Tame ordered structures

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Introduction

- A definably complete (DC) structure is a linearly ordered structure $\mathbb{K}$, such that every definable subset of $\mathbb{K}$ has a supremum in $\mathbb{K} \cup \left\{ \pm\infty \right\}$.
- For example, all expansions of $\mathbb{R}$ and all o-minimal structure are DC.
- All structures in this talk will be assumed to be DC expansions of ordered fields.

Introduction

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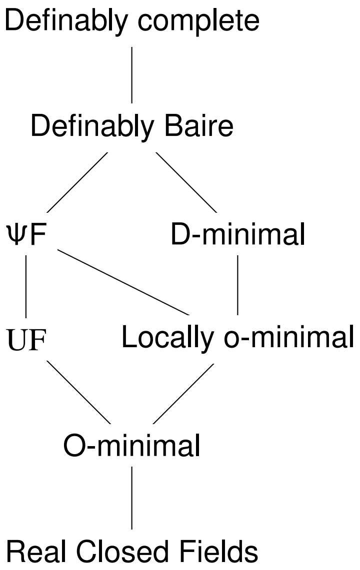
- For example, all expansions of $\mathbb{R}$ and all o-minimal structure are DC.
- All structures in this talk will be assumed to be DC expansions of ordered fields.
- We do not have yet a precise definition of "tame structures". We will describe some classes of tame structures (which are DC expansion of fields!) containing the class of o-minimal structures.
- Tame non-DC ordered structure (e.g., weakly o-minimal structures) are outside the scope of this talk.

NOTES

1. “Definable” means “definable with parameters”.
2. We will deal with linearly ordered structures only.
3. Being DC is a first-order property, and hence preserved under elementary equivalence and ultraproducts.
4. Various facts from elementary analysis can be proved in DC structures. For instance, a first order differential equation with definable coefficients and initial data has at most one definable solution.

5. We do not have a precise definition of “tame structure”. We will describe some classes of structures that are tame.
6. For example, if $\mathbb{K}$ is an expansion of $\mathbb{R}$ that defines $\mathbb{N}$, then $\mathbb{K}$ is **not** tame.
7. “Tame” means that definable sets are well-behaved from the geometric point of view.
8. $\mathbb{K}$ is **o-minimal** if every definable unary set is a finite union of points and intervals with endpoints in $\mathbb{K}$.

The overall picture



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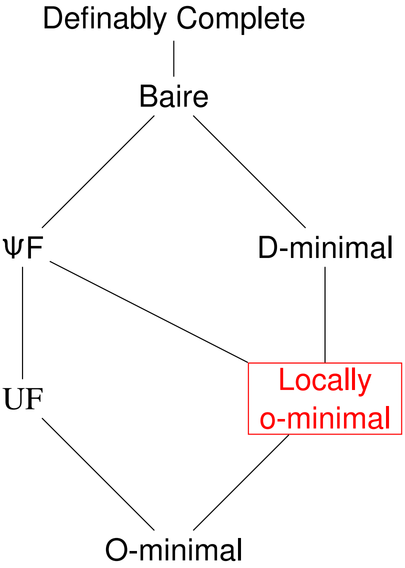


Locally o-minimal structures

Definition

$\mathbb{K}$ is locally o-minimal if it is o-minimal "around every point".
That is, for every X definable subset of $\mathbb{K}$ and every $a \in \mathbb{K}$, there exists $0 < \varepsilon \in \mathbb{K}$, such that $(a, a + \varepsilon)$ is either disjoint form X, or contained in X.

Example

Ultraproducts of o-minimal structures.

Remark

If $\mathbb{K}$ expands $\mathbb{R}$, then $\mathbb{K}$ is locally o-minimal iff it is o-minimal.
The same remains true if we only ask that $\mathbb{K}$ is Archimedean.

Notes

1. Being locally o-minimal is a first-order property, and hence preserve under elementary equivalence and ultraproducts.
2. Notice that we speak about **ultraproducts**, not ultrapowers! Ultrapowers of o-minimal structures are o-minimal, but it is easy to build ultraproducts of o-minimal structures which are not o-minimal.
3. Local o-minimality is not the same concept suggested by A. Onshuus.

Properties of locally o-minimal structures

Many facts from the theory of o-minimal structures remain true for locally o-minimal ones.

Monotonicity theorem Given a definable function $f : \mathbb{K} \to \mathbb{K}$, there exists a pseudo-finite set I such that, on each intervals of $K \setminus I$, f is continuous and either constant or strictly monotone. Same with $\mathcal{C}^p$ instead of continuous.

Constructibility Every definable subset of $\mathbb{K}^n$ is constructible (finite Boolean combination of definable open sets).

Definable Choice $\mathbb{K}$ has definable Skolem functions and elimination of imaginaries

Growth dichotomy $\mathbb{K}$ is either power-bounded, or defines an exponential function.

NOTES

1. Remember that we are assuming that $\mathbb{K}$ is a DC expansion of field.
2. We will define “pseudo-finite sets” later.
3. The monotonicity theorem can be generalized to higher dimension: given a definable function $f : \mathbb{K}^n \to \mathbb{K}$ there exists a definable set $U \subset \mathbb{K}^n$, such that U is nowhere dense, and f is $\mathcal{C}^p$ outside U .

Properties no longer true for locally o-minimal structures

Exchange principle Locally o-minimal structures do not satisfy the exchange principle for the algebraic closure.

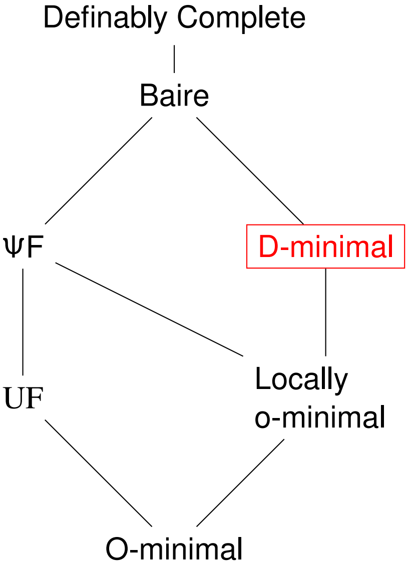
Theorem (A. Dolich)

If a locally o-minimal theory T satisfies the exchange principle, then T is o-minimal.

Rosiness A locally o-minimal non o-minimal theory is not rosy.

NIP There exists an ultraproduct of o-minimal structures (and hence locally o-minimal) which satisfies the Independence Property.

The overall picture



D-minimal structures

Definition (C. Miller)

$\mathbb{K}$ is d-minimal if, for every $\mathbb{K}'$ elementary extension of $\mathbb{K}$, and every X definable subset of $\mathbb{K}'$, X is the union of an open set and finitely many discrete sets.

Example (van den Dries)

$(\mathbb{R}, +, \cdot, <, 2^{\mathbb{Z}})$ is d-minimal, where $2^{\mathbb{Z}}$ is a predicate denoting the set of real integer powers of 2.

Properties of d-minimal structures

Many of the properties of locally minimal structures remain true for d-minimal ones. Let $\mathbb{K}$ be a d-minimal structure.

Monotonicity theorem Given a definable function $f : \mathbb{K} \to \mathbb{K}$, there exists a definable set I which is closed and with empty interior, and such that, on each intervals of $K \setminus I$, f is continuous and either constant or strictly monotone. Same with C^p instead of continuous.

Constructibility Every definable subset of $\mathbb{K}^n$ is constructible.

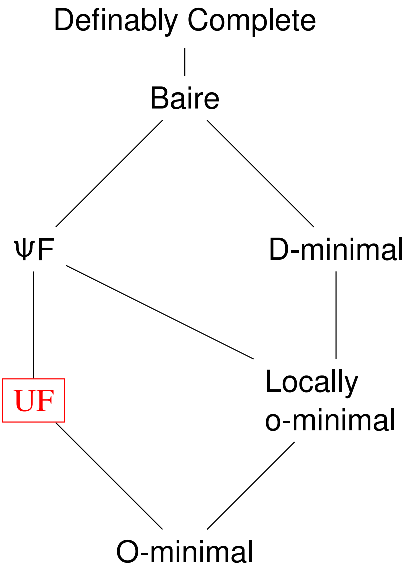
Definable Choice $\mathbb{K}$ has definable Skolem functions and elimination of imaginaries

(Un)rosiness If $\mathbb{K}$ is not o-minimal, then it is not rosy.

Notes

Being d-minimal is **not** a first-order property: it is not preserved under ultraproducts.

The overall picture



Open core and dense pairs

Definition

The open core of $\mathbb{K}$ is the reduct of $\mathbb{K}$ generated by all open sets definable in $\mathbb{K}$.

Definition

Given a theory T (expanding RCF), let T^d is the theory dense pairs of models of T: that is, the theory of pairs (A, B), such that

1. $A \prec B \models T$;
2. $A \neq B$;
3. A is dense in B:
   $\forall b < b' \in B \; \exists a \in A \; (b < a < b')$.

Structure satisfying Uniform Finiteness

Definition

$\mathbb{K}$ satisfies Uniform Finiteness (UF) if it eliminates the quantifier "there exist infinitely many".

Theorem (Dolich, Miller, Steinhorn)

$\mathbb{K}$ satisfies UF iff its open core is o-minimal.

Theorem (Dries, DMS)

If T is an o-minimal theory, then T^d is complete, it satisfies DC and UF, and T is an open core of T^d.

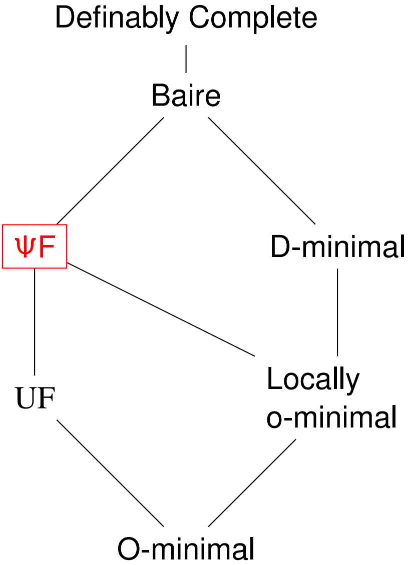
Theorem (F.)

If T is a d-minimal theory, then T^d is complete.
The Cauchy completion $\tilde{\mathbb{K}}$ of a d-minimal structure $\mathbb{K}$ has a unique structure such that $\mathbb{K} \preceq \tilde{\mathbb{K}}$; therefore, T^d is also consistent.

NOTES

1. $\mathbb{K}$ satisfies UF if, for every definable family $(A_t)_{t \in K^m}$ of subsets of $\mathbb{K}^n$ there exists $N \in \mathbb{N}$, such that, for every t, either A_t is infinite, or $|A_t| < N$.
2. Equivalently, $\mathbb{K}$ satisfies UF if, in every elementary extension of $\mathbb{K}$, the property "being finite" is definable.
3. UF is **not** preserved under ultraproducts.

The overall picture



Pseudo-finite sets

Definition

$X \subseteq \mathbb{K}$ is pseudo-finite if it is definable, discrete, closed, and bounded.

Remark

If $\mathbb{K}$ is an expansion of the real line, then any pseudo-finite subset of $\mathbb{K}^n$ is finite.

Lemma

If $X \subset \mathbb{K}^n$ is pseudo-finite and $f : \mathbb{K}^n \to \mathbb{K}^m$ is definable, then $f(X)$ is also pseudo-finite.

Pseudo-finite sets, continued

Example

If ${}^*\mathbb{R}$ is a non-standard model of analysis, and $X \subset {}^*\mathbb{R}$ is definable, then X is pseudo-finite iff its cardinality is a (standard or not) natural number.

Remark

$\mathbb{K}$ satisfies UF iff every pseudo-finite subset is finite.

Structures satisfying Pseudo-finiteness

Lemma

$\mathbb{K}$ satisfies Uniform Finiteness (UF) iff every definable discrete subset of $\mathbb{K}$ is finite.

Structures satisfying Pseudo-finiteness

Definition

$\mathbb{K}$ satisfies Pseudo-Finiteness (ΨF) iff every definable discrete subset of $\mathbb{K}$ is pseudo-finite (that is, closed and bounded).

Remark

1. If $\mathbb{K}$ satisfies UF, then it satisfies $\Psi\mathrm{F}$.
2. $\Psi\mathrm{F}$ is an elementary property: in particular, ultraproducts of structures with $\Psi\mathrm{F}$ do satisfy $\Psi\mathrm{F}$.

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Theorem (DMS)

$\mathbb{K}$ *satisfies* UF *iff its open core is o-minimal.*

Structures satisfying Pseudo-finiteness

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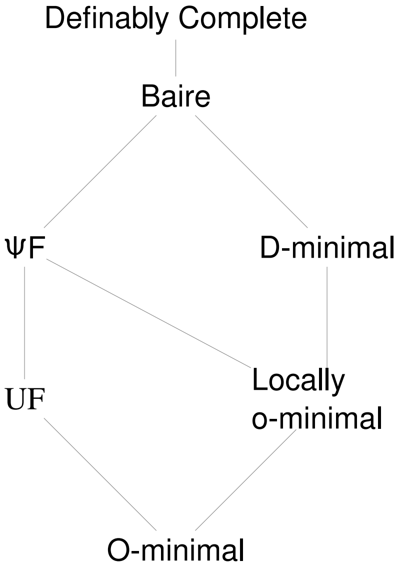
Theorem (F.)

$\mathbb{K}$ *satisfies* $\Psi\mathrm{F}$ *iff its open core is locally o-minimal.*

The overall picture



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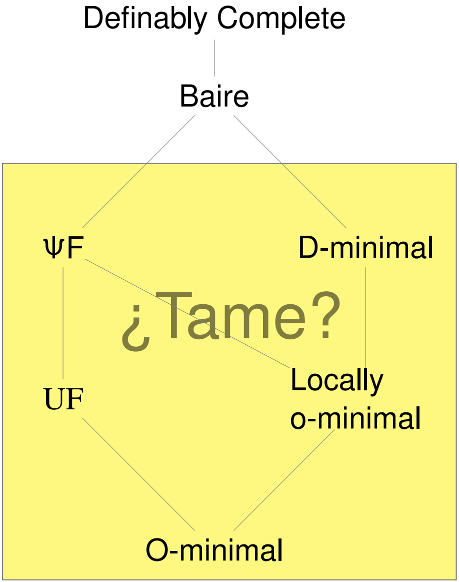

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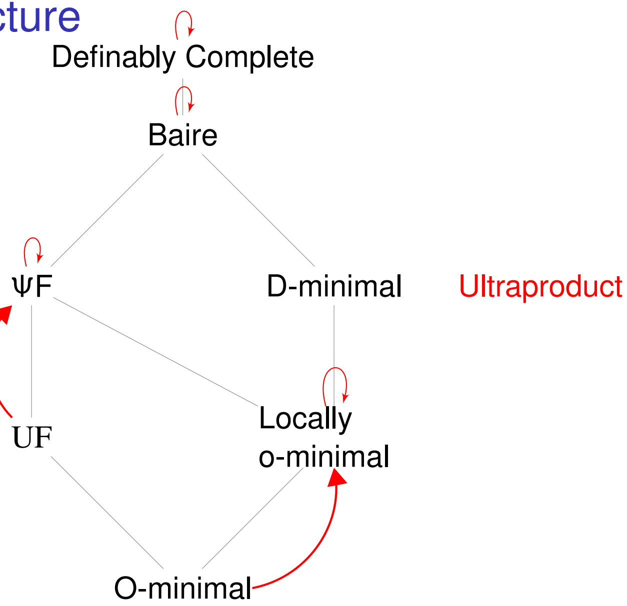

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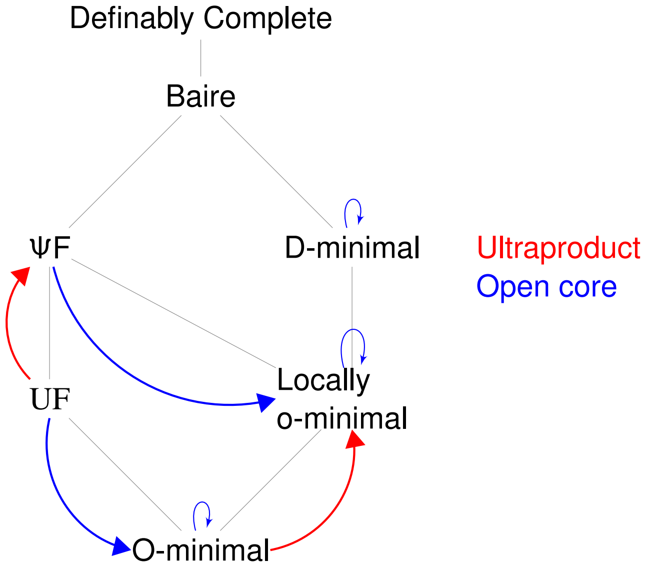

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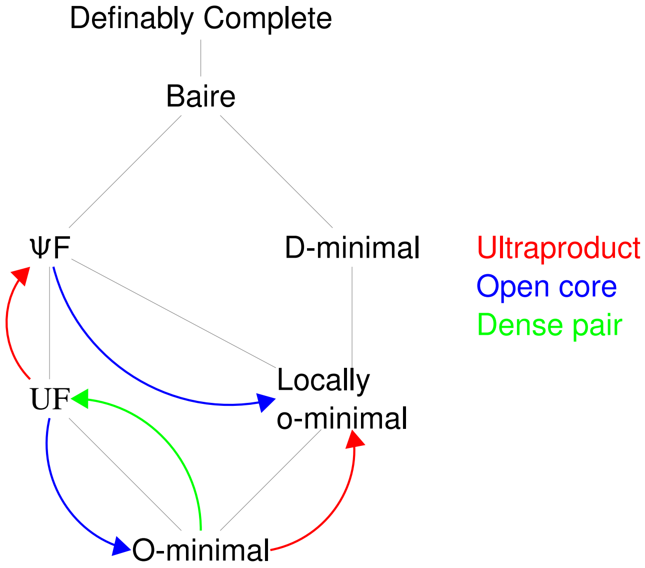

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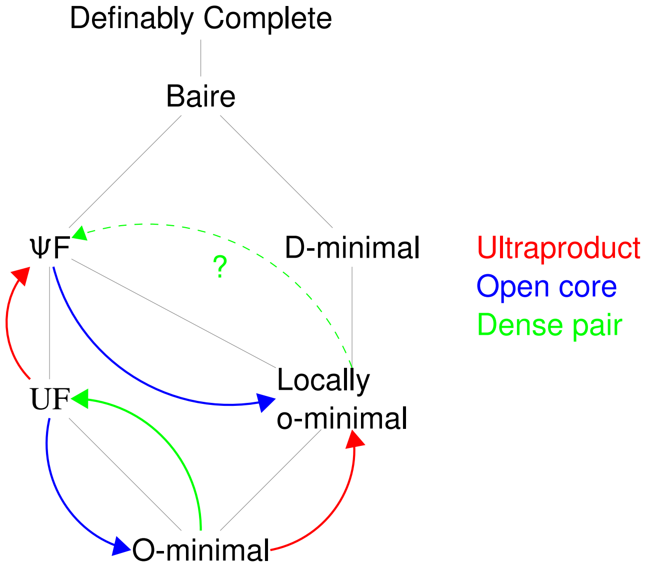
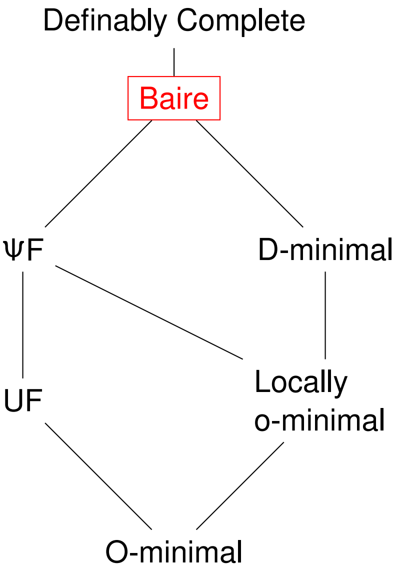


Notes

1. $\mathrm{UF} \cap$ d-minimal = o-minimal.
2. $\Psi\mathrm{F} \cap$ d-minimal = locally o-minimal.
3. In a d-minimal structure there are no definable subsets of $\mathbb{K}$ which are both dense and co-dense.
   In a structure satisfying $\Psi\mathrm{F}$ there are no definable unbounded discrete sets.
4. We have seen that if T is a complete d-minimal theory, then T^d is consistent and complete. However, we do not know if the open core of T^d is T itself, even in the case when T is locally o-minimal.
5. We conjecture that if T is locally o-minimal, then T^d satisfies Pseudo-Finiteness.

The overall picture



Baire structures

Definition

A subset X of $\mathbb{K}^n$ is nowhere dense if the closure of X has empty interior: $\mathring{\overline{X}} = \emptyset$.
$\mathbb{K}$ is definably Baire if (it is DC and) it is not the union of an increasing definable family of nowhere dense subsets.

Definition

$X \subseteq \mathbb{K}^n$ is definably meager if X is the union of an increasing definable family of nowhere dense subsets.

Remark

$\mathbb{K}$ is (definably) Baire iff it is not definably meager.
Being Baire is a a first-order property, and hence preserved under elementary equivalence and ultraproducts.

Notes

In Baire structures we have an analogue of Kuratowski-Ulam's Theorem: if $A \subseteq \mathbb{K}^{n+m}$ is meager, then A_x is meager for all x outside a meager subset of $\mathbb{K}^m$. Conversely, if $A \subset \mathbb{K}^{n+m}$ is **almost open** (that is, $A \Delta U$ is meager for some open definable set U), and A_x is meager for all x outside a meager subset of $\mathbb{K}^m$, then A itself is meager.

Baire structures, continued

Examples

- Every expansion of $\mathbb{R}$ is Baire.
- Every d-minimal structure is Baire.
- Every structure satisfying $\Psi\mathrm{F}$ is Baire.

Lemma

If $\mathbb{K}$ defines a discrete subring N, then $\mathbb{K}$ is Baire.

Proof.

N is a model of first-order Peano Axioms; hence, one can simulate in N a proof of the “classical” Baire Category Theorem. □

Conjecture

Every DC expansion of an ordered field is Baire.

Dimension

Definition

Let $X \subseteq \mathbb{K}^n$. The dimension of X is $\dim(X)$, the largest integer d such that, for some $\mathbb{K}$-linear subspace L of $\mathbb{K}^n$ of dimension d, $\Pi_L(X)$ has non-empty interior, where Π_L is the projection onto L.

Remark

- dim is monotone;
- if $X \subseteq \mathbb{K}^n$, then $\dim(X) \leq n$.

It is not true in general that $\dim(X \cup Y) = \max\big(\dim(X), \dim(Y)\big)$.

Dimension in d-minimal structures

Let $\mathbb{K}$ be d-minimal, and X and Y be definable sets.

Lemma (1)

- $\dim(X \cup Y) = \max\big(\dim(X), \dim(Y)\big)$;
- *if* $X \subseteq \mathbb{K}^{n+m}$, $Y := \Pi^{n+m}_{n}(X)$, $\dim(Y) = d$, *and* $\dim(X_a) = k$ *for every* $a \in Y$, *then* $\dim(X) = d + k$.

Lemma

- $\dim(\overline{X}) = \dim X$;
- *Sard's Lemma: if* $f : \mathbb{K}^n \to \mathbb{K}^m$ *is* C^1 *function definable in* $\mathbb{K}$, *then the set of critical points of* f *has empty interior.*
- *If moreover* $\mathbb{K}$ *is locally o-minimal, then* $\dim(\partial X) < \dim(X)$.

Dimension functions

Other kinds of structures have well-behaved notions of dimension too.

Definition

A dimension function (on some structure $\mathbb{K}$) is a function dim from definable sets into natural numbers, such that

1. dim is monotone;
2. the dimension of any singleton is 0;
3. $\dim(\mathbb{K}) = 1$;
4. dim is additive;
5. for every integer d and definable set $X \subseteq \mathbb{K}^{n+m}$,
   a. the set $\left\{ a \in \mathbb{K}^n : \dim(X_a) = d \right\}$ is definable;
   b. let $Y := \Pi_n^{n+m}(X)$; if $\dim(Y) = d$ and $\dim(X_a) = k$ for every $a \in Y$, then $\dim(X) = d + k$.

NOTES

1. Our definition of dimension functions is a minor modification of a definition by van den Dries. A dimension function is, essentially, a function on definable sets satisfying the Remark and Lemma 1.
2. Other kind of structures have well-behaved notions of dimension too For instance, an ultraproduct of d-minimal structures is not d-minima (in general), but has a dimension function satisfying also the additional lemma.
3. Every geometric theory has a dimension function.
4. Dense pairs of d-minimal structures also have a dimension function (but they do not satisfy the additional lemma!). We do not know in general if structures satisfying UF have a dimension function.
5. A dimension function is a very useful tool: many statements are proved by induction on the dimension of the sets.