, y^L)) := f(x^L, y) + f(x, y^L) - f(x^L, y^L).$$

2.8 Example. Let $f_0(x), \dots, f_n(x)$ be recursive functions, $f := f_0 + \dots + f_n$. Then,

$$f^L(x) = f(x) + f_i^L(x) - f_i(x)$$

$$f^R(x) = f(x) + f_i^R(x) - f_i(x),$$

where $0 \leq i \leq n$ and f_i^L, f_i^R are left and right options of f_i .

Proof. Induction on n , using the recursive definition of $x + y$. □

2.9 Example. Let $a \in \mathbf{No}$, $m \in \mathbb{N}$, $f(x) := ax^m$. Then,

$$f^\circ(x, x^L, x^R) = ax^m - (a - a^\circ)(x - x^L)^\alpha(x - x^R)^{m-\alpha},$$

where $0 \leq \alpha \leq m$. Moreover, f° is a left option iff $m - \alpha$ is even and $a^\circ < a$, or $m - \alpha$ is odd and $a^\circ > a$.

Proof. Let $x, y \in \mathbf{No}$. The recursive definition of xy implies that an option of xy is of the form

$$(xy)^\circ = xy - (x - x^\circ)(y - y^\circ),$$

i.e.

$$xy - (xy)^\circ = (x - x^\circ)(y - y^\circ).$$

Therefore, by induction on m ,

$$x_1 \cdots x_n - (x_1 \cdots x_n)^\circ = (x_1 - x_1^\circ) \cdots (x_n - x_n^\circ).$$

In particular,

$$(ax^m)^\circ = ax^m - (a - a^\circ)(x - (x^\circ)_1) \cdots (x - (x^\circ)_m),$$

where $(x^\circ)_1 \cdots (x^\circ)_m$ are options of x . Now, I can choose among $(x^\circ)_1 \dots, (x^\circ)_m$ the ‘best’ (i.e. the greatest) left option x^L , and the ‘best’ (i.e. the smallest) right option x^R , proving the result. \square

2.10 Lemma. Let

$$p(x) = \sum_{0 \leq i \leq n} a_i x^i \in \mathbf{No}[x].$$

Then,

$$p^\circ(x, x^L, x^R) = p(x) - (a_m - a_m^\circ)(x - x^L)^\alpha(x - x^R)^{m-\alpha},$$

where $0 \leq \alpha \leq m \leq n$. Moreover, p° is a left option iff $m - \alpha$ is even and $a^\circ < a$, or $m - \alpha$ is odd and $a^\circ > a$.

Proof. Put together 2.8 and 2.9. \square

The relation \preceq induces a partial order on $\mathbf{No}[x]$.

2.11 Definition. Let $p, q \in \mathbf{No}[x]$. The polynomial q is strictly simpler than p , in symbols $q \prec p$, iff

$$\begin{aligned} p &= \sum_{0 \leq i \leq n} a_i x^i \\ q &= \sum_{0 \leq i \leq n} b_i x^i, \end{aligned}$$

and there exists a (unique) $m \leq n$ such that

$$\begin{aligned} b_i &= a_i & i &= m + 1, \dots, n \\ b_m &\prec a_m. \end{aligned}$$

2.12 Remark. \preceq is a well-founded partial order on \mathbf{No} , therefore I can do induction on it. Moreover, if p is a polynomial and p° is one of its canonical options, as defined in Lemma 2.10, then, as a function of x , $p^\circ \prec p$. This means that by substituting any value b, b' for x^L and x^R in p° we obtain a polynomial $q(x) := p(x, b, b') \in \mathbf{No}[x]$ which is strictly simpler than $p(x)$, in the ordering of $\mathbf{No}[x]$.

Proof.

$$p^\circ(x, x^L, x^R) = p(x) - (a_m - a_m^\circ)(x - x^L)^\alpha(x - x^R)^{m-\alpha}.$$

As a polynomial in x , the i -coefficients of p° are equal to a_i for $i > m$, while the m -coefficient is a_m° , which is strictly simpler than a_m . \square

3 Main results

$\mathbf{No}^{\mathcal{D}}$, the Dedekind completion of \mathbf{No} , is not a class. Nevertheless, I can use it as an abbreviation of a well formed formula of NBG (with a free class variable).

$\overline{\mathbf{No}}$ is the class $\mathbf{No} \cup \{\pm\infty\}$. In the following definitions, \mathbb{K} is a class, \leq is a linear ordering on it and $\mathbb{K}^{\mathcal{D}}$ is the Dedekind completion of (\mathbb{K}, \leq) .

For the rest of this section, \mathfrak{A} will be some family of functions on \mathbf{No} . A function $f : \mathbf{No} \rightarrow \mathbf{No}$ is really a class, therefore a family of functions is not a class, but only an abbreviation for a well formed formula of NBG.

3.1 Definition (Tame). Let $n \geq 0 \in \mathbb{N}$, $f : \mathbb{K}^{n+1} \rightarrow \mathbb{K}$. f is *tame* iff for every $\vec{d} \in \mathbb{K}^n$ either $f(x, \vec{d})$ is constant, or for every $\zeta \in \mathbb{K}^{\mathcal{D}} \setminus \mathbb{K}$, $c \in \mathbb{K}$ there exist $a, b \in \mathbb{K}$ such that $a < \zeta < b$ and either

$$\begin{aligned} \forall x \in (a, \zeta) f(x, \vec{d}) > c \quad \text{or} \\ \forall x \in (a, \zeta) f(x, \vec{d}) < c, \end{aligned}$$

and similarly for (ζ, b) .

3.2 Definition (sup property). Let $n \geq 0 \in \mathbb{N}$, $f : \mathbb{K}^{n+1} \rightarrow \mathbb{K}$. f satisfies the *sup property* iff $\forall \vec{d} \in \mathbb{K}^n$, $\forall a < b \in \mathbb{K}$, $\forall c \in \mathbb{K}$, the infima and suprema of the following classes

$$\begin{aligned} \left\{ x \in \mathbb{K} : a < x < b \ \& \ f(x, \vec{d}) \leq c \right\} \\ \left\{ x \in \mathbb{K} : a < x < b \ \& \ f(x, \vec{d}) \geq c \right\} \end{aligned}$$

are in $\mathbb{K} \cup \{\pm\infty\}$.

3.3 Definition (Intermediate value). A function $f : \mathbb{K}^{n+1} \rightarrow \mathbb{K}$ satisfies the *intermediate value property* (I.V.P.) iff $\forall \vec{e} \in \mathbb{K}^n$ $\forall a < b \in \mathbb{K}$ $\forall d \in \mathbb{K}$

$$f(a, \vec{e}) < d < f(b, \vec{e}) \rightarrow \exists c \in \mathbb{K} \ a < c < b \ \& \ f(c, \vec{e}) = d.$$

Note that the tameness of a given f is not a well formed formula of NBG, because it involves a quantification over elements of $\mathbf{No}^{\mathcal{D}}$, i.e. over classes. Therefore in general it is not possible to speak about the collection of all tame functions in a given collection \mathfrak{A} . Moreover the theorems involving the tameness of \mathfrak{A} are actually meta-theorems.

On the other hand, the intermediate value and the sup properties correspond to well formed formulae of NBG, because they involve quantifications only on elements of \mathbf{No} .

3.4 Remark. If $f : \mathbb{K} \rightarrow \mathbb{K}$ satisfies the sup property and is continuous, then it satisfies the I.V.P.

Proof. $c = \sup \{ x \in (a, b) : f(x) \leq d \}$. □

Given a family of functions \mathfrak{A} and a property P of functions, I say that \mathfrak{A} satisfies P iff every function in \mathfrak{A} satisfies P ; for instance, I could say that \mathfrak{A} is tame.

Theorem 3. *Suppose that \mathfrak{A} is tame and satisfies the sup property. Let $f : \mathbf{No} \rightarrow \mathbf{No}$ be uniformly recursive over \mathfrak{A} and tame. Then, f satisfies the sup property.*

Proof. Let $a, b, d \in \mathbf{No}$, $a < b$. Define

$$\zeta := \sup \{ x \in \mathbf{No} : a < x < b \ \& \ f(x) \leq d \} \in \mathbf{No}^{\mathcal{D}} \cup \{ \pm\infty \}.$$

We have to prove that $\zeta \in \overline{\mathbf{No}}$. I will prove it by induction on d . Suppose not. Then, $a < \zeta < b$, and, by tameness, without loss of generality we can suppose that $f(x) < d$ in the interval (a, ζ) , while $f(x) > d$ in (ζ, b) . We will construct a $c \in (a, b)$ such that $f(c) = d$.

I will give the options of c . First, $a < c < b$, therefore a is a left option, b a right one. Assume that $f = \langle f^L \mid f^R \rangle$. Then, $f(c) = d$ is equivalent to

$$d^L < f(c) < d^R \tag{3.1a}$$

$$f^L(c, c^L, c^R) < d < f^R(c, c^L, c^R) \tag{3.1b}$$

(3.1a) By inductive hypothesis,

$$c^L := \sup \{ x \in (a, b) : f(x) \leq d^L \} \in \overline{\mathbf{No}}.$$

Moreover, $c^L \leq \zeta$, and $c^L \neq \zeta$ because $\zeta \notin \overline{\mathbf{No}}$. Therefore, we can add c^L to the left options of c . Similarly, we find c^R using d^R .

(3.1b) Assume that we have already found some options c^L, c^R of c . Let $U := (c^L, c^R)$. I will find some ‘new’ options $c^{L'}, c^{R'}$ such that $c^L \leq c^{L'} < \zeta < c^{R'} \leq c^R$ and if $x \in (c^{L'}, c^{R'})$, then

$$f^L(x, c^L, c^R) < d < f^R(x, c^L, c^R).$$

Then, I add $c^{L'}$ and $c^{R'}$ to the options of c , and repeat the process.

Let

$$c^{R'} := \inf \{ x \in U : f^L(x, c^L, c^R) \geq d \}.$$

The sup property implies that $c^{R'} \in \overline{\text{No}}$. If $c^{R'} = +\infty$, then $f^L(x, c^L, c^R) \leq d$ in all U , therefore I do not need to add any option to c . Otherwise, $\zeta \leq c^{R'} \leq c^R$, and $\zeta \neq c^{R'}$, therefore I can add $c^{R'}$ to the right options of c .

Similarly, let

$$c^{L'} := \sup \{ x \in I : f^R(x, c^L, c^R) \leq d \}.$$

Therefore, at the end of the process we find a $c \in (a, b)$ such that $f(c) = d$, a contradiction. \square

Note that in the previous theorem I am assuming that f is a function of only one variable.

3.5 Example. Conway proves that, with the already defined $<$, $+$ and \cdot , No is a linearly ordered field. Using the previous theorem, I will show that it is actually real closed. Conway proves the same thing, but with a quite different technique.

Proof. An linearly ordered field \mathbb{K} is real closed iff every polynomial in $\mathbb{K}[x]$ satisfies the I.V.P.. Therefore, by Remark 3.4, it is enough to prove that every $p(x) \in \text{No}[x]$ satisfies the sup property.

Moreover, since $\deg p' < \deg p$, the derivative p' is simpler than p . Therefore, by inductive hypothesis, every root of p' is in No , and hence p is tame.

By Remark 2.12 every option $p^\circ(x, x^\circ)$ of $p(x)$ is simpler than $p(x)$, therefore, by induction on p , it satisfies the sup property. The conclusion follows from Theorem 3. \square

Note that from the proof of Theorem 3 it is possible to extract an algorithm to compute the root of a polynomial in $\text{No}[x]$. This algorithm gives Conway's formula to compute $1/a$ if the polynomial in question is $ax - 1$, and C. Bach's formula for \sqrt{a} if we use the polynomial $x^2 - a$ instead. Higher degree polynomials yield quite complicate formulae.

3.1 Initial substructures of No

3.6 Definition (Initial). Let S be a subclass of No . S is *initial* iff

$$\forall x \in S \forall y \in \text{No} \ y \preceq x \rightarrow y \in S.$$

3.7 Lemma. Let $S \subseteq \text{No}$ be an initial subclass of No , $L < R$ be subclasses of S , $x := \langle L \mid R \rangle$ (if it exists). Let $z \prec x$. Then, $z \in S$.

Note that I cannot conclude that $x \in S$.

Proof. Without loss of generality, I can suppose $z < x$. Then, by the inverse cofinality theorem, there exists $y \in L$ such that $z \leq y < x$. So, $z \preceq y$.

However, $y \in L \subseteq S$ and S is initial, therefore $z \in S$. \square

3.8 Remark. The union of an arbitrary family of initial subclasses of No is initial. Therefore, given $S \subseteq \text{No}$, I can speak of the maximal initial subclass of S ; it is the union of all initial subsets of S , therefore it really exists.

3.9 Definition (Closure). Let $S \subseteq \text{No}$, $f : \text{No}^n \rightarrow \text{No}$. S is *closed* under f iff

$$f(S^n) \subseteq S.$$

S is closed under \mathfrak{A} iff it is closed under every f in \mathfrak{A} .

The *closure* of S under \mathfrak{A} , $S^{\mathfrak{A}}$, is the smallest $T \subseteq \text{No}$ closed under \mathfrak{A} and containing S .

Theorem 4. *Suppose that for every $T \subseteq \text{No}$ initial, $T^{\mathfrak{A}}$ is also initial. Let $f : \text{No}^n \rightarrow \text{No}$ be recursive over \mathfrak{A} . Let S_1, \dots, S_n be initial subclasses of No , $S := S_1 \times \dots \times S_n$. Then, $(S_1 \cup \dots \cup S_n \cup f(S))^{\mathfrak{A}}$ is also initial.*

Moreover, if every f° option of f is a function of $f(x^\circ)$ only, then $(f(S))^{\mathfrak{A}}$ is initial.

Note that I am not assuming that the definition of f is uniform.

Proof. Let $U := (S_1 \cup \dots \cup S_n \cup f(S))^{\mathfrak{A}}$, and let T be the maximal initial subclass of U . We need to prove that $T = U$.

Claim 1. For every $\bar{a} \in S$, $b := f(\bar{a}) \in T$.

The proof is by induction on \bar{a} . An option of b is

$$b^\circ = f^\circ(\bar{a}, \bar{a}^\circ, f(\bar{a}^\circ)).$$

Every \bar{a}° is strictly simpler than \bar{a} , therefore $\bar{a}^\circ \in S$. So, by inductive hypothesis, $f(\bar{a}^\circ) \in T$. But $S \subseteq T^n$ and, by hypothesis on \mathfrak{A} , T is closed under \mathfrak{A} , therefore $b^\circ \in T$. Thus, by Lemma 3.7, $b \in T$.

Claim 2. $T = U$.

It suffices to prove that T contains S_1, \dots, S_n and $f(S)$, and that T is closed under \mathfrak{A} :

- $T = T^{\mathfrak{A}}$ by hypothesis on \mathfrak{A} .
- $S_i \subseteq T$, $i = 1, \dots, n$ because the S_i are initial.
- $f(S) \subseteq T$ by Claim 1.

To prove the second point, define $U := (f(S))^{\mathfrak{A}}$, T its maximal initial subclass. As before, it is enough to prove the following claim:

Claim 3. For every $\bar{a} \in S$, $b := f(\bar{a}) \in T$.

The proof is by induction on \bar{a} . An option of b is

$$b^\circ = f^\circ(f(\bar{a}^\circ)).$$

$\bar{a}^\circ \prec \bar{a}$, therefore $\bar{a}^\circ \in S$. Thus, by inductive hypothesis, $f(\bar{a}^\circ) \in T$. By hypothesis on \mathfrak{A} , T is closed under \mathfrak{A} , implying that $b^\circ = f^\circ(f(\bar{a}^\circ)) \in T$. Hence, by Lemma 3.7, $b \in T$. \square

Theorem 5. *Suppose that for every $T \subseteq \text{No}$ initial, $T^{\mathfrak{A}}$ is also initial. Let $f : \text{No}^n \rightarrow \text{No}$ be recursive over \mathfrak{A} . Let S be an initial subclass of No . Then, $S^{\mathfrak{A} \cup \{f\}}$ is also initial.*

Proof. Let

$$\begin{aligned} T_0 &:= S \\ T_{i+1} &:= (T_i \cup f(T_i^n))^{\mathfrak{A}} \\ T &:= \bigcup_{i \in \mathbb{N}} T_i = S^{\mathfrak{A} \cup \{f\}}. \end{aligned}$$

Then, by Theorem 4 and induction on i , T_i is initial for every $i \in \mathbb{N}$, therefore T is also initial. \square

A different way of presenting the same reasoning is the following.

Proof. Let T the maximal initial subclass of $S^{\mathfrak{A} \cup \{f\}}$. By hypothesis on \mathfrak{A} , T is closed under \mathfrak{A} . I have to prove that T is also closed under f .

Let $\vec{a} \in T^n$. I will prove that $f(\vec{a}) \in T$ by induction on \vec{a} . By Lemma 3.7, it is enough to find $L < R \subseteq T$ such that

$$w := f(\vec{a}) = \langle L \mid R \rangle.$$

An option of w is of the form

$$f^\circ(\vec{a}, \vec{a}^\circ, f(\vec{a}^\circ)),$$

where $\vec{a}^\circ \in \mathfrak{A}$ and $\vec{a}^\circ \prec \vec{a}$. Therefore, $\vec{a}^\circ \in T^n$, and, by inductive hypothesis, $f(\vec{a}^\circ) \in T^n$. Thus, $f^\circ(\vec{a}, \vec{a}^\circ, f(\vec{a}^\circ)) \in T$. \square

3.10 Corollary. *Let S, U be initial subclasses of No. Then, the following subclasses of No are also initial:*

1. *The (additive) subgroup generated by S .*
2. *The subring generated by S .*
3. $-S := \{-x : x \in S\}$.
4. $S + U := \{x + y : x \in S, y \in U\}$.
5. *The subgroup generated by $SU := \{xy : x \in S, y \in U\}$.*

Proof. For the first two points, apply Theorem 5. The third point is obvious. For the other two points, apply Theorem 4. \square

3.11 Example. It is not true in general that if S, U are initial subgroups of No, then SU is also initial. For instance, take $S = U$ to be the subgroup generated by \mathbb{Z} and ω . Then, $\omega^2 + \omega = \omega(\omega + 1) \in SU$, but $\omega^2 + 1 \notin SU$.

3.12 Corollary. *Let \mathbb{K} be an initial subring of No. Let $L < R$ be subsets of \mathbb{K} , and let $c := \langle L \mid R \rangle$. Then, $\mathbb{K}[c]$ is also an initial subring of No.*

Proof. By Lemma 3.7, $\mathbb{K} \cup \{c\}$ is initial, therefore, by Corollary 3.10, the ring generated by it is also initial. \square

3.13 Definition (Strongly tame). A function $f : \mathbf{No}^{n+1} \rightarrow \mathbf{No}$ is *strongly tame* iff for all $a < b \in \mathbf{No}$, $\vec{e} \in \mathbf{No}^n$, $d \in \mathbf{No}$ either $f(x, \vec{e})$ is constant, or there exist $\zeta_0, \dots, \zeta_m \in \mathbf{No}^{\mathcal{D}}$ such that $a = \zeta_0 < \dots < \zeta_m = b$ and for $i = 0, \dots, m-1$

$$\begin{aligned} \forall x \in (\zeta_i, \zeta_{i+1}) f(x, \vec{e}) > d \quad \text{or} \\ \forall x \in (\zeta_i, \zeta_{i+1}) f(x, \vec{e}) < d. \end{aligned}$$

3.14 Definition. Let $f : \mathbf{No}^{n+1} \rightarrow \mathbf{No}$ be strongly tame, S be a subclass of \mathbf{No} . S is *closed under solutions* of f iff for all $\vec{e} \in S^n$ either $f(x, \vec{e})$ is constant, or for every $d \in S$

$$\forall c \in \mathbf{No} f(c, \vec{e}) = d \Rightarrow c \in S.$$

The *closure* of S under solutions of f is the smallest class containing S and closed under f and under its solutions.

Theorem 6. Suppose that \mathfrak{A} is strongly tame and satisfies the I.V.P. Let $f : \mathbf{No} \rightarrow \mathbf{No}$ uniformly recursive over \mathfrak{A} , strongly tame and satisfying the I.V.P.

Suppose that for every S initial subclass of \mathbf{No} , the closure of S under solutions of \mathfrak{A} and under f is also initial. Then, the closure of S under solutions of $\mathfrak{A} \cup \{f\}$ is initial.

The proof is quite similar to the one of Theorem 3.

Proof. Let T be the maximal initial subclass of the closure of S under solutions of $\mathfrak{A} \cup \{f\}$. By hypothesis, T is closed under solutions of \mathfrak{A} and under f . Therefore, it is enough to prove that T is closed under solutions of f . If f is constant, there is nothing to prove. Otherwise, let $d \in T$, $c \in \mathbf{No}$ such that $f(c) = d$. I will prove that $c \in T$ by induction on d .

I will give options of c in T .

Assume that $f = \langle f^L \mid f^R \rangle$. Then, $f(c) = d$ is equivalent to

$$d^L < f(c) < d^R \tag{3.2a}$$

$$f^L(c, c^L, c^R) < d < f^R(c, c^L, c^R) \tag{3.2b}$$

(3.2a) Let d° be an option of d .

$$\begin{aligned} c^L &:= \sup \{ x \in \mathbf{No} : x \leq c \ \& \ f(x) = d^\circ \} \\ c^R &:= \inf \{ x \in \mathbf{No} : x \geq c \ \& \ f(x) = d^\circ \}. \end{aligned}$$

The function f is strongly tame and satisfies the I.V.P., thus $c^L, c^R \in \overline{\mathbf{No}}$ and $c^L \leq c \leq c^R$. By the I.V.P., $c^L < c < c^R$. Moreover, $c^L, c^R \in T \cup \pm\infty$ by inductive hypothesis. By the I.V.P., $f(x) - d^\circ$ does not change sign in (c^L, c^R) ; in particular, if $d^\circ = d^L < d$, $f(x) > d^L$ in (c^L, c^R) , and similarly for $d^\circ = d^R$. Hence, I can take c^L and c^R as left and right options of c .

(3.2b) Suppose that I have already found c^L, c^R ‘old’ options of c . Let f° be an option of f , say $f^\circ = f^L < f$. I will construct $c^{L'}, c^{R'}$ ‘new’ options of c such that for $x \in (c^{L'}, c^{R'})$ $g(x) := f^L(x, c^L, c^R) < d$. Let

$$\begin{aligned} c^{L'} &:= \inf \{ x \in \mathbf{No} : x \leq c \ \& \ g(x) = d \} \\ c^{R'} &:= \sup \{ x \in \mathbf{No} : x \geq c \ \& \ g(x) = d \}. \end{aligned}$$

g is strongly tame, therefore $c^{L'}, c^{R'} \in \overline{\mathbf{No}}$ and $c^{L'} < c < c^{R'}$. By the I.V.P., $g(x) - d$ does not change sign in $(c^{L'}, c^{R'})$; consequently, $g(x) < d$ in $(c^{L'}, c^{R'})$. So, I can take $c^{L'}$ and $c^{R'}$ as left and right options of c .

At the end of the process, I obtain $L < R \subseteq T$ such that $c = \langle L \mid R \rangle$, so, by Lemma 3.7, $c \in T$. \square

3.15 Corollary. *The real closure of an initial subring of \mathbf{No} is initial. More in general, if $S \subseteq \mathbf{No}$ is initial, then the smallest real closed subfield of \mathbf{No} containing S is initial.*

Proof. \mathbb{Q} is an initial subfield of \mathbf{No} , therefore $\mathbb{Q} \cup S$ is also initial. Therefore, by Corollary 3.10, \mathbb{K} , the subring generated by it, is also initial.

It remains to prove that the real closure of \mathbb{K} is initial. Apply the Theorem 6 and induction on $\mathbf{No}[x]$. \square

If instead of considering *all* polynomials in $\mathbf{No}[x]$, we consider only the polynomials of degree up to a fixed degree n , the previous corollary is still valid, with the same proof. In particular, taking $n = 1$, we can conclude the following:

3.16 Corollary. *Let S be an initial subclass of \mathbf{No} . Then, the subfield of \mathbf{No} generated by S is initial.*

The following theorem was also proved in [5] with a different method.

Theorem 7. *Let \mathbb{K} be a real closed field and a proper set. Then, \mathbb{K} is isomorphic to an initial subfield of \mathbf{No} .*

Proof. If $\mathbb{K} \simeq \mathbb{Q}$, it is true.

Assume that \mathbb{F} is a real closed initial subfield of \mathbf{No} , and \mathbb{K} is (isomorphic to) the real closure of $\mathbb{F}(a)$ for some a transcendental over \mathbb{F} . Let $(L \mid R)$ be the cut determined by a over \mathbb{F} . For any $c \in (L \mid R)$, $\mathbb{F}(c)$ is isomorphic to $\mathbb{F}(a)$, and its real closure is isomorphic to \mathbb{K} . Moreover if $c = \langle L \mid R \rangle$ then, by Corollary 3.12, $\mathbb{K}[c]$ is initial, therefore, by Corollary 3.15, its real closure is also initial.

In general, let $(c_\beta)_{\beta < \alpha}$ be a transcendence basis of \mathbb{K} over \mathbb{Q} , for some $\alpha \in \mathbf{No}$. Let \mathbb{K}_0 be the real closure of \mathbb{Q} , and define \mathbb{K}_β to be the real closure of $\mathbb{Q}[c_i : i < \beta]$ for $\beta < \alpha$, i.e. $\mathbb{K}_0 := \mathbb{Q}$, and, for $0 < \beta \leq \alpha$

$$\mathbb{K}_\gamma := \begin{cases} \text{the real closure of } \mathbb{K}(c_\gamma) & \text{if } \beta = \gamma + 1 \\ \bigcup_{\gamma < \beta} \mathbb{K}_\gamma & \text{if } \beta \text{ is limit.} \end{cases}$$

In particular, $\mathbb{K}_\alpha := \mathbb{K}$. By the previous case and induction on β , each \mathbb{K}_β is isomorphic to an initial subfield of \mathbf{No} , and the conclusion follows. \square

It is not true that every ordered field (which is also a set) is isomorphic to an initial subfield of \mathbf{No} . For instance, take $\mathbb{K} := \mathbb{Q}(\sqrt{2} + 1/\omega) \subset \mathbf{No}$. Suppose, for contradiction, that there exists an isomorphism of ordered fields ψ between \mathbb{K} and an initial subfield of \mathbf{No} . Let $z = \psi(\sqrt{2} + 1/\omega)$. Then, $\sqrt{2} \prec z$, but $\sqrt{2} \notin \psi(\mathbb{K})$.

For more on the subject of initial embeddings of fields, see [7].

4 Examples and the concatenation function

4.1 Example. Let $f(x) := \langle x - 1 \mid x + 1 \rangle$. The image of f is the class of omnific integers. f satisfies the sup property, but not the I.V.P.. $f(x) = 0$ for $x \in (-1, 1)$, $f(x) = \alpha$ for $x \in [\alpha, \alpha + 1)$, $\alpha \in \mathbf{On}$, etc.

4.2 Example. Let

$$f(x) := [x] - x = \langle -1, [x] - x^R \mid 1, [x] - x^L \rangle.$$

Then, f is not tame.

Proof. Consider the cut $\omega : -\infty$ between positive finite numbers and infinite numbers. $f(x)$ changes sign infinitely many times in every neighbourhood of this cut. \square

4.3 Example. Let

$$f(x) := \langle -|x| \mid f_1^R(x^L), f_2^R(x^R) \rangle \quad \text{with}$$

$$f_1^R(z) := \begin{cases} 0 & \text{iff } z \geq 0 \\ 2 & \text{iff } z < 0 \end{cases}$$

$$f_2^R(z) := \begin{cases} 0 & \text{iff } z \leq 0 \\ 2 & \text{iff } z > 0. \end{cases}$$

Therefore,

$$f(x) = \begin{cases} \langle -|x| \mid 0 \rangle < 0 & \text{iff } x \neq 0 \\ 1 & \text{iff } x = 0 \end{cases}$$

4.4 Definition. Let $x, y > 0 \in \mathbf{No}$.

- $x \simeq y$ iff $\frac{x-y}{x}$ is infinitesimal.
- $x \lesssim y$ iff $x < y$ and $x \not\approx y$.
- $x \perp y$ iff $x \not\approx y$ and $y \not\approx x$.

4.5 Example. Let $d := 2/3$. For $a \in \mathbb{R}$, let

$$f_a^L(z) := \begin{cases} z & \text{if } z \leq a \\ a & \text{if } z \geq a \end{cases}$$

$$f_a^R(z) := \begin{cases} z & \text{if } z \geq a \\ a & \text{if } z \leq a. \end{cases}$$

Let

$$f(x) := \langle f_r^L(x^L) : d > r \in \mathbb{R} \mid f_s^R(x^R) : d < s \in \mathbb{R} \rangle.$$

Then, f is uniformly recursive. Moreover,

$$f(x) = x \quad \text{if } x \lesssim d$$

$$f(x) = d \quad \text{if } x \simeq d$$

$$f(x) = x \quad \text{if } d \lesssim x.$$

In particular, f does not satisfy the sup property, because

$$\sup \{ x \in \mathbf{No} : 0 < x < 1 \text{ \& } f(x) \leq d \} \notin \mathbf{No}$$

In the previous examples, the definition of f is uniform.

A surreal number z can be considered as a function from $\ell(z)$ into $\{+, -\}$, i.e. as a sequence of pluses and minuses, the *sign sequence* of z . Therefore, instead of 1 we can write the corresponding sequence $+$, instead of $1/2$, we can write $+-$, etc.

An element ζ of $\text{No}^{\mathcal{D}}$ has also a unique sign sequence, of length On iff $\zeta \notin \text{No}$. On the other hand, not every sign sequence corresponds to a element of $\text{No}^{\mathcal{D}}$; for instance, $+\infty$, the sequence of pluses of length On , is not in $\text{No}^{\mathcal{D}}$.

4.1 The concatenation function

4.6 Definition (Concatenation). Let $x, y \in \text{No}$. The *concatenation* of x, y , noted with $x : y$, is given by the sign sequence of x followed by the sign sequence of y .

The recursive definition of $x : y$ is

$$x : y = \langle x^L, x : y^L \mid x^R, x : y^R \rangle.$$

This definition is *not* uniform; while I can choose *any* representation of y , I must take the canonical representation of x (but see also [9]). In the following, I will prove that, given some hypothesis on \mathfrak{A} , it is never possible to find a uniform recursive definition of $x : y$ over \mathfrak{A} .

Let

$$f(x) := x : 1 = \langle x \mid x^R \rangle.$$

By definition, $f(x) < 0$ for $x < 0$, while $f(x) > 0$ for $x \geq 0$, therefore f does not satisfy the I.V.P.. Moreover, f is injective.

x	0	α	$-\alpha - 1$	$-\omega$	$1/2$	$1/\omega$	$\alpha \in \text{On}$.
$x : 1$	1	$\alpha + 1$	$-\alpha - 1/2$	$-\omega + 1$	$3/4$	$2/\omega$	

Table 1: Some values of $x : 1$

4.7 Lemma. Let $x \in \text{No}$. Then:

- $f(x) > x$.
- $x \prec f(x)$
- If $x < y$ and $y \prec x$, then $f(x) < y$.

Proof. Obvious. □

4.8 Lemma. Let $c, d \in \text{No}$. Then, there exists $a, b \in \overline{\text{No}}$ such that $a < c < b$ and

$$\begin{aligned} \forall x \in (a, c) f(x) < d \quad \text{or} \\ \forall x \in (a, c) f(x) > d, \end{aligned}$$

and similarly for (c, b) .

Proof. There are three cases, according to $d \stackrel{\leq}{\geq} c$.

$d < c$. Let $a := \langle d \mid c \rangle$, $b := +\infty$. Then, for every $x > a$ $f(x) > x > a > d$.

$d = c$. Let $a := c : -$, $b := +\infty$. For every $x \in (a, c)$ $c \prec x$, therefore $f(x) < c$. For every $x > c$ $f(x) > x > c$.

$d > c$. Let $a := c : -$. If $a < x < c$, then $c \prec x$, therefore $f(x) < c < d$.

Assume that $c \perp d$. Let $b := \langle c \mid d \rangle$. If $c < x < b$, then $b \prec x$, therefore $f(x) < b < d$.

Assume that $c \prec d$. Let $b := \langle c \mid d \rangle$. If $c < x < b$, then $b \prec x$, therefore $f(x) < b < d$.

Assume that $d \prec c$. Let $b := d$. If $c < x < b$, then $d \prec x$, therefore $f(x) < d$. \square

4.9 Remark. Let $d := 2/3 = +-+-+\dots$,

$$\begin{aligned} a_0 &:= 0, \\ a_1 &:= +- = 1/2, \\ a_2 &:= +-+- = 5/8, \\ a_3 &:= +-+-+-, \\ &\dots, \\ a_{n+1} &:= a_n : +-, \end{aligned}$$

and

$$\begin{aligned} b_n &:= a_n : + = +- \dots + -+ \\ c_n &:= a_n : - = +- \dots + - - . \end{aligned}$$

Then, for every $n \in \mathbb{N}$,

$$c_n < a_n < c_{n+1} < d < b_n.$$

Moreover, $f(a_n) = b_n > f(d) > d$, while $f(c_n) < a_n < d$. Besides,

$$d = \langle a_n \mid b_n \rangle_{n \in \mathbb{N}}$$

is the canonical representation of d . Moreover for every $x \in \mathbf{No}$

$$d \preceq f(x) \Leftrightarrow d \preceq x.$$

Finally, for every $x \in \mathbf{No}$ $d \preceq x$ iff $d - x$ is infinitesimal.

4.10 Remark. f is not continuous. In fact,

$$\lim_{\substack{x \rightarrow 0 \\ x \neq 0}} f(x) = 0,$$

while $f(0) = 1$.

4.11 Lemma. f is not tame.

Proof. Let d, a_n, b_n, c_n be as in Remark 4.9.

If $a < \zeta$, there exists $n \in \mathbb{N}$ such that $a < a_n < c_{n+1} < \zeta$. Therefore, $f(x) - d$ changes sign infinitely many times in every left neighbourhood of ζ .

However, note that in every infinitesimal neighbourhood of d , $f(x) \geq d$ iff $x \geq d$, and $f(x) \geq f(d)$ iff $x \geq f(d)$ or $x = d$. \square

4.12 Lemma. f does not satisfy the sup property.

Proof. Let d, a_n, b_n, c_n be as in Remark 4.9, and let $a := 0$.

Then, $f(a_i) > d$, while $f(c_i) < d$. Let

$$\zeta := \sup \{ x \in \mathbf{No} : a < x < d \text{ \& } f(x) \geq d \}.$$

Therefore,

$$\zeta \geq \sup \{ a_i : i \in \mathbb{N} \}.$$

On the other hand, if $d \preceq x$ and $x < d$, then $f(x) < d$, therefore

$$\zeta \leq \inf \{ x < d : d \preceq x \},$$

so $\zeta = \inf \{ x < d : d \preceq x \} \in \mathbf{No}^{\mathcal{D}} \setminus \mathbf{No}$. \square

4.13 Lemma. If \mathfrak{A} is a family of functions strongly tame and satisfying the I.V.P., then f cannot be uniformly recursive over \mathfrak{A} .

Proof. Suppose not, i.e. that $f = \langle f^L \mid f^R \rangle$, with $f^\circ \in \mathfrak{A}$.

Let d, a_n, b_n, c_n be as in Remark 4.9, and let ζ as in the previous proof.

I will show that there exists $c \in \mathbf{No}$ such that $f(c) = d$, which is clearly impossible. I will give the options of c .

$f(c) = d$ is equivalent to:

$$\begin{aligned} d \preceq f(c) \\ f^L(c, c^L, c^R) < d < f^R(c, c^L, c^R). \end{aligned}$$

By Remark 4.9, $d \preceq f(c)$ is equivalent to $d \preceq c$, and $d = \langle a_n \mid b_n \rangle_{n \in \mathbb{N}}$. Therefore, it is necessary and sufficient to add a_n to the left options of c and b_n to its right ones for every $n \in \mathbb{N}$ to ensure that $d \preceq f(c)$.

Let c^L, c^R be ‘old’ options of c such that $c^L < \zeta < c^R$, f^L be a left options of f . I will find $c^{L'}, c^{R'}$ ‘new’ options of c such that $c^{L'} < \zeta < c^{R'}$ and

$$\forall x \in (c^{L'}, c^{R'}) f^L(x, c^L, c^R) < d. \quad (4.1)$$

Let $g(x) := f^L(x, c^L, c^R)$,

$$\begin{aligned} c^{L'} &:= \sup \{ x \in \mathbf{No} : x < \zeta \text{ \& } g(x) = d \} \\ c^{R'} &:= \inf \{ x \in \mathbf{No} : x > \zeta \text{ \& } g(x) = d \}. \end{aligned}$$

\mathfrak{A} is strongly tame, therefore the previous sup and inf are actually a min and a max, unless $g(x)$ is constant.

For every left neighbourhood J of ζ there is $x \in J$ such that $f(x) < d$. Moreover, $g(x) < f(x)$ in (c^L, c^R) . Therefore, if $g(x)$ is constant, then $g(x) < d$, so $c^{L'} = -\infty$ and $c^{R'} = +\infty$.

Otherwise, $c^{L'}, c^{R'} \in \overline{\text{No}}$, while $\zeta \notin \overline{\text{No}}$, so $c^{L'} < \zeta < c^{R'}$. By the I.V.P., the sign of $g(x) - d$ is constant in $U := (c^{L'}, c^{R'})$. Again, in every left neighbourhood of ζ there is a x such that $f(x) < d$, therefore $g(x) < d$ in U .

Proceed similarly for f^R .

At the end of the process we have constructed a $c \in \text{No}$ such that $f(c) = d$, a contradiction. \square

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