Recursive definitions on surreal numbers

Antongiulio Fornasiero

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Abstract

Let $\mathbf{No}$ be Conway's class of surreal numbers. I will make explicit the notion of a function f on $\mathbf{No}$ recursively defined over some family of functions. Under some 'tameness' and uniformity condition, f must satisfy some interesting properties; in particular, the supremum of the class

$$\{\, x \in \mathbf{No} : f(x) \geq 0 \,\}$$

is actually an element of $\mathbf{No}$. As an application, I will prove that concatenation function $x : y$ cannot be defined recursively in a uniform way over polynomial functions.

Contents

1 Introduction

The class of surreal numbers $\mathbf{No}$ was introduced by Conway in [4]. I will present some results on regarding the properties of functions defined recursively on $\mathbf{No}$. As a particualar case, I will give a different proof of the fact that $\mathbf{On}$ is a real closed field.

I will assume on the part of the reader some familiarity with the theory of surreal numbers, as exposed in [4, 8, 1]; however, I will repeat some of its fundamental properties along the way.

1.1 Basic definitions and properties

I will recall some of the basic properties of the class of surreal numbers. The definitions and theorems of this section can be found in [4] and [8].

I will work in the set theory NBG of von Neumann, Bernays and Gödel with global choice. A well formed formula of NBG is a formula with set and class variables, without quantifications over classes.

I remind that **On** is the class of all ordinals.

1.1 Definition (Surreal numbers). Following Gonshor, I define a surreal number x as a function with domain an ordinal α and codomain the set $\{+, -\}$. The ordinal α is called the *length* of the x, in symbol $\ell(x)$. The collection of all surreal numbers is the proper class **No**.

1.2 Definition (Linear order). On **No** there is a linear order, defined according to the rule $x \leq y$ (x is less or equal to y) iff

$$\begin{array}{llll} x = y & \text{or} & & \\ x(\gamma) \text{ undefined} & \text{and} & y(\gamma) = + & \text{or} \\ x(\gamma) = - & \text{and} & y(\gamma) = + & \text{or} \\ x(\gamma) = - & \text{and} & y(\gamma) \text{ undefined}, & \end{array}$$

where

$$\gamma := \min \{ \beta \in \mathbf{On} : x(\beta) \neq y(\beta) \}.$$

1.3 Definition (Simpler). There is also a partial order $x \prec y$ (x is strictly *simpler* than y, or x is a *canonical option* of y), iff x is the restriction of y to an ordinal strictly smaller than $\ell(y)$.

1.4 Definition (Convex). A subclass $A \subseteq \mathbf{No}$ is *convex* iff

$$\forall x, y \in A\ \forall z \in \mathbf{No} \quad x < z < y \to z \in \mathbf{No}.$$

Given $L, R \subseteq \mathbf{No}$ and $z \in \mathbf{No}$, $L < z$ means that

$$\forall x \in L \quad x < z;$$

similar definition for $x < R$.
$L < R$ means that

$$\forall x \in L\ \forall y \in R \quad x < y.$$

The fundamental properties of $\prec$ and of $<$ are given by the following theorem.

Theorem 1.

- $(\mathbf{No}, \preceq)$ *is a well founded partial order.*
- $(\mathbf{No}, \leq)$ *is a dense linear order.*
- $\forall a \in \mathbf{No}$*,* $\mathcal{S}(a) := \{ x \in \mathbf{No} : a \preceq x \}$ *is a convex subclass of* **No**.
- *If* $A \subseteq \mathbf{No}$ *is convex and non-empty, then there is a unique simplest element* $a \in A$*, i.e.*

$$\forall x \in A\ a \preceq x.$$

- *If* $L < R$ *are subsets of* $\mathbf{No}$*, then the cut*

$$(\, L \mid R\,) := \{\, x \in \mathbf{No} : L < x < R\,\}$$

is convex and non-empty. Its simplest element is called $\langle\, L \mid R\,\rangle$*.*

- $\langle\, L \mid R\,\rangle = \langle\, L' \mid R'\,\rangle$ *iff for every* $x^L \in L$*,* $x^R \in R$*,* $x^{L'} \in L'$*,* $x^{R'} \in R'$*,*

$$x^L < \langle\, L' \mid R'\,\rangle < x^R$$
$$x^{L'} < \langle\, L \mid R\,\rangle < x^{R'}.$$

1.5 Definition (Canonical representation). Given $x \in \mathbf{No}$, let

$$L^x := \{\, y \in \mathbf{No} : y < x \,\&\, y \prec x\,\}$$
$$R^x := \{\, y \in \mathbf{No} : y > x \,\&\, y \prec x\,\}\,.$$

Then,

$$x = \langle\, L^x \mid R^x\,\rangle.$$

$\langle\, L^x \mid R^x\,\rangle$ is called the *canonical representation* of x.

Theorem 2 (Inverse cofinality theorem). *Let* $x, z \in \mathbf{No}$*,* $z \prec x$*. Let* $x = \langle\, L \mid R\,\rangle$ *be any representation of* x*. Then:*

- *If* $z < x$*, there exists* $y \in L$ *such that* $z \leq y < x$*.*
- *If* $z > x$*, there exists* $y \in R$ *such that* $z \geq y > x$*.*

2 Recursive definitions

2.1 Functions of one variable

2.1 Definition (Recursive functions). Let $f : \mathbf{No} \to \mathbf{No}$ be a function, L, R be two sets of functions. I write

$$f = \langle\, L \mid R\,\rangle$$

iff for all $x \in \mathbf{No}$

$$f(x) = \left\langle f^L\big(x, x^L, x^R, f(x^L), f(x^R)\big) \,\middle|\, f^R\,\big(x, x^L, x^R, f(x^L), f(x^R)\big) \right\rangle, \qquad (2.1)$$

where x^L, x^R vary in L^x and R^x respectively, and f^L, f^R vary in L and R respectively.

The formula (2.1) gives a recursive definition of f; in fact, if $f(x^{\circ})$ has already been defined for every x° canonical option of x, it defines uniquely $f(x)$ as the simplest element in the cut

$$\left(f^L\big(x, x^L, x^R, f(x^L), f(x^R)\big) \,\middle|\, f^R\big(x, x^L, x^R, f(x^L), f(x^R)\big) \right)_{x^L \in L^x, x^R \in R^x},$$

if it is non-empty.

The elements f° of $L \cup R$ are called *options* of f; they are functions with codomain **No** and domain classes containing $A \times B$, with

$$A = \{ (x, y, z) \in \mathbf{No}^3 : y < x < z \}$$
$$B = \{ (f(y), f(z)) : y < z \in \mathbf{No} \} .$$

I will often use the notations

$$f = \langle f^L \mid f^R \rangle$$

instead of $f = \langle L \mid R \rangle$, and

$$f^L(x, x^L, x^R)$$

instead of $f^L\big(x, x^L, x^R, f(x^L), f(x^R)\big)$, and similarly for f^R.

2.2 Definition (Uniform definitions). The recursive definition $f = \langle f^L \mid f^R \rangle$ is *uniform* iff the value of $f(x)$ does not depend on the chosen representation of x. This means that:

- $\forall x, y, z \in \mathbf{No}$ such that $y < x < z$
$$f^L\big(x, y, z, f(y), f(z)\big) < f(x) < f^R\big(x, y, z, f(y), f(z)\big).$$
- If $x \in \mathbf{No}$ and $x = \langle x^L \mid x^R \rangle$ is *any* representation of x, then
$$f(x) = \langle f^L(x, x^L, x^R) \mid f^R(x, x^L, x^R) \rangle.$$

If $\mathfrak{A}$ is a family of functions, and $f : \mathbf{No} \to \mathbf{No}$, then f is (*uniformly*) *recursive* over $\mathfrak{A}$ iff there exist two subsets L, R of $\mathfrak{A}$ such that $f = \langle L \mid R \rangle$ is a (uniform) recursive definition of f.

Analogous definitions can be given for f a function having as domain a convex subclass of **No** of the form $(L' \mid R')$, with $L' < R'$ sub*set*s of **No**.

If $a = \langle L \mid R \rangle$ then the variable a°, an option of a, will range in $L \cup R$, a^L will be an element of L, a^R of R. If do not specify otherwise, $\langle L \mid R \rangle$ will be the canonical representation of a, unless I am defining a, i.e. I am constructing L and R.

Similarly, if $f = \langle L \mid R \rangle$, f° will be ranging in $L \cup R$, f^L in L, f^R in R.

2.2 Functions of many variables

In this subsection, $n > 0$ is a fixed natural number. K is the set

$$K := \{ +, -, 0 \}^n \setminus \{(0, \ldots, 0)\}$$

and $k := |K| = 3^n - 1$.

2.3 Definition. The partial order $\preceq$ on **No** induces a well-founded partial order on $\mathbf{No}^n$ given by

$$(x_1, \ldots, x_n) \preceq (y_1 \ \ldots, y_n) \qquad \text{iff} \qquad \forall i = 1, \ldots, n \ x_i \preceq y_i.$$

Given $\sigma \in K$ and $\vec{x}, \vec{y} \in \mathbf{No}^n$, $\vec{y}$ is a σ-option of $\vec{x}$ iff $\vec{y} \prec \vec{x}$ and for all $i = 1, \ldots, n$

$$\begin{aligned} y_i < x_i \quad &\text{iff} \quad \sigma_i = - \\ y_i > x_i \quad &\text{iff} \quad \sigma_i = + \\ y_i = x_i \quad &\text{iff} \quad \sigma_i = 0. \end{aligned}$$

2.4 Definition. Let $f : \mathbf{No}^n \to \mathbf{No}$, L, R be sets of functions. I write $f = \langle\, L \mid R \,\rangle$ iff for all $\vec{x} \in \mathbf{No}^n$

$$f(x) = \big\langle\, f^L\big(\vec{x}, \vec{x}^{\sigma(1)}, \dots, \vec{x}^{\sigma(k)}, f(\vec{x}^{\sigma(1)}), \dots, f(\vec{x}^{\sigma(k)})\big) \mid f^R\big(\vec{x}, \vec{x}^{\sigma(1)}, \dots, \vec{x}^{\sigma(k)}, f(\vec{x}^{\sigma(1)}), \dots, f(\vec{x}^{\sigma(k)})\big) \,\big\rangle,$$

where f^L, f^R vary in L and R respectively, σ is a fixed enumeration of K, and the $\vec{x}^{\sigma(i)}$ vary among the $\sigma(i)$-options of $\vec{x}$.

For shorthand, I will write $f^L(x, x^{\circ}, f(x^{\circ}))$ instead of

$$f^L\big(\vec{x}, \vec{x}^{\sigma(1)}, \dots, \vec{x}^{\sigma(k)}, f(\vec{x}^{\sigma(1)}), \dots, f(\vec{x}^{\sigma(k)})\big).$$

Again, the definition of f is uniform iff $f(\vec{x})$ does not depend on the chosen representations of the components of $\vec{x}$.

2.3 Examples

2.5 Example. Let $f(x) := x + 1 = \langle\, x^L + 1, x \mid x^R + 1 \,\rangle$. Then,

$$\begin{aligned} f &= \langle\, f_1^L, f_2^L \mid f^R \,\rangle \quad \text{where} \\ f_1^L\big(f(x^L)\big) &:= f(x^L) \qquad f_2^L(x) := x \\ f^R\big(f(x^R)\big) &:= f(x^R). \end{aligned}$$

As we can see in the previous example, it might be that some option of f does not depend on some of the variables; for instance, f_1^L does not depend on x^R nor on $f(x^R)$. In this case, I have to impose that $f(x) > f_1^L\big(f(x^L)\big)$, even for those x that do not have canonical right options.

2.6 Example. In general, the recursive definition of $f(x, y) := x + y$ is

$$f(x, y) = \langle\, f(x^L, y), f(x, y^L) \mid f(x^R, y), f(x, y^R) \,\rangle.$$

2.7 Example. Let

$$f(x, y) := xy = \big\langle\, x^L y + x y^L - x^L y^L, x^R y + x y^R - x^R y^R \mid x^L y + x y^R - x^L y^R, x^R y + x y^L - x^R y^L \,\big\rangle.$$

Then, among the left options of f there is

$$f_1^L\big(f(x^L, y), f(x, y^L), f(x^L, y^L)\big) := f(x^L, y) + f(x, y^L) - f(x^L, y^L).$$

2.8 Example. Let $f_0(x), \dots, f_n(x)$ be recursive functions, $f := f_0 + \dots + f_n$. Then,

$$\begin{aligned} f^L(x) &= f(x) + f_i^L(x) - f_i(x) \\ f^R(x) &= f(x) + f_i^R(x) - f_i(x), \end{aligned}$$

where $0 \le i \le n$ and f_i^L, f_i^R are left and right options of f_i.

Proof. Induction on n, using the recursive definition of $x + y$. □

2.9 Example. Let $a \in \mathbf{No}$, $m \in \mathbb{N}$, $f(x) := ax^m$. Then,

$$f^{\mathbf{o}}(x, x^L, x^R) = ax^m - (a - a^{\mathbf{o}})(x - x^L)^{\alpha}(x - x^R)^{m-\alpha},$$

where $0 \leq \alpha \leq m$. Moreover, $f^{\mathbf{o}}$ is a left option iff $m - \alpha$ is even and $a^{\mathbf{o}} < a$, or $m - \alpha$ is odd and $a^{\mathbf{o}} > a$.

Proof. Let $x, y \in \mathbf{No}$. The recursive definition of xy implies that an option of xy is of the form

$$\big(xy\big)^{\mathbf{o}} = xy - (x - x^{\mathbf{o}})(y - y^{\mathbf{o}}),$$

i.e.

$$xy - \big(xy\big)^{\mathbf{o}} = (x - x^{\mathbf{o}})(y - y^{\mathbf{o}}).$$

Therefore, by induction on m,

$$x_1 \cdots x_n - \big(x_1 \cdots x_m\big)^{\mathbf{o}} = (x - x_1^{\mathbf{o}}) \cdots (x - x_m^{\mathbf{o}}).$$

In particular,

$$\big(ax^m\big)^{\mathbf{o}} = ax^m - (a - a^{\mathbf{o}})\big(x - (x^{\mathbf{o}})_1\big) \cdots \big(x - (x^{\mathbf{o}})_m\big),$$

where $(x^{\mathbf{o}})_1 \cdots (x^{\mathbf{o}})_m$ are options of x. Now, I can choose among $(x^{\mathbf{o}})_1 \ldots, (x^{\mathbf{o}})_m$ the 'best' (i.e. the greatest) left option x^L, and the 'best' (i.e. the smallest) right option x^R, proving the result. □

2.10 Lemma. *Let*

$$p(x) = \sum_{0 \leq i \leq n} a_i x^i \in \mathbf{No}[x].$$

Then,

$$p^{\mathbf{o}}(x, x^L, x^R) = p(x) - (a_m - a_m^{\mathbf{o}})(x - x^L)^{\alpha}(x - x^R)^{m-\alpha},$$

where $0 \leq \alpha \leq m \leq n$*. Moreover,* $p^{\mathbf{o}}$ *is a left option iff* $m - \alpha$ *is even and* $a^{\mathbf{o}} < a$*, or* $m - \alpha$ *is odd and* $a^{\mathbf{o}} > a$*.*

Proof. Put together 2.8 and 2.9. □

The relation $\preceq$ induces a partial order on $\mathbf{No}[x]$.

2.11 Definition. Let $p, q \in \mathbf{No}[x]$. The polynomial q is strictly simpler than p, in symbols $q \prec p$, iff

$$p = \sum_{0 \leq i \leq n} a_i x^i$$
$$q = \sum_{0 \leq i \leq n} b_i x^i,$$

and there exists a (unique) $m \leq n$ such that

$$b_i = a_i \qquad i = m+1, \ldots, n$$
$$b_m \prec a_m.$$

2.12 Remark. $\preceq$ is a well-founded partial order on **No**, therefore I can do induction on it. Moreover, if p is a polynomial and $p^{\mathbf{o}}$ is one of its canonical options, as defined in Lemma 2.10, then, as a function of x, $p^{\mathbf{o}} \prec p$. This means that by substituting any value b, b' for x^L and x^R in $p^{\mathbf{o}}$ we obtain a polynomial $q(x) := p(x, b, b') \in \mathbf{No}[x]$ which is strictly simpler than $p(x)$, in the ordering of $\mathbf{No}[x]$.

Proof.

$$p^{\mathbf{o}}(x, x^L, x^R) = p(x) - (a_m - a_m^{\mathbf{o}})(x - x^L)^{\alpha}(x - x^R)^{m-\alpha}.$$

As a polynomial in x, the i-coefficients of $p^{\mathbf{o}}$ are equal to a_i for $i > m$, while the m-coefficient is $a_m^{\mathbf{o}}$, which is strictly simpler than a_m. □

3 Main results

$\mathbf{No}^{\mathcal{D}}$, the Dedekind completion of **No**, is not a class. Nevertheless, I can use it as an abbreviation of a well formed formula of NBG (with a free class variable).

$\overline{\mathbf{No}}$ is the class $\mathbf{No} \cup \{\pm\infty\}$. In the following definitions, $\mathbb{K}$ is a class, $\leq$ is a linear ordering on it and $\mathbb{K}^{\mathcal{D}}$ is the Dedekind completion of $(\mathbb{K}, \leq)$.

For the rest of this section, $\mathfrak{A}$ will be some family of functions on **No**. A function $f : \mathbf{No} \to \mathbf{No}$ is really a class, therefore a family of functions is not a class, but only an abbreviation for a well formed formula of NBG.

3.1 Definition (Tame). Let $n \geq 0 \in \mathbb{N}$, $f : \mathbb{K}^{n+1} \to \mathbb{K}$. f is *tame* iff for every $\vec{d} \in \mathbb{K}^n$ either $f(x, \vec{d})$ is constant, or for every $\zeta \in \mathbb{K}^{\mathcal{D}} \setminus \mathbb{K}$, $c \in \mathbb{K}$ there exist $a, b \in \mathbb{K}$ such that $a < \zeta < b$ and either

$$\forall x \in (a, \zeta)\, f(x, \vec{d}) > c \quad \text{or}$$
$$\forall x \in (a, \zeta)\, f(x, \vec{d}) < c,$$

and similarly for (ζ, b).

3.2 Definition (sup **property).** Let $n \geq 0 \in \mathbb{N}$, $f : \mathbb{K}^{n+1} \to \mathbb{K}$. f satisfies the sup *property* iff $\forall \vec{d} \in \mathbb{K}^n$, $\forall a < b \in \mathbb{K}$, $\forall c \in \mathbb{K}$, the infima and suprema of the following classes

$$\left\{ x \in \mathbb{K} : a < x < b \;\&\; f(x, \vec{d}) \leq c \right\}$$
$$\left\{ x \in \mathbb{K} : a < x < b \;\&\; f(x, \vec{d}) \geq c \right\}$$

are in $\mathbb{K} \cup \{\pm\infty\}$.

3.3 Definition (Intermediate value). A function $f : \mathbb{K}^{n+1} \to \mathbb{K}$ satisfies the *intermediate value property* (I.V.P.) iff $\forall \vec{e} \in \mathbb{K}^n$ $\forall a < b \in \mathbb{K}$ $\forall d \in \mathbb{K}$

$$f(a, \vec{e}) < d < f(b, \vec{e}) \to \exists c \in \mathbb{K}\; a < c < b \;\&\; f(c, \vec{e}) = d.$$

Note that the tameness of a given f is not a well formed formula of NBG, because it involves a quantification over elements of $\mathbf{No}^{\mathcal{D}}$, i.e. over classes. Therefore in general it is not possible to speak about the collection of all tame functions in a given collection $\mathfrak{A}$. Moreover the theorems involving the tameness of $\mathfrak{A}$ are actually meta-theorems.

On the other hand, the intermediate value and the $\sup$ properties correspond to well formed formulae of NBG, because they involve quantifications only on elements of $\mathbf{No}$.

3.4 Remark. If $f : \mathbb{K} \to \mathbb{K}$ satisfies the $\sup$ property and is continuous, then it satisfies the I.V.P..

Proof. $c = \sup \{ x \in (a, b) : f(x) \leq d \}$. □

Given a family of functions $\mathfrak{A}$ and a property P of functions, I say that $\mathfrak{A}$ satisfies P iff every function in $\mathfrak{A}$ satisfies P; for instance, I could say that $\mathfrak{A}$ is tame.

Theorem 3. *Suppose that $\mathfrak{A}$ is tame and satisfies the* $\sup$ *property. Let $f : \mathbf{No} \to \mathbf{No}$ be uniformly recursive over $\mathfrak{A}$ and tame. Then, f satisfies the* $\sup$ *property.*

Proof. Let $a, b, d \in \mathbf{No}$, $a < b$. Define

$$\zeta := \sup \{ x \in \mathbf{No} : a < x < b \,\&\, f(x) \leq d \} \in \mathbf{No}^{\mathcal{D}} \cup \{ \pm\infty \} .$$

We have to prove that $\zeta \in \overline{\mathbf{No}}$. I will prove it by induction on d. Suppose not. Then, $a < \zeta < b$, and, by tameness, without loss of generality we can suppose that $f(x) < d$ in the interval (a, ζ), while $f(x) > d$ in (ζ, b). We will construct a $c \in (a, b)$ such that $f(c) = d$.

I will give the options of c. First, $a < c < b$, therefore a is a left option, b a right one. Assume that $f = \langle f^L \mid f^R \rangle$. Then, $f(c) = d$ is equivalent to

$$d^L < f(c) < d^R \tag{3.1a}$$
$$f^L(c, c^L, c^R) < d < f^R(c, c^L, c^R) \tag{3.1b}$$

(3.1a) By inductive hypothesis,

$$c^L := \sup \{ x \in (a, b) : f(x) \leq d^L \} \in \overline{\mathbf{No}}.$$

Moreover, $c^L \leq \zeta$, and $c^L \neq \zeta$ because $\zeta \notin \overline{\mathbf{No}}$. Therefore, we can add c^L to the left options of c. Similarly, we find c^R using d^R.

(3.1b) Assume that we have already found some options c^L, c^R of c. Let $U := (c^L, c^R)$. I will find some 'new' options $c^{L'}, c^{R'}$ such that $c^L \leq c^{L'} < \zeta < c^{R'} \leq c^R$ and if $x \in (c^{L'}, c^{R'})$, then

$$f^L(x, c^L, c^R) < d < f^R(x, c^L, c^R).$$

Then, I add $c^{L'}$ and $c^{R'}$ to the options of c, and repeat the process.

Let

$$c^{R'} := \inf \{ x \in U : f^L(x, c^L, c^R) \geq d \} .$$

The $\sup$ property implies that $c^{R'} \in \overline{\mathbf{No}}$. If $c^{R'} = +\infty$, then $f^L(x, c^L, c^R) \le d$ in all U, therefore I do not need to add any option to c. Otherwise, $\zeta \le c^{R'} \le c^R$, and $\zeta \ne c^{R'}$, therefore I can add $c^{R'}$ to the right options of c.

Similarly, let

$$c^{L'} := \sup \left\{ x \in I : f^R(x, c^L, c^R) \le d \right\}.$$

Therefore, at the end of the process we find a $c \in (a, b)$ such that $f(c) = d$, a contradiction. □

Note that in the previous theorem I am assuming that f is a function of only one variable.

3.5 Example. Conway proves that, with the already defined $<$, $+$ and $\cdot$, **No** is a linearly ordered field. Using the previous theorem, I will show that it is actually real closed. Conway proves the same thing, but with a quite different technique.

Proof. An linearly ordered field $\mathbb{K}$ is real closed iff every polynomial in $\mathbb{K}[x]$ satisfies the I.V.P.. Therefore, by Remark 3.4, it is enough to prove that every $p(x) \in \mathbf{No}[x]$ satisfies the $\sup$ property.

Moreover, since $\deg p' < \deg p$, the derivative p' is simpler than p. Therefore, by inductive hypothesis, every root of p' is in **No**, and hence p is tame.

By Remark 2.12 every option $p^{\mathbf{o}}(x, x^{\mathbf{o}})$ of $p(x)$ is simpler than $p(x)$, therefore, by induction on p, it satisfies the sup property. The conclusion follows from Theorem 3. □

Note that from the proof of Theorem 3 it is possible to extract an algorithm to compute the root of a polynomial in $\mathbf{No}[x]$. This algorithm gives Conway's formula to compute $1/a$ if the polynomial in question is $ax - 1$, and C. Bach's formula for $\sqrt{a}$ if we use the polynomial $x^2 - a$ instead. Higher degree polynomials yield quite complicate formulae.

3.1 Initial substructures of No

3.6 Definition (Initial). Let S be a subclass of **No**. S is *initial* iff

$$\forall x \in S\ \forall y \in \mathbf{No}\ y \preceq x \rightarrow y \in \mathbf{No}.$$

3.7 Lemma. *Let $S \subseteq$ No be an initial subclass of No, $L < R$ be subclasses of S, $x := \langle\, L \mid R \,\rangle$ (if it exists). Let $z \prec x$. Then, $z \in S$.*

Note that I cannot conclude that $x \in S$.

Proof. Without loss of generality, I can suppose $z < x$. Then, by the inverse cofinality theorem, there exists $y \in L$ such that $z \le y < x$. So, $z \preceq y$.

However, $y \in L \subseteq S$ and S is initial, therefore $z \in S$. □

3.8 Remark. The union of an arbitrary family of initial subclasses of **No** is initial. Therefore, given $S \subseteq$ **No**, I can speak of the maximal initial subclass of S; it is the union of all initial sub*set*s of S, therefore it really exists.

3.9 Definition (Closure). Let $S \subseteq \mathbf{No}$, $f : \mathbf{No}^n \to \mathbf{No}$. S is *closed* under f iff

$$f(S^n) \subseteq S.$$

S is closed under $\mathfrak{A}$ iff it is closed under every f in $\mathfrak{A}$.

The *closure* of S under $\mathfrak{A}$, $S^{\mathfrak{A}}$, is the smallest $T \subseteq \mathbf{No}$ closed under $\mathfrak{A}$ and containing S.

Theorem 4. *Suppose that for every $T \subseteq \mathbf{No}$ initial, $T^{\mathfrak{A}}$ is also initial. Let $f : \mathbf{No}^n \to \mathbf{No}$ be recursive over $\mathfrak{A}$. Let $S_1, \dots, S_n$ be initial subclasses of* No, *$S := S_1 \times \dots \times S_n$. Then, $\big(S_1 \cup \dots \cup S_n \cup f(S)\big)^{\mathfrak{A}}$ is also initial.*

Moreover, if every $f^{\mathbf{o}}$ option of f is a function of $f(x^{\mathbf{o}})$ only, then $\big(f(S)\big)^{\mathfrak{A}}$ is initial.

Note that I am not assuming that the definition of f is uniform.

Proof. Let $U := \big(S_1 \cup \dots \cup S_n \cup f(S)\big)^{\mathfrak{A}}$, and let T be the maximal initial subclass of U. We need to prove that $T = U$.

Claim 1. For every $\vec{a} \in S$, $b := f(\vec{a}) \in T$.

The proof is by induction on $\vec{a}$. An option of b is

$$b^{\mathbf{o}} = f^{\mathbf{o}}\big(\vec{a}, \vec{a}^{\mathbf{o}}, f(\vec{a}^{\mathbf{o}})\big).$$

Every $\vec{a}^{\mathbf{o}}$ is strictly simpler than $\vec{a}$, therefore $\vec{a}^{\mathbf{o}} \in S$. So, by inductive hypothesis, $f(\vec{a}^{\mathbf{o}}) \in T$. But $S \subseteq T^n$ and, by hypothesis on $\mathfrak{A}$, T is closed under $\mathfrak{A}$, therefore $b^{\mathbf{o}} \in T$. Thus, by Lemma 3.7, $b \in T$.

Claim 2. $T = U$.

It suffices to prove that T contains $S_1, \dots, S_n$ and $f(S)$, and that T is closed under $\mathfrak{A}$:

- $T = T^{\mathfrak{A}}$ by hypothesis on $\mathfrak{A}$.
- $S_i \subseteq T$, $i = 1, \dots, n$ because the S_i are initial.
- $f(S) \subseteq T$ by Claim 1.

To prove the second point, define $U := \big(f(S)\big)^{\mathfrak{A}}$, T its maximal initial subclass. As before, it is enough to prove the following claim:

Claim 3. For every $\vec{a} \in S$, $b := f(\vec{a}) \in T$.

The proof is by induction on $\vec{a}$. An option of b is

$$b^{\mathbf{o}} = f^{\mathbf{o}}\big(f(\vec{a}^{\mathbf{o}})\big).$$

$\vec{a}^{\mathbf{o}} \prec \vec{a}$, therefore $\vec{a}^{\mathbf{o}} \in S$. Thus, by inductive hypothesis, $f(\vec{a}^{\mathbf{o}}) \in T$. By hypothesis on $\mathfrak{A}$, T is closed under $\mathfrak{A}$, implying that $b^{\mathbf{o}} = f^{\mathbf{o}}\big(f(\vec{a}^{\mathbf{o}})\big) \in T$. Hence, by Lemma 3.7, $b \in T$. □

Theorem 5. *Suppose that for every $T \subseteq \mathbf{No}$ initial, $T^{\mathfrak{A}}$ is also initial. Let $f : \mathbf{No}^n \to \mathbf{No}$ be recursive over $\mathfrak{A}$. Let S be an initial subclass of* No. *Then, $S^{\mathfrak{A} \cup \{f\}}$ is also initial.*

Proof. Let

$$T_0 := S$$
$$T_{i+1} := \big(T_i \cup f({T_i}^n)\big)^{\mathfrak{A}}$$
$$T := \bigcup_{i \in \mathbb{N}} T_i = S^{\mathfrak{A} \cup \{f\}}.$$

Then, by Theorem 4 and induction on i, T_i is initial for every $i \in \mathbb{N}$, therefore T is also initial. □

A different way of presenting the same reasoning is the following.

Proof. Let T the maximal initial subclass of $S^{\mathfrak{A} \cup \{f\}}$. By hypothesis on $\mathfrak{A}$, T is closed under $\mathfrak{A}$. I have to prove that T is also closed under f.

Let $\vec{a} \in T^n$. I will prove that $f(\vec{a}) \in T$ by induction on $\vec{a}$. By Lemma 3.7, it is enough to find $L < R \subseteq T$ such that

$$w := f(\vec{a}) = \langle L \mid R \rangle.$$

An option of w is of the form

$$f^{\mathbf{o}}\big(\vec{a}, \vec{a}^{\mathbf{o}}, f(\vec{a}^{\mathbf{o}})\big),$$

where $\vec{a}^{\mathbf{o}} \in \mathfrak{A}$ and $\vec{a}^{\mathbf{o}} \prec \vec{a}$. Therefore, $\vec{a}^{\mathbf{o}} \in T^n$, and, by inductive hypothesis, $f(\vec{a}^{\mathbf{o}}) \in T^n$. Thus, $f^{\mathbf{o}}\left(\vec{a}, \vec{a}^{\mathbf{o}}, f(\vec{a}^{\mathbf{o}})\right) \in T$. □

3.10 Corollary. *Let S, U be initial subclasses of* No. *Then, the following subclasses of* No *are also initial:*

1. *The (additive) subgroup generated by S.*
2. *The subring generated by S .*
3. $-S := \{ -x : x \in S \}$.
4. $S + U := \{ x + y : x \in S, y \in U \}$.
5. *The subgroup generated by $SU := \{ xy : x \in S, y \in U \}$.*

Proof. For the first two points, apply Theorem 5. The third point is obvious. For the other two points, apply Theorem 4. □

3.11 Example. It is not true in general that if S, U are initial subgroups of No, then SU is also initial. For instance, take $S = U$ to be the subgroup generated by $\mathbb{Z}$ and ω. Then, $\omega^2 + \omega = \omega(\omega + 1) \in SU$, but $\omega^2 + 1 \notin SU$.

3.12 Corollary. *Let $\mathbb{K}$ be an initial subring of* No. *Let $L < R$ be subsets of $\mathbb{K}$, and let $c := \langle L \mid R \rangle$. Then, $\mathbb{K}[c]$ is also an initial subring of* No.

Proof. By Remma 3.7, $\mathbb{K} \cup \{ c \}$ is initial, therefore, by Corollary 3.10, the ring generated by it is also initial. □

3.13 Definition (Strongly tame). A function $f : \mathbf{No}^{n+1} \to \mathbf{No}$ is *strongly tame* iff for all $a < b \in \mathbf{No}$, $\vec{e} \in \mathbf{No}^n$, $d \in \mathbf{No}$ either $f(x, \vec{e})$ is constant, or there exist $\zeta_0, \dots, \zeta_m \in \mathbf{No}^{\mathcal{D}}$ such that $a = \zeta_0 < \cdots < \zeta_m = b$ and for $i = 0, \dots, m-1$

$$\forall x \in (\zeta_i, \zeta_{i+1}) \ f(x, \vec{e}) > d \quad \text{or}$$
$$\forall x \in (\zeta_i, \zeta_{i+1}) \ f(x, \vec{e}) < d.$$

3.14 Definition. Let $f : \mathbf{No}^{n+1} \to \mathbf{No}$ be strongly tame, S be a subclass of $\mathbf{No}$. S is *closed under solutions* of f iff for all $\vec{e} \in S^n$ either $f(x, \vec{e})$ is constant, or for every $d \in S$

$$\forall c \in \mathbf{No}\, f(c, \vec{e}) = d \ \Rightarrow \ c \in S.$$

The *closure* of S under solutions of f is the smallest class containing S and closed under f and under its solutions.

Theorem 6. *Suppose that $\mathfrak{A}$ is strongly tame and satisfies the I.V.P.. Let $f : \mathbf{No} \to \mathbf{No}$ uniformly recursive over $\mathfrak{A}$, strongly tame and satisfying the I.V.P..*

Suppose that for every S initial subclass of No, *the closure of S under solutions of $\mathfrak{A}$ and under f is also initial. Then, the closure of S under solutions of $\mathfrak{A} \cup \{f\}$ is initial.*

The proof is quite similar to the one of Theorem 3.

Proof. Let T be the maximal initial subclass of the closure of S under solutions of $\mathfrak{A} \cup \{f\}$. By hypothesis, T is closed under solutions of $\mathfrak{A}$ and under f. Therefore, it is enough to prove that T is closed under solutions of f. If f is constant, there is nothing to prove. Otherwise, let $d \in T$, $c \in \mathbf{No}$ such that $f(c) = d$. I will prove that $c \in T$ by induction on d.

I will give options of c in T.

Assume that $f = \langle f^L \mid f^R \rangle$. Then, $f(c) = d$ is equivalent to

$$d^L < f(c) < d^R \tag{3.2a}$$
$$f^L(c, c^L, c^R) < d < f^R(c, c^L, c^R) \tag{3.2b}$$

(3.2a) Let $d^{\mathbf{o}}$ be an option of d.

$$c^L := \sup \left\{ x \in \mathbf{No} : x \leq c \;\&\; f(x) = d^{\mathbf{o}} \right\}$$
$$c^R := \inf \left\{ x \in \mathbf{No} : x \geq c \;\&\; f(x) = d^{\mathbf{o}} \right\}.$$

The function f is strongly tame and satisfies the I.V.P., thus $c^L, c^R \in \overline{\mathbf{No}}$ and $c^L \leq c \leq c^R$. By the I.V.P., $c^L < c < c^R$. Moreover, $c^L, c^R \in T \cup \pm\infty$ by inductive hypothesis. By the I.V.P., $f(x) - d^{\mathbf{o}}$ does not change sign in (c^L, c^R); in particular, if $d^{\mathbf{o}} = d^L < d$, $f(x) > d^L$ in (c^L, c^R), and similarly for $d^{\mathbf{o}} = d^R$. Hence, I can take c^L and c^R as left and right options of c.

(3.2b) Suppose that I have already found c^L, c^R 'old' options of c. Let $f^{\mathbf{o}}$ be an option of f, say $f^{\mathbf{o}} = f^L < f$. I will construct $c^{L'}, c^{R'}$ 'new' options of c such that for $x \in (c^{L'}, c^{R'})$ $g(x) := f^L(x, c^L, c^R) < d$. Let

$$c^{L'} := \inf \left\{ x \in \mathbf{No} : x \leq c \;\&\; g(x) = d \right\}$$
$$c^{R'} := \sup \left\{ x \in \mathbf{No} : x \geq c \;\&\; g(x) = d \right\}.$$

g is strongly tame, therefore $c^{L'}, c^{R'} \in \overline{\mathbf{No}}$ and $c^{L'} < c < c^{R'}$. By the I.V.P., $g(x) - d$ does not change sign in $(c^{L'}, c^{R'})$; consequently, $g(x) < d$ in $(c^{L'}, c^{R'})$. So, I can take $c^{L'}$ and $c^{R'}$ as left and right options of c.

At the end of the process, I obtain $L < R \subseteq T$ such that $c = \langle L \mid R \rangle$, so, by Lemma 3.7, $c \in T$. □

3.15 Corollary. *The real closure of an initial subring of* $\mathbf{No}$ *is initial. More in general, if* $S \subseteq \mathbf{No}$ *is initial, then the smallest real closed subfield of* $\mathbf{No}$ *containing* S *is initial.*

Proof. $\mathbb{Q}$ is an initial subfield of $\mathbf{No}$, therefore $\mathbb{Q} \cup S$ is also initial. Therefore, by Corollary 3.10, $\mathbb{K}$, the subring generated by it, is also initial.

It remains to prove that the real closure of $\mathbb{K}$ is initial. Apply the Theorem 6 and induction on $\mathbf{No}[x]$. □

If instead of considering *all* polynomials in $\mathbf{No}[x]$, we consider only the polynomials of degree up to a fixed degree n, the previous corollary is still valid, with the same proof. In particular, taking $n = 1$, we can conclude the following:

3.16 Corollary. *Let* S *be an initial subclass of* $\mathbf{No}$*. Then, the subfield of* $\mathbf{No}$ *generated by* S *is initial.*

The following theorem was also proved in [5] with a different method.

Theorem 7. *Let* $\mathbb{K}$ *be a real closed field and a proper set. Then,* $\mathbb{K}$ *is isomorphic to an initial subfield of* $\mathbf{No}$*.*

Proof. If $\mathbb{K} \simeq \mathbb{Q}$, it is true.

Assume that $\mathbb{F}$ is a real closed initial subfield of $\mathbf{No}$, and $\mathbb{K}$ is (isomorphic to) the real closure of $\mathbb{F}(a)$ for some a transcendental over $\mathbb{F}$. Let $(\,L \mid R\,)$ be the cut determined by a over $\mathbb{F}$. For any $c \in (\,L \mid R\,)$, $\mathbb{F}(c)$ is isomorphic to $\mathbb{F}(a)$, and its real closure is isomorphic to $\mathbb{K}$. Moreover if $c = \langle L \mid R \rangle$ then, by Corollary 3.12, $\mathbb{K}[c]$ is initial, therefore, by Corollary 3.15, its real closure is also initial.

In general, let $(c_\beta)_{\beta<\alpha}$ be a transcendence basis of $\mathbb{K}$ over $\mathbb{Q}$, for some $\alpha \in \mathbf{No}$. Let $\mathbb{K}_0$ be the real closure of $\mathbb{Q}$, and define $\mathbb{K}_\beta$ to be the real closure of $\mathbb{Q}[c_i : i < \beta]$ for $\beta < \alpha$, i.e. $\mathbb{K}_0 := \mathbb{Q}$, and, for $0 < \beta \leq \alpha$

$$\mathbb{K}_\gamma := \begin{cases} \text{the real closure of } \mathbb{K}(c_\gamma) & \text{if } \beta = \gamma + 1 \\ \bigcup_{\gamma<\beta} \mathbb{K}_\gamma & \text{if } \beta \text{ is limit.} \end{cases}$$

In particular, $\mathbb{K}_\alpha := \mathbb{K}$. By the previous case and induction on β, each $\mathbb{K}_\beta$ is isomorphic to an initial subfield of $\mathbf{No}$, and the conclusion follows. □

It is not true that every ordered field (which is also a set) is isomorphic to an initial subfield of $\mathbf{No}$. For instance, take $\mathbb{K} := \mathbb{Q}(\sqrt{2} + {}^1\!/_\omega) \subset \mathbf{No}$. Suppose, for contradiction, that there exists an isomorphism of ordered fields ψ between $\mathbb{K}$ and an initial subfield of $\mathbf{No}$. Let $z = \psi(\sqrt{2} + {}^1\!/_\omega)$. Then, $\sqrt{2} \prec z$, but $\sqrt{2} \notin \psi(\mathbb{K})$.

For more on the subject of initial embeddings of fields, see [7].

4 Examples and the concatenation function

4.1 Example. Let $f(x) := \langle\, x - 1 \mid x + 1 \,\rangle$. The image of f is the class of omnific integers. f satisfies the $\sup$ property, but not the I.V.P.. $f(x) = 0$ for $x \in (-1, 1)$, $f(x) = \alpha$ for $x \in [\alpha, \alpha + 1)$, $\alpha \in \mathbf{On}$, etc.

4.2 Example. Let

$$f(x) := [x] - x = \langle\, -1, [x] - x^R \mid 1, [x] - x^L \,\rangle.$$

Then, f is not tame.

Proof. Consider the cut $\omega : -\infty$ between positive finite numbers and infinite numbers. $f(x)$ changes sign infinitely many times in every neighbourhood of this cut. □

4.3 Example. Let

$$f(x) := \left\langle -|x| \;\middle|\; f_1^R(x^L), f_2^R(x^R) \right\rangle \quad \text{with}$$

$$f_1^R(z) := \begin{cases} 0 & \text{iff } z \geq 0 \\ 2 & \text{iff } z < 0 \end{cases}$$

$$f_2^R(z) := \begin{cases} 0 & \text{iff } z \leq 0 \\ 2 & \text{iff } z > 0. \end{cases}$$

Therefore,

$$f(x) = \begin{cases} \langle\, -|x| \mid 0 \,\rangle < 0 & \text{iff } x \neq 0 \\ 1 & \text{iff } x = 0 \end{cases}$$

4.4 Definition. Let $x, y > 0 \in \mathbf{No}$.

- $x \simeq y$ iff $\frac{x-y}{x}$ is infinitesimal.
- $x \lnapprox y$ iff $x < y$ and $x \not\simeq y$.
- $x \perp y$ iff $x \not\lesssim y$ and $y \not\lesssim x$.

4.5 Example. Let $d := 2/3$. For $a \in \mathbb{R}$, let

$$f_a^L(z) := \begin{cases} z & \text{if } z \leq a \\ a & \text{if } z \geq a \end{cases}$$

$$f_a^R(z) := \begin{cases} z & \text{if } z \geq a \\ a & \text{if } z \leq a. \end{cases}$$

Let

$$f(x) := \langle\, f_r^L(x^L) : d > r \in \mathbb{R} \mid f_s^R(x^R) : d < s \in \mathbb{R} \,\rangle.$$

Then, f is uniformly recursive. Moreover,

$$\begin{aligned} f(x) &= x \quad \text{if } x \lnapprox d \\ f(x) &= d \quad \text{if } x \simeq d \\ f(x) &= x \quad \text{if } d \lnapprox x. \end{aligned}$$

In particular, f does not satisfy the $\sup$ property, because

$$\sup \left\{ x \in \mathbf{No} : 0 < x < 1 \;\&\; f(x) \leq d \right\} \notin \mathbf{No}$$

In the previous examples, the definition of f is uniform.

A surreal number z can be considered as a function from $\ell(z)$ into $\{+, -\}$, i.e. as a sequence of pluses and minuses, the *sign sequence* of z. Therefore, instead of 1 we can write the corresponding sequence $+$, instead of $1/2$, we can write $+-$, etc.

An element ζ of $\mathbf{No}^{\mathcal{D}}$ has also a unique sign sequence, of length $\mathbf{On}$ iff $\zeta \notin \mathbf{No}$. On the other hand, not every sign sequence corresponds to a element of $\mathbf{No}^{\mathcal{D}}$; for instance, $+\infty$, the sequence of pluses of length $\mathbf{On}$, is not in $\mathbf{No}^{\mathcal{D}}$.

4.1 The concatenation function

4.6 Definition (Concatenation). Let $x, y \in \mathbf{No}$. The *concatenation* of x, y, noted with $x : y$, is given by the sign sequence of x followed by the sign sequence of y.

The recursive definition of $x : y$ is

$$x : y = \langle x^L, x : y^L \mid x^R, x : y^R \rangle.$$

This definition is *not* uniform; while I can choose *any* representation of y, I must take the canonical representation of x (but see also [9]). In the following, I will prove that, given some hypothesis on $\mathfrak{A}$, it is never possible to find a uniform recursive definition of $x : y$ over $\mathfrak{A}$.

Let

$$f(x) := x : 1 = \langle x \mid x^R \rangle.$$

By definition, $f(x) < 0$ for $x < 0$, while $f(x) > 0$ for $x \geq 0$, therefore f does not satisfy the I.V.P.. Moreover, f is injective.

x	0	α	$-\alpha - 1$	$-\omega$	$1/2$	$1/\omega$	$\alpha \in \mathbf{On}$.
$x : 1$	1	$\alpha + 1$	$-\alpha - 1/2$	$-\omega + 1$	$3/4$	$2/\omega$	

Table 1: Some values of $x : 1$

4.7 Lemma. *Let* $x \in \mathbf{No}$. *Then:*

- $f(x) > x$.
- $x \prec f(x)$
- *If* $x < y$ *and* $y \prec x$, *then* $f(x) < y$.

Proof. Obvious. □

4.8 Lemma. *Let* $c, d \in \mathbf{No}$. *Then, there exists* $a, b \in \overline{\mathbf{No}}$ *such that* $a < c < b$ *and*

$$\forall x \in (a, c)\, f(x) < d \quad \text{or}$$
$$\forall x \in (a, c)\, f(x) > d,$$

and similarly for (c, b).

Proof. There are three cases, according to $d \lesseqgtr c$.

$d < c$. Let $a := \langle\, d \mid c \,\rangle$, $b := +\infty$. Then, for every $x > a$ $f(x) > x > a > d$.

$d = c$. Let $a := c : -$, $b := +\infty$. For every $x \in (a, c)$ $c \prec x$, therefore $f(x) < c$. For every $x > c$ $f(x) > x > c$.

$d > c$. Let $a := c : -$. If $a < x < c$, then $c \prec x$, therefore $f(x) < c < d$.

Assume that $c \perp d$. Let $b := \langle\, c \mid d \,\rangle$. If $c < x < b$, then $b \prec x$, therefore $f(x) < b < d$.

Assume that $c \prec d$. Let $b := \langle\, c \mid d \,\rangle$. If $c < x < b$, then $b \prec x$, therefore $f(x) < b < d$.

Assume that $d \prec c$. Let $b := d$. If $c < x < b$, then $d \prec x$, therefore $f(x) < d$. □

4.9 Remark. Let $d := {}^2\!/_3 = + - + - + - \dots$,

$$\begin{aligned} a_0 &:= 0, \\ a_1 &:= +- = {}^1\!/_2, \\ a_2 &:= + - +- = {}^5\!/_8, \\ a_3 &:= + - + - +-, \\ &\dots, \\ a_{n+1} &:= a_n : +-, \end{aligned}$$

and

$$\begin{aligned} b_n &:= a_n : + \;=\; + - \dots + -+ \\ c_n &:= a_n : - \;=\; + - \dots + - - \,. \end{aligned}$$

Then, for every $n \in \mathbb{N}$,

$$c_n < a_n < c_{n+1} < d < b_n.$$

Moreover, $f(a_n) = b_n > f(d) > d$, while $f(c_n) < a_n < d$. Besides,

$$d = \langle\, a_n \mid b_n \,\rangle_{n \in \mathbb{N}}$$

is the canonical representation of d. Moreover for every $x \in \mathbf{No}$

$$d \preceq f(x) \Leftrightarrow d \preceq x.$$

Finally, for every $x \in \mathbf{No}$ $d \preceq x$ iff $d - x$ is infinitesimal.

4.10 Remark. f is not continuous. In fact,

$$\lim_{\substack{x \to 0 \\ x \neq 0}} f(x) = 0,$$

while $f(0) = 1$.

4.11 Lemma. *f is not tame.*

Proof. Let d, a_n, b_n, c_n be as in Remark 4.9.

If $a < \zeta$, there exists $n \in \mathbb{N}$ such that $a < a_n < c_{n+1} < \zeta$. Therefore, $f(x) - d$ changes sign infinitely many times in every left neighbourhood of ζ.

However, note that in every infinitesimal neighbourhood of d, $f(x) \geq d$ iff $x \geq d$, and $f(x) \geq f(d)$ iff $x \geq f(d)$ or $x = d$. □

4.12 Lemma. *f does not satisfy the* sup *property.*

Proof. Let d, a_n, b_n, c_n be as in Remark 4.9, and let $a := 0$.

Then, $f(a_i) > d$, while $f(c_i) < d$. Let

$$\zeta := \sup \left\{ x \in \mathbf{No} : a < x < d \;\&\; f(x) \geq d \right\}.$$

Therefore,

$$\zeta \geq \sup \{ a_i : i \in \mathbb{N} \}.$$

On the other hand, if $d \preceq x$ and $x < d$, then $f(x) < d$, therefore

$$\zeta \leq \inf \{ x < d : d \preceq x \},$$

so $\zeta = \inf \{ x < d : d \preceq x \} \in \mathbf{No}^{\mathcal{D}} \setminus \mathbf{No}$. □

4.13 Lemma. *If $\mathfrak{A}$ is a family of functions strongly tame and satisfying the I.V.P., then f cannot be uniformly recursive over $\mathfrak{A}$.*

Proof. Suppose not, i.e. that $f = \langle f^L \mid f^R \rangle$, with $f^{\circ} \in \mathfrak{A}$.

Let d, a_n, b_n, c_n be as in Remark 4.9, and let ζ as in the previous proof.

I will show that there exists $c \in \mathbf{No}$ such that $f(c) = d$, which is clearly impossible. I will give the options of c.

$f(c) = d$ is equivalent to:

$$d \preceq f(c)$$
$$f^L(c, c^L, c^R) < d < f^R(c, c^L, c^R).$$

By Remark 4.9, $d \preceq f(c)$ is equivalent to $d \preceq c$, and $d = \langle a_n \mid b_n \rangle_{n \in \mathbb{N}}$. Therefore, it is necessary and sufficient to add a_n to the left options of c and b_n to its right ones for every $n \in \mathbb{N}$ to ensure that $d \preceq f(c)$.

Let c^L, c^R be 'old' options of c such that $c^L < \zeta < c^R$, f^L be a left options of f. I will find $c^{L'}, c^{R'}$ 'new' options of c such that $c^{L'} < \zeta < c^{R'}$ and

$$\forall x \in (c^{L'}, c^{R'})\; f^L(x, c^L, c^R) < d. \tag{4.1}$$

Let $g(x) := f^L(x, c^L, c^R)$,

$$c^{L'} := \sup \left\{ x \in \mathbf{No} : x < \zeta \;\&\; g(x) = d \right\}$$
$$c^{R'} := \inf \left\{ x \in \mathbf{No} : x > \zeta \;\&\; g(x) = d \right\}.$$

$\mathfrak{A}$ is strongly tame, therefore the previous sup and inf are actually a min and a max, unless $g(x)$ is constant.

For every left neighbourhood J of ζ there is $x \in J$ such that $f(x) < d$. Moreover, $g(x) < f(x)$ in (c^L, c^R). Therefore, if $g(x)$ is constant, then $g(x) < d$, so $c^{L'} = -\infty$ and $c^{R'} = +\infty$.

Otherwise, $c^{L'}, c^{R'} \in \overline{\mathbf{No}}$, while $\zeta \notin \overline{\mathbf{No}}$, so $c^{L'} < \zeta < c^{R'}$. By the I.V.P., the sign of $g(x) - d$ is constant in $U := (c^{L'}, c^{R'})$. Again, in every left neighbourhood of ζ there is a x such that $f(x) < d$, therefore $g(x) < d$ in U.

Proceed similarly for f^R.

At the end of the process we have constructed a $c \in \mathbf{No}$ such that $f(c) = d$, a contradiction. □

References

[1] N. L. Alling. *Foundations of Analysis over Surreal Number Fields*, volume 141 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1987.

[2] N. L. Alling and P. Ehrlich. An abstract characterization of a full class of surreal numbers. *C. R. Math. Rep. Acad. Sci. Canada*, 8(5):303–308, 1986.

[3] N. L. Alling and P. Ehrlich. An alternative construction of Conway's surreal numbers. *C. R. Math. Rep. Acad. Sci. Canada*, 8(4):241–246, 1986.

[4] J. H. Conway. *On Numbers and Games*. Number 6 in L.M.S. monographs. Academic Press, London & New York, 1976.

[5] P. Ehrlich. Number systems with simplicity hierarchies: a generalization of Conway's theory of surreal numbers. *J. Symbolic Logic*, 66(3):1231–1258, 2001.

[6] A. Fornasiero. *Integration on Surreal Numbers*. PhD thesis, University of Edinburgh, 2003. `http://www.dm.unipi.it/~fornasiero/phd_thesis/thesis_fornasiero_linearized.pdf`.

[7] A. Fornasiero. Embedding henselian fields into power series. Submitted. Preliminary version avaiable at: `http://www.dm.unipi.it/~fornasiero/ressayre.pdf`, 2004.

[8] H. Gonshor. *An Introduction to the Theory of Surreal Numbers*, volume 110 of *L.M.S. Lecture Note Series*. Cambridge University Press, Cambridge, 1986.

[9] P. Keddie. Ordinal operations on surreal numbers. *Bull. London Math. Soc.*, 26(6):531–53, 1994.

[10] D. E. Knuth. *Surreal Numbers*. Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1974.

[11] S. Kuhlmann. *Ordered Exponential Fields*, volume 12 of *Fields Institute Monographs*. The Fields Institute for Research in Mathematical Sciences, 2000.

[12] L. van den Dries. *Tame Topology and O-minimal Structures*, volume 248 of *L.M.S. Lecture Note Series*. Cambridge University Press, Cambridge, 1998.

[13] L. van den Dries and P. Ehrlich. Fields of surreal numbers and exponentiation. *Fund. Math.*, 2(167):173–188, 2001.