Contents

Introduction 5

Chapter 1. (Bounded) cohomology of groups 11
  1. Cohomology of groups 11
  2. Functoriality 12
  3. The topological interpretation of group cohomology 13
  4. Bounded cohomology of groups 14
  5. Functoriality and the comparison map 15
  6. Looking for a topological interpretation of bounded cohomology of groups 15
  7. The bar resolution 16

Chapter 2. (Bounded) cohomology of groups in low degree 19
  1. (Bounded) group cohomology in degree one 19
  2. Group cohomology in degree two 20
  3. Bounded group cohomology in degree two: quasimorphisms 21
  4. Homogeneous quasimorphisms 21
  5. Quasimorphisms on abelian groups 23
  6. The image of the comparison map 24
  7. The second bounded cohomology group of free groups 26

Chapter 3. Amenability 29
  1. Abelian groups are amenable 30
  2. Other amenable groups 31
  3. Amenability and bounded cohomology 33
  4. Johnson’s characterization of amenability 34
  5. A characterization of finite groups via bounded cohomology 36

Chapter 4. (Bounded) group cohomology via resolutions 39
  1. Relatively injectivity 39
  2. Resolutions of Γ-modules 40
  3. The classical approach to group cohomology via resolutions 42
  4. The topological interpretation of group cohomology revisited 43
  5. Bounded cohomology via resolutions 44
  6. Relatively injective normed Γ-modules 45
  7. Resolutions of normed Γ-modules 46
  8. More on amenability 49
9. Amenable spaces 51
10. Alternating cochains 54

Chapter 5. Bounded cohomology of topological spaces 57
1. Basic properties of bounded cohomology of spaces 57
2. Bounded singular cochains as relatively injective modules 58
3. The aspherical case 60
4. Ivanov’s contracting homotopy 61
5. Gromov’s Theorem 63
6. Alternating cochains 64
7. Relative bounded cohomology 65
8. Special cochains 65

Chapter 6. $\ell^1$-homology and duality 69
1. Normed chain complexes and their topological duals 69
2. $\ell^1$-homology of groups and spaces 70
3. Duality: first results 71
4. Some results by Matsumoto and Morita 72
5. Injectivity of the comparison map 74
6. The translation principle 76

Chapter 7. Simplicial volume 79
1. The case with non-empty boundary 79
2. Elementary properties of the simplicial volume 80
3. The simplicial volume of Riemannian manifolds 81
4. Simplicial volume and topological constructions 82
5. Simplicial volume and duality 84
6. The simplicial volume of products 85

Chapter 8. The proportionality principle 87
1. Continuous cohomology of topological spaces 87
2. Continuous cochains as relatively injective modules 88
3. Continuous cochains as strong resolutions of $\mathbb{R}$ 91
4. Straightening in non-positive curvature 92
5. Continuous cohomology versus singular cohomology 93
6. The transfer map 93
7. Straightening and the volume form 96
8. Further readings 98

Chapter 9. The simplicial volume of hyperbolic manifolds 99
1. Hyperbolic straight simplices 99
2. The seminorm of the volume form 100
3. The case of surfaces 101
4. Stable complexity of manifolds 102
5. The simplicial volume of negatively curved manifolds 104
6. The simplicial volume of flat manifolds 104
<table>
<thead>
<tr>
<th>CONTENTS</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chapter 10. Additivity of the simplicial volume</td>
<td>107</td>
</tr>
<tr>
<td>1. A cohomological proof of subadditivity</td>
<td>107</td>
</tr>
<tr>
<td>2. A cohomological proof of Gromov Additivity Theorem</td>
<td>109</td>
</tr>
<tr>
<td>Further readings</td>
<td>112</td>
</tr>
<tr>
<td>Chapter 11. Group actions on the circle</td>
<td>113</td>
</tr>
<tr>
<td>1. Homeomorphisms of the circle and the Euler class</td>
<td>113</td>
</tr>
<tr>
<td>2. The bounded Euler class</td>
<td>114</td>
</tr>
<tr>
<td>3. The (bounded) Euler class of a representation</td>
<td>115</td>
</tr>
<tr>
<td>4. The rotation number of a homeomorphism</td>
<td>116</td>
</tr>
<tr>
<td>5. Increasing degree one map of the circle</td>
<td>117</td>
</tr>
<tr>
<td>6. Semi-conjugation</td>
<td>120</td>
</tr>
<tr>
<td>Chapter 12. Milnor-Wood inequalities and the Chern conjecture</td>
<td>131</td>
</tr>
<tr>
<td>1. Topological, smooth and linear sphere bundles</td>
<td>131</td>
</tr>
<tr>
<td>2. The Euler class of a sphere bundle</td>
<td>134</td>
</tr>
<tr>
<td>3. Flat sphere bundles</td>
<td>143</td>
</tr>
<tr>
<td>4. The bounded Euler class of a flat circle bundle</td>
<td>148</td>
</tr>
<tr>
<td>5. Milnor-Wood inequalities</td>
<td>150</td>
</tr>
<tr>
<td>Chapter 13. The Euler class in higher dimension and the Chern Conjecture</td>
<td>155</td>
</tr>
<tr>
<td>1. Ivanov-Turaev cocycle</td>
<td>156</td>
</tr>
<tr>
<td>2. Representing cycles via simplicial cycles</td>
<td>159</td>
</tr>
<tr>
<td>3. The bounded Euler class of a linear sphere bundle</td>
<td>160</td>
</tr>
<tr>
<td>Bibliography</td>
<td>165</td>
</tr>
</tbody>
</table>
Introduction

Bounded cohomology of groups was first defined by Johnson [Joh72] and Trauber during the seventies. As an independent and very active research field, however, bounded cohomology was born in 1982, thanks to the pioneering paper “Volume and Bounded Cohomology” by M. Gromov [Gro82], where the definition of bounded cohomology was extended to deal also with topological spaces.

Let $C^n(M, \mathbb{R})$ denote the complex of real singular cochains with values in the topological space $M$. A cochain $\varphi \in C^n(M, \mathbb{R})$ is bounded if it takes uniformly bounded values on the set of singular $n$-simplices. Bounded cochains provide a subcomplex $C^\bullet_b(M, \mathbb{R})$ of singular cochains, and the bounded cohomology $H^\bullet_b(M, \mathbb{R})$ of $M$ (with trivial real coefficients) is just the (co)homology of the complex $C^\bullet_b(M, \mathbb{R})$. An analogous definition of boundedness applies to group cochains with (trivial) real coefficients, and the bounded cohomology $H^\bullet_b(\Gamma, \mathbb{R})$ of a group $\Gamma$ (with trivial real coefficients) is the (co)homology of the complex $C^\bullet_b(\Gamma, \mathbb{R})$ of the bounded cochains on $\Gamma$. A classical result which dates back to the forties ensures that the singular cohomology of an aspherical CW-complex is canonically isomorphic to the cohomology of its fundamental group. In the context of bounded cohomology a stronger result holds: the bounded cohomology of a countable CW-complex is canonically isomorphic to the bounded cohomology of its fundamental group, even without any assumption on the asphericity of the space [Gro82, Bro81, Iva87]. For example, the bounded cohomology of spheres is trivial in positive degree. On the other hand, the bounded cohomology of the wedge of two circles is infinite-dimensional in degrees 2 and 3, and still unknown in any degree bigger than 3. As we will see in this monograph, this phenomenon eventually depends on the fact that higher homotopy groups are abelian, and abelian groups are invisible to bounded cohomology, while “negatively curved” groups, such as non-abelian free groups, tend to have very big bounded cohomology modules.

The bounded cohomology of a group $\Gamma$ may be defined with coefficients in any normed (e.g. Banach) $\Gamma$-module, where the norm is needed to make sense of the notion of boundedness of cochains. Moreover, if $\Gamma$ is a topological group, then one may restrict to considering only continuous bounded cochains, thus obtaining the continuous bounded cohomology of $\Gamma$. In this monograph, we will often consider arbitrary normed $\Gamma$-modules, but we will restrict our attention to bounded cohomology of discrete groups. The reason
for this choice is twofold. First, the (very powerful) theory of continuous bounded cohomology is based on a quite sophisticated machinery, which is not needed in the case of discrete groups. Secondly, Monod’s book [Mon01] and Burger-Monod’s paper [BM02] already provide an excellent introduction to the continuous bounded cohomology, while to the author’s knowledge no reference is available where the fundamental properties of bounded cohomology of discrete groups are collected and proved in detail. However, we should emphasize that the theory of continuous bounded cohomology is essential in proving important results also in the context of discrete groups: many vanishing theorems for the bounded cohomology of lattices in Lie groups may be obtained by comparing the bounded cohomology of the lattice with the continuous bounded cohomology of the ambient group.

This monograph is devoted to provide a self-contained introduction to bounded cohomology of discrete groups and topological spaces. Several (by now classical) applications of the theory will be described in detail, while many others will be completely omitted. Of course, the choice of the topics discussed here is largely arbitrary, and based on the taste (and on the knowledge) of the author.

Before describing the content of each chapter, let us briefly recall some research fields which are closely related to bounded cohomology.

**Geometric group theory and quasification.** The bounded cohomology of a closed manifold is strictly related to the curvature of the metrics that the manifold can support. For example, if the closed manifold $M$ is flat or positively curved, then the fundamental group of $M$ is amenable, and $H^n_b(M, \mathbb{R}) = H^n_b(\pi_1(M), \mathbb{R}) = 0$ for every $n \geq 1$. On the other hand, if $M$ is negatively curved, then it is well-known that the comparison map $H^n_b(M, \mathbb{R}) \rightarrow H^n(M, \mathbb{R})$ induced by the inclusion $C^\infty_b(M, \mathbb{R}) \rightarrow C^\infty(M, \mathbb{R})$ is surjective in every degree bigger than one.

In fact, it turns out that the surjectivity of comparison map is related in a very clean way to a fundamental notion which represents the coarse geometric version of negative curvature: the notion of Gromov hyperbolicity. Namely, a group $\Gamma$ is Gromov hyperbolic if and only if the comparison map $H^n_b(\Gamma, V) \rightarrow H^n(\Gamma, V)$ is surjective for every $n \geq 2$ and for every Banach $\Gamma$-module $V$ (see [Min01, Min02]).

Coarse geometry comes into play also when studying the (non-)injectivity of the comparison map. In fact, let $EH^n_c(\Gamma, V)$ denote the kernel of the comparison map in degree $n$. It follows by the very definitions that an element $H^n_b(\Gamma, V)$ lies in $EH^n_c(\Gamma, V)$ if and only if any of its representatives is the coboundary of a (possibly unbounded) cocycle. More precisely, a cochain is usually called a *quasi-cocycle* if its differential is bounded, and a quasi-cocycle is *trivial* if it is the sum of a cocycle and a bounded cochain. Then $EH^n_b(\Gamma, V)$ is canonically isomorphic to the space of $(n-1)$-quasi-cocycles modulo trivial $(n-1)$-quasi-cocycles. When $V = \mathbb{R}$, quasi-cocycles of degree one are usually called *quasi-morphisms*. There exists a large literature
which concerns the construction of non-trivial quasi-morphisms in presence of (weaker and weaker versions of) negative curvature. Brooks [Bro81] first constructed infinitely many quasi-morphisms on the free group with two generators $\mathbb{F}_2$, that were shown to define linearly independent elements in $EH^2(\mathbb{F}_2, \mathbb{R})$ by Mitsumatsu [Mit84]. In particular, this proved that $EH^2_b(\mathbb{F}_2, \mathbb{R})$ (whence $H^2_b(\mathbb{F}_2, \mathbb{R})$) is infinite-dimensional.

Quasi-cocycles are cochains which satisfy the cocycle equation only up to a finite error, and geometric group theory provides tools that are particularly well-suited to study notions which involve finite errors in their definition. Therefore, it is not surprising that Brooks’ and Mitsumatsu’s result has been generalized to larger and larger classes of groups, which include now the class of non-elementary relatively hyperbolic groups, and most mapping class groups. We refer the reader to Remark 2.15 for a more detailed account on this issue. It is maybe worth mentioning that, even if in the cases cited above $EH^2_b(\Gamma, \mathbb{R})$ is always infinite-dimensional, there exist lattices $\Gamma$ in non-linear Lie groups for which $EH^2_b(\Gamma, \mathbb{R})$ is of finite non-zero dimension [MR06].

**Simplicial volume.** The $\ell^\infty$-norm of an $n$-cochain is the supremum of the values it takes on single singular $n$-simplices (or on single $(n+1)$-tuples of elements of the group, when dealing with group cochains rather than with singular cochains). So a cochain $\varphi$ is bounded if and only if it has a finite $\ell^\infty$-norm, and the $\ell^\infty$-norm induces a natural quotient $\ell^\infty$-seminorm on bounded cohomology. The $\ell^\infty$-norm on singular cochains arises as the dual of the $\ell^1$-norm on singular chains that associates to a linear combination of simplices the sum of the modules of the coefficients of the simplices. This $\ell^1$-norm induces an $\ell^1$-seminorm on homology. If $M$ is a closed oriented manifold, then Gromov defined the simplicial volume $\|M\|$ of $M$ as the $\ell^1$-seminorm of the real fundamental class of $M$ [Gro82]. Even if it depends only on the homotopy type of a manifold, the simplicial volume is deeply related to the geometric structures that a manifold can carry. For example, closed manifolds which support negatively curved Riemannian metrics have nonvanishing simplicial volume, while the simplicial volume of closed manifolds with non-negative Ricci tensor is null. In particular, flat or spherical closed manifolds have vanishing simplicial volume, while closed hyperbolic manifolds have positive simplicial volume. As one of the main motivations for its definition, Gromov himself showed that the simplicial volume provides a lower bound for the minimal volume of a manifold, which is the infimum of the volumes of the Riemannian metrics with suitable curvature bounds that are supported by the manifold.

An elementary duality result relates the simplicial volume of an $n$-dimensional manifold $M$ to the bounded cohomology module $H^n_b(M, \mathbb{R})$. For example, if $H^n_b(M, \mathbb{R}) = 0$ then also $\|M\| = 0$. In particular, the simplicial volume of simply connected manifolds (or, more in general, of manifolds with an amenable fundamental group) is vanishing. It is worth stressing the fact that no homological proof of this statement is available: in many
cases, the fact that $\|M\| = 0$ cannot be proved by exhibiting fundamental cycles with arbitrarily small norm. Moreover, the exact value of nonvanishing simplicial volumes is known only in a very few cases: hyperbolic manifolds $[\text{Gro82, Thu79}]$, some classes of 3-manifolds with boundary $[\text{BFP}]$, and the product of two surfaces $[\text{BK08b}]$. At least in the last case, it is not known to the author any description of a sequence of fundamental cycles whose $\ell^1$-norms approximate the simplicial volume. In fact, Bucher’s computation of the simplicial volume of the product of surfaces heavily relies on deep properties of bounded cohomology that have no counterpart in the context of singular homology. It should be stressed here that the seminorm on bounded cohomology plays a fundamental role in this computation: in order to exploit duality to reduce the computation of the $\ell^1$-seminorm of homology classes (whence, of the simplicial volume of a manifold) to computations in bounded cohomology, it is necessary to get a precise control on the $\ell^\infty$-seminorm of the bounded coclasses that are relevant for the problem.

**Characteristic classes.** A fundamental theorem by Gromov $[\text{Gro82}]$ states that, if $G$ is an algebraic subgroup of $\text{GL}_n(\mathbb{R})$, then every characteristic class of flat $G$-bundles lies in the image of the comparison map (i.e. it can be represented by a bounded cocycle). (See $[\text{Buc04}]$ for an alternative proof of a stronger result). Several natural questions arise from this result. First of all, one may ask whether such characteristic classes admit a canonical representative in bounded cohomology: since the comparison map is often non-injective, this would produce more refined invariants. Even when one is not able to find natural bounded representatives for a characteristic class, the seminorm on bounded cohomology (which induces a seminorm on the image of the comparison map by taking the infimum over the bounded representatives) may be used to produce numerical invariants, or to provide useful estimates.

In this context, the most famous example is certainly provided by the Euler class. To every orientable circle bundle there is associated an Euler class, which arises as an obstruction to the existence of a section, and completely classifies the topological type of the bundle. When restricting to flat circle bundles, a bounded Euler class may be defined, which now depends on the flat structure of the bundle (and, in fact, classifies the isomorphism type as flat bundles of bundles with minimal holonomy, see ...). Moreover, the seminorm of the bounded Euler class is equal to $1/2$. Now the Euler number of a circle bundle over a surface is obtained by evaluating the Euler class on the fundamental class of the surface. As a consequence, the Euler number of any flat circle bundle over a surface is bounded above by the product of $1/2$ times the simplicial volume of the surface. This yields the celebrated Milnor-Wood inequalities $[\text{Mil58a, Woo71}]$, which provided the first explicit and easily computable obstructions for a circle bundle to admit a flat structure. Of course, Milnor’s and Wood’s original proofs did not explicitly use bounded cohomology, which was not defined yet. However,
their arguments introduced concepts and techniques which are now at the base of the theory of quasi-morphisms, and they were implicitly based on the construction of a bounded representative for the Euler class.

These results basically extend also to higher dimensions. It was proved by Sullivan [Sul76] that the $\ell^\infty$-seminorm of the Euler class of an orientable flat vector bundle is bounded above by 1. A clever trick due to Smillie allowed to sharpen this bound from 1 to $2^{-n}$, where $n$ is the rank of the bundle. Then, Ivanov and Turaev [IT82] gave a different proof of Smillie’s result, also constructing a natural bounded Euler class in every dimension. A very clean characterization of this bounded cohomology class, as well as the proof that its seminorm is equal to $2^{-n}$, have recently been provided by Bucher and Monod in [BM12].

**Actions on the circle.** If $X$ is a topological space with fundamental group $\Gamma$, then any orientation-preserving topological action of $\Gamma$ on the circle gives rise to a flat circle bundle over $X$. Therefore, we may associate to every such action a bounded Euler class. It is a fundamental result of Ghys [Ghy87, Ghy01] that the bounded Euler class encodes the most essential features of the dynamics of an action. For example, an action admits a global fixed point if and only if its bounded Euler class vanishes. Furthermore, in the almost opposite case of minimal actions (i.e. for actions every orbit of which is dense), the bounded Euler class provides a complete invariant up to conjugation: two minimal circle actions share the same bounded Euler class if and only if they are topologically conjugate. These results establish a deep connection between bounded cohomology and a fundamental problem in one-dimensional dynamics.

**Representations and Rigidity.** As already mentioned above, bounded cohomology has proved to be very useful in proving rigidity results for representations. It is known that an epimorphism between discrete groups induces an injective map on 2-dimensional bounded cohomology with real coefficients. As a consequence, if $H^2_b(\Gamma, \mathbb{R})$ is finite-dimensional and $\rho: \Gamma \to G$ is any representation, then the second bounded cohomology of the image of $\rho$ must also be finite-dimensional. In some cases, this information suffices to ensure that $\rho$ is almost trivial. For example, if $\Gamma$ is a uniform irreducible lattice in a higher rank semisimple Lie group, then by work of Burger and Monod we have that $H^2_b(\Gamma, \mathbb{R})$ is finite-dimensional [BM99]. On the contrary, non virtually abelian subgroups of mapping class groups of hyperbolic surfaces have many non-trivial quasimorphisms [BF02], whence an infinite-dimensional second bounded cohomology. As a consequence, the image of any representation of a higher rank lattice into a mapping class group is virtually abelian, whence finite. This provides an independent proof of a result by Farb, Kaimanovich and Masur [FM98, KM96].

Rigidity results of a different nature arise when exploiting bounded cohomology to get more refined invariants with respect to the ones provided by usual cohomology. For example, we have seen that the norm of the
Euler class may be used to bound the Euler number of flat circle bundles. When considering representations into $PSL(2, \mathbb{R})$ of the fundamental group of closed surfaces of negative Euler characteristic, a celebrated result by Goldman [] implies that a representation has maximal Euler number if and only if it is faithful and discrete, i.e. if and only if it is the holonomy of a hyperbolic structure (in this case, one usually says that the representation is \textit{geometric}). A new proof of this fact has been recently provided in [BIW10], where it is proved that a representation into $PSL(2, \mathbb{R})$ is geometric if and only if its bounded Euler class has maximal norm (see also [Ioz02] for another proof of Goldman’s Theorem based on bounded cohomology, and [Bur11] for a complete characterization of classes of maximal norm in the case when $PSL(2, \mathbb{R})$ is replaced by the full group of homeomorphisms of the circle). In fact, one may define maximal representations of surface groups into a much more general class of Lie groups (see e.g. [BIW10]). This strategy has been followed e.g. in [BI09, BI07] to establish deformation rigidity for representations into $SU(m, 1)$ of lattices in $SU(n, 1)$ also in the non-uniform case (the uniform case having being proved by Goldman and Millson in [GM87]).
CHAPTER 1

(Bounded) cohomology of groups

1. Cohomology of groups

Let $\Gamma$ be a group (which has to be thought as endowed with the discrete topology). We begin by recalling the usual definition of the cohomology of $\Gamma$ with coefficients in a $\Gamma$-module $V$. All the results described here are classical, and date back to the works of Eilenberg and Mac Lane, Hopf, Eckmann, and Freudenthal in the 1940s, and to the contributions of Cartan and Eilenberg to the birth of the theory of homological algebra in the early 1950s.

Throughout the whole monograph, unless otherwise stated, group actions will always be on the left, and modules over (possibly non-commutative) rings will always be left modules. If $R$ is any commutative ring (in fact, we will be interested only in the cases $R = \mathbb{Z}$, $R = \mathbb{R}$), we denote by $R[\Gamma]$ the group ring associated to $R$ and $\Gamma$, i.e. the set of finite linear combinations of elements of $\Gamma$ with coefficients in $R$, endowed with the operations

$$\left( \sum_{g \in \Gamma} a_g g \right) + \left( \sum_{g \in \Gamma} b_g g \right) = \sum_{g \in \Gamma} (a_g + b_g) g,$$

$$\left( \sum_{g \in \Gamma} a_g g \right) \cdot \left( \sum_{g \in \Gamma} b_g g \right) = \sum_{g \in \Gamma} \sum_{h \in \Gamma} a_g h b_{h^{-1}} g$$

(when the sums on the left-hand sides of these equalities are finite, then the same is true for the corresponding right-hand sides). Observe that an $R[\Gamma]$-module $V$ is just an $R$-module $V$ endowed with an action of $\Gamma$ by $R$-linear maps, and that an $R[\Gamma]$-map between $R[\Gamma]$-modules is just an $R$-linear map which commutes with the actions of $\Gamma$. When $R$ is understood, we will often refer to $R[\Gamma]$-maps as to $\Gamma$-maps. If $V$ is an $R[\Gamma]$-module, then we denote by $V^\Gamma$ the subspace of $\Gamma$-invariants of $V$, i.e. the set

$$V^\Gamma = \{ v \in V \mid g \cdot v = v \text{ for every } g \in \Gamma \}.$$

We are now ready to describe the complex of cochains which defines the cohomology of $\Gamma$ with coefficients in $V$. For every $n \in \mathbb{N}$ we set

$$C^n(\Gamma, V) = \{ f : \Gamma^{n+1} \to V \}.$$
and we define \( \delta^n: C^n(\Gamma, V) \to C^{n+1}(\Gamma, V) \) as follows:

\[
\delta^n f(g_0, \ldots, g_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(g_0, \ldots, \hat{g}_i, \ldots, g_{n+1}) .
\]

It is immediate to check that \( \delta^{n+1} \circ \delta^n = 0 \) for every \( n \in \mathbb{N} \), so the pair \((C^\bullet(\Gamma, V), \delta^\bullet)\) is indeed a complex, which is usually known as the *homogeneous* complex associated to the pair \((\Gamma, V)\). The formula

\[
(g \cdot f)(g_0, \ldots, g_n) = g(f(g^{-1}g_0, \ldots, g^{-1}g_n))
\]

endows \( C^n(\Gamma, V) \) with an action of \( \Gamma \), whence with the structure of an \( R[\Gamma] \)-module. It is immediate to check that \( \delta^n \) is a \( \Gamma \)-map, so the \( \Gamma \)-invariants \( C^\bullet(\Gamma, V)^\Gamma \) provide a subcomplex of \( C^\bullet(\Gamma, V) \), whose homology is by definition the cohomology of \( \Gamma \) with coefficients in \( V \). More precisely, if

\[
Z^n(\Gamma, V) = C^n(\Gamma, V)^\Gamma \cap \ker \delta^n, \quad B^n(\Gamma, V) = \delta^{n-1}(C^{n-1}(\Gamma, V)^\Gamma)
\]

(where we understand that \( B^0(\Gamma, V) = 0 \)), then \( B^n(\Gamma, V) \subseteq Z^n(\Gamma, V) \), and

\[
H^n(\Gamma, V) = Z^n(\Gamma, V)/B^n(\Gamma, V) .
\]

**Definition 1.1.** The \( R \)-module \( H^n(\Gamma, V) \) is the \( n \)-th cohomology module of \( \Gamma \) with coefficients in \( V \).

By definition, we have \( H^0(\Gamma, V) = V^\Gamma \) for every \( R[\Gamma] \)-module \( V \). In this monograph we will be often concerned with the case when \( R = V = \mathbb{R} \) (or, less frequently, when \( R = V = \mathbb{Z} \)), and the action of \( \Gamma \) on \( V \) is trivial.

### 2. Functoriality

Group cohomology provides a binary functor. If \( \Gamma \) is a group and \( \alpha: V_1 \to V_2 \) is an \( R[\Gamma] \)-map, then \( f \) induces an obvious *change of coefficients map* \( \alpha^*: C^\bullet(\Gamma, V_1) \to C^\bullet(\Gamma, V_2) \) obtained by composing any cochain with values in \( V_1 \) with \( \alpha \). It is not difficult to show that, if

\[
0 \to V_1 \xrightarrow{\alpha} V_2 \xrightarrow{\beta} V_3 \to 0
\]

is an exact sequence of \( R[\Gamma] \)-modules, then the induced sequence of complexes

\[
0 \to C^\bullet(\Gamma, V_1)^\Gamma \xrightarrow{\alpha^*} C^\bullet(\Gamma, V_2)^\Gamma \xrightarrow{\beta^*} C^\bullet(\Gamma, V_3)^\Gamma \to 0
\]

is also exact, so there is a long exact sequence

\[
0 \to H^0(V_2, \Gamma) \to H^0(V_2, \Gamma) \to H^0(V_3, \Gamma) \to H^1(\Gamma, V_1) \to H^1(\Gamma, V_2) \to \ldots
\]

in cohomology.

Let us now consider the functoriality of cohomology with respect to the first variable. Let \( \psi: \Gamma_1 \to \Gamma_2 \) be a group homomorphism, and let \( V \) be an \( R[\Gamma_2] \)-module. Then \( \Gamma_1 \) acts on \( V \) via \( \psi \), so \( V \) is endowed with a natural structure of \( R[\Gamma_2] \)-module. If we denote this structure by \( \psi^{-1}V \), then the maps

\[
\psi^n: C^n(\Gamma_2, V) \to C^n(\Gamma_1, \psi^{-1}V) , \quad \psi^n(f)(g_0, \ldots, g_n) = f(\psi(g_0), \ldots, \psi(g_n))
\]
provide a chain map such that $\psi^n(C^n(\Gamma_2, V)^{\Gamma_2}) \subseteq C^n(\Gamma_1, \psi^{-1}V)^{\Gamma_1}$. As a consequence, we get a well-defined map

$$H^n(\psi^n) : H^n(\Gamma_2, V) \to H^n(\Gamma_1, \psi^{-1}V)$$

in cohomology. We will consider this map mainly in the case when $V$ is the trivial module $R$. In that context, the discussion above shows that every homomorphism $\psi : \Gamma_1 \to \Gamma_2$ induces a map

$$H^n(\psi^n) : H^n(\Gamma_2, R) \to H^n(\Gamma_1, R)$$

in cohomology.

### 3. The topological interpretation of group cohomology

Let us recall the well-known topological interpretation of group cohomology. We restrict our attention to the case when $V = R$ is a trivial $\Gamma$-module. If $X$ is any topological space, then we denote by $C_\bullet(X, R)$ (resp. $C^\bullet(X, R)$) the complex of singular chains (resp. cochains) with coefficients in $R$, and by $H_\bullet(X, R)$ (resp. $H^\bullet(X, R)$) the corresponding singular homology module.

Suppose now that $X$ is any path-connected topological space satisfying the following properties:

1. the fundamental group of $X$ is isomorphic to $\Gamma$,
2. the space $X$ admits a universal covering $\tilde{X}$, and
3. $\tilde{X}$ is $R$-acyclic, i.e. $H_n(\tilde{X}, R) = 0$ for every $n \geq 1$.

Then $H^\bullet(\Gamma, R)$ is canonically isomorphic to $H^\bullet(X, R)$ (see Section 4). If $X$ is a CW-complex, then condition (2) is automatically satisfied, and Whitehead Theorem implies that condition (3) may be replaced by one of the following equivalent conditions:

1. $\tilde{X}$ is contractible, or
2. $\pi_n(X) = 0$ for every $n \geq 2$.

If a CW-complex satisfies conditions (1), (2) and (3) (or (3'), or (3'')), then one usually says that $X$ is a $K(\Gamma, 1)$, or an Eilenberg-MacLane space. Whitehead Theorem implies that the homotopy type of a $K(\Gamma, 1)$ only depends on $\Gamma$, so if $X$ is any $K(\Gamma, 1)$, then it makes sense to define $H^\bullet(\Gamma, R)$ by setting $H^\bullet(\Gamma, R) = H^\bullet(X, R)$. It is not difficult to show that this definition agrees with the definition of group cohomology given above. In fact, associated to $\Gamma$ there is the $\Delta$-complex $B\Gamma$, having one $n$-simplex for every ordered $(n+1)$-tuple of elements of $\Gamma$, and obvious face operators (see e.g. [Hat02]). When endowed with the weak topology, $B\Gamma$ is clearly contractible. Moreover, the group $\Gamma$ acts freely and simplicially on $B\Gamma$, so the quotient $X_\Gamma$ of $B\Gamma$ by the action of $\Gamma$ inherits the structure of a $\Delta$-complex, and the projection $B\Gamma \to X_\Gamma$ is a universal covering. Therefore, $\pi_1(X_\Gamma) = \Gamma$ and $X_\Gamma$ is a $K(\Gamma, 1)$. By definition, the space of the simplicial $n$-cochains on $B\Gamma$
coincides with the module $C^n(\Gamma, R)$ introduced above, and the simplicial cohomology of $X_\Gamma$ is isomorphic to the cohomology of the $\Gamma$-invariant simplicial cochains on $B\Gamma$. Since the simplicial cohomology of $X_\Gamma$ is canonically isomorphic to the singular cohomology of $X_\Gamma$, this allows us to conclude that $H^\bullet(X_\Gamma, R)$ is canonically isomorphic to $H^\bullet(\Gamma, R)$.

4. Bounded cohomology of groups

Let us now shift our attention to bounded cohomology of groups. In order to do so we first need to define the notion of normed $\Gamma$-module. Let $R$ and $\Gamma$ be as above. For the sake of simplicity, we assume that $R = \mathbb{Z}$ or $R = \mathbb{R}$, and we denote by $| \cdot |$ the usual absolute value on $R$. A normed $R[\Gamma]$-module $V$ is an $R[\Gamma]$-module endowed with an invariant norm, i.e. a map $\| \cdot \| : V \to \mathbb{R}$ such that:

- $\| v \| = 0$ if and only if $v = 0$,
- $\| r \cdot v \| = |r| \cdot \| v \|$ for every $r \in R, v \in V$,
- $\| v + w \| \leq \| v \| + \| w \|$ for every $v, w \in V$,
- $\| g \cdot v \| = \| v \|$ for every $g \in \Gamma, v \in V$.

A $\Gamma$-map between normed $R[\Gamma]$-modules is a $\Gamma$-map between the underlying $R[\Gamma]$-modules, which is bounded with respect to the norms.

Let $V$ be a normed $R[\Gamma]$-module, and recall that $C^n(\Gamma, V)$ is endowed with the structure of an $R[\Gamma]$-module. For every $f \in C^m(\Gamma, V)$ one may consider the $\ell^\infty$-norm

$$\| f \|_\infty = \sup \{ \| f(g_0, \ldots, g_n) \| : (g_0, \ldots, g_n) \in \Gamma^{n+1} \} \in [0, +\infty].$$

We set

$$C^m_b(\Gamma, V) = \{ f \in C^m(\Gamma, V) : \| f \|_\infty < \infty \}$$

and we observe that $C^m_b(\Gamma, V)$ is an $R[\Gamma]$-submodule of $C^m(\Gamma, V)$. Therefore, $C^m_b(\Gamma, V)$ is a normed $R[\Gamma]$-module. The differential $\delta^n : C^n(\Gamma, V) \to C^{n+1}(\Gamma, V)$ restricts to a $\Gamma$-map of normed $R[\Gamma]$-modules $\delta^n : C^m_b(\Gamma, V) \to C^{m+1}_b(\Gamma, V)$, so one may define as usual

$$Z^m_b(\Gamma, V) = \ker \delta^n \cap C^n(\Gamma, V)^\Gamma, \quad B^m_b(\Gamma, V) = \delta^{n-1}(C^{n-1}_b(\Gamma, V)^\Gamma)$$

(where we understand that $B^0_b(\Gamma, V) = \{ 0 \}$), and set

$$H^m_b(\Gamma, V) = Z^m_b(\Gamma, V)/B^m_b(\Gamma, V).$$

The $\ell^\infty$-norm on $C^m_b(\Gamma, V)$ restricts to a norm on $Z^m_b(\Gamma, V)$, which descends to a seminorm on $H^m_b(\Gamma, V)$ by taking the infimum over all the representatives of a coclass: namely, for every $\alpha \in H^m_b(\Gamma, V)$ one sets

$$\| \alpha \|_\infty = \inf \{ \| f \|_\infty : f \in Z^m_b(\Gamma, V), [f] = \alpha \}.$$  

DEFINITION 1.2. The $R$-module $H^m_b(\Gamma, V)$ is the $n$-th bounded cohomology module of $\Gamma$ with coefficients in $V$. The seminorm $\| \cdot \|_\infty : H^m_b(\Gamma, V) \to \mathbb{R}$ is called the canonical seminorm of $H^m_b(\Gamma, V)$.
The canonical seminorm on $H^n_{b}(\Gamma, V)$ is a norm if and only if the subspace $B^n_{b}(\Gamma, V)$ is closed in $Z^n_{b}(\Gamma, V)$ (whence in $C^n_{b}(\Gamma, V)$). However, this is not always the case: for example, if $\Gamma$ is the fundamental group of a closed hyperbolic surface, then it is proved in [Som97, Som98] that $H^3_{b}(\Gamma, \mathbb{R})$ contains non-trivial elements with null seminorm. On the other hand, it was proved independently by Ivanov [Iva90] and Matsumoto and Morita [MM85] that $H^2_{b}(\Gamma, \mathbb{R})$ is a Banach space for every group $\Gamma$ (see Corollary 6.7).

5. Functoriality and the comparison map

Just as in the case of ordinary cohomology, also bounded cohomology provides a binary functor. The discussion carried out in Section 2 applies word by word in the bounded case. Namely, every exact sequence of normed $R[\Gamma]$-modules

$$0 \longrightarrow V_1 \overset{\alpha}{\longrightarrow} V_2 \overset{\beta}{\longrightarrow} V_3 \longrightarrow 0$$

induces a long exact sequence

$$0 \longrightarrow H^0_{b}(V_1, \Gamma) \longrightarrow H^0_{b}(V_2, \Gamma) \longrightarrow H^0_{b}(V_3, \Gamma) \longrightarrow H^1_{b}(\Gamma, V_1) \longrightarrow H^1_{b}(\Gamma, V_2) \longrightarrow \ldots$$

in cohomology. Moreover, if $\psi: \Gamma_1 \rightarrow \Gamma_2$ is a group homomorphism, and $V$ is a normed $R[\Gamma_2]$-module, then $V$ admits a natural structure of normed $R[\Gamma_1]$-module, which is denoted by $\psi^{-1}V$. Then, the homomorphism $\psi$ induces a well-defined map

$$H^n_{b}(\psi^n): H^n_{b}(\Gamma_2, V) \rightarrow H^n_{b}(\Gamma_1, \psi^{-1}V)$$

in bounded cohomology. In particular, in the case of trivial coefficients we get a map

$$H^n_{b}(\psi^n): H^n_{b}(\Gamma_2, R) \rightarrow H^n_{b}(\Gamma_1, R).$$

The inclusion $C^\bullet_b(\Gamma, V) \hookrightarrow C^\bullet(\Gamma, V)$ induces a map

$$c: H^\bullet_b(\Gamma, V) \rightarrow H^\bullet(\Gamma, V)$$

called the comparison map. We will see soon that the comparison map is neither injective nor surjective in general. One could approach the study of $H^n_{b}(\Gamma, V)$ by looking separately at the kernel and at the image of the comparison map. This strategy is described in Chapter 2 for bounded cohomology groups in low degree.

6. Looking for a topological interpretation of bounded cohomology of groups

Let us now restrict to the case when $V = R$ is a trivial normed $\Gamma$-module. We would like to compare the bounded cohomology of $\Gamma$ with a suitably defined singular bounded cohomology of a suitable topological model for $\Gamma$. There is a straightforward notion of boundedness for singular cochains, so the notion of bounded singular cohomology of a topological space is easily
1. (bounded) cohomology of groups

defined (see Chapter 5). Then, it is still true that the bounded cohomology of a group $\Gamma$ is canonically isomorphic to the bounded singular cohomology of a $K(\Gamma, 1)$ (in fact, even more is true: in the case of real coefficients, the bounded cohomology of $\Gamma$ is isometrically isomorphic to the bounded singular cohomology of any countable CW-complex $X$ such that $\pi_1(X) = \Gamma$ – see Theorem 5.9). However, the proof described in Section 3 for classical cohomology does not carry over to the bounded context. It is still true that homotopically equivalent spaces have isometrically isomorphic bounded cohomology, but, if $X_\Gamma$ is the $K(\Gamma, 1)$ defined in Section 3, then there is no clear reason why the bounded simplicial cohomology of $X_\Gamma$ (which coincides with the bounded cohomology of $\Gamma$) should be isomorphic to the bounded singular cohomology of $X_\Gamma$.

For example, if $X$ is any finite $\Delta$-complex, then every simplicial cochain on $X$ is obviously bounded, so the cohomology of the complex of bounded simplicial cochains on $X$ coincides with the classical singular cohomology of $X$, which in general is very different from the bounded singular cohomology of $X$.

7. the bar resolution

We have defined the cohomology (resp. the bounded cohomology) of $\Gamma$ as the cohomology of the complex $C^\bullet(\Gamma, V)^\Gamma$ (resp. $C^\bullet_b(\Gamma, V)^\Gamma$). Of course, an element $f \in C^\bullet(\Gamma, V)^\Gamma$ is completely determined by the values it takes on $(n+1)$-tuples having 1 as first entry. More precisely, if we denote by $C^n(\Gamma, V)^\Gamma = \{ f: \Gamma^n \to V \}$, and we consider $C^n(\Gamma, V)$ simply as an $R$-module for every $n \in \mathbb{N}$, then we have $R$-isomorphisms

$$C^n(\Gamma, V)^\Gamma \to C^n(\Gamma, V)$$

$$\varphi \mapsto ((g_1, \ldots, g_n) \mapsto \varphi(1, g_1 g_1 g_2, \ldots, g_1 \cdots g_n)) .$$

Under these isomorphisms, the differential $\delta^0: C^\bullet(\Gamma, V)^\Gamma \to C^{\bullet+1}(\Gamma, V)^\Gamma$ translates into the differential $\delta^0: C^\bullet(\Gamma, V) \to C^{\bullet+1}(\Gamma, V)$ such that

$$\delta^0(v)(g) = g \cdot v - v \quad v \in V = C^0(\Gamma, V), \; g \in \Gamma ,$$

and

$$\delta^n(f)(g_1, \ldots, g_{n+1}) = g_1 \cdot f(g_2, \ldots, g_{n+1})$$

$$+ \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_i g_{i+1}, \ldots, g_n)$$

$$+ (-1)^{n+1} f(g_1, \ldots, g_n) ,$$

for $n \geq 1$. The complex

$$0 \to C^0(\Gamma, V) \xrightarrow{\delta^0} C^1(\Gamma, V) \xrightarrow{\delta^1} \ldots \xrightarrow{\delta^{n-1}} C^n(\Gamma, V) \xrightarrow{\delta^n} \ldots$$
is usually known as the *inhomogeneous* complex associated to the pair \((\Gamma, V)\). By construction, the cohomology of this complex is canonically isomorphic to \(H^\bullet(\Gamma, V)\).

Just as we did for the homogeneous complex, if \(V\) is a normed \(R[\Gamma]\)-module, then we may define the submodule \(\overline{C}_b^n(\Gamma, V)\) of bounded elements of \(\overline{C}^n(\Gamma, V)\). For every \(n \in \mathbb{N}\), the isomorphism \(C^n(\Gamma, V)^\Gamma \cong \overline{C}^n(\Gamma, V)\) restricts to an isometric isomorphism \(C_b^n(\Gamma, V)^\Gamma \cong \overline{C}_b^n(\Gamma, V)\), so \(\overline{C}_b(\Gamma, V)\) is a subcomplex of \(\overline{C}^\bullet(\Gamma, V)\), whose cohomology is canonically isometrically isomorphic to \(H_b^\bullet(\Gamma, V)\).
CHAPTER 2

(Bounded) cohomology of groups in low degree

In this section we analyze the (bounded) cohomology modules of a group \( \Gamma \) in degree 0, 1, 2. We restrict our attention to the case when \( V = R \) is equal either to \( \mathbb{Z} \), or to \( \mathbb{R} \), both endowed with the structure of trivial \( R[\Gamma] \)-module. In order to simplify the computations, it will be convenient to work with the inhomogeneous complexes of (bounded) cochains.

By the very definitions, we have \( C^0(\Gamma, R) = C^0_b(\Gamma, R) = R \), and \( \delta^0 = 0 \), so
\[
H^0(\Gamma, R) = H^0_b(\Gamma, R) = R .
\]

1. (Bounded) group cohomology in degree one

Let us now describe what happens in degree one. By definition, for every \( \varphi \in C^1(\Gamma, R) \) we have
\[
\delta(f)(g_1, g_2) = f(g_1) + f(g_2) - f(g_1g_2)
\]
(recall that we are assuming that the action of \( \Gamma \) on \( R \) is trivial). In other words, if we denote by \( Z^1(\Gamma, R) \) and \( B^1(\Gamma, R) \) the spaces of cocycles and coboundaries of the inhomogeneous complex \( C^\bullet(\Gamma, R) \), then we have
\[
H^1(\Gamma, R) = Z^1(\Gamma, R) = \text{Hom}(\Gamma, R) .
\]
Moreover, every bounded homomorphism with values in \( \mathbb{Z} \) or in \( \mathbb{R} \) is obviously trivial, so
\[
H^1_b(\Gamma, R) = Z^1_b(\Gamma, R) = 0 .
\]
(here and henceforth, we denote by \( Z^*_{\bullet}(\Gamma, R) \) and \( B^*_{\bullet}(\Gamma, R) \) the spaces of cocycles and coboundaries of the bounded inhomogeneous complex \( C^\bullet_b(\Gamma, R) \)).

This result extends also to the case of non-constant coefficient modules, under suitable additional hypotheses. In fact, it is proved in [Joh72] (see also [Nos91, Section 7]) that \( H^1_b(\Gamma, V) = 0 \) for every group \( \Gamma \) and every reflexive normed \( R[\Gamma] \)-module \( V \). Things get much more interesting when considering non-reflexive coefficient modules. In fact, it turns out that a group \( \Gamma \) is amenable (see Definition 3.1) if and only if \( H^1_b(\Gamma, V) = 0 \) for every dual normed \( R[\Gamma] \)-module \( V \) (see Corollary 3.11). Therefore, for every non-amenable group \( \Gamma \) there exists a (dual) normed \( R[\Gamma] \)-module \( \Gamma \) such that \( H^1_b(\Gamma, V) \neq 0 \) (this module admits a very explicit description, see Section 4).
In fact, even when \( \Gamma \) is amenable, there can exist (non-dual) Banach \( \Gamma \)-modules \( V \) such that \( H^1_b(\Gamma, V) \neq 0 \): in [Nos91, Section 7] a Banach \( \mathbb{Z} \)-module \( \mathcal{A} \) is constructed such that \( H^1_b(\mathbb{Z}, \mathcal{A}) \) is infinite-dimensional.

2. Group cohomology in degree two

The situation in degree two is more interesting. A central extension of \( \Gamma \) by \( R \) is an exact sequence
\[ 1 \rightarrow R \xrightarrow{i} \Gamma' \xrightarrow{\pi} \Gamma \rightarrow 1 \]
such that \( i(R) \) is contained in the center of \( \Gamma' \). In what follows, in this situation we will identify \( R \) with \( i(R) \in \Gamma' \) via \( i \). Two such extensions are equivalent if they may be put into a commutative diagram as follows:
\[
\begin{array}{ccc}
1 & \longrightarrow & R \\
\downarrow{\text{Id}} & & \downarrow{f} \\
1 & \longrightarrow & \Gamma' \\
\end{array}
\]
(by the commutativity of the diagram, the map \( f \) is necessarily an isomorphism). Associated to an exact sequence
\[ 1 \rightarrow R \xrightarrow{i} \Gamma' \xrightarrow{\pi} \Gamma \rightarrow 1 \]
there is a cocyle \( \varphi \in \overline{C}^2(\Gamma, R) \) which is defined as follows: let \( s: \Gamma \rightarrow \Gamma' \) be any map such that \( \pi \circ s = \text{Id}_\Gamma \). Then we set
\[ \varphi(g_1, g_2) = s(g_1 g_2)^{-1} s(g_1)s(g_2) . \]
By construction we have \( \varphi(g_1, g_2) \in \ker \pi = R \), so \( \varphi \) is indeed an element in \( \overline{C}^2(\Gamma, R) \). It is easy to check that \( \delta(\varphi) = 0 \), so \( \varphi \in \overline{Z}^2(\Gamma, R) \). Moreover, different choices for the section \( s \) give rise to cocycles which differ one from the other by a coboundary, so any central extension of \( \Gamma \) by \( R \) defines an element in \( H^2(\Gamma, R) \). It is not difficult to reverse this construction to show that every element in \( H^2(\Gamma, R) \) is represented by a central extension, and two central extensions are equivalent if and only if they define the same element in \( H^2(\Gamma, R) \) (see e.g. [Bro82] for full details). Therefore:

**Proposition 2.1.** The group \( H^2(\Gamma, R) \) is in natural bijection with the set of equivalence classes of central extensions of \( \Gamma \) by \( R \).

**Remark 2.2.** Proposition 2.1 may be easily generalized to the case of arbitrary extensions of \( \Gamma \) by \( R \). Once such an extension is given, if \( \Gamma' \) is the middle term of the extension, then the action of \( \Gamma' \) on \( R = i(R) \) by conjugacy descends to a well-defined action of \( \Gamma \) on \( R \), thus defining on \( R \) the structure of a (possibly non-trivial) \( \Gamma \)-module (this structure is trivial precisely when \( R \) is central in \( \Gamma' \)). Then, one may associate to the extension an element of the cohomology group \( H^2(\Gamma, R) \) with coefficients in the (possibly non-trivial) \( R[\Gamma] \)-module \( R \).
3. Bounded group cohomology in degree two: quasimorphisms

As mentioned above, the study of $H^2_b(\Gamma, R)$ may be reduced to the study of the kernel and of the image of the comparison map

$$c: H^2_b(\Gamma, R) \rightarrow H^2(\Gamma, R).$$

We will describe in Chapter 6 a characterization of groups with injective comparison map due to Matsumoto and Morita [MM85]. In this section we describe the relationship between the kernel of the comparison map and the space of quasimorphisms on $\Gamma$.

**Definition 2.3.** A map $f: \Gamma \rightarrow R$ is a quasimorphism if there exists a constant $D \geq 0$ such that

$$|f(g_1) + f(g_2) - f(g_1g_2)| \leq D$$

for every $g_1, g_2 \in \Gamma$. The least $D \geq 0$ for which the above inequality is satisfied is the defect of $f$, and it is denoted by $D(f)$. The space of quasimorphisms is an $R$-module, and it is denoted by $Q(\Gamma, R)$.

By the very definition, a quasimorphism is an element of $C^1(\Gamma, R)$ having bounded differential. Of course, both bounded functions (i.e. elements in $C^1_b(\Gamma, R)$) and homomorphisms (i.e. elements in $Z^1(\Gamma, R) = \text{Hom}(\Gamma, R)$) are quasimorphisms.

If every quasimorphism could be obtained just by adding a bounded function to a homomorphism, the notion of quasimorphism would not really introduce something new. Observe that every bounded homomorphism with values in $R$ is necessarily trivial, so $C^1_b(\Gamma, R) \cap \text{Hom}(\Gamma, R) = \{0\}$, and we may think of $C^1_b(\Gamma, R) \oplus \text{Hom}(\Gamma, R)$ as of the space of “trivial” quasimorphisms on $\Gamma$.

The following result is an immediate consequence of the definitions, and shows that the existence of “non-trivial” quasimorphisms is equivalent to the non-injectivity of the comparison map $c: H^2_b(\Gamma, R) \rightarrow H^2(\Gamma, R)$.

**Proposition 2.4.** The differential $\delta: Q(\Gamma, R) \rightarrow C^2_b(\Gamma, R)$ induces an isomorphism

$$Q(\Gamma, R)/ \left( C^1_b(\Gamma, R) \oplus \text{Hom}_Z(\Gamma, R) \right) \cong \ker c.$$

Therefore, in order to show that $H^2_b(\Gamma, R)$ is non-trivial it is sufficient to construct quasimorphisms which do not stay at finite distance from a homomorphism.

4. Homogeneous quasimorphisms

Let us introduce the following:

**Definition 2.5.** A quasimorphism $f: \Gamma \rightarrow R$ is homogeneous if $f(g^n) = n \cdot f(g)$ for every $g \in \Gamma$, $n \in \mathbb{Z}$. The space of homogeneous quasimorphisms is a submodule of $Q(\Gamma, R)$, and it is denoted by $Q^h(\Gamma, R)$. 
Of course, there are no non-trivial bounded homogeneous quasimorphisms. In particular, for every quasimorphism $f$ there exists at most one homogeneous quasimorphism $\overline{f}$ such that $\|\overline{f} - f\|_{\infty} < +\infty$, so a homogeneous quasimorphism which is not a homomorphism cannot stay at finite distance from a homomorphism. When $R = \mathbb{Z}$, it may happen that homogeneous quasimorphisms are quite sparse in $Q(\Gamma, \mathbb{Z})$: for example, for every $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, if we denote by $\lfloor x \rfloor$ the largest integer which is not bigger than $x$, then the quasimorphism $f_\alpha : \mathbb{Z} \to \mathbb{Z}$ defined by

\begin{equation}
\quad f_\alpha(n) = \lfloor \alpha n \rfloor
\end{equation}

is not at finite distance from any element in $Q^h(\mathbb{Z}, \mathbb{Z}) = \text{Hom}(\mathbb{Z}, \mathbb{Z})$. When $R = \mathbb{R}$ homogeneous quasimorphisms play a much more important role, due to the following:

\textbf{Proposition 2.6.} Let $f \in Q(\Gamma, \mathbb{R})$ be a quasimorphism. Then, there exists a unique element $\overline{f} \in Q^h(\Gamma, \mathbb{R})$ that stays at a finite distance from $f$. Moreover, we have

$$\|f - \overline{f}\|_{\infty} \leq D(f) , \quad D(\overline{f}) \leq 4D(f) .$$

\textbf{Proof.} For every $g \in \Gamma$, $m,n \in \mathbb{N}$ we have

$$|f(g^{mn}) - nf(g^m)| \leq (n-1)D(f) ,$$

so

$$\left| \frac{f(g^n)}{n} - \frac{f(g^m)}{m} \right| \leq \left| \frac{f(g^n)}{n} - \frac{f(g^{mn})}{mn} \right| + \left| \frac{f(g^{mn})}{mn} - \frac{f(g^m)}{m} \right| \leq \left( \frac{1}{n} + \frac{1}{m} \right) D(f) .$$

Therefore, the sequence $f(g^n)/n$ is a Cauchy sequence, and the same is true for the sequence $f(g^{-n})/(-n)$. Since $f(g^n) + f(g^{-n}) \leq f(1) + D(f)$ for every $n$, we may conclude that the limit

$$\overline{f}(g) = \lim_{n \to \infty} \frac{f(g^n)}{n}$$

exists for every $g \in \Gamma$. Moreover, the inequality $|f(g^n) - nf(g)| \leq (n-1)D(f)$ implies that

$$\left| \frac{f(g^n)}{n} - \frac{f(g^n)}{n} \right| \leq D(f)$$

so by passing to the limit we obtain that $\|\overline{f} - f\|_{\infty} \leq D(f)$. This immediately implies that $\overline{f}$ is a quasimorphism such that $D(\overline{f}) \leq 4D(f)$. Finally, the fact that $\overline{f}$ is homogeneous is obvious. \qed

In fact, the stronger inequality $D(\overline{f}) \leq 2D(f)$ holds (see e.g. [Cal09] for a proof). Propositions 2.4 and 2.6 imply the following:

\textbf{Corollary 2.7.} The space $Q(\Gamma, \mathbb{R})$ decomposes as a direct sum

$$Q(\Gamma, \mathbb{R}) = Q^h(\Gamma, \mathbb{R}) \oplus \mathbb{C}_b^1(\Gamma, \mathbb{R}) .$$

Moreover, the restriction of $\overline{\delta}$ to $Q^h(\Gamma, \mathbb{R})$ induces an isomorphism

$$Q^h(\Gamma, \mathbb{R})/\text{Hom}(\Gamma, \mathbb{R}) \cong \ker c .$$
5. Quasimorphisms on abelian groups

Suppose now that $\Gamma$ is abelian. Then, for every $g_1, g_2 \in \Gamma$, every element $f \in Q^h(\Gamma, \mathbb{R})$, and every $n \in \mathbb{N}$ we have

$$|nf(g_1g_2) - nf(g_1) - nf(g_2)| = |f((g_1g_2)^n) - f(g_1^n) - f(g_2^n)| \leq D(f).$$

Dividing by $n$ this inequality and passing to the limit for $n \to \infty$ we get that $f(g_1g_2) = f(g_1) + f(g_2)$, i.e. $f$ is a homomorphism. Therefore, every homogeneous quasimorphism on $\Gamma$ is a homomorphism. Putting together Proposition 2.4 and Corollary 2.7 we obtain the following:

**Corollary 2.8.** If $\Gamma$ is abelian, then the comparison map $c: H^2_b(\Gamma, \mathbb{R}) \to H^2(\Gamma, \mathbb{R})$ is injective.

In fact, a much stronger result holds: if $\Gamma$ is abelian, then it is amenable (see Definition 3.1), so $H^2_n(\Gamma, \mathbb{R}) = 0$ for every $n \geq 1$ (see Corollary 3.7). We stress that Corollary 2.8 does not hold in the case with integer coefficients. For example, we have the following:

**Proposition 2.9.** We have $H^2_b(\mathbb{Z}, \mathbb{Z}) = \mathbb{R}/\mathbb{Z}$.

**Proof.** The short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow 0$$

induces a short exact sequence of complexes

$$0 \longrightarrow C^\bullet_b(Z, Z) \longrightarrow C^\bullet_b(Z, \mathbb{R}) \longrightarrow C^\bullet(Z, \mathbb{R}/\mathbb{Z}) \longrightarrow 0.$$

Let us consider the following portion of the long exact sequence induced in cohomology:

$$\ldots \longrightarrow H^1_b(Z, \mathbb{R}) \longrightarrow H^1(Z, \mathbb{R}/\mathbb{Z}) \longrightarrow H^2_b(Z, Z) \longrightarrow H^2_b(Z, \mathbb{R}) \longrightarrow \ldots$$

Recall from Section 1 that $H^1_b(Z, \mathbb{R}) = 0$. Moreover, we have $H^2(Z, \mathbb{R}) = H^2(S^1, \mathbb{R}) = 0$, so $H^2_b(Z, \mathbb{R})$ is equal to the kernel of the comparison map $c: H^2_b(Z, \mathbb{R}) \to H^2(Z, \mathbb{R})$, which vanishes by Corollary 2.8. Therefore, we have

$$H^2_b(Z, Z) \cong H^1(Z, \mathbb{R}/\mathbb{Z}) = \text{Hom}(\mathbb{Z}, \mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}.$$

**Remark 2.10.** It follows from the proof of the previous proposition that the isomorphism $\mathbb{R}/\mathbb{Z} \to H^2_b(Z, Z)$ is given by the map

$$\mathbb{R}/\mathbb{Z} \ni [\alpha] \mapsto [-\delta f_\alpha] \in H^2_b(Z, Z),$$

where $f_\alpha$ is the quasimorphism defined in (1).
It is interesting to notice that the module $H^2_b(\mathbb{Z}, \mathbb{Z})$ is \textit{not} finitely generated over $\mathbb{Z}$. This fact already shows that bounded cohomology may be very different from classical cohomology. The same phenomenon may occur in the case of real coefficients: in the following section we show that, if $F_2$ is the free group on 2 elements, then $H^2_b(F_2, \mathbb{R})$ is an infinite dimensional vector space.

6. The image of the comparison map

In Section 2 we have interpreted elements in $H^2(\Gamma, R)$ as equivalence classes of central extensions of $\Gamma$ by $R$. Of course, one may wonder whether elements of $H^2(\Gamma, R)$ admitting bounded representatives represent peculiar central extensions. It turns out that this is indeed the case: bounded classes represent central extensions which are quasi-isometrically trivial (i.e. quasi-isometrically equivalent to product extension). Of course, in order to give a sense to this statement we need to restrict our attention to the case when $\Gamma$ is finitely generated, and $R = \mathbb{Z}$ (or, more in general, $R$ is a finitely generated abelian group). Under these assumptions, let us consider the central extension

$$
1 \longrightarrow \mathbb{Z} \longrightarrow \Gamma' \overset{\pi}{\longrightarrow} \Gamma \longrightarrow 1,
$$

and the following condition:

(*) there exists a quasi-isometry $q: \Gamma' \rightarrow \Gamma \times \mathbb{Z}$ which makes the following diagram commute:

$$
\begin{array}{ccc}
\Gamma' & \overset{\pi}{\longrightarrow} & \Gamma \\
\downarrow{q} & & \downarrow{\text{Id}} \\
\Gamma \times \mathbb{Z} & \longrightarrow & \Gamma
\end{array}
$$

where the horizontal arrow on the bottom represents the obvious projection.

Condition (*) is equivalent to the existence of a Lipschitz section $s: \Gamma \rightarrow \Gamma'$ such that $\pi \circ s = \text{Id}_{\Gamma}$ (see e.g. [KL01, Proposition 8.2]). Gersten proved that a sufficient condition for a central extension to satisfy condition (*) is that its coclass in $H^2(\Gamma, \mathbb{Z})$ admits a \textit{bounded} representative (see [Ger92, Theorem 3.1]). Therefore, the image of the comparison map $H^2_b(\Gamma, \mathbb{Z}) \to H^2(\Gamma, \mathbb{Z})$ determines extensions which satisfy condition (*). As far as the author knows, it is not known whether the converse implication holds, i.e. if every central extension satisfying (*) is represented by a cohomology class lying in the image of the comparison map.

Let us now consider the case with real coefficients, and list some results about the surjectivity of the comparison map:

1. If $\Gamma$ is the fundamental group of a closed $n$-dimensional locally symmetric space of non-compact type, then the comparison map $c: H^2_b(\Gamma, \mathbb{R}) \to H^n(\Gamma, \mathbb{R}) \cong \mathbb{R}$ is surjective [LS06, BK07].
(2) If \( \Gamma \) is word hyperbolic, then the comparison map \( c: H^n_b(\Gamma, V) \to H^n(\Gamma, V) \) is surjective for every \( n \geq 2 \) when \( V \) is any normed \( \mathbb{R}[\Gamma] \)-module or any finitely generated group (considered as a trivial \( \mathbb{Z}[\Gamma] \)-module) [Min01].

(3) If \( \Gamma \) is finitely presented and the comparison map \( c: H^2_b(\Gamma, V) \to H^2(\Gamma, V) \) is surjective for every normed \( \mathbb{R}[\Gamma] \)-module, then \( \Gamma \) is word hyperbolic [Min02].

(4) The set of closed 3-manifolds \( M \) for which the comparison map \( H_b^*(\pi_1(M), \mathbb{R}) \to H_b^*(\pi_1(M), \mathbb{R}) \) is surjective in every degree is completely characterized in [FS02].

Observe that points (2) and (3) provide a characterization of word hyperbolicity in terms of bounded cohomology. Moreover, putting together point (2) with the discussion above, we see that every extension of a word hyperbolic group by \( \mathbb{Z} \) is quasi-isometrically trivial.

In this subsection we have concentrated our attention on the case of real coefficients. However, we have the following result [Min01, Theorem 15]:

**Proposition 2.11.** Let \( \Gamma \) be any group, and take an element \( \alpha \in H^n(\Gamma, \mathbb{Z}) \) such that \( \|\alpha\|_\infty < +\infty \), where \( \alpha \) denotes the image of \( \alpha \) via the change of coefficients homomorphism induced by the inclusion \( \mathbb{Z} \to \mathbb{R} \). Then \( \|\alpha\|_\infty < +\infty \).

**Proof.** Let \( \varphi \in C^n(\Gamma, \mathbb{Z})^\Gamma \) be a representative of \( \alpha \), which we also think of as a real cocycle. By our assumptions, we have

\[
\varphi = \varphi' + \delta \psi ,
\]

where \( \varphi' \in C^n_b(\Gamma, \mathbb{Z})^\Gamma \), \( \psi \in C^{n-1}(\Gamma, \mathbb{R})^\Gamma \). Let us denote by \( \overline{\psi} \) the cochain defined by \( \overline{\psi}(g_0, \ldots, g_{n-1}) = [\psi(g_0, \ldots, g_{n-1})] \), where for every \( x \in \mathbb{R} \) we denote by \( \lfloor x \rfloor \) the largest integer which does not exceed \( x \). Then \( \overline{\psi} \) is still \( \Gamma \)-invariant, so \( \overline{\psi} \in C^{n-1}(\Gamma, \mathbb{Z})^\Gamma \). Moreover, we obviously have \( \psi - \overline{\psi} \in C^{n-1}_b(\Gamma, \mathbb{R}) \). Let us now consider the equality

\[
\varphi - \delta \overline{\psi} = \varphi' + \delta(\psi - \overline{\psi}) .
\]

The left-hand side is a cocycle with integral coefficients, while the right-hand side is bounded. Therefore, the cocycle \( \varphi - \delta \overline{\psi} \) is a bounded representative of \( \alpha \), and we are done. \( \square \)

**Corollary 2.12.** Let \( \Gamma \) be any group. Then, the comparison map \( c_\mathbb{Z}^n: H^n_b(\Gamma, \mathbb{Z}) \to H^n(\Gamma, \mathbb{Z}) \) is surjective if and only if the comparison map \( c_\mathbb{R}^n: H^n_b(\Gamma, \mathbb{R}) \to H^n(\Gamma, \mathbb{R}) \) is.

**Proof.** Suppose first that \( c_\mathbb{Z}^n \) is surjective. Then, the image of \( c_\mathbb{R} \) is a vector subspace of \( H^n(\Gamma, \mathbb{R}) \) which contains the image of \( H^n_b(\Gamma, \mathbb{Z}) \) via the change of coefficients homomorphism. By the Universal Coefficients Theorem, this implies that \( c_\mathbb{R} \) is surjective. The converse implication readily follows from Proposition 2.11. \( \square \)
7. The second bounded cohomology group of free groups

Let $F_2$ be the free group on two generators. Since $H^2(F_2, \mathbb{R}) = H^2(S^1 \vee S^1, \mathbb{R}) = 0$, the computation of $H^2_b(F_2, \mathbb{R})$ is reduced to the analysis of the space $Q(\Gamma, \mathbb{R})$ of real quasimorphisms on $F_2$. There are several constructions of elements in $Q(\Gamma, \mathbb{R})$ available in the literature. The first such example is probably due to Johnson [Joh72], who proved that $H^2_b(F_2, \mathbb{R}) \neq 0$. Afterwords, Brooks [Bro81] produced an infinite family of quasimorphisms, which were shown to define independent elements $E H^2_b(F_2, \mathbb{R})$ by Mitsumatsu [Mit84]. Since then, other constructions have been provided by many authors for more general classes of groups (see Remark 2.15 below). We describe here a family of quasimorphisms which is due to Rolli [Rol].

Let $s_1, s_2$ be the generators of $F_2$, and let $\ell_\infty^{\text{odd}}(\mathbb{Z}) = \{ \alpha: \mathbb{Z} \to \mathbb{R} \mid \alpha(n) = -\alpha(-n) \text{ for every } n \in \mathbb{Z} \}$. For every $\alpha \in \ell_\infty^{\text{odd}}(\mathbb{Z})$ we consider the map

$$f_\alpha: F_2 \to \mathbb{R}$$

defined by

$$f_\alpha(s_{i_1}^{n_1} \cdots s_{i_k}^{n_k}) = \sum_{j=1}^k \alpha(n_j),$$

where we are identifying every element of $F_2$ with the unique reduced word representing it. It is elementary to check that $f_\alpha$ is a quasimorphism (see [Rol, Proposition 2.1]). Moreover, we have the following:

**Proposition 2.13 ([Rol]).** The map

$$\ell_\infty^{\text{odd}}(\mathbb{Z}) \to H^2_b(F_2, \mathbb{R}) \quad \alpha \mapsto [\delta(f_\alpha)]$$

is injective.

**Proof.** Suppose that $[\delta(f_\alpha)] = 0$. By Proposition 2.4, this implies that $f_\alpha = h + b$, where $h$ is a homomorphism and $b$ is bounded. Then, for $i = 1, 2, k \in \mathbb{N}$, we have $k \cdot h(s_i) = h(s_i^k) - b(s_i^k) = \alpha(k) - b(s_i^k)$. By dividing by $k$ and letting $k$ going to $\infty$ we get $h(s_i) = 0$, so $h = 0$, and $f_\alpha = b$ is bounded.

Observe now that for every $k, l \in \mathbb{Z}$ we have $f_\alpha((s_1 s_2)^k) = 2k \cdot \alpha(l)$. Since $f_\alpha$ is bounded, this implies that $\alpha = 0$, whence the conclusion. \qed

**Corollary 2.14.** Let $\Gamma$ be a group admitting an epimorphism

$$\varphi: \Gamma \to F_2.$$ 

Then the vector space $H^2_b(\Gamma, \mathbb{R})$ is infinite-dimensional.

**Proof.** Since $F_2$ is free, the epimorphism admits a right inverse $\psi$. The composition

$$H^2_b(F_2, \mathbb{R}) \xrightarrow{H^2_b(\varphi)} H^2_b(\Gamma, \mathbb{R}) \xrightarrow{H^2_b(\psi)} H^2_b(F_2, \mathbb{R})$$
of the induced maps in bounded cohomology is the identity, and Proposition 2.13 ensures that $H^2_b(F_2, \mathbb{R})$ is infinite-dimensional. The conclusion follows.

**Remark 2.15.** The previous corollary implies that a large class of groups has infinite-dimensional second bounded cohomology module. In fact, this holds true for every group belonging to one of the following families:

1. surface groups (for closed surfaces of genus $g \geq 2$) [BS84, Mit84, BG88]
2. non-elementary Gromov hyperbolic groups [EF97];
3. groups acting properly discontinuously via isometries on Gromov hyperbolic spaces with limit set consisting of at least three points [Fuj98];
4. groups admitting a non-elementary *weakly proper discontinuous* action on a Gromov hyperbolic space [BF02] (see also [Ham08]);
5. groups admitting acylindrically via isometries on Gromov hyperbolic spaces with limit set consisting of at least three points [HO13]
   (we refer to [Osi13] for the definition of acylindrical action, and to [BBF13] for closely related results);
6. groups having infinitely many ends [Fuj00].

Conditions (1), (2), (3) and (4) define bigger and bigger classes of groups: the corresponding cited papers generalize one the results of the other. An important application of point (3) is that every subgroup of the mapping class group of a compact surface either is virtually abelian, or has infinite-dimensional second bounded cohomology. A nice geometric interpretation of (homogeneous) quasimorphisms is given in [Man05], where it is proved that, if $\Gamma$ admits a quasimorphism with is *bushy* according to his terminology, then $EH^2_b(\Gamma, \mathbb{R})$ is infinite-dimensional. However, contrary to what has been somewhat expected, there exist lattices $\Gamma$ in non-linear Lie groups for which $EH^2_b(\Gamma, \mathbb{R})$ is of finite non-zero dimension [MR06].
Amenability

The original definition of amenable group is due to von Neumann [vN29], who introduced the class of amenable groups while studying the Banach-Tarski paradox. As usual, we will restrict our attention to the definition of amenability in the context of discrete groups, referring the reader e.g. to [Pie84, Pat88] for a thorough account on amenability in the wider context of locally compact groups.

Let \( \Gamma \) be a group. We denote by \( \ell^\infty(\Gamma) = C^0_b(\Gamma, \mathbb{R}) \) the space of bounded real functions on \( \Gamma \), endowed with the usual structure of normed \( \mathbb{R}[\Gamma] \)-module (in fact, of Banach \( \Gamma \)-module). A mean on \( \Gamma \) is a map \( m: \ell^\infty(\Gamma) \to \mathbb{R} \) satisfying the following properties:

1. \( m \) is linear;
2. if \( 1_\Gamma \) denotes the map taking every element of \( \Gamma \) to \( 1 \in \mathbb{R} \), then \( m(1_\Gamma) = 1 \);
3. \( m(f) \geq 0 \) for every non-negative \( f \in \ell^\infty(\Gamma) \).

If conditions (1) and (2) are satisfied, then condition (3) is equivalent to

\[ (3') \inf_{g \in \Gamma} f(g) \leq m(f) \leq \sup_{g \in \Gamma} f(g) \]

for every \( f \in \ell^\infty(\Gamma) \).

In particular, the dual norm of a mean on \( \Gamma \), when considered as a functional on \( \ell^\infty(\Gamma) \), is equal to 1. A mean \( m \) is left invariant (or simply invariant) if it satisfies the following additional condition:

4. \( m(g \cdot f) = m(f) \) for every \( g \in \Gamma, f \in \ell^\infty(\Gamma) \).

**Definition 3.1.** A group \( \Gamma \) is amenable if it admits an invariant mean.

The following lemma describes some equivalent definitions of amenability that will prove useful in the sequel.

**Lemma 3.2.** Let \( \Gamma \) be a group. Then, the following conditions are equivalent:

1. \( \Gamma \) is amenable;
2. there exists a non-trivial left invariant continuous functional \( \varphi \in \ell^\infty(\Gamma)' \);
3. \( \Gamma \) admits a left invariant finitely additive probability measure.

**Proof.** If \( A \) is a subset of \( \Gamma \), we denote by \( H_{iA} \) the characteristic function of \( A \).

(1) \( \Rightarrow \) (2): If \( m \) is an invariant mean on \( \ell^\infty(\Gamma) \), then the map \( f \mapsto m(f) \) defines the desired functional on \( \ell^\infty(\Gamma) \).
(2) ⇒ (3): Let \( \varphi: \ell^\infty(\Gamma) \to \mathbb{R} \) be a non-trivial continuous functional. For every \( A \subseteq \Gamma \) we define a non-negative real number \( \mu(A) \) as follows. For every partition \( \mathcal{P} \) of \( A \) into a finite number of subsets \( A_1, \ldots, A_n \), we set
\[
\mu_\mathcal{P}(A) = |\varphi(Hi_{A_1})| + \ldots + |\varphi(Hi_{A_n})|.
\]
Observe that, if \( \varepsilon_i \) is the sign of \( \varphi(Hi_{A_i}) \), then
\[
\mu_\mathcal{P}(A) = \varphi(\sum_i \varepsilon_i Hi_{A_i}) \leq \|\varphi\|,
\]
so
\[
\mu(A) = \sup_\mathcal{P} \mu_\mathcal{P}(A)
\]
is a finite non-negative number for every \( A \subseteq \Gamma \). By the linearity of \( \varphi \), if \( \mathcal{P}' \) is a refinement of \( \mathcal{P} \), then \( \mu_\mathcal{P}(A) \leq \mu_{\mathcal{P}'}(A) \), and this easily implies that \( \mu \) is a finitely additive measure on \( \Gamma \). Recall now that the set of characteristic functions generates a dense subspace of \( \ell^\infty(\Gamma) \). As a consequence, since \( \varphi \) is non-trivial, there exists a subset \( A \subseteq \Gamma \) such that \( \mu(A) \geq |\varphi(Hi_A)| > 0 \). In particular, we have \( \mu(\Gamma) > 0 \), so after rescaling we may assume that \( \mu \) is a probability measure on \( \Gamma \). The fact that \( \mu \) is \( \Gamma \)-invariant is now a consequence of the \( \Gamma \)-invariance of \( \varphi \).

(3) ⇒ (1): Let \( \mu \) be a left invariant finitely additive measure on \( \Gamma \), and let \( \mathcal{Z} \subseteq \ell^\infty(\Gamma) \) be the subspace generated by the functions \( Hi_A, A \subseteq \Gamma \). Using the finite additivity of \( \mu \), it is not difficult to show that there exists a linear functional \( m_\mathcal{Z}: \mathcal{Z} \to \mathbb{R} \) such that \( m(Hi_A) = \mu(A) \) for every \( A \subseteq \Gamma \). Moreover, \( m_\mathcal{Z} \) has operator norm equal to one, so it is continuous. Since \( \mathcal{Z} \) is dense in \( \ell^\infty(\Gamma) \), the functional \( m_\mathcal{Z} \) uniquely extends to a continuous functional \( m \in \ell^\infty(\Gamma)' \). The fact that \( m \) is a mean is an immediate consequence of the fact that every non-negative function in \( \ell^\infty(\Gamma) \) may be approximated by linear combinations of characteristic functions with positive coefficients. Finally, the \( \Gamma \)-invariance of \( \mu \) implies that \( m_\mathcal{Z} \), whence \( m \), is also invariant.

\[\Box\]

The action of \( \Gamma \) on itself by right translations endows \( \ell^\infty(\Gamma) \) also with the structure of a right Banach \( \Gamma \)-module, and it is well-known that the existence of a left-invariant mean on \( \Gamma \) implies the existence of a bi-invariant mean on \( \Gamma \); in other words, \( \Gamma \) is amenable if and only if it admits a bi-invariant mean. However, we won’t use this fact in the sequel.

Finite groups are obviously amenable: an invariant mean \( m \) is obtained by setting \( m(f) = (1/|\Gamma|) \sum_{g \in \Gamma} f(g) \) for every \( f \in \ell^\infty(\Gamma) \). Of course, we are interested in finding less obvious examples of amenable groups.

1. Abelian groups are amenable

A key result in the theory of amenable groups is the following theorem, which is originally due to von Neumann:

**Theorem 3.3 ([vN29]).** Every abelian group is amenable.

**Proof.** We follow here the proof given in [Pat88], which is based on the Markov-Kakutani Fixed Point Theorem.
Let $\Gamma$ be an abelian group, and let $\ell^\infty(\Gamma)'$ be the topological dual of $\ell^\infty(\Gamma)$, endowed with the weak$^*$ topology. We consider the subset $K \subseteq \ell^\infty(\Gamma)'$ given by (not necessarily invariant) means on $\Gamma$, i.e. we set

$$K = \{ \varphi \in \ell^\infty(\Gamma)' \mid \varphi(1_\Gamma) = 1, \, \varphi(f) \geq 0 \text{ for every } f \geq 0 \}.$$

The set $K$ is non-empty (it contains the map $\varphi$ sending every element of $\ell^\infty(\Gamma)$ to the value it takes on the identity of $\Gamma$). Moreover, it is closed, convex, and contained in the closed ball of radius one in $\ell^\infty(\Gamma)'$. Therefore, $K$ is compact by the Banach-Alaouglu Theorem.

For every $g \in \Gamma$ let us now consider the map $L_g : \ell^\infty(\Gamma)' \to \ell^\infty(\Gamma)'$ defined by

$$L_g(\varphi)(f) = \varphi(g \cdot f).$$

We want to show that the $L_g$ admit a common fixed point in $K$. To this aim, one may apply the Markov-Kakutani Theorem cited above, which we prove here (in the case we are interested in) for completeness.

We first show that, if $H$ is any non-empty compact convex subset of $K$ which is left invariant by a single $L_g$, then $L_g$ admits a fixed point in $H$. Let us fix $\varphi \in H$. For every $n \in \mathbb{N}$ we set

$$\varphi_n = \frac{1}{n+1} \sum_{i=0}^{n} L_g^i(\varphi).$$

By convexity, $\varphi_n \in H$ for every $n$, so there exists a subsequence $\varphi_{n_i}$ tending to $\overline{\varphi} \in H$. Recall that $\|\varphi\| = \|L_g^{n_i+1}(\varphi)\| = 1$, so for every $f \in \ell^\infty(\Gamma)$ we have

$$|\varphi_{n_i}(g \cdot f) - \varphi_{n_i}(f)| = |(L_g(\varphi_{n_i}) - \varphi_{n_i})(f)| = \frac{|(L_g^{n_i+1}(\varphi) - \varphi)(f)|}{n_i+1} \leq \frac{2\|f\|}{n_i+1}.$$

By passing to the limit for $n_i$ tending to $\infty$, we obtain that $L_g(\overline{\varphi})(f) = \overline{\varphi}(g \cdot f) = \overline{\varphi}(f)$ for every $f \in \ell^\infty(\Gamma)$, so $L_g$ has a fixed point in $H$.

Let us now observe that $K$ is $L_g$-invariant for every $g \in \Gamma$. Therefore, the set $K_g$ of points of $K$ which are fixed by $L_g$ is non-empty. Moreover, it is easily seen that $K_g$ is closed (whence compact) and convex. Since $\Gamma$ is abelian, if $g_1, g_2$ are elements of $\Gamma$, then the maps $L_{g_1}$ and $L_{g_2}$ commute with each other, so $K_{g_1}$ is $L_{g_2}$-invariant. The argument above shows that $L_{g_2}$ has a fixed point in $K_{g_1}$, so $K_{g_1} \cap K_{g_2} \neq \emptyset$. One may iterate this argument to show that every finite intersection $K_{g_1} \cap \ldots \cap K_{g_n}$ is non-empty. In other words, the family $K_g, \ g \in \Gamma$ satisfies the finite intersection property, so $K_\Gamma = \bigcap_{g \in \Gamma} K_g$ is non-empty. But $K_\Gamma$ is precisely the set of invariant means on $\Gamma$.

2. Other amenable groups

Starting from abelian groups, it is possible to construct larger classes of amenable groups:

**Proposition 3.4.** Let $\Gamma, H$ be amenable groups. Then:
(1) Every subgroup of \( \Gamma \) is amenable.

(2) If the sequence
\[
1 \longrightarrow H \longrightarrow \Gamma' \longrightarrow \Gamma \longrightarrow 1
\]
is exact, then \( \Gamma' \) is amenable (in particular, the direct product of a finite number of amenable groups is amenable).

(3) Every direct union of amenable groups is amenable.

Proof. (1): Let \( K \) be a subgroup of \( \Gamma \). Let \( S \) be a set of representatives in \( \Gamma \) for the set \( K \backslash \Gamma \) of right lateral classes of \( K \) in \( \Gamma \). For every \( f \in \ell^\infty(K) \), we define \( \hat{f} \in \ell^\infty(\Gamma) \) by setting \( \hat{f}(g) = f(k) \), where \( g = ks, s \in S, k \in K \).

If \( m \) is an invariant mean on \( \Gamma \), then we set \( m_K(f) = m(\hat{f}) \). It is easy to check that \( m_K \) is an invariant mean on \( K \), so \( K \) is amenable.

(2): Let \( m_H, m_\Gamma \) be invariant means on \( H, \Gamma \) respectively. For every \( f \in \ell^\infty(\Gamma') \) we construct a map \( f_\Gamma \in \ell^\infty(\Gamma) \) as follows. For \( g \in \Gamma \), we take \( g' \) in the preimage of \( g \) in \( \Gamma' \), and we define \( f_{g'} \in \ell^\infty(H) \) by setting \( f_{g'}(h) = f(g'h) \). Then, we set \( f_\Gamma(g) = m_H(f_{g'}) \). Using that \( m_H \) is \( H \)-invariant one may show that \( f_\Gamma(g) \) does not depend on the choice of \( g' \), so \( f_\Gamma \) is well-defined. We obtain the desired mean on \( \Gamma' \) by setting \( m(f) = m_H(f_{\Gamma'}) \).

(3): Let \( \Gamma \) be the direct union of the amenable groups \( \Gamma_i, i \in I \). For every \( i \) we consider the set
\[
K_i = \{ \varphi \in \ell^\infty(\Gamma_i)' | \varphi(1_\Gamma) = 1, \varphi(f) \geq 0 \text{ for every } f \geq 0, \varphi \text{ is } \Gamma_i - \text{invariant} \}.
\]
If \( m_i \) is an invariant mean on \( \Gamma_i \), then the map \( f \mapsto m_i(f|_{\Gamma_i}) \) defines an element of \( K_i \). Therefore, each \( K_i \) is non-empty. Moreover, the fact that the union of the \( \Gamma_i \) is direct implies that the family \( K_i, i \in I \) satisfies the finite intersection property. Finally, each \( K_i \) is closed and compact in the weak* topology on \( \ell^\infty(\Gamma_i)' \), so the intersection \( K = \bigcap_{i \in I} K_i \) is non-empty. But \( K \) is precisely the set of \( \Gamma \)-invariants means on \( \Gamma \), whence the conclusion.

The class of elementary amenable groups is the smallest class of groups which contains all finite and all abelian groups, and is closed under the operations of taking subgroups, forming quotients, forming extensions, and taking direct unions [Day57]. For example, virtually solvable groups are elementary amenable. Proposition 3.4 (together with Theorem 3.3) shows that every elementary amenable group is indeed amenable. However, any group of intermediate growth is amenable [Pat88] but is not elementary amenable [Cho80]. In particular, there exist amenable groups which are not virtually solvable.

Remark 3.5. We have already noticed that, if \( \Gamma_1, \Gamma_2 \) are amenable groups, then so is \( \Gamma_1 \times \Gamma_2 \). More precisely, in the proof of Proposition 3.4-(2) we have shown how to construct an invariant mean \( m \) on \( \Gamma_1 \times \Gamma_2 \) starting from invariant means \( m_i \) on \( \Gamma_i, i = 1, 2 \). Under the identification of means with finitely additive probability measures, the mean \( m \) corresponds to the product measure \( m_1 \times m_2 \), so it makes sense to say that \( m \) is the product of \( m_1 \) and \( m_2 \). If \( \Gamma_1, \ldots, \Gamma_n \) are amenable groups with invariant means
3. Amenability and bounded cohomology

In this section we show that amenable groups are somewhat invisible to bounded cohomology with coefficients in normed \( \mathbb{R}[\Gamma] \)-modules (as already pointed out in Proposition 2.9, things are quite different in the case of \( \mathbb{Z}[\Gamma] \)-modules). We say that an \( \mathbb{R}[\Gamma] \)-module \( V \) is a dual normed \( \mathbb{R}[\Gamma] \)-module if \( V \) is isomorphic (as a normed \( \mathbb{R}[\Gamma] \)-module) to the topological dual of some normed \( \mathbb{R}[\Gamma] \)-module \( W \). In other words, \( V \) is the space of bounded linear functionals on the normed \( \mathbb{R}[\Gamma] \)-module \( W \), endowed with the action defined by \( g \cdot f(w) = f(g^{-1}w) \), \( g \in \Gamma \), \( w \in W \).

**Theorem 3.6.** Let \( \Gamma \) be an amenable group, and let \( V \) be a dual normed \( \mathbb{R}[\Gamma] \)-module. Then \( H^b_n(\Gamma, V) = 0 \) for every \( n \geq 1 \).

**Proof.** Recall that the bounded cohomology \( H^b_b(\Gamma, V) \) is defined as the cohomology of the \( \Gamma \)-invariants of the homogeneous complex

\[
0 \rightarrow C^0_b(\Gamma, V) \xrightarrow{\delta^0} C^1_b(\Gamma, V) \rightarrow \cdots \rightarrow C^n_b(\Gamma, V) \rightarrow \cdots
\]

Of course, the cohomology of the complex \( C^\bullet_b(\Gamma, V) \) of possibly non-invariant cochains vanishes in positive degree, since for \( n \geq 0 \) the maps

\[
k^{n+1} : C_b^{n+1}(\Gamma, V) \rightarrow C_b^n(\Gamma, V), \quad k^{n+1}(f)(g_0, \ldots, g_n) = f(1, g_0, \ldots, g_n)
\]

provide a (partial) homotopy between the identity and the zero map of \( C^\bullet_b(\Gamma, V) \) in positive degree. In order to prove the theorem, we are going to show that, under the assumption that \( \Gamma \) is amenable, a similar homotopy can be defined on the complex of invariant cochains. Roughly speaking, while \( k^\bullet \) is obtained by coning over the identity of \( \Gamma \), we will define a \( \Gamma \)-invariant homotopy \( j^\bullet \) by averaging the cone operator over all the elements of \( \Gamma \). Let us fix an invariant mean \( m \) on \( \Gamma \), and let \( f \) be an element of \( C_b^{n+1}(\Gamma, V) \). Recall that \( V = W' \) for some \( \mathbb{R}[\Gamma] \)-module \( W \), so \( f(g_0, \ldots, g_n) \) is a bounded functional on \( W \) for every \( (g, g_0, \ldots, g_n) \in \Gamma^{n+2} \). For every \( (g_0, \ldots, g_n) \in \Gamma^{n+1} \), \( w \in W \), we consider the function

\[
f_w : \Gamma \rightarrow \mathbb{R}, \quad f_w(g) = f(g, g_0, \ldots, g_n)(w).
\]

It follows from the definitions that \( f_w \) is an element of \( \ell^\infty(\Gamma) \), so we may set \( (j^{n+1}(f))(g_0, \ldots, g_n)(w) = m(f_w) \). It is immediate to check that this formula defines a continuous functional on \( W \), whose norm is bounded in terms of the norm of \( f \). In other words, \( j^{n+1}(f)(g_0, \ldots, g_n) \) is indeed an element of \( V \), and the map \( j^{n+1} : C_b^{n+1}(\Gamma, V) \rightarrow C_b^n(\Gamma, V) \) is bounded. The \( \Gamma \)-invariance of the mean \( m \) implies that \( j^{n+1} \) is \( \Gamma \)-equivariant. The collection...
of maps $j_{n+1}^n$, $n \in \mathbb{N}$ provides the required partial $\Gamma$-equivariant homotopy between the identity and the zero map of $C_b^n(\Gamma, V)$, $n \geq 1$.

One may wonder whether the assumption that $V$ is a dual normed $\mathbb{R}[\Gamma]$-module is really necessary in Theorem 3.6. It turns out that this is indeed the case: the vanishing of the bounded cohomology of $\Gamma$ with coefficients in any normed $\mathbb{R}[\Gamma]$-module is equivalent to the fact that $\Gamma$ is finite (see Theorem 3.12).

**Corollary 3.7.** Let $\Gamma$ be an amenable group. Then $H^n_b(\Gamma, \mathbb{R}) = 0$ for every $n \geq 1$.

Recall from Chapter 3 that the second bounded cohomology group with trivial real coefficients is strictly related to the space of real quasimorphisms. Putting together Corollary 3.7 with Proposition 2.4 and Corollary 2.7 we get the following result, which extends to amenable groups the characterization of real quasimorphisms on abelian groups given Section 5:

**Corollary 3.8.** Let $\Gamma$ be an amenable group. Then every real quasimorphism on $\Gamma$ is at bounded distance from a homomorphism. Equivalently, every homogeneous real quasimorphism on $\Gamma$ is a homomorphism.

From Corollary 2.14 and Corollary 3.7 we deduce that there exist many groups which are not amenable:

**Corollary 3.9.** Suppose that the group $\Gamma$ contains a non-abelian free subgroup. Then $\Gamma$ is not amenable.

**Proof.** We know from Corollary 2.14 that non-abelian free groups have non-trivial second bounded cohomology group with real coefficients, so they cannot be amenable. The conclusion follows from the fact that the subgroup of an amenable group is amenable.

It was a long-standing problem to understand whether the previous corollary could be sharpened into a characterization of amenable groups as those groups which do not contain any non-abelian free subgroup. This question is usually attributed to von Neumann, and appeared first in [Day57]. Ol’shanskii answered von Neumann’s question in the negative in [O’80]. The first examples of finitely presented groups which are not amenable but do not contain any non-abelian free group are due to Ol’shanskii and Sapir [OS02].

### 4. Johnson’s characterization of amenability

We have just seen that the bounded cohomology of any amenable group with values in any dual coefficient module vanishes. In fact, also the converse implication holds. More precisely, amenability of $\Gamma$ is ensured by the vanishing of a specific coclass in a specific bounded cohomology module (of degree one). We denote by $\ell^\infty(\Gamma)/\mathbb{R}$ the quotient of $\ell^\infty(\Gamma)$ by the (trivial) $\mathbb{R}[\Gamma]$-submodule of constant functions. Such a quotient is itself endowed
with the structure of a normed $\mathbb{R} [\Gamma]$-module. Note that the topological dual $(\ell^\infty(\Gamma)/\mathbb{R})'$ may be identified with the subspace of elements of $\ell^\infty(\Gamma)'$ that vanish on constant functions. For example, if $\delta g$ is the element of $\ell^\infty(\Gamma)'$ such that $\delta g(f) = f(g)$, then for every pair $(g_0, g_1) \in \Gamma^2$ the map $\delta g_0 - \delta g_1$ defines an element in $(\ell^\infty(\Gamma)/\mathbb{R})'$. For later purposes, we observe that the action of $\Gamma$ on $\ell^\infty(\Gamma)'$ is such that

$$g \cdot \delta g_0 (f) = \delta g_0 (g^{-1} \cdot f) = f(gg_0) = \delta g_0 (f).$$

Let us consider the element

$$J \in C^*_b(\Gamma, (\ell^\infty(\Gamma)/\mathbb{R})'), \quad J(g_0, g_1) = \delta g_1 - \delta g_0.$$ Of course we have $\delta J = 0$. Moreover, for every $g, g_0, g_1 \in \Gamma$ we have

$$(g \cdot J)(g_0, g_1) = g(J(g^{-1}g_0, g^{-1}g_1)) = g(\delta g^{-1}g_0 - \delta g^{-1}g_1) = \delta g_0 - \delta g_1 = J(g_0, g_1)$$ so $J$ is $\Gamma$-invariant, and defines an element $[J] \in H^1_b(\Gamma, (\ell^\infty(\Gamma)/\mathbb{R})')$, called the Johnson class of $\Gamma$.

**Theorem 3.10.** Suppose that the Johnson class of $\Gamma$ vanishes. Then $\Gamma$ is amenable.

**Proof.** By Lemma 3.2, it is sufficient to show that the topological dual $\ell^\infty(\Gamma)'$ contains a non-trivial $\Gamma$-invariant element.

We keep notation from the preceding paragraph. If $[J] = 0$, then $J = \delta \psi$ for some $\psi \in C^*_b(\Gamma, (\ell^\infty(\Gamma)/\mathbb{R})')^\Gamma$. For every $g \in \Gamma$ we denote by $\hat{\psi}(g) \in \ell^\infty(\Gamma)'$ the pullback of $\psi(g)$ via the projection map $\ell^\infty(\Gamma) \to \ell^\infty(\Gamma)/\mathbb{R}$. We consider the element $\varphi \in \ell^\infty(\Gamma)'$ defined by $\varphi = \delta_1 - \hat{\psi}(1)$. Since $\hat{\psi}(1)$ vanishes on constant functions, we have $\varphi(1) = 1$, so $\varphi$ is non-trivial, and we are left to show that $\varphi$ is $\Gamma$-invariant.

The $\Gamma$-invariance of $\psi$ implies the $\Gamma$-invariance of $\hat{\psi}$, so

$$\hat{\psi}(g) = (g \cdot \hat{\psi})(g) = g \cdot (\hat{\psi}(1)) \quad \text{for every } g \in \Gamma. \quad (2)$$

From $J = \delta \psi$ we deduce that

$$\delta g_1 - \delta g_0 = \hat{\psi}(g_1) - \hat{\psi}(g_0) \quad \text{for every } g_0, g_1 \in \Gamma. \quad (3)$$

In particular, we have

$$\delta g - \hat{\psi}(g) = \delta_1 - \hat{\psi}(1) \quad \text{for every } g \in \Gamma. \quad (3)$$

Therefore, using Equations (2), (3), for every $g \in \Gamma$ we get

$$g \cdot \varphi = g \cdot (\delta_1 - \hat{\psi}(1)) = (g \cdot \delta_1) - g \cdot (\hat{\psi}(1)) = \delta g - \hat{\psi}(g) = \delta_1 - \hat{\psi}(1) = \varphi,$$

and this concludes the proof. $\square$

Putting together Theorems 3.6 and 3.10 we get the following result, which characterizes amenability in terms of bounded cohomology:

**Corollary 3.11.** Let $\Gamma$ be a group. Then the following conditions are equivalent:

- $\Gamma$ is amenable,
• $H^1_b(\Gamma, V) = 0$ for every dual normed $\mathbb{R}[\Gamma]$-module $V$,
• $H^n_b(\Gamma, V) = 0$ for every dual normed $\mathbb{R}[\Gamma]$-module $V$ and every $n \geq 1$.

5. A characterization of finite groups via bounded cohomology

As already mentioned above, amenability is not sufficient to guarantee the vanishing of bounded cohomology with coefficients in any normed $\mathbb{R}[\Gamma]$-module. Let $\ell^1(\Gamma) = C^0_b(\Gamma)$ be the space of summable real functions on $\Gamma$, endowed with the usual structure of normed $\mathbb{R}[G]$-module. Let us denote by $e: \ell^1(\Gamma) \to \mathbb{R}$ the evaluation map such that $e(f) = \sum_{g \in \Gamma} f(g)$. Since $e$ is clearly $\Gamma$-invariant, the space $\ell^1_0(\Gamma) = \ker e$ is a normed $\mathbb{R}[G]$-space. What is more, since $\ell^1(\Gamma)$ is Banach and $e$ is continuous, $\ell^1_0(\Gamma)$ is itself a Banach $\Gamma$-module.

With an abuse, for $g \in \Gamma$ we denote by $\delta_g \in \ell^1(\Gamma)$ characteristic function of the singleton $\{g\}$. Let us now consider the element $K \in C^1_b(\Gamma, \ell^1_0(\Gamma))$, $K(g_0, g_1) = \delta_{g_0} - \delta_{g_1}$.

Of course we have $\delta K = 0$, and it is immediate to check that $g \cdot \delta_{g'} = \delta_{gg'}$ for every $g, g' \in \Gamma$, so

$$(g \cdot K)(g_0, g_1) = g(K(g_0^{-1}g_0, g_0^{-1}g_1)) = g(\delta_{g_0} - \delta_{g_1}) = \delta_{g_0} - \delta_{g_1} = K(g_0, g_1)$$

so $K$ defines a bounded coclass $[K] \in H^1_b(\Gamma, \ell^1_0(\Gamma))$, which we call the characteristic coclass of $\Gamma$.

We are now ready to prove that finite groups may be characterized by the vanishing of bounded cohomology with coefficients in any normed vector space.

**Theorem 3.12.** Let $\Gamma$ be a group. Then the following conditions are equivalent:

1. $\Gamma$ is finite;
2. $H^n_b(\Gamma, V) = 0$ for every normed $\mathbb{R}[\Gamma]$-module $V$ and every $n \geq 1$;
3. $H^n_b(\Gamma, V) = 0$ for every Banach $\Gamma$-module $V$ and every $n \geq 1$;
4. the characteristic coclass of $\Gamma$ vanishes.

**Proof.** (1) $\Rightarrow$ (2): Let $n \geq 1$. For every cochain $f \in C^n_b(\Gamma, V)$ we define a cochain $K^n(f) \in C^{n-1}_b(\Gamma, V)$ by setting

$$k^n(f)(g_1, \ldots, g_n) = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} f(g, g_1, \ldots, g_n),$$

where $|\Gamma|$ denotes the cardinality of $\Gamma$. The maps $k^n$, $n \geq 1$, provide a (partial) equivariant $\mathbb{R}[\Gamma]$-homotopy between the identity and the null map of $C^n_b(\Gamma, V)$, $n \geq 1$, whence the conclusion.
The implication $(3) \Rightarrow (4)$ is obvious, while $(3)$ implies $(4)$ since $\ell^1_0(\Gamma)$ is Banach, so we are left to show that the vanishing of the characteristic coclass implies that $\Gamma$ is finite.

So, suppose that $[K] = 0$, i.e. there exists a $\Gamma$-equivariant map $\psi: \Gamma \to \ell^1_0(\Gamma)$ such that

$$\delta_{g_1} - \delta_{g_0} = \psi(g_1) - \psi(g_0)$$

for every $g_0, g_1 \in \Gamma$.

In particular, we have

$$\delta_{g} - \psi(g) = \delta_1 - \psi(1)$$

for every $g \in \Gamma$.

Therefore, using Equations (4), and the $\Gamma$-equivariance of $\psi$ we get

$$g \cdot f = g \cdot (\delta_1 - \psi(1)) = (g \cdot \delta_1) - (g \cdot \psi(1)) = \delta_g - \psi(g) = \delta_1 - \psi(1) = f,$$

which implies that $f$ is constant. Since $f$ is not null and summable, this implies in turn that $G$ is finite. $\square$
Computing group cohomology by means of its very definition is usually very hard. The topological interpretation of group cohomology already provides a powerful tool for computations: for example, one may estimate the cohomological dimension of a group \( \Gamma \) (i.e. the maximal \( n \in \mathbb{N} \) such that \( H^n(\Gamma, R) \neq 0 \)) in terms of the dimension of a \( K(\Gamma, 1) \) as a CW-complex. We have already mentioned that a topological interpretation for the bounded cohomology of a group is still available, but in order to prove this fact more machinery has to be introduced. Before going into the bounded case, we describe some well-known results which hold in the classical case: namely, we will show that the cohomology of \( \Gamma \) may be computed by looking at several complexes of cochains. This crucial fact was already observed in the pioneering works on group cohomology of the 1940s and the 1950s. There are several ways to define group cohomology in terms of resolutions. We privilege here an approach that better extends to the case of bounded cohomology. We briefly compare our approach to more traditional ones in Section 3.

1. Relatively injectivity

We begin by introducing the notions of relatively injective \( R[\Gamma] \)-module and of strong \( \Gamma \)-resolution of an \( R[\Gamma] \)-module. The counterpart of these notions in the context of normed \( R[\Gamma] \)-modules will play an important role in the theory of bounded cohomology of groups. The importance of these notions is due to the fact that the cohomology of \( \Gamma \) may be computed by looking at any strong \( \Gamma \)-resolution of the coefficient module by relatively injective modules (see Corollary 4.5 below).

A \( \Gamma \)-map \( \iota: A \rightarrow B \) between \( R[\Gamma] \)-modules is strongly injective if there is an \( R \)-linear map \( \sigma: B \rightarrow A \) such that \( \sigma \circ \iota = \text{Id}_A \) (in particular, \( \iota \) is injective). We emphasize that, even if \( A \) and \( B \) are \( \Gamma \)-modules, the map \( \sigma \) is not required to be \( \Gamma \)-equivariant.

**Definition 4.1.** An \( R[\Gamma] \)-module \( U \) is relatively injective (over \( R[\Gamma] \)) if the following holds: whenever \( A, B \) are \( \Gamma \)-modules, \( \iota: A \rightarrow B \) is a strongly injective \( \Gamma \)-map and \( \alpha: A \rightarrow U \) is a \( \Gamma \)-map, there exists a \( \Gamma \)-map \( \beta: B \rightarrow U \) such that \( \beta \circ \iota = \alpha \).
4. (BOUNDED) GROUP COHOMOLOGY VIA RESOLUTIONS

\[
0 \rightarrow A \xleftarrow{\sigma} B \xrightarrow{\iota} C \xrightarrow{\alpha} D \rightarrow 0,
\]

In the case when \( R = \mathbb{R} \), a map is strongly injective if and only if it is injective, so in the context of \( \mathbb{R}[\Gamma] \)-modules the notions of relative injectivity and the traditional notion of injectivity coincide. However, relative injectivity is rather weaker than injectivity in general. For example, if \( R = \mathbb{Z} \) and \( \Gamma = \{1\} \), then \( C^0(\Gamma, R) \) is isomorphic to the trivial \( \Gamma \)-module \( \mathbb{Z} \), which of course is not injective over \( \mathbb{Z}[\Gamma] = \mathbb{Z} \). On the other hand, we have the following:

**Lemma 4.2.** For every \( R[\Gamma] \)-module \( V \) and every \( n \in \mathbb{N} \), the \( R[\Gamma] \)-module \( C^n(\Gamma, V) \) is relatively injective.

**Proof.** Let us consider the extension problem described in Definition 4.1, with \( U = C^n(\Gamma, V) \). Then we define \( \beta \) as follows:

\[
\beta(b)(g_0, \ldots, g_n) = \alpha(g_0\sigma(g_0^{-1}b))(g_0, \ldots, g_n).
\]

The fact that \( \beta \) is \( R \)-linear and that \( \beta \circ \iota = \alpha \) is straightforward, so we just need to show that \( \beta \) is \( \Gamma \)-equivariant. But for every \( g \in \Gamma \), \( b \in B \) and \( (g_0, \ldots, g_n) \in \Gamma^{n+1} \) we have

\[
g(\beta(b))(g_0, \ldots, g_n) = g(\beta(b)(g_0^{-1}g_0, \ldots, g_0^{-1}g_n)) = g(\alpha(g_0^{-1}g_0\sigma(g_0^{-1}gb))(g_0, \ldots, g_0^{-1}g_n)) = g((g_0^{-1}(\alpha(g_0\sigma(g_0^{-1}gb))))(g_0, \ldots, g_0^{-1}g_n)) = \alpha(g_0\sigma(g_0^{-1}gb))(g_0, \ldots, g_n) = \beta(gb)(g_0, \ldots, g_n).
\]

\[\square\]

2. Resolutions of \( \Gamma \)-modules

An \( R[\Gamma] \)-complex (or simply a \( \Gamma \)-complex or a complex) is a sequence of \( R[\Gamma] \)-modules \( E^i \) and \( \Gamma \)-maps \( \delta^i: E^i \rightarrow E^{i+1} \), \( i \in \mathbb{N} \), such that \( \delta^{i+1} \circ \delta^i = 0 \) for every \( i \):

\[
0 \rightarrow E^0 \xrightarrow{\delta^0} E^1 \xrightarrow{\delta^1} \ldots \xrightarrow{\delta^n} E^{n+1} \xrightarrow{\delta^{n+1}} \ldots
\]

Such a sequence will be denoted by \( (E^\bullet, \delta^\bullet) \). Moreover, we set \( Z^n(E^\bullet) = \ker \delta^n \cap (E^n)^\Gamma \), \( B^n(E^\bullet) = \delta^{n-1}(E^{n-1})^\Gamma \) (where again we understand that \( B^0(E^\bullet) = 0 \)), and we define the cohomology of the complex \( E^\bullet \) by setting

\[
H^n(E^\bullet) = Z^n(E^\bullet)/B^n(E^\bullet).
\]

A chain map between \( \Gamma \)-complexes \( (E^\bullet, \delta^E_\bullet) \) and \( (F^\bullet, \delta^F_\bullet) \) is a sequence of \( \Gamma \)-maps \( \{\alpha^i: E^i \rightarrow F^i \mid i \in \mathbb{N}\} \) such that \( \delta^F_\bullet \circ \alpha^i = \alpha^{i+1} \circ \delta^E_\bullet \) for every \( i \in \mathbb{N} \). If \( \alpha^\bullet, \beta^\bullet \) are chain maps between \( (E^\bullet, \delta^E_\bullet) \) and \( (F^\bullet, \delta^F_\bullet) \), a \( \Gamma \)-homotopy
between $\alpha^\bullet$ and $\beta^\bullet$ is a sequence of $\Gamma$-maps \( \{T^i: E_i \to F_{i-1}^i | i \geq 0 \} \) such that $T^1 \circ \delta_{E}^0 = \alpha^0 - \beta^0$ and $\delta_{E}^{-1} \circ T^i + T^{i+1} \circ \delta_{E}^i = \alpha^i - \beta^i$ for every $i \geq 1$. Every chain map induces a well-defined map in cohomology, and $\Gamma$-homotopic chain maps induce the same map in cohomology.

If $E$ is an $R[\Gamma]$-module, an augmented $\Gamma$-complex $(E, E^\bullet, \delta^\bullet)$ with augmentation map $\varepsilon: E \to E^0$ is a complex

\[
0 \to E \xrightarrow{\varepsilon} E^0 \xrightarrow{\delta^0} E^1 \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^n} E^{n+1} \xrightarrow{\delta^{n+1}} \cdots
\]

A resolution of $E$ (over $\Gamma$) is an exact augmented complex $(E, E^\bullet, \delta^\bullet)$ (over $\Gamma$). A resolution $(E, E^\bullet, \delta^\bullet)$ is relatively injective if $E^n$ is relatively injective for every $n \geq 0$. It is well-known that any map between modules extends to a chain map between injective resolutions of the modules. Unfortunately, the same result for relatively injective resolutions does not hold. The point is that relative injectivity guarantees the needed extension property only for strongly injective maps. Therefore, we need to introduce the notion of strong resolution.

A contracting homotopy for a resolution $(E, E^\bullet, \delta^\bullet)$ is a sequence of $R$-linear maps $k^i: E_i \to E_{i-1}$ such that $\delta_{E}^{i-1} \circ k^i + k^{i+1} \circ \delta^i = \Id_{E_i}$ if $i \geq 0$, and $k^0 \circ \varepsilon = \Id_{E^0}$:

\[
0 \xrightarrow{k^0} E \xrightarrow{\varepsilon} E^0 \xrightarrow{\delta^0} E^1 \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^n} E^n \xrightarrow{\delta^{n+1}} \cdots
\]

Note however that it is not required that $k^i$ be $\Gamma$-equivariant. A resolution is strong if it admits a contracting homotopy.

The following proposition shows that the chain complex $C^\bullet(\Gamma, V)$ provides a relatively injective strong resolution of $V$:

**Proposition 4.3.** Suppose that $R = \mathbb{R}$. Let $\varepsilon: V \to C^0(\Gamma, V)$ be defined by $\varepsilon(v)(g) = v$ for every $v \in V$, $g \in \Gamma$. Then the augmented complex

\[
0 \to V \xrightarrow{\varepsilon} C^0(\Gamma, V) \xrightarrow{\delta^0} C^1(\Gamma, V) \to \cdots \to C^n(\Gamma, V) \to \cdots
\]

provides a relatively injective strong resolution of $V$.

**Proof.** We already know that each $C^i(\Gamma, V)$ is relatively injective. In order to show that the augmented complex $(V, C^\bullet(\Gamma, V), \delta^\bullet)$ is a strong resolution it is sufficient to observe that the maps

\[
k^{n+1}: C^{n+1}(\Gamma, V) \to C^n(\Gamma, V) \quad k^{n+1}(f)(g_0, \ldots, g_n) = f(1, g_0, \ldots, g_n)
\]

provide the required contracting homotopy. \hfill \Box

The resolution of $V$ described in the previous proposition is obtained just by augmenting the homogeneous complex associated to $(\Gamma, V)$, and it is usually called the standard resolution of $V$ (over $\Gamma$).

The following result may be proved by means of standard homological algebra arguments (see e.g. [Iva87], [Mon01, Lemmas 7.2.4 and 7.2.6] for a detailed proof in the bounded case – the argument there extends to
this context just by forgetting any reference to the norms). It implies that any relatively injective strong resolution of a $\Gamma$-module $V$ may be used to compute the cohomology modules $H^\bullet(\Gamma, V)$ (see Corollary 4.5).

**Theorem 4.4.** Let $\alpha : E \to F$ be a $\Gamma$-map between $R[\Gamma]$-modules, let $(E, E^\bullet, \delta^E)$ be a strong resolution of $E$, and suppose that $(F, F^\bullet, \delta^F)$ is an augmented complex such that $F^i$ is relatively injective for every $i \geq 0$. Then $\alpha$ extends to a chain map $\alpha^\bullet$, and any two extensions of $\alpha$ to chain maps are $\Gamma$-homotopic.

**Corollary 4.5.** Let $(V, V^\bullet, \delta^V)$ be a relatively injective strong resolution of $V$. Then for every $n \in \mathbb{N}$ there is a canonical isomorphism

$$H^n(\Gamma, V) \cong H^n(V^\bullet).$$

**Proof.** By Proposition 4.3, both $(V, V^\bullet, \delta^V)$ and the standard resolution of $V$ are relatively injective strong resolutions of $V$ over $\Gamma$. Therefore, Theorem 4.4 provides chain maps between $C^\bullet(\Gamma, V)$ and $V^\bullet$, which are one the $\Gamma$-homotopy inverse of the other. Therefore, these chain maps induce isomorphisms in cohomology. \□

3. The classical approach to group cohomology via resolutions

In this section we describe the relationship between the description of group cohomology via resolutions given above and more traditional approaches to the subject (see e.g. [Bro82]). The reader who is not interested can safely skip the section, since the results cited below will not be used elsewhere in this monograph.

If $V, W$ are $\mathbb{Z}[\Gamma]$-modules, then the space $\text{Hom}_{\mathbb{Z}}(V, W)$ is endowed with the structure of a $\mathbb{Z}[\Gamma]$-module by setting $(g \cdot f)(v) = g(f(g^{-1}v))$ for every $g \in \Gamma$, $f \in \text{Hom}_{\mathbb{Z}}(V, W)$, $v \in V$. Then, the cohomology $H^\bullet(\Gamma, V)$ is often defined in the following equivalent ways (we refer e.g. to [Bro82] for the definition of projective module (over a ring $R$); for our discussion, it is sufficient to know that any free $R$-module is projective over $R$, so any free resolution is projective):

1. (Via injective resolutions of $V$): Let $(V, V^\bullet, \delta^V)$ be an injective resolution of $V$ over $\mathbb{Z}[\Gamma]$, and take the complex $W^\bullet = \text{Hom}_{\mathbb{Z}}(V^\bullet, \mathbb{Z})$, endowed with the action $g : f(v) = f(g^{-1}v)$. Then $(W^\bullet)^{\Gamma} = \text{Hom}_{\mathbb{Z}[\Gamma]}(V^\bullet, \mathbb{Z})$ (where $\mathbb{Z}$ is endowed with the structure of trivial $\Gamma$-module), and one may define $H^\bullet(\Gamma, V)$ as the homology of the $\Gamma$-invariants of $W^\bullet$.

2. (Via projective resolutions of $\mathbb{Z}$): Let $(\mathbb{Z}, P_\bullet, d_\bullet)$ be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}[\Gamma]$, and take the complex $Z^\bullet = \text{Hom}_{\mathbb{Z}}(P_\bullet, V)$. We have again that $(Z^\bullet)^{\Gamma} = \text{Hom}_{\mathbb{Z}[\Gamma]}(P_\bullet, V)$, and again we may define $H^\bullet(\Gamma, V)$ as the homology of the $\Gamma$-invariants of the complex $Z^\bullet$.

The fact that these two definitions are indeed equivalent is proved e.g. in [Bro82].
We have already observed that, if $V$ is a $\mathbb{Z}[\Gamma]$-module, the module $C^n(\Gamma, V)$ is not injective in general. However, this is not really a problem, since the complex $C^\bullet(\Gamma, V)$ may be recovered from a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}[\Gamma]$. Namely, let $C_n(\Gamma, \mathbb{Z})$ be the free $\mathbb{Z}$-module admitting the set $\Gamma^{n+1}$ as a basis. The diagonal action of $\Gamma$ onto $\Gamma^{n+1}$ endows $C_n(\Gamma, \mathbb{Z})$ with the structure of a $\mathbb{Z}[\Gamma]$-module. The modules $C_n(\Gamma, \mathbb{Z})$ may be arranged into a resolution

$$0 \leftarrow \mathbb{Z} \leftarrow C_0(\Gamma, \mathbb{Z}) \leftarrow C_1(\Gamma, \mathbb{Z}) \leftarrow \ldots \leftarrow C_n(\Gamma, \mathbb{Z}) \leftarrow C_{n+1} \leftarrow \ldots$$

of the trivial $\mathbb{Z}[\Gamma]$-module $\mathbb{Z}$ over $\mathbb{Z}[\Gamma]$ (the homology of the $\Gamma$-coinvariants of this resolution is by definition the homology of $\Gamma$). Now, it is easy to check that $C_n(\Gamma, \mathbb{Z})$ is free, whence projective, as a $\mathbb{Z}[\Gamma]$-module. Moreover, the module $\text{Hom}_\mathbb{Z}(C_n(\Gamma, \mathbb{Z}), V)$ is $\mathbb{Z}[\Gamma]$-isomorphic to $C_n(\Gamma, V)$. This shows that, in the context of the traditional definition of group cohomology via resolutions, the complex $C^\bullet(\Gamma, V)$ arises from a projective resolution of $\mathbb{Z}$, rather than from an injective resolution of $V$.

4. The topological interpretation of group cohomology revisited

Let us show how Corollary 4.5 may be exploited to prove that the cohomology of $\Gamma$ is isomorphic to the singular cohomology of any path-connected topological space $X$ satisfying conditions (1), (2) and (3) described in Section 3. We fix an identification between $\Gamma$ and the group of the covering automorphisms of the universal covering $\tilde{X}$ of $X$. The action of $\Gamma$ on $\tilde{X}$ induces an action of $\Gamma$ on $C^\bullet(\tilde{X}, R)$, whence an action of $\Gamma$ on $C^\bullet(\tilde{X}, R)$, which is defined by $(g \cdot \varphi)(c) = \varphi(g^{-1}c)$ for every $c \in C^\bullet(\tilde{X}, R)$, $\varphi \in C^\bullet(\tilde{X}, R)$, $g \in \Gamma$. Therefore, for $n \in \mathbb{N}$ both $C_n(\tilde{X}, R)$ and $C^n(\tilde{X}, R)$ are endowed with the structure of $R[\Gamma]$-modules. We have a natural identification $C^\bullet(\tilde{X}, R)^\Gamma \cong C^\bullet(X, R)$.

**Lemma 4.6.** For every $n \in \mathbb{N}$, the singular cochain module $C^n(\tilde{X}, R)$ is relatively injective.

**Proof.** For every topological space $Y$, let us denote by $S_n(Y)$ the set of singular simplices with values in $Y$.

We denote by $L_n(\tilde{X})$ a set of representatives for the action of $\Gamma$ on $S_n(\tilde{X})$ (for example, if $F$ is a set of representatives for the action of $\Gamma$ on $\tilde{X}$, we may define $L_n(\tilde{X})$ as the set of singular $n$-simplices whose first vertex lies in $F$). Then, for every $n$-simplex $s \in S_n(\tilde{X})$, there exist a unique $g_s \in \Gamma$, and a unique $\tilde{s} \in L_n(\tilde{X})$ such that $g_s \cdot \tilde{s} = s$. Let us now consider the extension problem:

$$
\begin{array}{ccc}
0 & \leftarrow & A \\
& \downarrow \alpha & \leftarrow B \\
& \sigma & \leftarrow & \beta \\
C^n(\tilde{X}, R) & \leftarrow &
\end{array}
$$
We define the desired extension $\beta$ by setting
\[
\beta(b)(s) = \alpha(g_s \sigma(g_s^{-1} \cdot b))(s) = \alpha(\sigma(g_s^{-1} \cdot b))(\bar{s})
\]
for every $s \in S_n(\bar{X})$. It is easy to verify that the map $\beta$ is an $R[\Gamma]$-map, and that $\alpha = \beta \circ \iota$.

An alternative proof (providing the same solution to the extension problem) is the following. Using that the action of $\Gamma$ on $\bar{X}$ is free, it is easy to show that $L_n(\bar{X})$, when considered as a subset of $C_n(\bar{X}, R)$, is a free basis of $C_n(\bar{X}, R)$ over $R[\Gamma]$. In other words, every $c \in C_n(\bar{X}, R)$ may be expressed uniquely as a sum of the form $c = \sum_{i=1}^{k} a_i g_i s_i$, $a_i \in R$, $g_i \in \Gamma$, $s_i \in L_n(\bar{X})$. Therefore, the map
\[
\psi: C^0(\Gamma, C^0(L_n(\bar{X}), R)) \to C^n(\bar{X}, R), \quad \psi(f) \left( \sum_{i=1}^{k} a_i g_i s_i \right) = \sum_{i=1}^{k} a_i f(g_i)(s_i)
\]
is well-defined. If we endow $C^0(L_n(\bar{X}), R)$ with the structure of trivial $\Gamma$-module, then a straightforward computation shows that $\psi$ is in fact a $\Gamma$-isomorphism, so the conclusion follows from Lemma 4.2. \hfill \Box

**Proposition 4.7.** Let $\varepsilon: R \to C^0(\bar{X}, R)$ be defined by $\varepsilon(t)(s) = t$ for every singular 0-simplex $s$ in $\bar{X}$. Suppose that $H^i(\bar{X}, R) = 0$ for every $i \geq 1$. Then the augmented complex
\[
0 \to R \xrightarrow{\varepsilon} C^0(\bar{X}, R) \xrightarrow{\delta^1} C^1(\bar{X}, R) \to \cdots \xrightarrow{\delta^{n-1}} C^n(\bar{X}, R) \xrightarrow{\delta^n} C^{n+1}(\bar{X}, R)
\]
is a relatively injective strong resolution of the trivial $R[\Gamma]$-module $R$.

**Proof.** Since $\bar{X}$ is path-connected, we have that $\text{Im} \varepsilon = \ker \delta^0$. Observe now that $C_n(\bar{X}, R)$ is $R$-free for every $n \in \mathbb{N}$. As a consequence, the obvious augmented complex associated to $C_\bullet(\bar{X}, R)$, being acyclic, is homotopically trivial over $R$. Since $C^n(\bar{X}, R) \cong \text{Hom}_R(C_n(\bar{X}, R), R)$, we may conclude that the augmented complex described in the statement is a strong resolution of $R$ over $R[\Gamma]$. Then the conclusion follows from Lemma 4.6. \hfill \Box

Putting together Propositions 4.3, 4.7 and Corollary 4.5 we may provide the following topological characterization of $H^n(\Gamma, R)$:

**Corollary 4.8.** Let $X$ be a path-connected space admitting a universal covering $\bar{X}$, and suppose that $H_i(\bar{X}, R) = 0$ for every $i \geq 1$. Then $H^i(X, R)$ is canonically isomorphic to $H^i(\pi_1(X), R)$ for every $i \in \mathbb{N}$.

5. Bounded cohomology via resolutions

Just as in the case of classical cohomology, it is often useful to have alternative ways to compute the bounded cohomology of a group. This section is devoted to an approach to bounded cohomology which closely follows the traditional approach to classical cohomology via homological algebra. The circle of ideas we are going to describe first appeared (in the case with trivial
real coefficients) in a paper by Brooks [Bro81], where it was exploited to give an independent proof of Gromov’s result that the isomorphism type of the bounded cohomology of a space (with real coefficients) only depends on its fundamental group [Gro82]. Brooks’ theory was then developed by Ivanov in his foundational paper [Iva87] (see also [Nos91] for the case of coefficients in general normed $\Gamma$-modules). Ivanov gave a new proof of the vanishing of the bounded cohomology (with real coefficients) of a simply connected space (this result played an important role in Brooks’ argument, and was originally due to Gromov [Gro82]), and managed to incorporate the seminorm into the homological algebra approach to bounded cohomology, thus proving that the bounded cohomology of a space is isometrically isomorphic to the bounded cohomology of its fundamental group (in the case with real coefficients). Ivanov-Noskov’s theory was further developed by Burger and Monod [BM99, BM02, Mon01], who paid a particular attention to the continuous bounded cohomology of topological groups.

Both Ivanov-Noskov’s and Monod’s theory are concerned with Banach $\Gamma$-modules, which are in particular $\mathbb{R}[\Gamma]$-modules. For the moment, we prefer to consider also the (quite different) case with integral coefficients. Therefore, we let $R$ be either $\mathbb{Z}$ or $\mathbb{R}$, and we concentrate our attention on the category of normed $R[\Gamma]$-modules introduced in Section 4. In the next sections we will see that relative injective modules and strong resolutions may be defined in this context just by adapting to normed $R[\Gamma]$-modules the analogous definitions for generic $R[\Gamma]$-modules.

6. Relatively injective normed $\Gamma$-modules

Throughout the whole section, unless otherwise stated, we will deal only with normed $R[\Gamma]$-modules. Therefore, $\Gamma$-morphisms will be always assumed to be bounded.

The following definitions are taken from [Iva87] (where only the case when $R = \mathbb{R}$ and $V$ is Banach is dealt with). A bounded linear map $\iota: A \to B$ of normed $R$-modules is strongly injective if there is an $R$-linear map $\sigma: B \to A$ with $\|\sigma\| \leq 1$ and $\sigma \circ \iota = \text{Id}_A$ (in particular, $\iota$ is injective). We emphasize that, even when $A$ and $B$ are $R[\Gamma]$-modules, the map $\sigma$ is not required to be $\Gamma$-equivariant.

**Definition 4.9.** A normed $R[\Gamma]$-module $E$ is relatively injective if for every strongly injective $\Gamma$-morphism $\iota: A \to B$ of normed $R[\Gamma]$-modules and every $\Gamma$-morphism $\alpha: A \to E$ there is a $\Gamma$-morphism $\beta: B \to E$ satisfying
$\beta \circ \iota = \alpha$ and $\|\beta\| \leq \|\alpha\|$.

\[
\begin{array}{c}
0 \longrightarrow A \xrightarrow{\sigma} B \\
\downarrow \alpha \quad \downarrow \beta \\
E \end{array}
\]

**Remark 4.10.** Let $E$ be a normed $R[\Gamma]$-module, and let $\hat{E}$ be the underlying $R[\Gamma]$-module. Then no obvious implication exists between the fact that $E$ is relatively injective (in the category of normed $R[\Gamma]$-modules, i.e. according to Definition 4.9), and the fact that $\hat{E}$ is (in the category of $R[\Gamma]$-modules, i.e. according to Definition 4.1). This could suggest that the use of the same name for these different notions could indeed be an abuse. However, unless otherwise stated, henceforth we will deal with relatively injective modules only in the context of normed $R[\Gamma]$-modules, so the reader may safely take Definition 4.9 as the only definition of relative injectivity.

The following result is due to Ivanov [Iva87] in the case of real coefficients, and to Monod [Mon01] in the general case (see also Remark 4.13), and shows that the modules involved in the definition of bounded cohomology are relatively injective.

**Lemma 4.11.** Let $V$ be a normed $R[\Gamma]$-module. Then the normed $R[\Gamma]$-module $C^0_b(\Gamma, V)$ is relatively injective.

**Proof.** Let us consider the extension problem described in Definition 4.9, with $E = C^0_b(\Gamma, V)$. Then we define $\beta$ as follows:

$\beta(b)(g_0, \ldots, g_n) = \alpha(g_0 \sigma(g_0^{-1} b))(g_0, \ldots, g_n)$.

It is immediate to check that $\beta \circ \iota = \alpha$. Moreover, since $\|\sigma\| \leq 1$, we have $\|\beta\| \leq \|\alpha\|$. Finally, the fact that $\beta$ commutes with the actions of $\Gamma$ may be proved by the very same computation given in the proof of Lemma 4.2. □

### 7. Resolutions of normed $\Gamma$-modules

A normed $R[\Gamma]$-complex is an $R[\Gamma]$-complex whose modules are normed $R[\Gamma]$-spaces, and whose differential is a bounded $R[\Gamma]$-map in every degree. A chain map between $(E^\bullet, \delta_E^\bullet)$ and $(F^\bullet, \delta_F^\bullet)$ is a chain map between the underlying $R[\Gamma]$-complexes which is bounded in every degree, and a $\Gamma$-homotopy between two such chain maps is just a $\Gamma$-homotopy between the underlying maps of $R[\Gamma]$-modules, which is bounded in every degree. The cohomology $H^*_b(E^\bullet)$ of the normed $\Gamma$-complex $(E^\bullet, \delta_E^\bullet)$ is defined as usual by taking the cohomology of the subcomplex of $\Gamma$-invariants. The norm on $E^n$ restricts to a norm on $\Gamma$-invariant cocycles, which induces in turn a seminorm on $H^*_b(E^\bullet)$ for every $n \in \mathbb{N}$.

An augmented normed $\Gamma$-complex $(E, E^\bullet, \delta^\bullet)$ with augmentation map $\varepsilon: E \to E^0$ is a $\Gamma$-complex

$0 \longrightarrow E \xrightarrow{\varepsilon} E^0 \xrightarrow{\delta^0} E^1 \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^n} E^{n+1} \xrightarrow{\delta^{n+1}} \cdots$
We also ask that $\varepsilon$ is an isometric embedding. A resolution of $E$ (as a normed $R[\Gamma]$-complex) is an exact augmented normed complex $(E,E^*,\delta^*)$. It is \textit{relatively injective} if $E^n$ is relatively injective for every $n \geq 0$. From now on, we will call simply \textit{complex} any normed complex.

Let $(E,E^*,\delta^*_E)$ be an augmented complex, and suppose that $(F,F^*,\delta^*_F)$ is a relatively injective resolution of $F$. We would like to be able to extend any $\Gamma$-map $E \to F$ to a chain map between $E^*$ and $F^*$. As observed in the preceding section, to this aim we need to require the resolution $(E,E^*,\delta^*_E)$ to be strong, according to the following definition.

A \textit{contracting homotopy} for a resolution $(E,E^*,\delta^*)$ is a sequence of linear maps $k^i: E^i \to E^{i-1}$ such that $\|k^i\| \leq 1$ for every $i \in \mathbb{N}$, $\delta^{i-1} \circ k^i + k^{i+1} \circ \delta^i = \text{Id}_{E^i}$ if $i \geq 0$, and $k^0 \circ \varepsilon = \text{Id}_{E^0}$:

$$0 \longrightarrow E \underset{\varepsilon}{\longleftarrow} E^0 \overset{k^1}{\longrightarrow} E^1 \overset{\delta^1}{\longrightarrow} \cdots \overset{k^n}{\longrightarrow} E^n \overset{\delta^n}{\longrightarrow} \cdots$$

Note however that it is not required that $k^i$ be $\Gamma$-equivariant. A resolution is \textit{strong} if it admits a contracting homotopy.

**Proposition 4.12.** Let $V$ be a normed $R[\Gamma]$-space, and let $\varepsilon: V \to C^0_b(\Gamma,V)$ be defined by $\varepsilon(v)(g) = v$ for every $v \in V$, $g \in \Gamma$. Then the augmented complex

$$0 \longrightarrow V \overset{\varepsilon}{\longrightarrow} C^0_b(\Gamma,V) \overset{\delta^0}{\longrightarrow} C^1_b(\Gamma,V) \to \cdots \longrightarrow C^n_b(\Gamma,V) \to \cdots$$

provides a relatively injective strong resolution of $V$.

**Proof.** We already know that each $C^i_b(\Gamma,V)$ is relatively injective, so in order to conclude it is sufficient to observe that the map

$$k^{n+1}: C^{n+1}_b(\Gamma,V) \to C^n_b(\Gamma,V) \quad k^{n+1}(f)(g_0,\ldots,g_n) = f(1,g_0,\ldots,g_n)$$

provides a contracting homotopy for the resolution $(V,C^*_b(\Gamma,V),\delta^*)$. \hfill $\square$

The resolution described in Proposition 4.12 is the \textit{standard resolution} of $V$ as a normed $R[\Gamma]$-module.

**Remark 4.13.** Let us briefly compare our notion of standard resolution with Ivanov’s and Monod’s ones. In [Iva87], for every $n \in \mathbb{N}$ the set $C^n_b(\Gamma,\mathbb{R})$ is denoted by $B(\Gamma^{n+1})$, and it is endowed with the structure of a right Banach $\Gamma$-module by the action $g \cdot f(g_0,\ldots,g_n) = f(g_0,\ldots,g_n \cdot g)$. Moreover, the sequence of modules $B(\Gamma^n)$, $n \in \mathbb{N}$, is equipped with a structure of $\Gamma$-complex

$$0 \longrightarrow \mathbb{R} \overset{d_{-1}}{\longrightarrow} B(\Gamma) \overset{d_0}{\longrightarrow} B(\Gamma^2) \overset{d_1}{\longrightarrow} \cdots \overset{d_n}{\longrightarrow} B(\Gamma^{n+2}) \overset{d_{n+1}}{\longrightarrow} \cdots,$$

where $d_{-1}(t)(g) = t$ and

$$d_n(f)(g_0,\ldots,g_{n+1}) = (-1)^{n+1}f(g_1,\ldots,g_{n+1})$$

$$+ \sum_{i=0}^{n}(-1)^{n-i}f(g_0,\ldots,g_ig_{i+1},\ldots,g_{n+1})$$
for every $n \geq 0$ (here we are using Ivanov’s notation also for the differential). Now, it is readily seen that Ivanov’s resolution is isomorphic to our standard resolution via the isometric $\Gamma$-chain isomorphism $\varphi^\cdot: B^\cdot(\Gamma) \to B(\Gamma^{n+1})$ defined by

$$\varphi^n(f)(g_0, \ldots, g_n) = f(g_{n-1}^{-1}g_{n-1}^{-1}, \ldots, g_n^{-1}g_{n-1}^{-1}g_1^{-1}g_0^{-1})$$

(with inverse $(\varphi^n)^{-1}(f)(g_0, \ldots, g_n) = f(g_n^{-1}g_{n-1}^{-1}g_{n-2}, \ldots, g_1^{-1}g_0^{-1})$).

We also observe that the contracting homotopy described in Proposition 4.12 is conjugated by $\varphi^\cdot$ into Ivanov’s contracting homotopy for the complex $(B(\Gamma^\cdot), d^\cdot)$ (which is defined in [Iva87]).

Our notation is much closer to Monod’s one. In fact, in [Mon01] the more general case of a topological group $\Gamma$ is addressed, and the $n$-th module of the standard $\Gamma$-resolution of $\mathbb{R}$ is inductively defined by setting

$$C^0_b(\Gamma, \mathbb{R}) = C_b(\Gamma, \mathbb{R}), \quad C^n_b(\Gamma, \mathbb{R}) = C_b(\Gamma, C^{n-1}_b(\Gamma, \mathbb{R})),$$

where $C_b(\Gamma, E)$ denotes the space of continuous bounded maps from $\Gamma$ to the Banach space $E$.

However, as observed in [Mon01, Remarks 6.1.2 and 6.1.3], the case when $\Gamma$ is an abstract group may be recovered from the general case just by equipping $\Gamma$ with the discrete topology. In that case, our notion of standard resolution coincides with Monod’s one (see also [Mon01, Remark 7.4.9]).

The following result can be proved by means of standard homological algebra arguments (see [Iva87], [Mon01, Lemmas 7.2.4 and 7.2.6] for full details):

**Theorem 4.14.** Let $\alpha: E \to F$ be a $\Gamma$-map between normed $R[\Gamma]$-modules, let $(E, E^\cdot, \delta_E^\cdot)$ be a strong resolution of $E$, and suppose $(F, F^\cdot, \delta_F^\cdot)$ is an augmented complex such that $F^i$ is relatively injective for every $i \geq 0$. Then $\alpha$ extends to a chain map $\alpha^\cdot$, and any two extensions of $\alpha$ to chain maps are $\Gamma$-homotopic.

**Corollary 4.15.** Let $V$ be a normed $R[\Gamma]$-module, and let $(V, V^\cdot, \delta_V^\cdot)$ be a relatively injective strong resolution of $V$. Then for every $n \in \mathbb{N}$ there is a canonical isomorphism

$$H^n_b(\Gamma, V) \cong H^n_b(\Gamma^\cdot) .$$

Moreover, this isomorphism is bi-Lipschitz with respect to the seminorms of $H^n_b(\Gamma, V)$ and $H^n_b(\Gamma^\cdot)$.

**Proof.** By Proposition 4.12, the standard resolution of $V$ is also a relatively injective strong resolution of $V$ over $\Gamma$. Therefore, Theorem 4.4 provides chain maps between $C^\cdot_b(\Gamma, V)$ and $V^\cdot$, which are one the $\Gamma$-homotopy inverse of the other. Therefore, these chain maps induce isomorphisms in cohomology. The conclusion follows from the fact that bounded chain maps induce bounded maps in cohomology. □
By Corollary 4.15, every relatively injective strong resolution of \( V \) induces a seminorm on \( H^\bullet_b(\Gamma, V) \). Moreover, the seminorms defined in this way are pairwise equivalent. However, in many applications, it is important to be able to compute the exact \textit{canonical} seminorm of elements in \( H^\bullet_b(\Gamma, V) \), i.e. the seminorm induced on \( H^\bullet_b(\Gamma, V) \) by the standard resolution \( C^\bullet_b(\Gamma, V) \). Unfortunately, it is not possible to capture the isometry type of \( H^\bullet_b(\Gamma, V) \) via arbitrary relatively injective strong resolutions. Therefore, a special role is played by those resolutions which compute the canonical seminorm. The following fundamental result is due to Ivanov, and implies that these distinguished resolutions are in some sense extremal:

**Theorem 4.16.** Let \( V \) be a normed \( R[\Gamma] \)-module, and let \((V, V^\bullet, \delta^\bullet)\) be any strong resolution of \( V \). Then the identity of \( V \) can be extended to a chain map \( \alpha^\bullet \) between \( V^\bullet \) and the standard resolution of \( V \), in such a way that \( \| \alpha^n \| \leq 1 \) for every \( n \geq 0 \). In particular, the canonical seminorm of \( H^\bullet_b(\Gamma, V) \) is not bigger than the seminorm induced on \( H^\bullet_b(\Gamma, V) \) by any relatively injective strong resolution.

**Proof.** One can define \( \alpha^n \) by induction setting, for every \( v \in E^n \) and \( g_j \in \Gamma \):

\[
\alpha^n(v)(g_0, \ldots, g_n) = \alpha^{n-1}(g_0\left(k^n(g_0^{-1}(v))\right))(g_1, \ldots, g_n),
\]

where \( k^\bullet \) is a contracting homotopy for the given resolution \((V, V^\bullet, \delta^\bullet)\). It is not difficult to prove by induction that \( \alpha^\bullet \) is indeed a norm non-increasing chain \( \Gamma \)-map (see [Iva87], [Mon01, Theorem 7.3.1] for the details).

**Corollary 4.17.** Let \( V \) be a normed \( R[\Gamma] \)-module, let \((V, V^\bullet, \delta^\bullet)\) be a relatively injective strong resolution of \( V \), and suppose that the identity of \( V \) may be extended to a chain map \( \alpha^\bullet \): \( C^\bullet_b(\Gamma, V) \rightarrow V^\bullet \) such that \( \| \alpha^n \| \leq 1 \) for every \( n \in \mathbb{N} \). Then \( \alpha^\bullet \) induces an isometric isomorphism between \( H^\bullet_b(\Gamma, V) \) and \( H^\bullet_b(V^\bullet) \). In particular, the seminorm induced by the resolution \((V, V^\bullet, \delta^\bullet)\) coincides with the canonical seminorm on \( H^\bullet_b(\Gamma, V) \).

8. More on amenability

The following result establishes an interesting relationship between the amenability of \( \Gamma \) and the relative injectivity of normed \( R[\Gamma] \)-modules.

**Proposition 4.18.** The following facts are equivalent:

1. The group \( \Gamma \) is amenable.
2. Every dual normed \( R[\Gamma] \)-module is relatively injective.
3. The trivial \( R[\Gamma] \)-module \( R \) is relatively injective.

**Proof.** (1) \( \Rightarrow \) (2): Let \( W \) be a normed \( R[\Gamma] \)-module, and let \( V = W' \) be the dual normed \( R[\Gamma] \)-module of \( W \). We first construct a left inverse (over \( R[\Gamma] \)) of the augmentation map \( \varepsilon: V \rightarrow C^0_b(\Gamma, V) \). We fix an invariant mean \( m \) on \( \Gamma \). For \( f \in C^0_b(\Gamma, V) \) and \( w \in W \) we consider the function

\[
f_w: \Gamma \rightarrow \mathbb{R}, \quad f_w(g) = f(g)(w).
\]
It follows from the definitions that $f_w$ is an element of $\ell^\infty(\Gamma)$, so we may define a map $r: C^0_b(\Gamma, V) \to V$ by setting $r(f)(w) = m(f_w)$. It is immediate to check that $r(f)$ is indeed a bounded functional on $W$, whose norm is bounded by $\|f\|_\infty$. In other words, the map $r$ is well-defined and norm non-increasing. The $\Gamma$-invariance of the mean $m$ implies that $r$ is $\Gamma$-equivariant, and an easy computation shows that $r \circ \varepsilon = \text{Id}_V$.

Let us now consider the diagram

![Diagram](image)

By Lemma 4.11, $C^0_b(\Gamma, V)$ is relatively injective, so there exists a bounded $\mathbb{R}[\Gamma]$-map $\beta'$ such that $\|\beta'\| \leq \|\varepsilon \circ \alpha\| \leq \|\alpha\|$ and $\beta' \circ \iota = \varepsilon \circ \alpha$. The $\mathbb{R}[\Gamma]$-map $\beta := r \circ \beta'$ satisfies $\|\beta\| \leq \|\alpha\|$ and $\beta \circ \iota = \alpha$. This shows that $V$ is relatively injective.

(2) $\Rightarrow$ (3) is obvious, so we are left to show that (3) implies (1). If $\mathbb{R}$ is relatively injective and $\sigma: \ell^\infty(\Gamma) \to \mathbb{R}$ is the map defined by $\sigma(f) = f(1)$, then there exists an $\mathbb{R}[\Gamma]$-map $\beta: \ell^\infty(\Gamma) \to \mathbb{R}$ such that $\|\beta\| \leq 1$ and the following diagram commutes:

![Diagram](image)

Using that $\beta(1\Gamma) = 1$ and $\|\beta\| \leq 1$ it is easy to show that $\beta$ is a mean. Being an $\mathbb{R}[\Gamma]$-map, $\beta$ is $\Gamma$-invariant, whence the conclusion.

The previous proposition allows us to provide an alternative proof of Theorem 3.6, which we recall here for the convenience of the reader:

**Theorem 4.19.** Let $\Gamma$ be an amenable group, and let $V$ be a dual normed $\mathbb{R}[\Gamma]$-module. Then $H^n_b(\Gamma, V) = 0$ for every $n \geq 1$.

**Proof.** The complex

$$0 \longrightarrow V \xrightarrow{\text{Id}} V \longrightarrow 0$$

provides a relatively injective strong resolution of $V$, so the conclusion follows from Corollary 4.15.
9. Amenable spaces

The notion of amenable space was introduced by Zimmer [Zim78] in the context of actions of topological groups on standard measure spaces (see e.g. [Mon01, Section 5.3] for several equivalent definitions). In our case of interest, i.e. when $\Gamma$ is a discrete group acting on a set $S$ (which may be thought as endowed with the discrete topology), the amenability of $S$ as a $\Gamma$-space amounts to the amenability of the stabilizers in $\Gamma$ of elements of $S$ [AEG94, Theorem 5.1]. Therefore, we may take this characterization as a definition:

**Definition 4.20.** A left action $\Gamma \times S \to S$ of a group $\Gamma$ on a set $S$ is *amenable* if the stabilizer of every $s \in S$ is an amenable subgroup of $\Gamma$. In this case, we equivalently say that $S$ is an amenable $\Gamma$-set.

The importance of amenable $\Gamma$-sets is due to the fact that they may be exploited to isometrically compute the bounded cohomology of $\Gamma$. If $S$ is any $\Gamma$-set and $V$ is any normed $\mathbb{R}[\Gamma]$-module, then we denote by $\ell^\infty(S^{n+1},V)$ the space of bounded functions from $S^{n+1}$ to $V$. This space may be endowed with the a structure of a normed $\mathbb{R}[\Gamma]$-module via the action

$$(g \cdot f)(s_0, \ldots, s_n) = g \cdot (f(g^{-1}s_0, \ldots, g^{-1}s_n)).$$

The differential $\delta^n: \ell^\infty(S^{n+1},V) \to \ell^\infty(S^{n+1},V)$ defined by

$$\delta^n(f)(s_0, \ldots, s_{n+1}) = \sum_{i=0}^n (-1)^i f(s_0, \ldots, \hat{s_i}, \ldots, s_n)$$

endows the pair $(\ell^\infty(S^{n+1},V), \delta^*)$ with the structure of a normed $\mathbb{R}[\Gamma]$-complex. Together with the augmentation $\varepsilon: V \to \ell^\infty(S,V)$ given by $\varepsilon(v)(s) = v$ for every $s \in S$, such a complex provides a strong resolution of $V$:

**Lemma 4.21.** The augmented complex

$$0 \to V \to \ell^\infty(S,V) \xrightarrow{\delta^0} \ell^\infty(S^2,V) \xrightarrow{\delta^1} \ell^\infty(S^3,V) \xrightarrow{\delta^2} \ldots$$

provides a strong resolution of $V$.

**Proof.** Let $s_0$ be a fixed element of $S$. Then the maps $k^n: \ell^\infty(S^{n+1},V) \to \ell^\infty(S^n,V)$, $k^n(f)(s_1, \ldots, s_n) = f(s_0, s_1, \ldots, s_n)$ provide the required contracting homotopy. \hfill $\square$

**Lemma 4.22.** Suppose that $S$ is an amenable $\Gamma$-space, and that $V$ is a dual normed $\mathbb{R}[\Gamma]$-module. Then $\ell^\infty(S^{n+1},V)$ is relatively injective for every $n \geq 0$.

**Proof.** Since any intersection of amenable subgroups is amenable, the $\Gamma$-set $S$ is amenable if and only if $S^n$ is. Therefore, it is sufficient to deal with the case $n = 0$. 
Let $W$ be the normed $\mathbb{R}[\Gamma]$-module such that $V = W'$, and consider the extension problem described in Definition 4.9, with $E = \ell^\infty(S,V)$:

$$
\begin{array}{ccc}
0 & \rightarrow & A \\
\downarrow \alpha & & \downarrow \beta \\
E^\infty(S,V) & \rightarrow & B
\end{array}
$$

We denote by $R \subseteq S$ a set of representatives for the action of $\Gamma$ on $S$, and for every $r \in R$ we denote by $\Gamma_r$ the stabilizer of $r$, endowed with the invariant mean $\mu_r$. Moreover, for every $s \in S$ we choose an element $g_s \in \Gamma$ such that $g_s^{-1}(s) = r_s \in R$. Then $g_s$ is uniquely determined up to right multiplication by elements in $\Gamma_{r_s}$.

Let us fix an element $b \in B$. In order to define $\beta(b)$, for every $s \in S$ we need to know the value taken by $\beta(b)(s)$ on every $w \in W$. Therefore, we fix $s \in S$, $w \in W$, and we set

$$
(\beta(b)(s))(w) = \mu_{\Gamma_s}(f_b)
$$

where $f_b \in \ell^\infty(\Gamma_{r_s}, \mathbb{R})$ is defined by

$$
f_b(g) = \left((g_s g) \cdot \alpha(g^{-1} g_s^{-1} b)\right)(s)(w).
$$

Since $\|\sigma\| \leq 1$ we have that $\|\beta\| \leq \|\alpha\|$, and the behaviour of means on constant functions implies that $\beta \circ \iota = \alpha$.

Observe that the element $\beta(b)(s)$ does not depend on the choice of $g_s \in \Gamma$. In fact, if we replace $g_s$ by $g_s g'$ for some $g' \in \Gamma_{r_s}$, then the function $f_b$ defined above is replaced by the function

$$
f'_b(g) = \left((g_s g' g) \cdot \alpha(g^{-1} g_s^{-1} g' b)\right)(s)(w) = f_b(g' g),
$$

and $\mu_{\Gamma_s}(f'_b) = \mu_{\Gamma_s}(f_b)$ by the invariance of the mean $\mu_{\Gamma_s}$. This fact allows us to prove that $\beta$ is a $\Gamma$-map. In fact, let us fix $\overline{g} \in \Gamma$ and let $\overline{s} = \overline{g}^{-1}(s)$. Then we may assume that $g_s = \overline{g}^{-1} g_s$, so

$$
(\overline{g}(\beta(b)))(s)(w) = \beta(b)(\overline{s})(\overline{g}^{-1} w) = \mu_{\Gamma_s}(\overline{f}_b),
$$

where $\overline{f}_b \in \ell^\infty(\Gamma_{r_s}, \mathbb{R})$ is given by

$$
\overline{f}_b(g) = \left((\overline{g}^{-1} g_s g) \cdot \alpha(g^{-1} g_s^{-1} \overline{g} b)\right)(\overline{s})(\overline{g}^{-1} w)
= \left((g_s g) \cdot \alpha(g^{-1} g_s^{-1} \overline{g} b)\right)(s)(w)
= f_{\overline{g}b}(g).
$$

This concludes the proof. 

As anticipated above, we are now able to show that amenable spaces may be exploited to compute bounded cohomology:

**Theorem 4.23.** Let $S$ be an amenable $\Gamma$-set and let $V$ be a dual normed $\mathbb{R}[\Gamma]$-module. Then the homology of the complex

$$
0 \rightarrow \ell^\infty(S,V)^\Gamma \xrightarrow{\delta^0} \ell^\infty(S^2,V)^\Gamma \xrightarrow{\delta^1} \ell^\infty(S^3,V)^\Gamma \xrightarrow{\delta^2} \ldots
$$
9. AMENABLE SPACES 53

is canonically isometrically isomorphic to $H^\bullet_b(\Gamma, V)$.

**Proof.** The previous lemmas imply that the augmented complex $(V, \ell^\infty(S^{n+1}, V), \delta^\bullet)$ provides a relatively injective strong resolution of $V$, so Corollary 4.15 implies that the homology of the $\Gamma$-invariants of $(V, \ell^\infty(S^{n+1}), \delta^\bullet)$ is isomorphic to the bounded cohomology of $\Gamma$ with coefficients in $V$. By Corollary 4.17, in order to prove that such an isomorphism is isometric we are left to construct a norm non-increasing chain map

$$\alpha^\bullet : C_b^\bullet(\Gamma, V) \to \ell^\infty(S^{n+1}, V).$$

We keep notation from the proof of the previous lemma, i.e. we fix a set of representatives $R$ for the action of $\Gamma$ on $S$, and for every $s \in S$ we choose an element $g_s \in \Gamma$ such that $g_s^{-1} s = r_s \in R$. For every $r \in R$ we also fix an invariant mean $\mu_r$ on the stabilizer $\Gamma_r$.

Let us fix an element $f \in C_b^n(\Gamma, V)$, and take $(s_0, \ldots, s_n) \in S^{n+1}$. If $V = W'$, we also fix an element $w \in W$. For every $i$ we denote by $r_i \in R$ the representative of the orbit of $s_i$, and we consider the invariant mean $\mu_{r_0} \times \ldots \times \mu_{r_n}$ on $\Gamma_{r_0} \times \ldots \times \Gamma_{r_n}$ (see Remark 3.5). Then we consider the function $f_{s_0, \ldots, s_n} \in \ell^\infty(\Gamma_{r_0} \times \ldots \times \Gamma_{r_n}, \mathbb{R})$ defined by

$$f_{s_0, \ldots, s_n}(g_0, \ldots, g_n) = f(g_{s_0}g_0, \ldots, g_{s_n}g_n)(w).$$

By construction we have $\|f_{s_0, \ldots, s_n}\|_\infty \leq \|f\|_\infty \cdot \|w\|_W$, so we may set

$$(\alpha^n(f)(s_0, \ldots, s_n))(w) = (\mu_{r_0} \times \ldots \times \mu_{r_n})(f_{s_0, \ldots, s_n}),$$

thus defining an element $\alpha^n(f)(s_0, \ldots, s_n) \in W' = V$ such that $\|\alpha^n(f)\|_V \leq \|f\|_\infty$. We have thus shown that $\alpha^n : C_b^n(\Gamma, V) \to \ell^\infty(S^{n+1}, V)$ is a well-defined norm non-increasing linear map. The fact that $\alpha^n$ commutes with the action of $\Gamma$ follows from the invariance of the means $\mu_r$, $r \in R$, and the fact that $\alpha^\bullet$ is a chain map is obvious.

Let us prove some direct corollaries of the previous results. Observe that, if $W$ is a dual normed $\mathbb{R}[K]$-module and $\psi : \Gamma \to K$ is a homomorphism, then the induced module $\psi^{-1}(W)$ is a dual normed $\mathbb{R}[\Gamma]$-module.

**Theorem 4.24.** Let $\psi : \Gamma \to K$ be a surjective homomorphism with amenable kernel, and let $W$ be a dual normed $\mathbb{R}[K]$-module. Then the induced map

$$H^\bullet_b(K, W) \to H^\bullet_b(\Gamma, \psi^{-1}W)$$

is an isometric isomorphism.

**Proof.** The action $\Gamma \times K \to K$ defined by $(g, k) \mapsto \psi(g)k$ endows $K$ with the structure of an amenable $\Gamma$-set. Therefore, Theorem 4.23 implies that the bounded cohomology of $\Gamma$ with coefficients in $\psi^{-1}W$ is isometrically isomorphic to the cohomology of the complex $\ell^\infty(K^{n+1}, \psi^{-1}W)$. However, since $\psi$ is surjective, we have a (tautological) isometric identification between $\ell^\infty(K^{n+1}, \psi^{-1}W)$ and $C_b^\bullet(K, W)^K$, whence the conclusion. □
Corollary 4.25. Let $\psi: \Gamma \to K$ be a surjective homomorphism with amenable kernel. Then the induced map
\[ H^n_b(K, \mathbb{R}) \to H^n_b(\Gamma, \mathbb{R}) \]
is an isometric isomorphism for every $n \in \mathbb{N}$.

10. Alternating cochains

A cochain $\varphi \in C^n(\Gamma, \mathbb{R})$ is alternating if it satisfies the following condition: for every permutation $\sigma \in S_{n+1}$ of the set $\{0, \ldots, n\}$, if we denote by $\text{sgn}(\sigma) = \pm 1$ the sign of $\sigma$, then the equality
\[ \varphi(g_{\sigma(0)}, \ldots, g_{\sigma(n)}) = \text{sgn}(\sigma) \cdot \varphi(g_0, \ldots, g_n) \]
holds for every $(g_0, \ldots, g_n) \in \Gamma^{n+1}$. We denote by $C_n^\text{alt}(\Gamma, \mathbb{R}) \subseteq C_n(\Gamma, \mathbb{R})$ the subset of alternating cochains, and we set $C_n^{b,\text{alt}}(\Gamma, \mathbb{R}) = C_n^\text{alt}(\Gamma, \mathbb{R}) \cap C_n^b(\Gamma, \mathbb{R})$. It is well-known that (bounded) alternating cochains provide a $\Gamma$-subcomplex of general (bounded) cochains. In fact, it turns out that, in the case of real coefficients, one can compute the (bounded) cohomology of $\Gamma$ via the complex of alternating cochains:

Proposition 4.26. The complex $C_{\text{alt}}^\bullet(\Gamma, \mathbb{R})$ (resp. $C_{b,\text{alt}}^\bullet(\Gamma, \mathbb{R})$) isometrically computes the cohomology (resp. the bounded cohomology) of $\Gamma$ with real coefficients.

Proof. We concentrate our attention on bounded cohomology, the case of ordinary cohomology being very similar. The inclusion $j^\bullet: C_{b,\text{alt}}^\bullet(\Gamma, \mathbb{R}) \to C_b^\bullet(\Gamma, \mathbb{R})$ induces a norm non-increasing map on bounded cohomology, so in order to prove the proposition it is sufficient to construct a norm non-increasing $\Gamma$-chain map $\text{alt}^\bullet_b: C_b^\bullet(\Gamma, \mathbb{R}) \to C_{b,\text{alt}}^\bullet(\Gamma, \mathbb{R})$ which satisfies the following properties:

1. $\text{alt}^n_b$ is a retraction onto the subcomplex of alternating cochains, i.e. $\text{alt}^n_b \circ j^n = \text{Id}$ for every $n \geq 0$;
2. $j^\bullet \circ \text{alt}^\bullet_b$ is $G$-homotopic to the identity of $C_b^\bullet(\Gamma, \mathbb{R})$ (as usual, via a homotopy which is bounded in every degree).

For every $\varphi \in C_b^n(\Gamma, \mathbb{R})$, $(g_0, \ldots, g_n) \in \Gamma^{n+1}$ we set
\[ \text{alt}^n_b(\varphi)(g_0, \ldots, g_n) = \frac{1}{(n+1)!} \sum_{\sigma \in S_{n+1}} (-1)^{\text{sgn}(\sigma)} \cdot \varphi(g_{\sigma(0)}, \ldots, g_{\sigma(n)}) . \]

It is easy to check that $\text{alt}^\bullet_b$ satisfies the required properties (the fact that it is indeed homotopic to the identity of $C_b^\bullet(\Gamma, \mathbb{R})$ may be deduced, for example, by the computations carried out in the context of singular chains in [FM11, Appendix B]).

Let $S$ be an amenable $\Gamma$-set. We have seen in Theorem 4.23 that the bounded cohomology of $\Gamma$ with coefficients in the normed $\mathbb{R}[G]$-module $V$ is isometrically isomorphic to the cohomology of the complex $\ell^\infty(S^{\bullet+1}, V)$. The very same argument described in the proof of Proposition 4.26 shows
that the bounded cohomology of $\Gamma$ is computed also by the subcomplex of alternating elements of $\ell^\infty(S^{\bullet+1}, V)^\Gamma$. More precisely, let us denote by $\ell^\infty_{\text{alt}}(S^n, V)$ the submodule of alternating elements of $\ell^\infty(S^n, V)$ (the definition of alternating being obvious). Then we have the following:

**Theorem 4.27.** Let $S$ be an amenable $\Gamma$-set and let $V$ be a dual normed $\mathbb{R}[\Gamma]$-module. Then the homology of the complex

\[ 0 \longrightarrow \ell^\infty_{\text{alt}}(S, V)^\Gamma \xrightarrow{\delta^0} \ell^\infty_{\text{alt}}(S^2, V)^\Gamma \xrightarrow{\delta^1} \ell^\infty_{\text{alt}}(S^3, V)^\Gamma \xrightarrow{\delta^2} \ldots \]

is canonically isometrically isomorphic to $H^\bullet_b(\Gamma, V)$. 
Bounded cohomology of topological spaces

Let \( X \) be a topological space, and let \( R = \mathbb{Z}, \mathbb{R} \). Recall that \( C_\bullet(X, R) \) (resp. \( C^\bullet(X, R) \)) denotes the usual complex of singular chains (resp. cochains) on \( X \) with coefficients in \( R \), and \( S_i(X) \) is the set of singular \( i \)-simplices in \( X \). We also regard \( S_i(X) \) as a subset of \( C_\bullet(X, R) \), so that for any cochain \( \varphi \in C^i(X, R) \) it makes sense to consider its restriction \( \varphi|_{S_i(X)} \). For every \( \varphi \in C^i(X, R) \), we set

\[
\|\varphi\| = \|\varphi\|_\infty = \sup\{ |\varphi(s)| \mid s \in S_i(X) \} \in [0, \infty].
\]

We denote by \( C^\bullet_b(X, R) \) the submodule of bounded cochains, i.e. we set

\[
C^\bullet_b(X, R) = \{ \varphi \in C^\bullet(X, R) \mid \|\varphi\| < \infty \}.
\]

Since the differential takes bounded cochains into bounded cochains, \( C^\bullet_b(X, R) \) is a subcomplex of \( C^\bullet(X, R) \). We denote by \( H^\bullet(X, R) \) and \( H^\bullet_b(X, R) \) respectively the homology of the complexes \( C^\bullet(X, R) \) and \( C^\bullet_b(X, R) \). Of course, \( H^\bullet(X, R) \) is the usual singular cohomology module of \( X \) with coefficients in \( R \), while \( H^\bullet_b(X, R) \) is the bounded cohomology module of \( X \) with coefficients in \( R \). Just as in the case of groups, the norm on \( C^i(X, R) \) descends (after the suitable restrictions) to a seminorm on each of the modules \( H^\bullet(X, R) \), \( H^\bullet_b(X, R) \). More precisely, if \( \varphi \in H \) is a class in one of these modules, which is obtained as a quotient of the corresponding module of cocycles \( Z \), then we set

\[
\|\varphi\| = \inf \{ \|\psi\| \mid \psi \in Z, [\psi] = \varphi \text{ in } H \}.
\]

This seminorm may take the value \( \infty \) on elements in \( H^\bullet(X, R) \) and may be null on non-zero elements in \( H^\bullet_b(X, R) \) (but not on non-zero elements in \( H^\bullet(X, R) \)): this is clear in the case with integer coefficients, and it is a consequence of the Universal Coefficient Theorem in the case with real coefficients, since a real cohomology class with vanishing seminorm has to be null on any cycle, whence null in \( H^\bullet(X, \mathbb{R}) \). The inclusion of bounded cochains into possibly unbounded cochains induces the comparison map

\[
c^\bullet : H^\bullet_b(X, R) \to H^\bullet(X, R).
\]

1. Basic properties of bounded cohomology of spaces

Bounded cohomology enjoys some of the fundamental properties of classical singular cohomology: for example, \( H^0_b(\{\text{pt.}\}, R) = 0 \) if \( i > 0 \), and \( H^0_b(\{\text{pt.}\}, R) = R \) (more in general, \( H^0_b(X, R) \) is canonically isomorphic to
\( \ell^\infty(S) \), where \( S \) is the set of the path connected components of \( X \). The usual proof of the homotopy invariance of singular homology is based on the construction of an algebraic homotopy which maps every \( n \)-simplex to the sum of at most \( n + 1 \) \((n + 1)\)-dimensional simplices. As a consequence, the homotopy operator induced in cohomology preserves bounded cochains. This implies that bounded cohomology is a homotopy invariant of topological spaces. Moreover, if \((X, Y)\) is a topological pair, then there is an obvious definition of \( H^\bullet_b(X, Y) \), and it is immediate to check that the analogous of the long exact sequence of the pair in classical singular cohomology also holds in the bounded case.

Perhaps the most important peculiarity of bounded cohomology with respect to classical singular cohomology is the lacking of any Mayer-Vietoris sequence (or, equivalently, of any excision theorem). In particular, spaces with finite-dimensional bounded cohomology may be tamely glued to each other to get spaces with infinite-dimensional bounded cohomology (see Remark 5.6 below).

Recall from Section 3 that \( H^n(X, R) \cong H^n(\pi_1(X), R) \) for every aspherical CW-complex \( X \). As anticipated in Section 6, a fundamental result by Gromov provides an isometric isomorphism \( H^\bullet_b(X, \mathbb{R}) \cong H^\bullet_b(\pi_1(X), \mathbb{R}) \) even without any assumption on the asphericity of \( X \). This section is devoted to a proof of Gromov’s result. We will closely follow Ivanov’s argument \([Iva87]\), which deals with the case when \( X \) is (homotopically equivalent to) a countable CW-complex. Before going into Ivanov’s argument, we will concentrate our attention on the easier case of aspherical spaces.

2. Bounded singular cochains as relatively injective modules

Henceforth, we assume that \( X \) is a path connected topological space admitting the universal covering \( \widetilde{X} \), we denote by \( \Gamma \) the fundamental group of \( X \), and we fix an identification of \( \Gamma \) with the group of covering automorphisms of \( \widetilde{X} \). Just as we did in Section 4 for \( C^\bullet(\widetilde{X}, R) \), we endow \( C^\bullet_b(\widetilde{X}, R) \) with the structure of a normed \( \mathbb{R}[\Gamma]\)-module. Our arguments are based on the obvious but fundamental isometric identification

\[
C^\bullet(X, R) \cong C^\bullet_b(\widetilde{X}, R)^\Gamma,
\]

which induces a canonical isometric identification

\[
H^\bullet_b(X, R) \cong H^\bullet_b(C^\bullet_b(\widetilde{X}, R)^\Gamma).
\]

As a consequence, in order to prove the isomorphism \( H^\bullet_b(X, R) \cong H^\bullet_b(\Gamma, R) \) it is sufficient to show that the complex \( C^\bullet_b(\widetilde{X}, R) \) provides a relatively injective strong resolution of \( R \) (an additional argument shows that this isomorphism is also isometric). The relative injectivity of the modules \( C^\bullet_n(\widetilde{X}, R) \) can be easily deduced from the argument described in the proof of Lemma 4.6, which applies verbatim in the context of bounded singular cochains.
Lemma 5.1. For every $n \in \mathbb{N}$, the bounded cochain module $C^n_b(\tilde{X}, R)$ is relatively injective.

Therefore, in order to show that the bounded cohomology of $X$ is isomorphic to the bounded cohomology of $\Gamma$ we need to show that the (augmented) complex $C^\bullet_b(\tilde{X}, R)$ provides a strong resolution of $R$. We will show that this is the case if $X$ is aspherical. In the general case, it is not even true that $C^\bullet_b(\tilde{X}, R)$ is acyclic (see Remark 5.12). However, in the case when $R = \mathbb{R}$ a deep result by Ivanov shows that the complex $C^\bullet_b(\tilde{X}, \mathbb{R})$ indeed provides a strong resolution of $\mathbb{R}$. A sketch of Ivanov’s proof will be given in Section 4.

Before going on, we point out that we always have a norm non-increasing map from the bounded cohomology of $\Gamma$ to the bounded cohomology of $X$:

Lemma 5.2. Let us endow the complexes $C^\bullet_b(\Gamma, R)$ and $C^\bullet_b(\tilde{X}, R)$ with the obvious augmentations. Then, there exists a norm non-increasing chain map

$$r^\bullet : C^\bullet_b(\Gamma, R) \to C^\bullet_b(\tilde{X}, R)$$

extending the identity of $R$.

Proof. Let us choose a set of representatives $R$ for the action of $\Gamma$ on $\tilde{X}$. We consider the map $r_0 : S_0(\tilde{X}) = \tilde{X} \to \Gamma$ taking a point $x$ to the unique $g \in \Gamma$ such that $x \in g(R)$. For $n \geq 1$, we define $r_n : S_n(\tilde{X}) \to \Gamma^{n+1}$ by setting $r_n(s) = (r_0(s(e_0)), \ldots, r_n(s(e_n)))$, where $s(e_i)$ is the $i$-th vertex of $s$. Finally, we extend $r_n$ to $C_n(\tilde{X}, R)$ by $R$-linearity and define $r^\bullet$ as the dual map of $r_n$. Since $r_n$ takes any single simplex to a single $(n+1)$-tuple, it is readily seen that $r^\bullet$ is norm non-increasing (in particular, it takes bounded cochains into bounded cochains). The fact that $r^\bullet$ is a $\Gamma$-equivariant chain map is obvious. □

Corollary 5.3. Let $X$ be a path connected topological space. Then, for every $n \in \mathbb{N}$ there exists a natural norm non-increasing map

$$H^n_b(\Gamma, R) \to H^n_b(X, R).$$

Proof. By Lemma 5.2, there exist a norm non-increasing chain map

$$r^\bullet : C^\bullet_b(\Gamma, R) \to C^\bullet_b(\tilde{X}, R)$$

extending the identity of $R$, so we may define the desired map $H^n_b(\Gamma, R) \to H^n_b(X, R)$ to be equal $H^n_b(r^\bullet)$. We know that every $C^\bullet_b(\tilde{X}, R)$ is relatively injective as an $R[\Gamma]$-module, while $C^\bullet_b(\Gamma, R)$ provides a strong resolution of $R$ over $R[\Gamma]$, so Theorem 4.14 ensures that $H^n_b(r^\bullet)$ does not depend on the choice of the particular chain map $r^\bullet$ which extends the identity of $R$. □

A very natural question is whether the map provided by Corollary 5.3 is an (isometric) isomorphism. In the following sections we will see that this holds true when $X$ is aspherical or when $R$ is equal to the field of real numbers. The fact that $H^n_b(\Gamma, R)$ is isometrically isomorphic to $H^n_b(X, R)$ when $X$ is aspherical is not surprising, and just generalizes the analogous
result for classical cohomology: the (bounded) cohomology of a group Γ may be defined as the (bounded) cohomology of any aspherical space having Γ as fundamental group; this is a well-posed definition since any two such spaces are homotopically equivalent, and (bounded) cohomology is a homotopy invariant. On the other hand, the fact that the isometric isomorphism holds even without the assumption that X is aspherical is a very deep result due to Gromov [Gro82].

We begin by proving the following:

**Lemma 5.4.** Let Y be a path connected topological space. If Y is aspherical, then the augmented complex $C^*_b(Y, R)$ admits a contracting homotopy (so it is a strong resolution of $R$). If $\pi_i(Y) = 0$ for every $i \leq n$, then there exists a partial contracting homotopy

$$\mathbb{R} \xleftarrow{k^0} C^0_b(Y, R) \xleftarrow{k^1} C^1_b(Y, R) \xleftarrow{k^2} \cdots \xleftarrow{k^n} C^n_b(Y, R) \xleftarrow{k^{n+1}} C^{n+1}_b(Y, R)$$

(where we require that the equality $\delta^{m-1}k^m + k^{m+1}\delta^m = \text{Id}_{C^m_b(Y, R)}$ holds for every $m \leq n$, and $\delta^{-1} = \varepsilon$ is the usual augmentation).

**Proof.** For every $-1 \leq m \leq n$ we construct a map $T_m: C_m(Y, R) \to C_{m+1}(Y, R)$ taking any single simplex into a single simplex in such a way that $d_{m+1}T_m + T_{m-1}d_m = \text{Id}_{C_m(Y, R)}$, where we understand that $C_{-1}(Y, R) = R$ and $d_0: C_0(Y, R) \to R$ is the augmentation map $d_0(\sum r_i y_i) = \sum r_i$. Let us fix a point $y_0 \in Y$, and define $T_{-1}: R \to C_0(Y, R)$ by setting $T_{-1}(r) = ry_0$.

For $m \geq 0$ we define $T_m$ as the $R$-linear extension of a map $T_m: S_m(Y) \to S_{m+1}(Y)$ having the following property: for every $s \in S_m(Y)$, the 0-th vertex of $T_m(s)$ is equal to $y_0$, and has $s$ as opposite face. We proceed by induction, and suppose that $T_i$ has been defined for every $-1 \leq i \leq m$. Take $s \in S_m(Y)$. Then, using the fact that $\pi_m(Y) = 0$ and the properties of $T_{m-1}$, it is not difficult to show that a simplex $s' \in S_{m+1}(Y)$ exists which satisfies both the equality $d_{m+1}s' = s - T_{m-1}(d_ms)$ and the additional requirement described above. We set $T_{m+1}(s) = s'$, and we are done.

Since $T_{m-1}$ sends any single simplex to a single simplex, its dual map $k^m$ sends bounded cochains into bounded cochains, and has operator norm equal to one. Therefore, the maps $k^m: C^m_b(Y, R) \to C^{m-1}_b(Y, R)$, $m \leq n+1$, provide the desired (partial) contracting homotopy.

**3. The aspherical case**

We are now ready to show that, under the assumption that X is aspherical, the bounded cohomology of X is isometrically isomorphic to the bounded cohomology of Γ for any ring of coefficients:

**Theorem 5.5.** Let X be an aspherical space, i.e. a path connected topological space such that $\pi_n(X) = 0$ for every $n \geq 2$. Then $H^*_b(X, R)$ is isometrically isomorphic to $H^*_b(\Gamma, R)$ for every $n \in \mathbb{N}$. 


4. Ivanov’s contracting homotopy

We now come back to the general case, i.e. we do not assume that $\widetilde{X}$ is contractible. In order to show that $H^b_\text{b}(X, R)$ is isometrically isomorphic to $H^b_{\text{b}}(\Gamma, R)$ we need to prove that the complex of singular bounded cochains on $\widetilde{X}$ provides a strong resolution of $R$. In the case when $R = \mathbb{Z}$, this is false in general, since the complex $C_b^*(\widetilde{X}, \mathbb{Z})$ may even be non-exact (see Remark 5.12). On the other hand, a deep result by Ivanov ensures that $C_b^*(\widetilde{X}, \mathbb{R})$ indeed provides a strong resolution of $\mathbb{R}$.

Ivanov’s argument makes use of sophisticated techniques from algebraic topology, which work under the assumption that $X$, whence $\widetilde{X}$, is a countable CW-complex (but see Remark 5.10). We begin with the following:

**Lemma 5.7** ([Iva87], Theorem 2.2). Let $p: Z \to Y$ be a principal $H$-bundle, where $H$ is an abelian topological group. Then there exists a chain map $A^*: C_b^*(Z, \mathbb{R}) \to C_b^*(Y, \mathbb{R})$ such that $\|A^n\| \leq 1$ for every $n \in \mathbb{N}$ and $A^* \circ p^*$ is the identity of $C_b^*(Y, \mathbb{R})$.

**Proof.** For $\varphi \in C_b^n(Z, \mathbb{R})$, $s \in S_n(Y)$, the value $A(\varphi)(s)$ is obtained by suitably averaging the value of $\varphi(s')$, where $s'$ ranges over the set $P_s = \{s' \in S_n(Z) | p \circ s' = s\}$.

More precisely, let $K_n = S_n(H)$ be the space of continuous functions from the standard $n$-simplex to $H$, and define on $K_n$ the operation given by pointwise multiplication of functions. With this structure, $K_n$ is an abelian group, so it admits an invariant mean $\mu_n$. Observe that the permutation group $\mathfrak{S}_{n+1}$ acts on $\Delta^n$ via affine transformations. This action induces an action on $K_n$, whence on $\ell^\infty(K_n)$, and on the space of means on $K_n$, so there is an obvious notion of $\mathfrak{S}_{n+1}$-invariant mean on $K_n$. By averaging over the action of $\mathfrak{S}_{n+1}$, the space of $K_n$-invariant means may be retracted onto the space $\mathcal{M}_n$ of $\mathfrak{S}_{n+1}$-invariant $K_n$-invariant means on $K_n$, which, in particular, is non-empty. Finally, observe that any affine identification of $\Delta^{n-1}$ with a face of $\Delta^n$ induces a map $\mathcal{M}_n \to \mathcal{M}_{n-1}$. Since elements of $\mathcal{M}_n$...
are $\mathcal{S}_{n+1}$-invariant, this map does not depend on the chosen identification. Therefore, we get a sequence of maps

$$\mathcal{M}_0 \leftarrow \mathcal{M}_1 \leftarrow \mathcal{M}_2 \leftarrow \mathcal{M}_3 \leftarrow \cdots$$

Recall now that the Banach-Alaouglu Theorem implies that every $\mathcal{M}_n$ is compact (with respect to the weak* topology on $\ell^\infty(K_n)'$). This easily implies that there exists a sequence $\{\mu_n\}$ of means such that $\mu_n \in \mathcal{M}_n$ and $\mu_n \mapsto \mu_{n-1}$ under the map $\mathcal{M}_n \to \mathcal{M}_{n-1}$. We say that such a sequence in compatible.

Let us now fix $s \in S_n(Y)$, and observe that there is a bijection between $P_s$ and $K_n$. This bijection is uniquely determined up to the choice of an element in $P_s$, i.e. up to left multiplication by an element of $K_n$. In particular, if $f \in \ell^\infty(P_s)$, and $\mu$ is a left invariant mean on $K_n$, then there is a well-defined value $\mu(f)$.

Let us now choose a compatible sequence of means $\{\mu_n\}_{n \in \mathbb{N}}$. We define the operator $A^n$ by setting

$$A^n(\varphi)(s) = \mu_n(\varphi|_{P_s}) \quad \text{for every } \varphi \in C^*_b(Z), \ s \in S_n(Y).$$

Since the sequence $\{\mu_n\}$ is compatible, the sequence of maps $A^\bullet$ is a chain map. The inequality $\|A^n\| \leq 1$ is obvious, and the fact that $A^\bullet \circ p^\bullet$ is the identity of $C^*_b(Y, \mathbb{R})$ may be deduced from the behaviour of means on constant functions. \hfill $\Box$

**Theorem 5.8 ([Iva87]).** Let $X$ be a path connected countable CW-complex with universal covering $\tilde{X}$. Then the (augmented) complex $C^*_b(\tilde{X}, \mathbb{R})$ provides a relatively injective strong resolution of $\mathbb{R}$.

**Proof.** We only sketch Ivanov’s argument, referring the reader to [Iva87] for full details. Building on results by Dold and Thom [DT58], Ivanov constructs an infinite tower of bundles

$$X_1 \leftarrow p_1 X_2 \leftarrow p_2 X_3 \leftarrow \cdots \leftarrow X_n \leftarrow \cdots$$

where $X_1 = \tilde{X}$, $\pi_i(X_m) = 0$ for every $i \leq m$, $\pi_i(X_m) = \pi_i(X)$ for every $i > m$ and each map $p_m : X_{m+1} \to X_m$ is a principal $H_m$-bundle for some topological connected abelian group $H_m$, which has the homotopy type of a $K(\pi_{n+1}(X), n)$.

By Lemma 5.4, for every $n$ we may construct a partial contracting homotopy

$$\mathbb{R} \leftarrow C^0_b(X_n, \mathbb{R}) \xrightarrow{k^0_n} C^1_b(X_n, \mathbb{R}) \xrightarrow{k^1_n} \cdots \xrightarrow{k^{n+1}_n} C^{n+1}_b(X_n, \mathbb{R}).$$

Moreover, Lemma 5.7 implies that for every $m \in \mathbb{N}$ the chain map $p^\bullet_m : C^*_b(X_m, \mathbb{R}) \to C^*_b(X_{m+1}, \mathbb{R})$ admits a left inverse chain map $A^\bullet_m : C^*_b(X_{m+1}, \mathbb{R}) \to C^*_b(X_m, \mathbb{R})$ which is norm non-increasing. This allows us to define a partial contracting homotopy

$$\mathbb{R} \leftarrow C^0_b(X, \mathbb{R}) \xrightarrow{k^0} C^1_b(X, \mathbb{R}) \xrightarrow{k^1} \cdots \xrightarrow{k^{n+1}} C^{n+1}_b(X, \mathbb{R}).$$
5. Gromov’s Theorem

The discussion in the preceding section implies the following:

**Theorem 5.9 ([Gro82, Iva87]).** Let $X$ be a countable CW-complex. Then $H^n_\ast(X, \mathbb{R})$ is canonically isometrically isomorphic to $H^n_\ast(\Gamma, \mathbb{R})$. An explicit isometric isomorphism is induced by the map $r^\bullet : C^\bullet(\Gamma, \mathbb{R})^\Gamma \to C^\bullet(\tilde{X}, \mathbb{R})^\Gamma = C^\bullet(X, \mathbb{R})$ described in Lemma 5.2.

**Proof.** Observe that, if $X$ is a countable CW-complex, then $\Gamma = \pi_1(X)$ is countable, so $\tilde{X}$ is also a countable CW-complex. Therefore, Lemma 5.1 and Theorem 5.8 imply that the complex

$$0 \to R \xrightarrow{\varepsilon} C_0^\bullet(\tilde{X}, \mathbb{R}) \xrightarrow{\delta^0} C_1^\bullet(\tilde{X}, \mathbb{R}) \xrightarrow{\delta^1} \ldots$$

provides a relatively injective strong resolution of $\mathbb{R}$ as a trivial $\mathbb{R}[\Gamma]$-module, so $H^n_\ast(X, \mathbb{R})$ is canonically isomorphic to $H^n_\ast(\Gamma, \mathbb{R})$ for every $n \in \mathbb{N}$. The fact that the isomorphism $H^n_\ast(X, \mathbb{R}) \cong H^n_\ast(\Gamma, \mathbb{R})$ is isometric is a consequence of Corollary 4.17 and Lemma 5.2.

**Remark 5.10.** Theo Bülher recently proved that Ivanov’s argument may be generalized to show that $C^\bullet(Y, \mathbb{R})$ is a strong resolution of $\mathbb{R}$ whenever $Y$ is a simply connected topological space [Büh]. As a consequence, Theorem 5.9 holds even when the assumption that $X$ is a countable CW-complex is replaced by the weaker condition that $X$ is path connected and admits a universal covering.

**Corollary 5.11 (Gromov Mapping Theorem).** Let $X, Y$ be path connected countable CW-complexes and let $f : X \to Y$ be a continuous mapping inducing an epimorphism $f_* : \pi_1(X) \to \pi_1(Y)$ with amenable kernel. Then

$$H^n_\ast(f) : H^n_\ast(Y, \mathbb{R}) \to H^n_\ast(X, \mathbb{R})$$

is an isometric isomorphism for every $n \in \mathbb{N}$.

**Proof.** The explicit description of the isomorphism between the bounded cohomology of a space and the one of its fundamental group implies that
the diagram

\[
\begin{array}{ccc}
H^b_n(Y) & \xrightarrow{H^b_n(f)} & H^b_n(X) \\
\uparrow & & \uparrow \\
H^b_n(\pi_1(Y)) & \xrightarrow{H^b_n(f_\ast)} & H^b_n(\pi_1(X))
\end{array}
\]

is commutative, where the vertical arrows represent the isometric isomorphism of Theorem 5.9. Therefore, the conclusion follows from Corollary 4.25.

Remark 5.12. Theorem 5.9 does not hold for bounded cohomology with integer coefficients. In fact, if \( X \) is any topological space, then the short exact sequence \( 0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0 \) induces an exact sequence

\[
H^b_n(X, \mathbb{R}) \to H^n(X, \mathbb{R}/\mathbb{Z}) \to H^{n+1}_b(X, \mathbb{Z}) \to H^{n+1}_b(X, \mathbb{R})
\]

(see the proof of Proposition 2.9). If \( X \) is a simply connected CW-complex, then \( H^b_n(X, \mathbb{R}) = 0 \) for every \( n \geq 1 \), so we have

\[
H^b_n(X, \mathbb{Z}) \cong H^{n-1}_b(X, \mathbb{R}/\mathbb{Z}) \quad \text{for every } n \geq 2.
\]

For example, in the case of the 2-dimensional sphere we have \( H^3_b(S^2, \mathbb{Z}) \cong H^2(S^2, \mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z} \).

6. Alternating cochains

We have seen in Section 10 that (bounded) cohomology of group may be computed via the complex of alternating cochains. The same holds true also in the context of (bounded) singular cohomology of topological space.

If \( \sigma \in \mathfrak{S}_{n+1} \) is any permutation of \( \{0, \ldots, n\} \), then we denote by \( \sigma : \Delta^n \to \Delta^n \) the affine automorphism of the standard simplex which induces the permutation \( \sigma \) on the vertices of \( \Delta^n \). Then we say that a cochain \( \varphi \in C^n(X, \mathbb{R}) \) is alternating if

\[
\varphi(s) = \text{sgn}(\sigma) \cdot \varphi(s \circ \sigma)
\]

for every \( s \in S_n(X), \sigma \in \mathfrak{S}_{n+1} \). We also denote by \( C_{\text{alt}}^n(X, \mathbb{R}) \subseteq C^b(X, \mathbb{R}) \) the subcomplex of alternating cochains, and we set \( C_{b,\text{alt}}^n(X, \mathbb{R}) = C^b(X, \mathbb{R}) \cap C_{\text{alt}}^n(X, \mathbb{R}) \). Once a generic cochain \( \varphi \in C^n(X, \mathbb{R}) \) is given, we may alternate it by setting

\[
\text{alt}^n(\varphi)(s) = \frac{1}{(n+1)!} \sum_{\sigma \in \mathfrak{S}_{n+1}} (-1)^{\text{sgn}(\sigma)} \cdot \varphi(s \circ \sigma)
\]

for every \( s \in S_n(X) \). Then the very same argument exploited in the proof of Proposition 4.26 applies in this context to give the following:

Proposition 5.13. The complex \( C_{\text{alt}}^n(X, \mathbb{R}) \) (resp. \( C_{b,\text{alt}}^n(X, \mathbb{R}) \)) isometrically computes the cohomology (resp. the bounded cohomology) of \( X \) with real coefficients.
7. Relative bounded cohomology

When dealing with manifolds with boundary, it is often useful to study relative homology and cohomology. For example, in Section 5 we will show how the simplicial volume of a manifold with boundary $M$ can be computed via the analysis of the relative bounded cohomology module of the pair $(M, \partial M)$. This will prove useful to show that the simplicial volume is additive with respect to gluings along boundary components with amenable fundamental groups. Until the end of the chapter, all the cochain and cohomology modules will be assumed to be with real coefficients.

Let $Y$ be a subspace of the topological space $X$. We denote by $C^n(X,Y)$ the submodule of cochains which vanish on simplices supported in $Y$. In other words, $C^n(X,Y)$ is the kernel of the map $C^n(X) \to C^n(Y)$ induced by the inclusion $Y \hookrightarrow X$. We also set $C_b^n(X,Y) = C^n(X,Y) \cap C_b^n(X)$, and we denote by $H^\bullet_b(X,Y)$ (resp. $H^\bullet_b(X)$) the cohomology of the complex $C^\bullet(X,Y)$ (resp. $C^\bullet_b(X)$). The well-known short exact sequence of the pair for ordinary cohomology also holds for bounded cohomology: the short exact sequence of complexes

$$0 \longrightarrow C_b^n(X,Y) \longrightarrow C_b^n(X) \longrightarrow C_b^n(Y) \longrightarrow 0$$

induces the long exact sequence

$$(5) \quad \ldots \longrightarrow H_b^n(Y) \longrightarrow H_b^{n+1}(X,Y) \longrightarrow H_b^{n+1}(X) \longrightarrow H_b^{n+1}(Y) \longrightarrow \ldots$$

Recall now that, if the fundamental group of every component of $Y$ is amenable, then $H_b^n(Y) = 0$ for every $n \geq 1$, so the inclusion $j^n : C_b^n(X,Y) \to C_b^n(X)$ induces a norm non-increasing isomorphism

$$H_b^n(j^n) : H_b^n(X,Y) \to H_b^n(X)$$

for every $n \geq 2$. The following result is proved in [BBF+14], and shows that this isomorphism is in fact isometric:

**Theorem 5.14.** Let $(X,Y)$ be a pair of countable CW-complexes, and suppose that the fundamental group of each component of $Y$ is amenable. Then the map

$$H_b^n(j^n) : H_b^n(X,Y) \to H_b^n(X)$$

is an isometric isomorphism for every $n \geq 2$.

The rest of this chapter is devoted to the proof of Theorem 5.14.

8. Special cochains

Henceforth we assume that $(X,Y)$ is a pair of countable CW-complexes such that the fundamental group of every component of $Y$ is amenable. We denote by $\tilde{X}$ the universal covering of $X$, let $p : \tilde{X} \to X$ be a universal covering of $X$, and set $\tilde{Y} = p^{-1}(Y)$. As usual, we denote by $\Gamma$ the fundamental
group of \( X \), we fix an identification of \( \Gamma \) with the group of the covering automorphisms of \( p \), and we consider the induced identification

\[
C^n_b(X) = C^n_b(\tilde{X})^\Gamma.
\]

We say that a cochain \( \varphi \in C^n_b(\tilde{X}) \) is \emph{special} (with respect to \( \tilde{Y} \)) if the following conditions hold:

- \( \varphi \) is alternating;
- let \( s, s' \) be singular \( n \)-simplices with values in \( \tilde{X} \) and suppose that, for every \( i = 0, \ldots, n \), either \( s(w_i) = s'(w_i) \), or \( s(w_i) \) and \( s'(w_i) \) belong to the same connected component of \( \tilde{Y} \), where \( w_0, \ldots, w_n \) are the vertices of the standard \( n \)-simplex. Then \( \varphi(s) = \varphi(s') \).

We denote by \( C_{bs}^\bullet(\tilde{X}, \tilde{Y}) \subseteq C^\bullet_b(\tilde{X}) \) the subcomplex of special cochains, and we set

\[
C_{bs}^\bullet(X,Y) = C_{bs}^\bullet(\tilde{X}, \tilde{Y}) \cap C^\bullet_b(\tilde{X})^\Gamma = C^\bullet_b(\tilde{X})^\Gamma = C^\bullet_b(X).
\]

Any cochain \( \varphi \in C^\bullet_b(\tilde{X}, \tilde{Y}) \) vanishes on every simplex having two vertices on the same connected component of \( \tilde{Y} \). In particular

\[
C^n_b(X,Y) \subseteq C^n_b(X,Y) \subseteq C^n_b(X)
\]

for every \( n \geq 1 \). We denote by \( l^\bullet : C_{bs}^\bullet(X,Y) \to C^\bullet_b(X) \) the natural inclusion.

**Proposition 5.15.** There exists a norm non-increasing chain map

\[
\eta^\bullet : C^\bullet_b(X) \to C_{bs}^\bullet(X,Y)
\]

such that the composition \( l^\bullet \circ \eta^\bullet \) is chain-homotopic to the identity of \( C^\bullet_b(X) \).

**Proof.** Let us briefly describe the strategy of the proof. First of all, we will define a \( \Gamma \)-set \( S \) which provides a sort of discrete approximation of the pair \( (\tilde{X}, \tilde{Y}) \). As usual, the group \( \Gamma \) already provides an approximation of \( \tilde{X} \). However, in order to prove that \( Y \) is completely irrelevant from the point of view of bounded cohomology, we need to approximate every component of \( \tilde{Y} \) by a single point, and this implies that the set \( S \) cannot coincide with \( \Gamma \) itself. Basically, we add one point for each component of \( \tilde{Y} \). Since the fundamental group of every component of \( Y \) is amenable, the so obtained \( \Gamma \)-set \( S \) is amenable: therefore, the bounded cohomology of \( \Gamma \), whence of \( X \), may be isometrically computed using the complex of alternating cochains on \( S \). Finally, alternating cochains on \( S \) can be isometrically translated into special cochains on \( X \). But, in degree greater than one, special cochains are relative cochains, and this concludes the proof.

Let us now give some more details. Let \( Y = \sqcup_{i \in I} C_i \) be the decomposition of \( Y \) into the union of its connected components. If \( \tilde{C}_i \) is a choice of a connected component of \( p^{-1}(C_i) \) and \( \Gamma_i \) denotes the stabilizer of \( \tilde{C}_i \) in \( \Gamma \) then

\[
p^{-1}(C_i) = \bigcup_{\gamma \in \Gamma/\Gamma_i} \gamma \tilde{C}_i.
\]
We endow the set
\[ S = \Gamma \sqcup \bigsqcup_{i \in I} \Gamma / \Gamma_i \]
with the obvious structure of \( \Gamma \)-set and we choose a fundamental domain
\( \mathcal{F} \subseteq \tilde{X} \setminus \tilde{Y} \) for the \( \Gamma \)-action on \( \tilde{X} \setminus \tilde{Y} \). We define a \( \Gamma \)-equivariant map
\[ r: \tilde{X} \to S \]
as follows:
\[ r(\gamma x) = \begin{cases} 
\gamma & \text{if } x \in \mathcal{F}, \\
\gamma \Gamma_i & \text{if } x \in \tilde{C}_i.
\end{cases} \]
For every \( n \geq 0 \) we set \( \ell^\infty_{\text{alt}}(S^{n+1}) = \ell^\infty_{\text{alt}}(S^{n+1}, \mathbb{R}) \) and we define
\[ r^n: \ell^\infty_{\text{alt}}(S^{n+1}) \to C_{b_s}(\tilde{X}, \tilde{Y}), \quad r^n(f)(s) = f(r(s(e_0)), \ldots, r(s(e_n))). \]
The fact that \( r^n \) takes values in the module of special cochains is immediate, and clearly \( r^n \) is a norm non-increasing \( \Gamma \)-equivariant chain map extending the identity on \( \mathbb{R} \).

Recall now that, by Theorem 5.8, the complex \( C^*_b(\tilde{X}) \) provides a relatively injective strong resolution of \( \mathbb{R} \), so there exists a norm non-increasing \( \Gamma \)-equivariant chain map \( C^*_b(\tilde{X}) \to C^*_b(\Gamma, \mathbb{R}) \). Moreover, by composing the map provided by Theorem 4.23 and the obvious alternating operator we get a norm non-increasing \( \Gamma \)-equivariant chain map \( C^*_b(\Gamma, \mathbb{R}) \to \ell^\infty_{\text{alt}}(S^{*+1}) \). By composing these morphisms of normed \( \Gamma \)-complexes we finally get a norm non-increasing \( \Gamma \)-chain map
\[ \zeta^*: C^*_b(\tilde{X}) \to \ell^\infty_{\text{alt}}(S^{*+1}) \]
which extends the identity of \( \mathbb{R} \).

Let us now consider the composition \( \theta^* = r^* \circ \zeta^*: C^*_b(\tilde{X}) \to C^*_b(\tilde{X}, \tilde{Y}) \): it is a norm non-increasing \( \Gamma \)-chain map which extends the identity of \( \mathbb{R} \). It is now easy to check that the chain map \( \eta^*: C^*_b(X) \to C^*_b(X, Y) \) induced by \( \theta^* \) satisfies the required properties. In fact, \( \eta^* \) is obviously norm non-increasing. Moreover, the composition of \( \theta^* \) with the inclusion \( C^*_b(\tilde{X}, \tilde{Y}) \to C^*_b(\tilde{X}) \) extends the identity of \( \mathbb{R} \). Since \( C^*_b(\tilde{X}) \) is a relatively injective strong resolution of \( \mathbb{R} \), this implies in turn that this composition is \( \Gamma \)-homotopic to the identity of \( \mathbb{R} \), and this concludes the proof. \( \square \)

**Corollary 5.16.** Take \( \alpha \in H^n_b(X) \) and let \( \varepsilon > 0 \) be given. Then there exists a special cocycle \( f \in C^n_{b_s}(X, Y) \) such that
\[ [f] = \alpha, \quad \|f\|_{\infty} \leq \|\alpha\|_{\infty} + \varepsilon. \]
Recall now that \( C^n_{b_s}(X, Y) \subseteq C^n_b(X, Y) \) for every \( n \geq 2 \). Therefore, Corollary 5.16 implies that, for \( n \geq 2 \), the norm of every coclass in \( H^n_b(X) \) may be computed by taking the infimum over relative cocycles. This concludes the proof of Theorem 5.14.
CHAPTER 6

ℓ1-homology and duality

Complexes of cochains naturally arise by taking duals of complexes of chains. Moreover, the Universal Coefficient Theorem ensures that, at least when working with real coefficients, taking (co)homology commutes with taking duals. Therefore, cohomology with real coefficients is canonically isomorphic with the dual of homology with real coefficients. This result does not extend to bounded cohomology. However, even in the bounded case duality plays an important role in the study of the relations between homology and cohomology. We restrict our attention to the case with real coefficients.

1. Normed chain complexes and their topological duals

Before going into the study of the cases we are interested in, we introduce some general terminology and recall some general results proved in [MM85, L"oh08]. A normed chain complex is a complex $0 \leftarrow C_0 \leftarrow C_1 \leftarrow C_2 \leftarrow \cdots$ where every $C_i$ is a normed real vector space, and $d_i$ is bounded for every $i \in \mathbb{N}$. For notational convenience, in this section we will denote such a complex by the symbol $(C,d)$, rather than by $(C\_\bullet,d\_\bullet)$. If $C_n$ is complete for every $n$, then we say that $(C,d)$ is a Banach chain complex (or simply a Banach complex). Let $(C'\_\bullet) = (C\_\bullet)'$ be the topological dual of $C\_\bullet$, endowed with the operator norm, and denote by $\delta^i: (C'\_i) \rightarrow (C'\_{i+1})$ the dual map of $d_{i+1}$. Then the normed cochain complex

$0 \longrightarrow (C'^0) \xrightarrow{\delta^0} (C'^1) \xrightarrow{\delta^1} (C'^2) \xrightarrow{\delta^2} \cdots$

is called the normed dual complex of $(C,d)$, and it is denoted by $(C',\delta)$. Observe that the normed dual complex of any normed chain complex is Banach.

In several interesting cases, the normed spaces $C_i$ are not complete (for example, this is the case of singular chains on topological spaces, endowed with the $\ell^1$-norm – see below). Let us denote by $\hat{C}_i$ the completion of $C_i$. Being bounded, the differential $d_i: C_i \rightarrow C_{i-1}$ extends to a bounded map $\hat{d}_i: \hat{C}_i \rightarrow \hat{C}_{i-1}$, and it is obvious that $\hat{d}_i \circ \hat{d}_{i-1} = 0$ for every $i \in \mathbb{N}$, so we may consider the normed chain complex $(\hat{C},\hat{d})$. The topological dual of $\hat{C}_i$ is again $(C')^i$, and $\delta^i$ is the dual map of $\hat{d}_{i+1}$, so $(C',\delta)$ is the normed dual
6. $\ell^1$-HOMOLOGY AND DUALITY

The homology of the complex $(\hat{C}, \hat{d})$ is very different in general from the homology of $(C, d)$; however the inclusion $C \rightarrow C'$ induces a seminorm-preserving map in homology (see Corollary 6.3).

The homology (resp. cohomology) of the complex $(\hat{C}, \hat{d})$ (resp. $(C', \delta)$) is denoted by $H_\bullet(C)$ (resp. $H^\ast(C')$). As usual, the norms on $C_n$ and $(C')^n$ induce seminorms on $H_n(C)$ and $H^\ast_n(C')$ respectively. We will denote these (semi)norms respectively by $\| \cdot \|_1$ and $\| \cdot \|_\infty$. The duality pairing between $(C')^n$ and $C_n$ induces the Kronecker product

\[ \langle \cdot, \cdot \rangle : H^n_b(C') \times H_n(C) \rightarrow \mathbb{R}. \]

2. $\ell^1$-HOMOLOGY OF GROUPS AND SPACES

Let us introduce some natural examples of normed dual cochain complexes. Since we are restricting our attention to the case of real coefficients, for every group $\Gamma$ (resp. space $X$) we simply denote by $H^\ast_b(\Gamma)$ (resp. by $H^\ast_b(X)$) the bounded cohomology of $\Gamma$ (resp. of $X$) with real coefficients.

For every $n \geq 0$, the space $C_n(X) = C_n(X, \mathbb{R})$ may be endowed with the $\ell^1$-norm

\[ \| \sum a_i s_i \| = \sum |a_i| \]

(where the above sums are finite), which descends to a seminorm $\| \cdot \|_1$ on $H_n(X)$. The complex $C^\ast_b(X)$, endowed with the usual $\ell^\infty$-norm, coincides with the normed dual complex of $C_\bullet(X)$.

As explained above, to the normed chain complex $C_\bullet(X)$ there is associated the normed chain complex obtained by taking the completion of $C_n(X)$ for every $n \in \mathbb{N}$. Such a complex is denoted by $C^\ell_\bullet(X)$. An element $c \in C^\ell_n(X)$ is a sum

\[ \sum_{s \in S_n(X)} a_s s \]

such that

\[ \sum_{s \in S_n(X)} |a_s| < +\infty \]

and it is called an $\ell^1$-chain. The $\ell^1$-homology of $X$ is just the homology of the complex $C^\ell_\bullet(X)$, and it is denoted by $H^\ell_\bullet(X)$. The inclusion of singular chains into $\ell^1$-chains induces a norm non-increasing map

\[ H_\bullet(X) \rightarrow H^\ell_\bullet(X), \]

which is in general neither injective nor surjective.

The same definitions may be given in the context of groups. If $\Gamma$ is a group, for every $n \in \mathbb{N}$ we denote by $C_n(\Gamma) = C_n(\Gamma, \mathbb{R})$ the $\mathbb{R}$-vector space having $\Gamma^n$ as a basis, where we understand that $C_0(\Gamma) = \mathbb{R}$. For every $n \geq 2$
we define \( d_n : C_n(\Gamma) \to C_{n-1}(\Gamma) \) as the linear extension of the map

\[
(g_1, \ldots, g_n) \mapsto (g_2, \ldots, g_n) + \sum_{i=1}^{n-1} (-1)^i (g_1, \ldots, g_i, g_{i+1}, \ldots, g_n) + (-1)^n (g_1, \ldots, g_{n-1})
\]

and we set \( d_1 = d_0 = 0 \). It is easily seen that \( d_{n-1}d_n = 0 \) for every \( n \geq 1 \), and the homology \( H_\bullet(\Gamma) \) of \( \Gamma \) (with real coefficient) is defined as the homology of the complex \( (C_\bullet(\Gamma), d_\bullet) \).

Observe that \( C_i(\Gamma) \) admits the \( \ell^1 \)-norm defined by

\[
\| \sum a_{g_1, \ldots, g_i}(g_1, \ldots, g_i) \|_1 = \sum |a_{g_1, \ldots, g_i}|
\]

(where all the sums in the expressions above are finite), which descends in turn to a seminorm \( \| \cdot \|_1 \) on \( H_i(\Gamma) \). Moreover, we denote by \( C_i^\ell(\Gamma) \) the completion of \( C_i(\Gamma) \) with respect to the \( \ell^1 \)-norm, and by \( H_i^\ell(\Gamma) = H_i(C_i^\ell(\Gamma)) \) the corresponding \( \ell^1 \)-homology module, which is endowed with the induced seminorm.

It follows from the very definitions that topological duals of \( (C_i(\Gamma), \| \cdot \|_1) \) and \( (C_i^\ell(\Gamma), \| \cdot \|_1) \) is the Banach space \( (C_i^\ell(\Gamma), \| \cdot \|_\infty) \) of inhomogeneous cochains introduced in Section 7. Moreover, the dual map of \( d_i \) coincides with the differential \( \delta_i+1 \), so \( C_i^\bullet(\Gamma) \) is the dual normed cochain complex both of \( C_\bullet(\Gamma) \) and of \( C_i^\ell(\Gamma) \).

3. Duality: first results

As mentioned above, by the Universal Coefficient Theorem taking (co)homology commutes with taking algebraic duals. However, this is no more true when replacing algebraic duals with topological duals, so \( H^n_b(C') \) is not isomorphic to the topological dual of \( H_n(C) \) in general (but see Theorems 6.5 and 6.10 for the case when \( C \) is Banach). Nevertheless, the following results establish several useful relationships between \( H^n_b(C') \) and \( H_n(C) \).

If \( H \) is any seminormed vector space, then we denote by \( H' \) the space of bounded linear functionals on \( H \). So \( H' \) is canonically identified with the topological dual of the quotient of \( H \) by the subspace of elements with vanishing seminorm.

**Lemma 6.1.** Let \( (C,d) \) be a normed chain complex with dual normed chain complex \( (C',\delta) \). Then, the map

\[
H_b^q(C') \to (H_q(C))'
\]

induced by the Kronecker product is surjective. Moreover, for every \( \alpha \in H_n(C) \) we have

\[
\| \alpha \|_1 = \max \{ \langle \beta, \alpha \rangle \mid \beta \in H_b^n(C'), \| \beta \|_\infty \leq 1 \}.
\]
Proof. Let $B_q, Z_q$ be the spaces of cycles and boundaries in $C_q$. The space $(H_q(C))^\prime$ is isomorphic to the space of bounded functionals $Z_q \to \mathbb{R}$ that vanish on $B_q$. Any such element admits a bounded extension to $C_q$ by Hahn-Banach, and this implies the first statement of the lemma.

Let us come to the second statement. The inequality $\geq$ is obvious. Let $a \in C_n$ be a representative of $\alpha$. In order to conclude it is enough to find an element $b \in (C'_n)$ such that $\delta^n b = 0$, $b(a) = \|\alpha\|_1$ and $\|b\|_\infty \leq 1$. If $\|\alpha\|_1 = 0$ we may take $b = 0$. Otherwise, let $V \subseteq C_n$ be the closure of $d_{n+1} C_{n+1}$ in $C_n$, and put on the quotient $W := C_n/V$ the induced seminorm $\|\cdot\|_W$. Since $V$ is closed, such seminorm is in fact a norm. By construction, $\|\alpha\|_1 = \|[a]\|_W$. Therefore, Hahn-Banach Theorem provides a functional $\overline{b}: W \to \mathbb{R}$ with operator norm one such that $\overline{b}([a]) = \|\alpha\|_1$. We obtain the desired element $b \in (C'_n)$ by pre-composing $\overline{b}$ with the projection $C_n \to W$. □

Corollary 6.2. Let $(C,d_C)$, $(D,d_D)$ be normed chain complexes and let $\alpha: C \to D$ be a chain map of normed chain complexes (so $\alpha$ is bounded in every degree). If the induced map in bounded cohomology

$$H^n_b(\alpha): H^n_b(D') \to H^n_b(C')$$

is an isometric isomorphism, then the induced map in homology

$$H_n(\alpha): H_n(C) \to H_n(D)$$

preserves the seminorm.

Proof. From the naturality of the Kronecker product, for every $\alpha \in H_n(C)$, $\varphi \in H^n_b(D')$ we get

$$\langle H^n_b(\alpha)(\varphi), \alpha \rangle = \langle \varphi, H_n(\alpha)(\alpha) \rangle,$$

so the conclusion follows from Lemma 6.1. □

Corollary 6.3. Let $(C,d)$ be a normed chain complex and denote by $i: C \to \hat{C}$ the inclusion of $C$ in its metric completion. Then the induced map in homology

$$H_n(i): H_n(C) \to H_n(\hat{C})$$

preserves the seminorm for every $n \in \mathbb{N}$.

Proof. Since $i(C)$ is dense in $\hat{C}$, the map $i$ induces the identity on $C'$, whence on $H^*_b(C')$. The conclusion follows from Corollary 6.2. □

Corollary 6.3 implies that, in order to compute seminorms, one may usually reduce to the study of Banach chain complexes.

4. Some results by Matsumoto and Morita

In this section we describe some results taken from [MM85].
Definition 6.4. Let \((C, d)\) be a normed chain complex. Then \((C, d)\) satisfies the \(q\)-uniform boundary condition (or, for short, the \(q\)-UBC condition) if the following condition holds: there exists \(K \geq 0\) such that, if \(c\) is a boundary in \(C_q\), then there exists a chain \(z \in C_{q+1}\) with the property that 
\[d_{q+1}z = c \quad \text{and} \quad \|z\|_1 \leq K \cdot \|c\|_1.\]

Theorem 6.5 ([MM85], Theorem 2.3). Let \((C, d)\) be a Banach chain complex with Banach dual cochain complex \((C', \delta)\). Then the following conditions are equivalent:

1. \((C, d)\) satisfies \(q\)-UBC;
2. the seminorm on \(H_q(C)\) is a norm;
3. the seminorm on \(H_b^{q+1}(C')\) is a norm;
4. the Kronecker product induces an isomorphism \(H_b^{q+1}(C') \cong (H_{q+1}(C))'\).

Proof. Let us denote by \(Z_i\) and \(B_i\) the spaces of cycles and boundaries of degree \(i\) in \(C_i\), and by \(Z_i^I\) and \(B_i^I\) the spaces of cocycles and coboundaries of degree \(i\) in \((C')^I\).

1. \(\Leftrightarrow\) 2: We endow the space \(C_{q+1}/Z_{q+1}\) with the quotient seminorm. Being the kernel of a bounded map, the space \(Z_{q+1}\) is closed in \(C_{q+1}\), so the seminorm is a norm, and the completeness of \(C_{q+1}\) implies the completeness of \(C_{q+1}/Z_{q+1}\). Condition 1 is equivalent to the fact that the isomorphism \(C_{q+1}/Z_{q+1} \cong B_q\) induced by \(d_{q+1}\) is bi-Lipschitz. By the open mapping Theorem, this is in turn equivalent to the fact that \(B_q\) is complete. On the other hand, \(B_q\) is complete if and only if it is closed in \(Z_q\), i.e. if and only if condition 2 holds.

2. \(\Leftrightarrow\) 3: Condition 2 is equivalent to the fact that the range of \(d_{q+1}\) is closed, while (3) holds if and only if the range of \(\delta^q\) is closed. Now the closed range Theorem implies that, if \(f: V \to W\) is a bounded map between Banach spaces, then the range of \(f\) is closed if and only if the range of the dual map \(f': W' \to V'\) is closed, and this concludes the proof.

3. \(\Rightarrow\) 4: The surjectivity of the map \(H_b^{q+1}(C') \to (H_{q+1}(C))'\) is proved in Lemma 6.1. Let \(f \in Z_b^{q+1}(C')\) be such that \([f] = 0\) in \((H_{q+1}(C))'\). Then \(f|_{Z_{q+1}} = 0\). If (2) holds, then \(B_q\) is Banach, so by the open mapping Theorem the differential \(d_{q+1}\) induces a bi-Lipschitz isomorphism \(C_{q+1}/Z_{q+1} \cong B_q\). Therefore, we have \(f = g'd_{q+1}\), where \(g' \in (B_q)'\). By Hahn-Banach, \(g'\) admits a continuous extension to a map \(g \in (C')^q\), so \(f = \delta^q g\), and \([f] = 0\) in \(H_b^{q+1}(C')\).

The implication (4) \(\Rightarrow\) (3) follows from the fact that any element of \(H_b^{q+1}(C')\) with vanishing seminorm lies in the kernel of the map \(H_b^{q+1}(C') \to (H_{q+1}(C))'\). \(\square\)

Corollary 6.6 ([MM85], Corollary 2.4). Let \((C, d)\) be a Banach chain complex with Banach dual cochain complex \((C', \delta)\), and let \(q \in \mathbb{N}\). Then:

1. If \(H_q(C) = H_{q+1}(C) = 0\), then \(H_b^{q+1}(C') = 0\).
6. \(\ell^1\)-HOMOLOGY AND DUALITY

(2) If \(H_0^2(C') = H_1^2(C') = 0\), then \(H_q(C) = 0\).

(3) \(H_\bullet(C) = 0\) if and only if \(H_b^*(C') = 0\).

**Proof.** (1): By Theorem 6.5, if \(H_q(C) = 0\) then \(H_b^{q+1}(C') = 0\).

(2): By Theorem 6.5, if \(H_b^{q+1}(C') = 0\), then \(H_q(C)\) is Banach, so by

Hahn-Banach \(H_q(C)\) vanishes if and only if \((H_q(C))' = 0\). But \(H_b^0(C') = 0\)

implies \((H_q(C))' = 0\) by Lemma 6.1.

(3): By points (1) and (2), we only have to show that \(H_\bullet(C) = 0\) implies \(H_0^0(C') = 0\). But \(H_0(C) = 0\) implies that \(d_1: C_1 \to C_0\) is surjective, so

\(\delta^0: (C')^0 \to (C')^1\) is injective and \(H_b^0(C') = 0\).

□

As observed in [MM85], a direct application of Corollary 6.6 implies the vanishing of the \(\ell^1\)-homology of a countable CW-complex with amenable fundamental group. In fact, a stronger result holds: the \(\ell^1\)-homology of a countable CW-complex only depends on its fundamental group. The first proof of this fact is due to Bouarich [Bou04]. In Section 6 we will describe an approach to this result which is due to Loeh [Löh08], and which is very close in spirit to Matsumoto and Morita’s arguments.

We now show how Theorem 6.5 can be exploited to prove that, for every group \(\Gamma\), the seminorm of \(H^2_\bullet(\Gamma)\) is in fact a norm. An alternative proof of this fact is given in [Iva90].

**Corollary 6.7 ([MM85, Iva90]).** Let \(\Gamma\) be any group. Then \(H^2_\bullet(\Gamma)\)

is a Banach space, i.e. the canonical seminorm of \(H^2_\bullet(\Gamma)\) is a norm. If \(X\) is any countable CW-complex, then \(H^2_\bullet(X)\) is a Banach space.

**Proof.** By Theorem 5.9, the second statement is a consequence of the first one. Following [Mit84], let us consider the map

\[
S: \Gamma \to C^\ell_2(\Gamma), \quad S(g) = \sum_{k=1}^{\infty} 2^{-k}(g^k, g^k).
\]

We have \(\|S(g)\|_1 = 1\) and \(d_2S(g) = g\) for every \(g \in \Gamma\), so \(S\) extends to a bounded map \(S: C^\ell_1(\Gamma) \to C^\ell_2(\Gamma)\) such that \(d_2S = \text{Id}_{C^\ell_1(\Gamma)}\). This shows that \(C^\ell_\bullet(\Gamma)\) satisfies 1-UBC (and that \(H^\ell_1(\Gamma) = 0\)). Then the conclusion follows from Theorem 6.5. □

5. Injectivity of the comparison map

Let us now come back to the case when \((C, d)\) is a (possibly non-Banach) normed chain complex, and let \((\hat{C}, \hat{d})\) be the completion of \((C, d)\). For every \(i \in \mathbb{N}\) we denote by \((C^i)^* = (C_i)^*\) the algebraic dual of \(C_i\), and we consider the algebraic dual complex \((C^*, \delta)\) of \((C, d)\) (since this will not raise any ambiguity, we denote by \(\delta\) both the differential of the algebraic dual complex and the differential of the normed dual complex). We also denote by \(H^\bullet(C^*)\) the cohomology of \((C^*, \delta)\). In the case when \(C = C^\bullet(X)\) or \(C = C^\bullet(\Gamma)\) we have \(H^\bullet(C^*) = H^\bullet(X)\) or \(H^\bullet(C^*) = H^\bullet(\Gamma)\) respectively.
The inclusion of complexes \((C', \delta) \to (C^*, \delta)\) induces the comparison map

\[ c: H_b^*(C') \to H^*(C^*) \]

**Theorem 6.8 ([MM85], Theorem 2.8).** *Let us keep notation from the preceding paragraph. Then the following conditions are equivalent:

1. \((C, d)\) satisfies q-UBC.
2. \((\hat{C}, \hat{d})\) satisfies q-UBC and the space of \((q + 1)\)-cycles of \(C_{q+1}\) is dense in the space of \((q + 1)\)-cycles of \(\hat{C}_{q+1}\).
3. The comparison map \(c_{q+1}: H_b^{q+1}(C') \to H^{q+1}(C^*)\) is injective.

**Proof.** Let us denote by \(Z_i\) and \(B_i\) (resp. \(\hat{Z}_i\) and \(\hat{B}_i\)) the spaces of cycles and boundaries in \(C_i\) (resp. in \(\hat{C}_i\)), and by \(Z_{b_i}^i\) and \(B_{b_i}^i\) the spaces of cocycles and coboundaries in \((C^i)\).

1. \(\Rightarrow\) (2): Suppose that \((C, d)\) satisfies q-UBC with respect to the constant \(K \geq 0\), and fix an element \(b \in \hat{B}_q\). Since \(B_q\) is dense in \(\hat{B}_q\), it is easy to construct a sequence \(\{b_n\}_{n \in \mathbb{N}} \subseteq B_q\) such that \(\sum_{i=1}^{\infty} b_i = b\) and \(\sum_{i=1}^{\infty} ||b_i||_1 \leq 2||b||_1\). By (1), for every \(i\) we may choose \(c_i \in C_{q+1}\) such that \(d\hat{c}_i = b_i\) and \(||c_i||_1 \leq K||b_i||_1\) for a uniform \(K \geq 0\). In particular, the sum \(\sum_{i=1}^{\infty} c_i\) converges to an element \(c \in \hat{C}_{q+1}\) such that \(\hat{d}c = b\) and \(||c||_1 \leq 2K||b||_1\), so \((\hat{C}, \hat{d})\) satisfies q-UBC.

Let us now fix an element \(z = \lim_{i \to \infty} c_i \in \hat{Z}_{q+1}\), where \(c_i \in C_{q+1}\) for every \(i\). Choose an element \(c'_i \in C_{q+1}\) such that \(d\hat{c}'_i = -dc_i\) (so \(c_i + c'_i \in Z_{q+1}\)) and \(||c'_i||_1 \leq K||dc_i||_1\). Observe that

\[
||c'_i||_1 \leq K||dc_i||_1 = K||d(c_i - z)||_1 \leq (q + 2)K||c_i - z||_1,
\]

so \(\lim_{i \to \infty} c'_i = 0\). Therefore, we have \(z = \lim_{i \to \infty} (c_i + c'_i)\) and \(Z_{q+1}\) is dense in \(\hat{Z}_{q+1}\).

2. \(\Rightarrow\) (1): Suppose that \((\hat{C}, \hat{d})\) satisfies q-UBC with respect to the constant \(K \geq 0\), and fix an element \(b \in B_q\). Then there exist \(c \in C_{q+1}\) such that \(dc = b\) and \(c' \in \hat{C}_{q+1}\) such that \(d\hat{c}' = b\) and \(||c'||_1 \leq K||b||_1\). Since \(Z_{q+1}\) is dense in \(\hat{Z}_{q+1}\) we may find \(z \in Z_{q+1}\) such that \(||c - c' - z||_1 \leq ||b||_1\). Then we have \(d(c - z) = dc = b\) and

\[
||c - z||_1 \leq ||c - c' - z||_1 + ||c'||_1 \leq (K + 1)||b||_1.
\]

2. \(\Rightarrow\) (3): Take \(f \in Z_b^{q+1}\) such that \([f] = 0\) in \(H_b^{q+1}(C^*)\). By the Universal Coefficient Theorem, \(f\) vanishes on \(Z_{q+1}\). By density, \(f\) vanishes on \(\hat{Z}_{q+1}\), so it defines the null element of \((H_{q+1}(\hat{C}))'\). But Theorem 6.5 implies that, under our assumptions, the natural map \(H_b^{q+1}(C') \to (H_{q+1}(\hat{C}))'\) is injective, so \([f] = 0\) in \(H_b^{q+1}(C')\).

3. \(\Rightarrow\) (2): By the Universal Coefficient Theorem, the kernel of the map \(H_b^{q+1}(C') \to (H_{q+1}(C))'\) is contained in the kernel of the comparison map \(H_b^{q+1}(C') \to H^{q+1}(C^*)\), so Theorem 6.5 implies that \((\hat{C}, \hat{d})\) satisfies q-UBC.
Suppose now that $Z_{q+1}$ is not dense in $\hat{Z}_{q+1}$. Then there exists an element $f \in (C')^{q+1}$ which vanishes on $Z_{q+1}$ and is not null on $\hat{Z}_{q+1}$. The element $f$ vanishes on $B_{q+1}$, so it belongs to $Z_{b+1}$. Moreover, since $f|_{Z_{q+1}} = 0$ we have $[f] = 0$ in $H^{q+1}(C')$. On the other hand, since $f|_{\hat{Z}_{q+1}} \neq 0$ we have $[f] \neq 0$ in $H^{q+1}_b(C')$.

**Corollary 6.9.** If $\Gamma$ is a group and $q \geq 1$, then the comparison map $H^q_b(\Gamma) \to H^q(\Gamma)$ is injective if and only if $C_\bullet(\Gamma)$ satisfies $(q-1)$-UBC (observe that, if $q = 0, 1$, then the comparison map is obviously injective).

If $X$ is a topological space and $q \geq 1$, then the comparison map $H^q_b(X) \to H^q(X)$ is injective if and only if $C_\bullet(X)$ satisfies $(q-1)$-UBC (again, if $q = 0, 1$, then the comparison map is obviously injective).

### 6. The translation principle

We are now ready to prove that the homology of a Banach chain complex is completely determined by the bounded cohomology of its dual chain complex.

**Theorem 6.10 ([Löh08]).** Let $\alpha: C \to D$ be a chain map between Banach chain complexes, and let $H_\bullet(\alpha): H_\bullet(C) \to H_\bullet(D)$, $H^\bullet_b(\alpha): H^\bullet_b(D') \to H^\bullet_b(C')$ be the induced maps in homology and in bounded cohomology. Then:

1. The map $H_n(\alpha): H_n(C) \to H_n(D)$ is an isomorphism for every $n \in \mathbb{N}$ if and only if $H^\bullet_b(\alpha): H^\bullet_b(D') \to H^\bullet_b(C')$ is an isomorphism for every $n \in \mathbb{N}$.
2. If $H^\bullet_b(\alpha): H^\bullet_b(D') \to H^\bullet_b(C')$ is an isometric isomorphism for every $n \in \mathbb{N}$, then also $H_n(\alpha): H_n(C) \to H_n(D)$ is.

**Proof.** The proof is based on the fact that the question whether a given chain map induces an isomorphism in (co)homology may be translated into a question about the vanishing of the corresponding mapping cone, that we are now going to define.

For every $n \in \mathbb{N}$ we set

$$\text{Cone}(\alpha)_n = D_n + C_{n-1}$$

and we define a boundary operator $\partial_n: \text{Cone}(\alpha)_n \to \text{Cone}(\alpha)_{n-1}$ by setting

$$\partial_n(v,w) = (d^Dv + \alpha_{n-1}(w), -d^Cw),$$

where $d^C, d^D$ are the differentials of $C, D$ respectively. We also put on $\text{Cone}(\alpha)_n$ the norm obtained by summing the norms of its summands, thus endowing $\text{Cone}(\alpha)_\bullet$ with the structure of a Banach complex.

Dually, we set

$$\text{Cone}(\alpha)^n = (D')^n + (C')^{n-1}$$

and

$$\overline{\partial}_n(\varphi, \psi) = (-\delta_D\varphi, -\delta_C\psi - \alpha^n(\varphi)), $$
where \( \alpha^n : (D')^n \rightarrow (C')^n \) is the dual map of \( \alpha_n : C_n \rightarrow D_n \), and \( \delta_C, \delta_D \) are the differentials of \( C', D' \) respectively. Observe that \( \text{Cone}(\alpha)^n \), when endowed with the norm given by the maximum of the norms of its summands, is canonically isomorphic to the topological dual of \( \text{Cone}(\alpha)_n \) via the pairing

\[
(\varphi, \psi)(v, w) = \varphi(v) - \psi(w),
\]

and \( (\text{Cone}(\alpha)^\bullet, \delta^\bullet) \) coincides with the normed dual cochain complex of \( (\text{Cone}(\alpha)_\bullet, \delta_\bullet) \).

As a consequence, by Corollary 6.6 we have

\[
H^*_n(\text{Cone}(\alpha)_\bullet) = 0 \iff H^*_b(\text{Cone}(\alpha)^\bullet) = 0.
\]

Let us denote by \( \Sigma C \) the suspension of \( C \), i.e. the complex obtained by setting \( (\Sigma C)^n = C^{n-1} \), and consider the short exact sequence of complexes

\[
0 \rightarrow D \xrightarrow{i} \text{Cone}(\alpha)_\bullet \xrightarrow{\eta} \Sigma C \rightarrow 0
\]

where \( \iota(w) = (0, w) \) and \( \eta(v, w) = v \). Of course we have \( H_n(\Sigma C) = H_{n-1}(C) \), so the corresponding long exact sequence is given by

\[
\ldots \rightarrow H_n(\text{Cone}(\alpha)_\bullet) \rightarrow H_{n-1}(C) \xrightarrow{\partial} H_{n-1}(D) \rightarrow H_{n-1}(\text{Cone}(\alpha)_\bullet) \rightarrow \ldots
\]

Moreover, an easy computation shows that the connecting homomorphism \( \partial \) coincides with the map \( H_{n-1}(\alpha) \). As a consequence, the map \( H^*_n(\alpha) \) is an isomorphism in every degree if and only if \( H^*_n(\text{Cone}(\alpha)_\bullet) = 0 \).

A similar argument shows that \( H^*_b(\alpha) \) is an isomorphism in every degree if and only if \( H^*_b(\text{Cone}(\alpha)^\bullet) = 0 \), so point (1) of the theorem follows from (6).

Point (2) is now an immediate consequence of Corollary 6.2.

The converse of point (2) of the previous theorem does not hold in general (see [Löh08, Remark 4.6] for a counterexample).

**Corollary 6.11.** Let \( X \) be a path connected countable CW-complex with fundamental group \( \Gamma \). Then \( H^*_b(X) \) is isometrically isomorphic to \( H^*_b(\Gamma) \).

Therefore, the \( \ell^1 \)-homology of \( X \) only depends on its fundamental group.

**Proof.** We retrace the proof of Lemma 5.2 to construct a chain map of normed chain complexes \( r_\bullet : C^*_b(X) \rightarrow C^*_b(\Gamma) \). Let \( s \in S_n(X) \) be a singular simplex, let \( \tilde{X} \) be the universal covering of \( X \), and let \( \tilde{s} \) be a lift of \( s \) in \( S_n(\tilde{X}) \). We choose a set of representatives \( R \) for the action of \( \Gamma \) on \( \tilde{X} \), and we set

\[
r_n(s) = (g_0^{-1}g_1, \ldots, g_{n-1}^{-1}g_n) \in \Gamma^n,
\]

where \( g_i \in \Gamma \) is such that the \( i \)-th vertex of \( \tilde{s} \) lies in \( g_i(R) \). It is easily seen that \( r_n(s) \) does not depend on the chosen lift \( \tilde{s} \). Moreover, \( r_n \) extends to a well-defined chain map \( r_\bullet : C^*_b(X) \rightarrow C^*_b(\Gamma) \), whose dual map \( r^\bullet : C^*_b(\Gamma) \rightarrow C^*_b(X) \) coincides with (the restriction to \( \Gamma \)-invariants) of the map described in Lemma 5.2. Theorem 5.9 implies that \( r^\bullet \) induces an isometric isomorphism on bounded cohomology, so the conclusion follows from Theorem 6.10.
Corollary 6.12. Let $X, Y$ be countable CW-complexes and let $f: X \to Y$ be a continuous map inducing an epimorphism with amenable kernel on fundamental groups. Then

$$H^\ell_n(f): H^\ell_n(X) \to H^\ell_n(Y)$$

is an isometric isomorphism for every $n \in \mathbb{N}$.

Proof. The conclusion follows from Corollary 5.11 and Theorem 6.10. \qed
CHAPTER 7

Simplicial volume

The simplicial volume is an invariant of manifolds introduced by Gromov in his seminal paper [Gro82]. If \( M \) is a closed oriented manifold, then \( H_n(M, \mathbb{Z}) \) is infinite cyclic generated by the fundamental class \([M]\mathbb{Z}\) of \( M \). If \([M] = [M]_\mathbb{R} \in H_n(M, \mathbb{R})\) denotes the image of \([M]\mathbb{Z}\) via the change of coefficient map \( H_n(M, \mathbb{Z}) \to H_n(M, \mathbb{R})\), then the simplicial volume of \( M \) is defined as

\[
\|M\| = \|[M]\|_1 ,
\]

where \( \| \cdot \|_1 \) denotes the \( \ell^1 \)-norm on singular homology introduced in Chapter 6. The simplicial volume of \( M \) does not depend on the choice of the orientation, so it is defined for every orientable manifold. Moreover, if \( M \) is non-orientable and \( \widetilde{M} \) is the orientable double covering of \( M \), then the simplicial volume of \( M \) is defined by \( \|M\| = \|\widetilde{M}\|/2 \).

1. The case with non-empty boundary

Let \( M \) be a manifold with boundary. We identify \( C_\bullet(\partial M, \mathbb{R}) \) with its image in \( C_\bullet(M, \mathbb{R}) \) via the morphism induced by the inclusion, and observe that \( C_\bullet(\partial M, \mathbb{R}) \) is closed in \( C_\bullet(M, \mathbb{R}) \) with respect to the \( \ell^1 \)-norm. Therefore, the quotient seminorm \( \| \cdot \|_1 \) on \( C_n(M, \partial M, \mathbb{R}) = C_n(M, \mathbb{R})/C_n(\partial M, \mathbb{R}) \) is a norm, which endows \( C_n(M, \partial M, \mathbb{R}) \) with the structure of a normed chain complex.

If \( M \) is connected and orientable, then \( H_n(M, \partial M, \mathbb{Z}) \) is infinite cyclic generated by the element \([M, \partial M]\mathbb{Z}\), which is well-defined up to the sign. If \([M, \partial M] = [M, \partial M]_\mathbb{R} \) is the image of \([M, \partial M]\mathbb{Z}\) via the change of coefficient morphism, then the simplicial volume of \( M \) is defined by

\[
\|M, \partial M\| = \|[M, \partial M]\|_1 .
\]

Just as in the closed case we may extend the definition of \( \|M, \partial M\| \) to the case when \( M \) is not orientable.

The simplicial volume may be defined also for non-compact manifolds. However, we restrict here to the compact case. Unless otherwise stated, henceforth every manifold will be assumed to be compact and orientable. Moreover, since we will be mainly dealing with (co)homology with real coefficients, unless otherwise stated henceforth we simply denote by \( H_i(M, \partial M) \), \( H^i(M, \partial M) \) and \( H_b^i(M, \partial M) \) the modules \( H_i(M, \partial M, \mathbb{R}) \), \( H^i(M, \partial M, \mathbb{R}) \) and \( H_b^i(M, \partial M, \mathbb{R}) \). We adopt the corresponding notation for group (co)homology.
Even if it depends only on the homotopy type of a manifold, the simplicial volume is deeply related to the geometric structures that a manifold can carry. For example, closed manifolds which support negatively curved Riemannian metrics have nonvanishing simplicial volume, while the simplicial volume of closed manifolds with non-negative Ricci tensor is null (see [Gro82]). In particular, flat or spherical closed manifolds have vanishing simplicial volume, while closed hyperbolic manifolds have positive simplicial volume (see Section 5).

Several vanishing and nonvanishing results for the simplicial volume are available by now, but the exact value of nonvanishing simplicial volumes is known only in a very few cases. If $M$ is (the natural compactification of) a complete finite-volume hyperbolic $n$-manifold without boundary, then a celebrated result by Gromov and Thurston implies that the simplicial volume of $M$ is equal to the Riemannian volume of $M$ divided by the volume $v_n$ of the regular ideal geodesic $n$-simplex in hyperbolic space. The only other exact computation of nonvanishing simplicial volume of closed manifolds is for the product of two closed hyperbolic surfaces or more generally manifolds locally isometric to the product of two hyperbolic planes [BK08b]. The first exact computations of $\|M, \partial M\|$ for classes of 3-manifolds for which $\|\partial M\| > 0$ are given in [BFP]. Building on these examples, more values for the simplicial volume can be obtained by taking connected sums or amalgamated sums over submanifolds with amenable fundamental group.

In this monograph we will restrict ourselves to those results about the simplicial volume that more heavily depend on the dual theory of bounded cohomology. In doing so, we will compute the simplicial volume of closed hyperbolic manifolds (Theorem 7.4), prove Gromov Proportionality Principle for closed Riemannian manifolds (Theorem 7.3), and prove Gromov Additivity Theorem for manifolds obtained by gluings along boundary components with amenable fundamental group (Theorem 7.5).

2. Elementary properties of the simplicial volume

Let $M, N$ be $n$-manifolds and let $f: (M, \partial M) \to (N, \partial N)$ be a continuous map of pairs of degree $d$. The induced map on singular chains sends any single simplex to a single simplex, so it induces a norm non-increasing map

$H_\bullet(f): H_\bullet(M, \partial M) \to H_\bullet(N, \partial N)$.

As a consequence we have

\[
\|N, \partial N\| = \|[N, \partial N]\|_1 = \frac{\|H_n(f)([M, \partial M])\|_1}{|d|} \leq \frac{\|[M, \partial M]\|_1}{|d|} = \frac{\|M, \partial M\|}{|d|}.
\]

This shows that, if $\|N, \partial N\| > 0$, then the set of possible degrees of maps from $M$ to $N$ is bounded. If $M = N$, then iterating a map of degree
$d, d \notin \{-1, 0, 1\}$, we obtain maps of arbitrarily large degree, so we get the following:

**Proposition 7.1.** If the manifold $M$ admits a self-map of degree different from $-1, 0, 1$, then $\|M, \partial M\| = 0$. In particular, if $n \geq 1$ then $\|S^2\| = \|(S^1)^n\| = 0$, and if $n \geq 2$ then $\|D^n, \partial D^n\| = 0$.

The following result shows that the simplicial volume is multiplicative with respect to finite coverings:

**Proposition 7.2.** Let $M \to N$ be a covering of degree $d$ between manifolds (possibly with boundary). Then

$$\|M, \partial M\| = d \cdot \|N, \partial N\|.$$

**Proof.** If $\sum_{i \in I} a_i s_i \in C_*(N, \partial N, \mathbb{R})$ is a fundamental cycle for $N$ and $\overline{s}_j^d, j = 1, \ldots, d$ are the lifts of $s_i$ to $M$ (these lifts exist since $s_i$ is defined on a simply connected space), then $\sum_{i \in I} \sum_{j=1}^d a_i \overline{s}_j^d$ is a fundamental cycle for $M$, so taking the infimum over the representatives of $[N, \partial N]$ we get

$$\|M, \partial M\| \leq d \cdot \|N, \partial N\|.$$

Since a degree-$d$ covering is a map of degree $d$, the conclusion follows from (7). \qed

3. The simplicial volume of Riemannian manifolds

A Riemannian covering between Riemannian manifolds is a locally isometric topological covering. Recall that two Riemannian manifolds $M_1, M_2$ are *commensurable* if there exists a Riemannian manifold which is the total space of a finite Riemannian covering of $M_i$ for $i = 1, 2$. Since the Riemannian volume is multiplicative with respect to coverings, Proposition 7.2 implies that

$$\frac{\|M_1, \partial M_1\|}{\text{Vol}(M_1)} = \frac{\|M_2, \partial M_2\|}{\text{Vol}(M_2)}$$

for every pair $M_1, M_2$ of commensurable Riemannian manifolds. Gromov’s Proportionality Principle extends this property to pairs of manifolds which share the same Riemannian universal covering:

**Theorem 7.3 (Proportionality Principle [Gro82]).** Let $M$ be a Riemannian manifold. Then the ratio

$$\frac{\|M, \partial M\|}{\text{Vol}(M)}$$

only depends on the isometry type of the Riemannian universal covering of $M$.

In the case of hyperbolic manifolds, Gromov and Thurston computed the exact value of the proportionality constant appearing in Theorem 7.3:
Theorem 7.4 ([Thu79, Gro82]). Let $M$ be a closed hyperbolic $n$-manifold. Then
\[ \|M\| = \frac{\text{Vol}(M)}{v_n}, \]
where $v_n$ is the maximal value of the volumes of geodesic simplices in the hyperbolic space $\mathbb{H}^n$.

As mentioned above, Theorem 7.4 also holds in the case when $M$ is the natural compactification of a complete finite-volume hyperbolic $n$-manifold [Gro82, Thu79, Fra04, FP10, FM11, BBI] (such a compactification is homeomorphic to a compact topological manifold with boundary, so it also has a well-defined simplicial volume).

4. Simplicial volume and topological constructions

Let us now describe the behaviour of simplicial volume with respect to some standard operations, like taking connected sums or products, or performing surgery. The simplicial volume is additive with respect to connected sums [Gro82]. In fact, this is a consequence of a more general result concerning the simplicial volume of manifolds obtained by gluing manifolds along boundary components with amenable fundamental group.

Let $M_1, \ldots, M_k$ be oriented $n$-manifolds, and let us fix a pairing $(S^+_1, S^-_1), \ldots, (S^+_h, S^-_h)$ of some boundary components of $\sqcup_{j=1}^k M_j$, in such a way that every boundary component of $\sqcup_{j=1}^k M_j$ appears at most once among the $S^+_i$. For every $i = 1, \ldots, h$, let also $f_i: S^+_i \to S^-_i$ be a fixed orientation-reversing homeomorphism (since every $M_j$ is oriented, every $S^+_i$ inherits a well-defined orientation). We denote by $M$ the oriented manifold obtained by gluing $M_1, \ldots, M_k$ along $f_1, \ldots, f_h$, and we suppose that $M$ is connected.

For every $i = 1, \ldots, h$ we denote by $j^+(i)$ the index such that $S^+_i \subseteq M_{j^+(i)}$, and by $K_i^+$ the kernel of the map $\pi_1(S^+_i) \to \pi_1(M_{j^+(i)})$ induced by the inclusion. We say that the gluings $f_1, \ldots, f_h$ are compatible if the equality
\[ (f_i)_* (K_i^+) = K_i^- \]
holds for every $i = 1, \ldots, h$. Then we have the following:

Theorem 7.5 (Gromov Additivity Theorem [Gro82, BBF+14]). Let $M_1, \ldots, M_k$ be $n$-dimensional manifolds, $n \geq 2$, suppose that the fundamental group of every boundary component of every $M_j$ is amenable, and let $M$ be obtained by gluing $M_1, \ldots, M_k$ along (some of) their boundary components. Then
\[ \|M, \partial M\| \leq \|M_1, \partial M_1\| + \ldots + \|M_k, \partial M_k\|. \]
In addition, if the gluings defining $M$ are compatible, then
\[ \|M, \partial M\| = \|M_1, \partial M_1\| + \ldots + \|M_k, \partial M_k\|. \]
Of course, the gluings defining $M$ are automatically compatible if each $S^+_i$ is $\pi_1$-injective in $M_{j^+_i}$ (in fact, this is the case in the most relevant applications of Theorem 7.5). Even in this special case, no inequality between $\|M, \partial M\|$ and $\sum_{j=1}^k \|M_j, \partial M_j\|$ holds in general if we drop the requirement that the fundamental group of every $S^+_i$ be amenable (see Remark 7.8). On the other hand, even if the fundamental group of every $S^+_i$ is amenable, the equality in Theorem 7.5 does not hold in general for non-compatible gluings (see again Remark 7.8).

Let us mention two important corollaries of Theorem 7.5.

**Corollary 7.6 (Additivity for connected sums).** Let $M_1, M_2$ be closed $n$-dimensional manifolds, $n \geq 3$. Then

$$\|M_1 \# M_2\| = \|M_1\| + \|M_2\|,$$

where $M_1 \# M_2$ denotes the connected sum of $M_1$ and $M_2$.

**Proof.** Let $M'_i$ be $M_i$ with one open ball removed. Since the $n$-dimensional disk has vanishing simplicial volume and $S^{n-1}$ is simply connected, Theorem 7.5 implies that $\|M'_i, \partial M'_i\| = \|M_i\|$. Moreover, Theorem 7.5 also implies that $\|M_1 \# M_2\| = \|M'_1, \partial M'_1\| + \|M'_2, \partial M'_2\|$, and this concludes the proof. $\square$

Let now $M$ be a closed 3-dimensional manifold. The prime decomposition Theorem, the JSJ decomposition Theorem and Perelman’s proof of Thurston’s geometrization Conjecture imply that $M$ can be canonically cut along spheres and incompressible tori into the union of hyperbolic and Seifert fibered pieces. Since the simplicial volume of Seifert fibered spaces vanishes, Theorem 7.5 and Corollary 7.6 imply the following:

**Corollary 7.7 ([Gro82]).** Let $M$ be a 3-dimensional manifold, and suppose that either $M$ is closed, or it is bounded by incompressible tori. Then the simplicial volume of $M$ is equal to the sum of the simplicial volumes of its hyperbolic pieces.

A proof of Corollary 7.7 (based on results from [Thu79]) may be found also in [Som81].

**Remark 7.8.** The following examples show that the hypotheses of Theorem 7.5 should not be too far from being the weakest possible.

Let $M$ be a hyperbolic 3-manifold with connected geodesic boundary. It is well-known that $\partial M$ is $\pi_1$-injective in $M$. We fix a pseudo-Anosov homeomorphism $f: \partial M \to \partial M$, and for every $m \in \mathbb{N}$ we denote by $D_m M$ the twisted double obtained by gluing two copies of $M$ along the homeomorphism $f^m: \partial M \to \partial M$ (so $D_0 M$ is the usual double of $M$). It is shown in [Jun97] that

$$\|D_0 M\| < 2 \cdot \|M, \partial M\|.$$
On the other hand, by [Som98] we have \( \lim_{m \to \infty} \text{Vol} D_{m}M = \infty \), so \( \lim_{m \to \infty} \| D_{m}M \| = \infty \), and the inequality

\[
\| D_{m}M \| > 2 \cdot \| M, \partial M \|
\]

holds for infinitely many \( m \in \mathbb{N} \). This shows that, even in the case when each \( S_{\pm}^{i} \) is \( \pi_{1} \)-injective in \( M_{j \pm (i)} \), no inequality between \( \| M, \partial M \| \) and \( \sum_{j=1}^{k} \| M_{j}, \partial M_{j} \| \) holds in general if one drops the requirement that the fundamental group of every \( S_{\pm}^{i} \) be amenable.

On the other hand, if \( M_{1} \) is (the natural compactification of) the once-punctured torus and \( M_{2} \) is the 2-dimensional disk, then the manifold \( M \) obtained by gluing \( M_{1} \) with \( M_{2} \) along \( \partial M_{1} \cong \partial M_{2} \cong S^{1} \) is a torus, so

\[
\| M \| = 0 < 2 + 0 = \| M_{1}, \partial M_{1} \| + \| M_{2}, \partial M_{2} \| .
\]

This shows that, even in the case when the fundamental group of every \( S_{\pm}^{i} \) is amenable, the equality \( \| M, \partial M \| = \sum_{j=1}^{k} \| M_{j}, \partial M_{j} \| \) does not hold in general if one drops the requirement that the gluings be compatible.

Coming to products, it is readily seen that the product \( \Delta^{n} \times \Delta^{m} \) of two standard simplices of dimension \( n \) and \( m \) can be triangulated by \( \binom{n+m}{m} \) simplices of dimension \( n + m \). Using this fact it is easy to prove that, if \( M, N \) are closed manifolds of dimension \( m, n \) respectively, then

\[
(9) \quad \| M \times N \| \leq \binom{n+m}{m} \| M \| \cdot \| N \| .
\]

In fact, an easy application of duality implies the following stronger result:

**Proposition 7.9 ([Gro82]).** Let \( M, N \) be closed manifolds of dimension \( m, n \) respectively. Then

\[
\| M \| \cdot \| N \| \leq \| M \times N \| \leq \binom{n+m}{m} \| M \| \cdot \| N \| .
\]

In the following Chapters we will see how bounded cohomology may be profitably exploited in the proofs of Proposition 7.9 and Theorems 7.3, 7.4 and 7.5. More precisely, we prove Proposition 7.9 in Section 6 here below, Theorem 7.3 in Chapter 8, Theorem 7.4 in Chapter 9 and Theorem 7.5 in Chapter 10.

5. Simplicial volume and duality

Let \( M \) be an \( n \)-manifold with (possibly empty) boundary (recall that every manifold is assumed to be orientable and compact). Then, the topological dual of the relative chain module \( C_{i}(M, \partial M) \) is the relative cochain module \( C_{b}^{i}(M, \partial M) \), which may be canonically identified with the kernel of the map \( C_{i}^{b}(M) \to C_{i}^{b}(\partial M) \), i.e. with the submodule of \( C_{i}^{b}(M) \) given by those cochains that are null on chains supported in \( \partial M \). We denote by \( H_{b}^{i}(M, \partial M) \) the corresponding relative bounded cohomology module.
6. The simplicial volume of products

If \( \varphi \in H^i(M, \partial M) \) is a singular (unbounded) coclass, then we denote by \( \|\varphi\|_\infty \in [0, +\infty] \) the infimum of the norms of the representatives of \( \varphi \) in \( Z^n(M, \partial M) \). If \( c^i : H_b^i(M, \partial M) \to H^i(M, \partial M) \) is the comparison map induced by the inclusion of bounded relative cochains into relative cochains, then it is clear that, for every \( \varphi \in H^i(M, \partial M) \), we have

\[
\|\varphi\|_\infty = \inf \{\|\varphi_b\|_\infty \mid \varphi_b \in H_b^i(M, \partial M), \ c^i(\varphi_b) = \varphi\}
\]

We denote by \([M, \partial M]^* \in H^n(M, \partial M)\) the fundamental coclass of \( M \), i.e. the element \([M, \partial M]^* \in H^n(M, \partial M)\) such that

\[
\langle [M, \partial M]^*, [M, \partial M] \rangle = 1
\]

where \( \langle \cdot , \cdot \rangle \) denotes the Kronecker product (in Chapter 6 we defined the Kronecker product only in the context of bounded cohomology, but of course the same definition makes sense in the case of the usual singular cohomology).

**Proposition 7.10.** We have

\[
\|M, \partial M\| = \max \{\langle \varphi_b, [M, \partial M] \rangle \mid \varphi_b \in H_b^n(M, \partial M), \ |\varphi_b|_\infty \leq 1\},
\]

so

\[
\|M, \partial M\| = \begin{cases} 0 & \text{if } \| [M, \partial M]^* \|_\infty = +\infty \\ \| [M, \partial M]^* \|^{-1} & \text{otherwise.} \end{cases}
\]

**Proof.** The proposition follows from Lemma 6.1. \(\square\)

Proposition 7.10 implies the following non-trivial vanishing result:

**Corollary 7.11.** Let \( M \) be a closed \( n \)-manifold with amenable fundamental group. Then

\[
\|M\| = 0.
\]

In particular, any closed simply connected manifold has vanishing simplicial volume.

6. The simplicial volume of products

Let \( M, N \) be closed manifolds of dimension \( m, n \) respectively. As an easy application of duality, we now prove Proposition 7.9, i.e. we establish the inequalities

\[
\|M\| \cdot \|N\| \leq \|M \times N\| \leq \binom{n + m}{m} \|M\| \cdot \|N\|.
\]

Let \([M]^* \in H^m(M), [N]^* \in H^n(N)\) be the fundamental coclasses of \( M, N \) respectively. Then it is readily seen that

\[
[M]^* \cup [N]^* = [M \times N]^* \in H^n(M \times N),
\]

\[
\| [M \times N]^* \|_\infty = \| [M]^* \cup [N]^* \|_\infty \leq \| [M]^* \|_\infty \cdot \| [N]^* \|_\infty,
\]

where \( \cup \) denotes the cup product. So the inequality

\[
\|M \times N\| \geq \|M\| \cdot \|N\|
\]

follows from Proposition 7.10, and the conclusion follows from inequality (9).
CHAPTER 8

The proportionality principle

This section is devoted to the proofs of Theorems 7.3 and 7.4. The Proportionality Principle for the simplicial volume of Riemannian manifolds is due to Gromov [Gro82]. A detailed proof first appeared in [Löh06], where Löh exploited the approach via “measure homology” introduced by Thurston [Thu79]. Another proof is given in [BK08a], where Bucher follows an approach which is based on the use of bounded cohomology, and is closer in spirit to the original argument by Gromov (however, it may be worth mentioning that, in [Löh06], the proof of the fact that measure homology is isometric to the standard singular homology, which is the key step towards the proportionality principle, still relies on results about bounded cohomology). Here we closely follow the account on Gromov’s and Bucher’s approach to the proportionality principle described in [Fri11]. However, in order to avoid some technical subtleties, we restrict our attention to the study of non-positively curved compact Riemannian manifolds (see Section 8 for a brief discussion of the general case). Gromov’s approach to the proportionality principle exploits an averaging process which can be defined only on sufficiently regular cochains. As a consequence, we will consider only cochains which are continuous with respect to the compact-open topology on the space of singular simplices.

1. Continuous cohomology of topological spaces

Let $M$ be an $n$-dimensional manifold. For every $i \in \mathbb{N}$, we endow the space of singular $i$-simplices $S_i(M)$ with the compact-open topology (since the standard simplex is compact, this topology coincides with the topology of the uniform convergence with respect to any metric which induces the topology of $M$). For later reference we point out the following elementary property of the compact-open topology (see e.g. [Dug66, page 259]):

**Lemma 8.1.** Let $X, Y, Z$ be topological spaces, and $f: Y \to Z$, $g: X \to Y$ be continuous. The maps $f_*: F(X, Y) \to F(X, Z)$, $g^*: F(Y, Z) \to F(X, Z)$ defined by $f_*(h) = f \circ h$, $g^*(h) = h \circ g$ are continuous.

We say that a cochain $\varphi \in C^i(M)$ is continuous if it restricts to a continuous map on $S_i(M)$, and we denote by $C^*_c(M)$ the subcomplex of continuous cochains in $C^*(M)$ (the fact that $C^*_c(M)$ is indeed a subcomplex of $C^*(M)$ is a consequence of Lemma 8.1). We also denote by $C^*_{b,c}(M) = C^*_c(M) \cap C^*_b(M)$
the complex of bounded continuous cochains. The corresponding cohomology modules will be denoted by $H^\bullet_c(M)$ and $H^\bullet_{b,c}(M)$. The natural inclusions of cochains induce maps

$$H^\bullet(i^\bullet): H^\bullet_c(M) \to H^\bullet(M), \quad H^\bullet_b(i^\bullet): H^\bullet_{b,c}(M) \to H^\bullet_b(M).$$

Bott stated in [Bot75] that, at least for “reasonable spaces”, the map $i^\bullet$ is an isomorphism. However, Mostow asserted in [Mos76, Remark 2 at p. 27] that the natural proof of this fact seems to raise some difficulties. More precisely, it is quite natural to ask whether continuous cohomology satisfies Eilenberg-Steenrod axioms for cohomology. First of all, continuous cohomology is functorial thanks to Lemma 8.1. Moreover, it is not difficult to show that continuous cohomology satisfies the so-called “dimension axiom” and “homotopy axiom”. However, if $Y$ is a subspace of $X$ it is in general not possible to extend cochains in $C^\bullet_c(Y)$ to cochains in $C^\bullet_c(X)$, so that it is not clear if a natural long exact sequence for pairs actually exists in the realm of continuous cohomology. This difficulty can be overcome either by considering only pairs $(X,Y)$ where $X$ is metrizable and $Y$ is closed in $X$, or by exploiting a cone construction, as described in [Mdz09]. A still harder issue arises about excision: even if the barycentric subdivision operator consists of a finite sum (with signs) of continuous self-maps of $S^\bullet_s(X)$, the number of times a simplex should be subdivided in order to become “small” with respect to a given open cover depends in a decisive way on the simplex itself. These difficulties have been overcome independently by Mdzinarishvili and the author (see respectively [Mdz09] and [Fri11]). In fact, it turns out that the inclusion induces an isomorphism between continuous cohomology and singular cohomology for every space having the homotopy type of a metrizable and locally contractible topological space.

The question whether the map $H^\bullet_b(i^\bullet)$ is an isometric isomorphism is even more difficult. An affirmative answer is provided in [Fri11] in the case of spaces having the homotopy type of an aspherical CW-complex.

As anticipated above, we will restrict our attention to the case when $M$ is a manifold supporting a non-positively curved Riemannian structure. In this case the existence of a straightening procedure for simplices makes things much easier, and allows us to prove that $H^\bullet(i^\bullet)$ and $H^\bullet_b(i^\bullet)$ are both isometric isomorphisms via a rather elementary argument.

2. Continuous cochains as relatively injective modules

Until the end of the chapter we denote by $p: \tilde{M} \to M$ the universal covering of the closed smooth manifold $M$, and we fix an identification of $\Gamma = \pi_1(M)$ with the group of the covering automorphisms of $p$.

Recall that the complex $C^\bullet(\tilde{M})$ (resp. $C^\bullet_b(\tilde{M})$) is naturally endowed with the structure of an $\mathbb{R}[\Gamma]$-module (resp. normed $\mathbb{R}[\Gamma]$-module). For every $g \in \Gamma$, $i \in \mathbb{N}$, the map $S_i(\tilde{M}) \to S_i(M)$ sending the singular simplex $s$ to $g \cdot s$ is continuous with respect to the compact-open topology (see
Lemma 8.1. Therefore, the action of $\Gamma$ on (bounded) singular cochains on $\tilde{M}$ preserves continuous cochains, and $C^*_c(\tilde{M})$ (resp. $C^*_b,c(\tilde{M})$) inherits the structure of an $\mathbb{R}[\Gamma]$-module (resp. normed $\mathbb{R}[\Gamma]$-module). In this section we show that, for every $i \in \mathbb{N}$, the module $C^*_c(\tilde{M})$ (resp. $C^*_b,c(\tilde{M})$) is relatively injective according to Definition 4.1 (resp. Definition 4.9). The same result was proved for ordinary (i.e. possibly non-continuous) cochains in Chapter 4. The following lemma ensures that continuous cochains on $M$ canonically correspond to $G$-invariant continuous cochains on $\tilde{M}$.

**Lemma 8.2.** The chain map $p^*: C^*(M) \to C^*(\tilde{M})$ restricts to the following isometric isomorphisms of complexes:

$$p^*: C^*(M) \to C^*(\tilde{M})^\Gamma,$$

which, therefore, induce isometric isomorphisms

$$H^*_c(M) \cong H^*(C^*_c(\tilde{M})^\Gamma), \quad H^*_b,c(M) \cong H^*(C^*_b,c(\tilde{M})^\Gamma).$$

**Proof.** We have already used the obvious fact that $p^*$ is an isometric embedding on the space of $\Gamma$-invariant cochains, thus the only non-trivial issue to prove is the fact that $p^*(\varphi)$ is continuous if and only if $\varphi$ is continuous. However, it is not difficult to show that the map $p_n: s_n(\tilde{M}) \to s_n(M)$ induced by $p$ is a covering (see [Fri11, Lemma A.4] for the details). In particular, it is continuous, open and surjective, and this readily implies that if $\varphi: s_n(M) \to \mathbb{R}$ is any map, then $\varphi$ is continuous if and only if $\varphi \circ p_n: s_n(\tilde{M}) \to \mathbb{R}$ is continuous, whence the conclusion.

It is a standard fact of algebraic topology that the action of $\Gamma$ on $\tilde{M}$ is wandering, i.e. any $x \in \tilde{M}$ admits a neighbourhood $U_x$ such that $g(U_x) \cap U_x = \emptyset$ for every $g \in \Gamma \setminus \{1\}$. In the following lemma we describe a particular instance of generalized Bruhat function (see [Mon01, Lemma 4.5.4] for a more general result based on [Bou63, Proposition 8 in VII §2 N° 4]).

**Lemma 8.3.** There exists a continuous map $h_{\tilde{M}}: \tilde{M} \to [0,1]$ with the following properties:

1. For every $x \in \tilde{M}$ there exists a neighbourhood $W_x$ of $x$ in $\tilde{M}$ such that the set $\{g \in \Gamma | g(W_x) \cap W_x \neq \emptyset\}$ is finite.

2. For every $x \in \tilde{M}$, we have $\sum_{g \in \Gamma} h_{\tilde{M}}(g \cdot x) = 1$ (note that the sum on the left-hand side is finite by (1)).

**Proof.** Let us take a locally finite open cover $\{U_i\}_{i \in I}$ of $X$ such that for every $i \in I$ there exists $V_i \subseteq M$ with $p^{-1}(U_i) = \bigcup_{g \in \Gamma} g(V_i)$ and $g(V_i) \cap g'(V_i) = \emptyset$ whenever $g \neq g'$. Let $\{\varphi_i\}_{i \in I}$ be a partition of unity adapted to $\{U_i\}_{i \in I}$. It is easily seen that the map $\psi_i: \tilde{M} \to \mathbb{R}$ which coincides with $\varphi_i \circ p$ on $V_i$ and is null elsewhere is continuous. We can now set $h_{\tilde{M}} = \sum_{i \in I} \psi_i$. Since $\{U_i\}_{i \in I}$ is locally finite, also $\{V_i\}_{i \in I}$, whence $\{\text{supp} \psi_i\}_{i \in I}$, is locally finite, so $h_{\tilde{M}}$ is indeed well-defined and continuous.
8. THE PROPORTIONALITY PRINCIPLE

It is easy to check that $h_{\tilde{M}}$ indeed satisfies the required properties (see [Fri11, Lemma 5.1] for the details).

□

**Proposition 8.4.** For every $n \geq 0$ the modules $C^n_c(\tilde{M})$ and $C^n_{b,c}(\tilde{M})$ are relatively injective (resp. as an $\mathbb{R}[\Gamma]$-module and as a normed $\mathbb{R}[\Gamma]$-module).

**Proof.** Let $\iota: A \to B$ be an injective map between $\mathbb{R}[\Gamma]$-modules, with left inverse $\sigma: B \to A$, and suppose we are given a $\Gamma$-map $\alpha: A \to C^n_c(\tilde{M})$. We denote by $e_0, \ldots, e_n$ the vertices of the standard $n$-simplex, and define $\beta: B \to C^n_c(\tilde{M})$ as follows: given $b \in B$, the cochain $\beta(b)$ is the unique linear extension of the map that on the singular simplex $s$ takes the following value:

$$\beta(b)(s) = \sum_{g \in \Gamma} h_{\tilde{M}}\left(g^{-1}(s(e_0))\right) \cdot (\alpha(g(\sigma(g^{-1}(b))))(s)),$$

where $h_{\tilde{M}}$ is the map provided by Lemma 8.3. By Lemma 8.3–(1), the sum involved is in fact finite, so $\beta$ is well-defined. Moreover, for every $b \in B$, $g_0 \in \Gamma$ and $s \in S_n(\tilde{M})$ we have

$$\beta(g_0 \cdot b)(s) = \sum_{g \in \Gamma} h_{\tilde{M}}\left(g^{-1}(s(e_0))\right) \cdot \alpha\left(g(\sigma(g^{-1}(g_0(b))))\right)(s)$$

$$= \sum_{g \in \Gamma} h_{\tilde{M}}\left(g^{-1}(g_0^{-1} \cdot s(e_0))\right) \cdot \alpha\left(g_0 g_0^{-1} g(\sigma(g^{-1}(b))))\right)(s)$$

$$= \sum_{k \in \Gamma} h_{\tilde{M}}\left(k^{-1}(g_0^{-1} \cdot s(e_0))\right) \cdot \alpha\left(g_0 k(\sigma(k^{-1}(b))))\right)(s)$$

$$= \beta(b)(g_0^{-1} \cdot s) = (g_0 \cdot \beta(b))(s),$$

so $\beta$ is a $\Gamma$-map. Finally,

$$\beta(\iota(b))(s) = \sum_{g \in \Gamma} h_{\tilde{M}}\left(g^{-1}(s(e_0))\right) \cdot \alpha\left(g(\sigma(g^{-1}(\iota(b))))\right)(s)$$

$$= \sum_{g \in \Gamma} h_{\tilde{M}}\left(g^{-1}(s(e_0))\right) \cdot \alpha\left(g(\sigma(\iota(g^{-1} \cdot b)))\right)(s)$$

$$= \sum_{g \in \Gamma} h_{\tilde{M}}\left(g^{-1}(s(e_0))\right) \cdot \alpha(b)(s)$$

$$= \left(\sum_{g \in \Gamma} h_{\tilde{M}}\left(g^{-1}(s(e_0))\right)\right) \cdot \alpha(b)(s) = \alpha(b)(s),$$

so $\beta \circ \iota = \alpha$. In order to conclude that $C^n_c(\tilde{M})$ is relatively injective we need to show that $\beta(b)$ is indeed continuous. However, if $s \in S_n(\tilde{M})$ is a singular $n$-simplex, then by Lemma 8.3–(1) there exists a neighbourhood $U$ of $s$ in $S_n(\tilde{M})$ such that the set $\{g \in \Gamma | h_{\tilde{M}}(g^{-1}(s'(e_0))) \neq 0 \text{ for some } s' \in U\}$ is finite. This readily implies that if $\alpha(A) \subseteq C^n_c(\tilde{M})$, then also $\beta(B) \subseteq C^n_c(\tilde{M})$.

The same argument applies verbatim if $C^n(\tilde{M})$ is replaced by $C^n_b(\tilde{M})$, and $A, B$ are normed modules: in fact, if $\alpha$ is bounded and $\|\sigma\| \leq 1$, then also $\beta$ is bounded, and $\|\beta\| \leq \|\alpha\|$. Thus $C^n_{b,c}(\tilde{M})$ is a relatively injective normed $\Gamma$-module. □
3. Continuous cochains as strong resolutions of $\mathbb{R}$

Recall from Section 4 that, if $\tilde{M}$ is contractible, then the complex $C^\bullet(\tilde{M})$, endowed with the obvious augmentation, provides a strong resolution of the trivial $\Gamma$-module $\mathbb{R}$. Moreover, thanks to Ivanov’s Theorem 5.8, the augmented normed complex $C^\bullet_b(\tilde{M})$ is a strong resolution of the normed $\Gamma$-module $\mathbb{R}$ even without the assumption that $\tilde{M}$ is contractible. The situation is a bit different when dealing with continuous cochains. Unfortunately, we are not able to prove that Ivanov’s contracting homotopy (see Section 4) preserves continuous cochains, so it is not clear whether $C^\bullet_{b,c}(\tilde{M})$ always provides a strong resolution of $\mathbb{R}$ in the general case.

In order to prove that the augmented complexes $C^\bullet_c(\tilde{M})$ and $C^\bullet_{b,c}(\tilde{M})$ are strong we need to make some further assumption on $\tilde{M}$. Namely, asking that $\tilde{M}$ is contractible (i.e. that $\tilde{M}$ is a $K(\Gamma,1)$) would be sufficient. However, to our purposes it is sufficient to concentrate our attention on the easier case when $M$ supports a non-positively curved Riemannian metric. In that case, Cartan-Hadamard Theorem implies that, for every point $x \in \tilde{M}$, the exponential map at $x$ establishes a diffeomorphism between the tangent space at $x$ and the manifold $\tilde{M}$. As a consequence, every pair of points in $\tilde{M}$ are joined by a unique geodesic, and (constant-speed parameterizations of) geodesics continuously depend on their endpoints: we say that $\tilde{M}$ is continuously uniquely geodesic.

For $i \in \mathbb{N}$ we denote by $e_i \in \mathbb{R}^N$ the point $(0, \ldots, 1, \ldots)$, where the unique non-zero coefficient is at the $i$-th entry (entries are indexed by $\mathbb{N}$, so $(1,0,\ldots) = e_0$). We denote by $\Delta^p$ the standard $p$-simplex, i.e. the convex hull of $e_0, \ldots, e_p$, and we observe that with these notations we have $\Delta^p \subseteq \Delta^{p+1}$.

**Proposition 8.5.** The complexes $C^\bullet_c(\tilde{M})$ and $C^\bullet_{b,c}(\tilde{M})$ are strong resolutions of $\mathbb{R}$ (resp. as an unbounded $\Gamma$-module and as a normed $\mathbb{R}$-module).

**Proof.** Let us choose a basepoint $x_0 \in \tilde{M}$. For $n \geq 0$, we define an operator $T_n: C_n(\tilde{M}) \to C_{n+1}(\tilde{M})$ which sends any singular $n$-simplex $s$ to the $(n+1)$-simplex obtained by coning $s$ over $x_0$. In order to properly define the needed coning procedure we will exploit the fact that $\tilde{M}$ is uniquely continuously geodesic.

For $x \in \tilde{M}$ we denote by $\gamma_x: [0,1] \to \tilde{M}$ the constant-speed parameterization of the geodesic joining $x_0$ with $x$. Let $n \geq 0$. We denote by $Q_0$ the face of $\Delta^{n+1}$ opposite to $e_0$, and we consider the identification $r: Q_0 \to \Delta^n$ given by $r(t_1 e_1 + \ldots + t_{n+1} e_{n+1}) = t_1 e_0 + \ldots + t_{n+1} e_n$. We now define $T_n: C_n(\tilde{M}) \to C_{n+1}(\tilde{M})$ as the unique linear map such that, if $s \in S_n(\tilde{M})$, then the following holds: if $p = t e_0 + (1-t)q \in \Delta^{n+1}$, where $q \in Q_0$, then $(T_n(s))(p) = \gamma_{s(q)}(t)$. In other words, $T_n(s)$ is just the geodesic cone over $s$ with vertex $x_0$. Using that $\tilde{M}$ is uniquely continuously geodesic, one may easily check that $T_n(s)$ is well-defined and continuous. Moreover,
the restriction of $T_n$ to $S_n(\widetilde{M})$ defines a map

$$T_n: S_n(\widetilde{M}) \rightarrow S_{n+1}(\widetilde{M})$$

which is continuous with respect to the compact-open topology. We finally define $T_{-1}: \mathbb{R} \rightarrow C_0(\widetilde{M})$ by $T_{-1}(t) = tx_0$. It is readily seen that, if $d_\bullet$ is the usual (augmented) differential on singular chains, then $d_0 T_{-1} = \text{Id}_{\mathbb{R}}$, and for every $n \geq 0$ we have $T_{n-1} \circ d_n + d_{n+1} \circ T_n = \text{Id}_{C_n(\widetilde{M})}$.

For every $n \geq 0$, let now $k^n: C^n(\widetilde{M}) \rightarrow C^{n-1}(\widetilde{M})$ be defined by $k^n(\varphi)(c) = \varphi(T_{n-1}(c))$. Since $T_n: S_n(\widetilde{M}) \rightarrow S_{n+1}(\widetilde{M})$ is continuous, the map $k^n$ preserves continuous cochains, so $\{k^n\}_{n \in \mathbb{N}}$ provides a contracting homotopy for the complex $C_\bullet(\widetilde{M})$, which is therefore a strong resolution of $\mathbb{R}$ as an unbounded $\mathbb{R}[\Gamma]$-module.

Finally, since $\|T_n\| = 1$ sends a simplex to a simplex, if $\alpha \in C^n(\widetilde{M})$ then $\|k^n(\alpha)\| \leq \|\alpha\|$. Thus $k^n$ restricts to a contracting homotopy for the complex of normed $\Gamma$-modules $C_{b,c}(\widetilde{M})$. Therefore, this complex gives a strong resolution of $\mathbb{R}$ as a normed $\Gamma$-module.

We have thus proved that, if $M$ supports a non-positively curved metric, then $C_\bullet^c(\widetilde{M})$ and $C_{b,c}(\widetilde{M})$ provide relatively injective strong resolutions of $\mathbb{R}$ (as an unbounded $\Gamma$-module and as a normed $\Gamma$-module, respectively). As proved in Lemma 4.6, Proposition 4.7, Lemma 5.1 and Lemma 5.4, the same is true for the complexes $C_\bullet^c(\widetilde{M})$ and $C_{b,c}(\widetilde{M})$. Since the inclusions $C_\bullet^c(\widetilde{M}) \rightarrow C_\bullet(\widetilde{M}), C_{b,c}(\widetilde{M}) \rightarrow C_{b,c}(\widetilde{M})$ are norm non-increasing chain maps which extend the identity of $\mathbb{R}$, Theorems 4.4 and 4.14 and Lemma 8.2 imply the following:

**Proposition 8.6.** Let $M$ be a closed manifold supporting a non-positively curved metric. Then the maps

$$H^\bullet(i^\bullet): H^\bullet_c(M) \rightarrow H^\bullet(M), \quad H^\bullet_b(i^\bullet): H^\bullet_{b,c}(M) \rightarrow H^\bullet_b(M)$$

are norm non-increasing isomorphisms.

In order to promote these isomorphisms to isometries it is sufficient to exhibit norm non-increasing chain $\Gamma$-maps $\theta^\bullet: C_\bullet(\widetilde{M}) \rightarrow C_\bullet^c(\widetilde{M}), \theta^\bullet_b: C^*_b(\widetilde{M}) \rightarrow C^*_b(\widetilde{M})$ which extend the identity of $\mathbb{R}$. To this aim we exploit a straightening procedure, which will prove useful several times in this book.

4. **Straightening in non-positive curvature**

The *straightening procedure* for simplices was introduced by Thurston in [Thu79]. It was originally defined on hyperbolic manifolds, but, as may authors have already noticed, it may be performed in the more general context of non-positively curved Riemannian manifolds.

We assume that the closed manifold $M$ is endowed with a fixed non-positively curved Riemannian metric, and we recall that, under this assumption, the universal covering $\widetilde{M}$ is uniquely continuously geodesic. Let $k \in \mathbb{N}$,
and let \( x_0, \ldots, x_k \) be points in \( \tilde{M} \). The straight simplex \([x_0, \ldots, x_k] \in S_k(\tilde{M})\) with vertices \( x_0, \ldots, x_k \) is defined as follows: if \( k = 0 \), then \([x_0]\) is the 0-simplex with image \( x_0 \); if straight simplices have been defined for every \( h \leq k \), then \([x_0, \ldots, x_{k+1}] : \Delta^{k+1} \to \tilde{M} \) is determined by the following condition: for every \( z \in \Delta^k \subseteq \Delta^{k+1} \), the restriction of \([x_0, \ldots, x_{k+1}] \) to the segment with endpoints \( z, e_{k+1} \) is a constant speed parameterization of the geodesic joining \([x_0, \ldots, x_k](z)\) to \( x_{k+1} \). The fact that \([x_0, \ldots, x_{k+1}] \) is well-defined and continuous is an immediate consequence of the fact that \( \tilde{M} \) is continuously uniquely geodesic.

5. Continuous cohomology versus singular cohomology

We are now ready to prove that (bounded) continuous cohomology is isometrically isomorphic to (bounded) cohomology at least for non-positively curved manifolds:

**Proposition 8.7.** Let \( M \) be a closed manifold supporting a non-positively curved metric. Then the maps

\[
H^\bullet(i) : C^\bullet_c(M) \to C^\bullet_c(M), \quad H^\bullet_b(i) : C^\bullet_{b,c}(M) \to C^\bullet_b(M),
\]

are isometric isomorphisms.

**Proof.** As observed at the end of Section 3, it is sufficient to exhibit norm non-increasing chain \( \Gamma \)-maps \( \theta^\bullet : C^\bullet(\tilde{M}) \to C^\bullet_c(\tilde{M}), \theta^\bullet_b : C^\bullet_{b,c}(\tilde{M}) \to C^\bullet_b(\tilde{M}) \) which extend the identity of \( \mathbb{R} \). To this aim, we choose a basepoint \( x_0 \in \tilde{M} \). If \( \varphi \in C^n(\tilde{M}) \) and \( s \in S_n(\tilde{M}) \), then we set

\[
\theta^n(\varphi)(s) = \sum_{(g_0, \ldots, g_n) \in \Gamma^{n+1}} h_{\tilde{M}}(g_0^{-1} s(e_0)) \cdots h_{\tilde{M}}(g_n^{-1} s(e_n)) \cdot \varphi([g_0 x_0, \ldots, g_n x_0]).
\]

In other words, the value of \( \theta^n(\varphi) \) is the weighted sum of the values taken by \( \varphi \) on straight simplices with vertices in the orbit of \( x_0 \), where weights continuously depend on the position of the vertices of \( s \). Using the properties of \( h_{\tilde{M}} \) described in Lemma 8.3 it is easy to check that \( \theta^\bullet \) is a well-defined chain \( \Gamma \)-map which extends the identity of \( \mathbb{R} \). Moreover, \( \theta^\bullet \) is obviously norm non-increasing in every degree, so it restricts a norm non increasing chain \( \Gamma \)-map \( \theta^\bullet_c \), and this concludes the proof.

\[\square\]

6. The transfer map

Until the end of the chapter, we assume that the closed manifold \( M \) is endowed with a non-positively curved Riemannian metric. Moreover, we denote by \( G \) the group of orientation–preserving isometries of \( \tilde{M} \). It is well-known that \( G \) admits a Lie group structure inducing the compact–open topology. Moreover, there exists on \( G \) a left-invariant regular Borel measure \( \mu_G \), which is called Haar measure of \( G \) and is unique up to scalars. Since \( G \) contains a cocompact subgroup, its Haar measure is in fact also right-invariant [Sau02, Lemma 2.32]. Since \( \Gamma \) is discrete in \( G \) and \( M \cong \tilde{M}/\Gamma \) is
compact, there exists a Borel subset $F \subseteq G$ with the following properties:
the family $\{\gamma \cdot F\}_{\gamma \in \Gamma}$ provides a locally finite partition of $G$ (in particular,
$F$ contains exactly one representative for each left coset of $\Gamma$ in $G$), and $F$
is relatively compact in $G$. From now on, we normalize the Haar measure
$\mu_G$ in such a way that $\mu_G(F) = 1$.

For $H = \Gamma, G$, we will consider the homology $H^\bullet(C^\bullet_\gamma(\tilde{M})^H)$ of the
complex given by $H$-invariant cochains in $C^\bullet_\gamma(\tilde{M})$. We also endow $H^\bullet(C^\bullet_\gamma(\tilde{M})^H)$
with the seminorm induced by $C^\bullet_\gamma(\tilde{M})^H$ (this seminorm is not finite in
general). Recall that by Lemma 8.2 we have an isometric isomorphism
$H^\bullet(C^\bullet_\gamma(\tilde{M})^\Gamma) \cong H^\bullet_\gamma(\tilde{M})$. The chain inclusion $C^\bullet_\gamma(\tilde{M})^\Gamma \rightarrow C^\bullet_\gamma(\tilde{M})^\gamma$
induces a norm non-increasing map

$$\text{res}^\bullet : H^\bullet(C^\bullet_\gamma(\tilde{M})^G) \rightarrow H^\bullet(C^\bullet_\gamma(\tilde{M})^\Gamma) \cong H^\bullet_\gamma(\tilde{M}).$$

Following [BK08a], we will now construct a norm non-increasing left
inverse of $\text{res}^\bullet$. Take $\varphi \in C^\bullet_\gamma(\tilde{M})$ and $s \in C^\bullet(\tilde{M})$, and consider the function
$f^s_\gamma : G \rightarrow \mathbb{R}$ defined by $f^s_\gamma(g) = \varphi(g \cdot s)$. By Lemma 8.1 $f^s_\gamma$ is continuous,
whence bounded on the relatively compact subset $F \subseteq G$. Therefore, a well-
de fined cochain $\text{trans}^i(\varphi) \in C^\bullet(\tilde{M})$ exists such that for every $s \in S_i(\tilde{M})$ we have

$$\text{trans}^i(\varphi)(s) = \int_F f^s_\gamma(g) d\mu_G(g) = \int_F \varphi(g \cdot s) d\mu_G(g).$$

**Proposition 8.8.** The cochain $\text{trans}^i(\varphi)$ is continuous. Moreover, if
$\varphi$ is $\Gamma$–invariant, then $\text{trans}^i(\varphi)$ is $G$–invariant, while if $\varphi$ is $G$–invariant,
then $\text{trans}^i(\varphi) = \varphi$.

**Proof:** For $s, s' \in S_i(\tilde{M})$ we set $d(s, s') = \sup_{x \in \Delta_i} d_{\tilde{M}}(s(q), s'(q))$. Since
$\Delta_i$ is compact, $d_S$ is a well-defined distance on $S_i(\tilde{M})$, and it induces the
compact-open topology on $S_i(\tilde{M})$.

Let now $s_0 \in S_i(\tilde{M})$ and $\varepsilon > 0$ be fixed. By Lemma 8.1, the set $\mathcal{F} \cdot s_0 \subseteq
S_i(\tilde{M})$ is compact. Since $\varphi$ is continuous, this easily implies that there exists
$\eta > 0$ such that $|\varphi(s_1) - \varphi(s_2)| \leq \varepsilon$ for every $s_1 \in \mathcal{F} \cdot s_0$, $s_2 \in B_{d_S}(s_1, \eta)$,
where $B_{d_S}(s_1, \eta)$ is the open ball of radius $\eta$ centered at $s_1$. Take now
$s \in B_{d_S}(s_0, \eta)$. Since $G$ acts isometrically on $S_i(X)$, for every $g \in F$ we have
distance of radius $\eta$ centered at $s_1$. Take now
$s \in B_{d_S}(s_0, \eta)$. Since $G$ acts isometrically on $S_i(X)$, for every $g \in F$ we have
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distance of radius $\eta$ centered at $s_1$. Take now
$s \in B_{d_S}(s_0, \eta)$. Since $G$ acts isometrically on $S_i(X)$, for every $g \in F$ we have

$$|\text{trans}^i(\varphi)(s) - \text{trans}^i(\varphi)(s_0)| \leq \int_F |\varphi(g \cdot s) - \varphi(g \cdot s_0)| d\mu_G(g) \leq \varepsilon.$$
\{γ \cdot F \cdot g_0\}_{γ ∈ Γ} is locally finite in G. Together with the compactness of F, this readily implies that a finite number of distinct elements γ_1, ..., γ_r ∈ Γ exist such that F admits the finite partition

\[ F = \bigsqcup_{i=1}^{r} F_i, \]

where \( F_i = F \cap (γ_i^{-1} \cdot F \cdot g_0) \) for every \( i = 1, \ldots, r \). As a consequence we also have

\[ F \cdot g_0 = \bigsqcup_{i=1}^{r} γ_i \cdot F_i. \]

Let us now fix an element \( ϕ \in C^q_c(\tilde{M})^Γ \) and a simplex \( s ∈ S_q(\tilde{M}) \). Using the \( Γ \)-invariance of \( ϕ \) and the (left) \( G \)-invariance of \( dμ_G \), for every \( i = 1, \ldots, r \) we obtain

\[ \int_{F_i} ϕ(g \cdot s) \, dμ_G(g) = \int_{γ_i F_i} ϕ(g \cdot s) \, dμ_G(g) = \int_{γ_i F_i} ϕ(g \cdot s) \, dμ_G(g). \]

Therefore we get

\[ \text{trans}^q(ϕ)(g_0 \cdot s) = \int_F ϕ(g g_0 \cdot s) \, dμ_G(g) = \int_{F \cdot g_0} ϕ(g \cdot s) \, dμ_G(g) \]

\[ = \sum_{i=1}^{r} \int_{γ_i F_i} ϕ(g \cdot s) \, dμ_G(g) = \sum_{i=1}^{r} \int_{F_i} ϕ(g \cdot s) \, dμ_G(g) \]

\[ = \int_F ϕ(g \cdot s) \, dμ_G(g) = \text{trans}^q(ϕ)(s), \]

where the second equality is due to the (right) \( G \)-invariance of \( dμ_G \), and the fourth equality is due to equation (10). We have thus shown that, if \( ϕ \) is \( Γ \)-invariant, then \( \text{trans}^q(ϕ) \) is \( G \)-invariant, whence the conclusion. \( \square \)

Proposition 8.8 provides a well-defined map \( \text{trans}^* : C^*(\tilde{M})^Γ \to C^*(\tilde{M})^G \). It is readily seen that \( \text{trans}^* \) is a chain map. With an abuse, we still denote by \( \text{trans}^* \) the map \( H^*(\text{trans}^*) : H^*(C^*(\tilde{M})^Γ) \to H^*(C^*(\tilde{M})^G) \) induced in cohomology. Since \( \text{trans}^* \) restricts to the identity on \( G \)-invariant cochains, we have the following commutative diagram:

\[ \begin{array}{ccc}
\text{Id} & \xrightarrow{\text{res}} & H^*(C^*(\tilde{M})^G) \\
\downarrow & & \downarrow \\
H^*(\text{trans}^*) & \xrightarrow{\text{trans}^*} & H^*(C^*(\tilde{M})^G) \\
H^c_c(M) & & \\
\end{array} \]

where the vertical arrow describes the isomorphism provided by Lemma 8.2. Since \( \text{trans}^* \) is obviously norm non-increasing, we get the following:
8. THE PROPORTIONALITY PRINCIPLE

Proposition 8.9. The map $\text{res}^\bullet: H^\bullet(C^\bullet_c(\widetilde{M})^G) \to H^\bullet(C^\bullet_c(\widetilde{M})^\Gamma)$ is an isometric embedding.

7. Straightening and the volume form

We are now going to describe an explicit representative of the volume coclass on $M$. Since differential forms may be integrated only on smooth simplices, we first need to replace generic singular simplices with smooth ones. In general, this can be done by means of well-known smoothing operators (see e.g. [Lee03, Theorem 16.6]). However, under the assumption that $M$ is non-positively curved we may as well exploit the straightening procedure described above.

Recall that a singular $k$-simplex $s$ with values in a smooth manifold is smooth if, for every $q \in \Delta^k$, the map $s$ may be smoothly extended to an open neighbourhood of $q$ in the $k$-dimensional affine subspace of $\mathbb{R}^N$ containing $\Delta^k$. The space $sS_k(\tilde{M})$ (resp. $sS_k(M)$) of smooth simplices with values in $\tilde{M}$ (resp. in $M$) may be naturally endowed with the $C^1$-topology.

Lemma 8.10. For every $(k+1)$-tuple $(x_0,\ldots,x_k)$, the singular simplex $[x_0,\ldots,x_k]$ is smooth. Moreover, if we endow $\tilde{M}^{k+1}$ with the product topology and $sS_k(\tilde{M})$ with the $C^1$-topology, then the map

$$\tilde{M}^{k+1} \to sS_k(\tilde{M}), \quad (x_0,\ldots,x_k) \mapsto [x_0,\ldots,x_k]$$

is continuous.

Proof. We have already mentioned the fact that, since $\tilde{M}$ is non-positively curved, for every $x \in \tilde{M}$ the exponential map $\exp_x : T_x \tilde{M} \to \tilde{M}$ is a diffeomorphism. Moreover, if $z \in \Delta^{k-1}$ and $q = tz + (1-t)e_k \in \Delta^k$, then by definition we have

$$[x_0,\ldots,x_k](q) = \exp_{x_k}(t \exp^{-1}_{x_k}([x_0,\ldots,x_{k-1}](z))).$$

Now the conclusion follows by an easy inductive argument.

Let us now concentrate on some important homological properties of the straightening. We define $\tilde{\text{str}}_k : C_k(\tilde{M}) \to C_k(\tilde{M})$ as the unique linear map such that for every $s \in S_k(\tilde{M})$

$$\tilde{\text{str}}_k(s) = [s(e_0),\ldots,s(e_k)] \in S_k(\tilde{M}).$$

Proposition 8.11. The map $\tilde{\text{str}}_\bullet : C_\bullet(\tilde{M}) \to C_\bullet(\tilde{M})$ satisfies the following properties:

1. $d_{k+1} \circ \tilde{\text{str}}_{k+1} = \tilde{\text{str}}_k \circ d_{k+1}$ for every $n \in \mathbb{N}$;
2. $\tilde{\text{str}}_k(\gamma \circ s) = \gamma \circ \tilde{\text{str}}_k(s)$ for every $k \in \mathbb{N}$, $\gamma \in \Gamma$, $s \in S_k(\tilde{M})$;
3. the chain map $\tilde{\text{str}}_\bullet : C_\bullet(\tilde{M}) \to C_\bullet(\tilde{M})$ is $\Gamma$-equivariantly homotopic to the identity, via a homotopy which takes any smooth simplex into the sum of a finite number of smooth simplices.
7. Straightening and the Volume Form

Proof. If \( x_0, \ldots, x_k \in \tilde{M} \), then it is easily seen that for every \( i \leq k \) the \( i \)-th face of \([x_0, \ldots, x_k]\) is given by \([x_0, \ldots, \hat{x}_i, \ldots, x_k]\); moreover since isometries preserve geodesics we have \( \gamma \circ [x_0, \ldots, x_k] = [\gamma(x_0), \ldots, \gamma(x_k)] \) for every \( \gamma \in \text{Isom}(\tilde{M}) \). These facts readily imply points (1) and (2) of the proposition.

Finally, for \( s \in S_k(\tilde{M}) \), let \( F_s: \Delta^k \times [0, 1] \to \tilde{M} \) be defined by \( F_s(x, t) = \beta_x(t) \), where \( \beta_x: [0, 1] \to \tilde{M} \) is the constant-speed parameterization of the geodesic segment joining \( s(x) \) with \( \text{str}(s)(x) \). We set \( T_k(s) = (F_k)_\bullet(c) \), where \( c \) is the standard chain triangulating the prism \( \Delta^k \times [0, 1] \) by \((k+1)\)-simplices. The fact that \( d_{k+1} T_k + T_{k-1} d_k = \text{Id} - \text{str}_k \) is now easily checked, while the \( \Gamma \)-equivariance of \( T_* \) is a consequence of the fact that geodesics are preserved by isometries. The final statement of the proposition easily follows from the definition of \( T_* \).

As a consequence of the previous proposition, the chain map \( \text{str}_* \) induces a chain mapping

\[
\text{str}_*: C_\bullet(M) \to C_\bullet(M)
\]

which is homotopic to the identity. We say that a simplex \( s \in S_k(M) \) is straight if it is equal to \( \text{str}_k(s') \) for some \( s' \in S_k(M) \), i.e. if it may be obtained by composing a straight simplex in \( \tilde{M} \) with the covering projection \( p \). By Lemma 8.10, any straight simplex in \( M \) is smooth, and the map \( \text{str}_k: S_k(M) \to sS_k(M) \) is continuous, if we endow \( S_k(M) \) (resp. \( sS_k(M) \)) with the compact-open topology (resp. with the \( C^1 \)-topology).

We are now ready to define the volume cocycle. Let \( n \) be the dimension of \( M \), and suppose that \( M \) is oriented. We define a map \( \text{Vol}_M: S_n(M) \to \mathbb{R} \) by setting

\[
\text{Vol}_M(s) = \int_{\text{str}_n(s)} \omega_M,
\]

where \( \omega_M \in \Omega^n(M) \) is the volume form of \( M \). Since straight simplices are smooth, this map is well-defined. Moreover, since integration is continuous with respect to the \( C^1 \)-topology, the linear extension of \( \text{Vol}_M \), which will still be denoted by \( \text{Vol}_M \), defines an element in \( C^n_\bullet(M) \subseteq C^n(M) \). Using Stokes’ Theorem and the fact that \( \text{str}_* \) is a chain map, one may easily check that the cochain \( \text{Vol}_M \) is a cocycle, and defines therefore elements \([\text{Vol}_M] \in H^n(M), [\text{Vol}_M]_c \in H^n_c(M)\). Recall that \([M]^* \in H^n(M) \) denotes the fundamental coclass of \( M \).

Lemma 8.12. We have \([\text{Vol}_M] = \text{Vol}(M) \cdot [M]^* \).

Proof. Since \( H^n(M) \cong \mathbb{R} \), we have \([\text{Vol}_M] = ([\text{Vol}_M], [M]_\mathbb{R}) \cdot [M]^\mathbb{R} \). Moreover, it is well-known that the fundamental class of \( M \) can be represented by the sum \( c = \sum a_i s_i \) of the simplices in a positively oriented smooth triangulation of \( M \). By Proposition 8.11, the difference \( c - \text{str}_n(c) \) is the
boundary of a smooth \((n + 1)\)-chain, so Stokes' Theorem implies that
\[
\text{Vol}(M) = \int_c \omega_M = \int_{\text{str}_n(c)} \omega_M = \text{Vol}_M(c) = \langle [\text{Vol}_M], [M]_\mathbb{R} \rangle.
\]

Observe now that, under the identification \(C^n_c(M) \cong C^n_c(\tilde{M})^G\), the volume cocycle \(\text{Vol}_M\) corresponds to the cocycle
\[
\text{Vol}_{\tilde{M}}: C^n(\tilde{M}) \to \mathbb{R} \quad \text{Vol}_{\tilde{M}}(s) = \int_{\text{str}_n(s)} \omega_{\tilde{M}},
\]
where \(\omega_{\tilde{M}}\) is the volume form of the universal covering \(\tilde{M}\). Of course, \(\text{Vol}_{\tilde{M}}\) is \(G\)-invariant, so it defines an element \(\hat{[\text{Vol}_M]}^G\) in \(H^n_c(C^\bullet(\tilde{M})^G)\) such that \(\text{res}^n(\hat{[\text{Vol}_M]}^G) = \hat{[\text{Vol}_M]}^c \in H^n_c(M)\) (recall that we have an isometric identification \(H^n_c(M) \cong H^n_c(C^\bullet(\tilde{M})^F)\)). We are now ready to prove the main result of this section:

**Theorem 8.13 (Gromov Proportionality Theorem).** Let \(M\) be a closed non-positively curved Riemannian manifold. Then
\[
\|M\| = \frac{\text{Vol}(M)}{\|\hat{[\text{Vol}_M]}^G\|_\infty}
\]
(where we understand that \(k/\infty = 0\) for every real number \(k\)). In particular, the proportionality constant between the simplicial volume and the Riemannian volume of \(M\) only depends on the isometry type of the universal covering of \(M\).

**Proof.** The maps \(\text{res}^n: H^n_c(C^\bullet(\tilde{M})^G) \to H^n_c(M)\) and \(i^n: H^n_c(M) \to H^n_c(M)\) are isometric embedding, so putting together the duality principle (see Proposition 7.10) and Lemma 8.12 we get
\[
\|M\| = \frac{1}{\|M^*\|_\infty} = \frac{\text{Vol}(M)}{\|\hat{[\text{Vol}_M]}^G\|_\infty} = \frac{\text{Vol}(M)}{\|i^n(\text{res}^n(\hat{[\text{Vol}_M]}^G))\|_\infty} = \frac{\text{Vol}(M)}{\|\hat{[\text{Vol}_M]}^G\|_\infty}.
\]

**8. Further readings**
CHAPTER 9

The simplicial volume of hyperbolic manifolds

In this section we compute the simplicial volume of closed hyperbolic manifolds. Gromov and Thurston’s original argument is based on the fact that singular homology is isometrically isomorphic to measure homology (the first detailed proof of this fact appeared in [Löh06], and makes an essential use of Monod’s results about continuous bounded cohomology). Another proof of Gromov-Thurston’s Theorem, which closely follows the original approach but avoids the use of measure homology, may be found in [BP92] (see also [Rat94]). Here we present (a small variation of) the recent proof due to Bucher [BK08a], which exploits the description of the proportionality constant between the simplicial volume and the Riemannian volume described in Theorem 8.13.

Let $M$ be a closed oriented hyperbolic $n$-manifold. The universal covering of $M$ is isometric to the $n$-dimensional hyperbolic space $H^n$. As usual, we identify the fundamental group of $M$ with the group $\Gamma$ of the automorphisms of the covering $p: H^n \to M$, and we denote by $G$ the group of the orientation-preserving isometries of $H^n$. By Theorem 8.13, we are left to compute the $\ell^\infty$-seminorm of the element $[\text{Vol}_{H^n}]^G_c$ of $H^n(C^*_c(M))^G$. We denote by $\text{Vol}_{H^n}$ the representative of $[\text{Vol}_{H^n}]^G_c$ described in the previous section, i.e. the cocycle such that

$$\text{Vol}_{H^n}(s) = \int_{\text{str}_n(s)} \omega_{H^n},$$

where $\omega_{H^n}$ is the hyperbolic volume form. Of course, in order to estimate the seminorm of the volume coclass we first need to understand the geometry of hyperbolic straight simplices.

1. Hyperbolic straight simplices

We denote by $\overline{H^n} = H^n \cup \partial H^n$ the natural compactification of hyperbolic space, obtained by adding to $H^n$ one point for each class of asymptotic geodesic rays in $H^n$. We recall that every pair of points $\overline{H^n}$ is connected by a unique geodesic segment (which has infinite length if any of its endpoints lies in $\partial H^n$). A subset in $\overline{H^n}$ is convex if whenever it contains a pair of points it also contains the geodesic segment connecting them. The convex hull of a set $A$ is defined as usual as the intersection of all convex sets containing $A$. 

99
A \((\text{geodesic})\) \(k\)-simplex \(\Delta\) in \(\mathbb{H}^n\) is the convex hull of \(k + 1\) points in \(\mathbb{H}^n\), called vertices. A geodesic \(k\)-simplex is \(\text{finite}\) if all its vertices lie in \(\mathbb{H}^n\), \(\text{ideal}\) if all its vertices lie in \(\partial \mathbb{H}^n\), and \(\text{regular}\) if every permutation of its vertices is induced by an isometry of \(\mathbb{H}^n\). Since \(\mathbb{H}^n\) has constant curvature, \(\text{finite}\) geodesic simplices are exactly the images of straight simplices. Moreover, for every \(\ell > 0\) there exists, up to isometry, exactly one \(\text{finite regular}\) geodesic \(n\)-simplex of edgelength \(\ell\), which will be denoted by \(\tau_\ell\). In the same way, there exists, up to isometry, exactly one \(\text{ideal regular}\) geodesic \(n\)-simplex, which will be denoted by \(\tau_\infty\).

It is well-known that the supremum \(v_n\) of the volumes of geodesic \(n\)-simplices is finite for every \(n \geq 2\). However, computing \(v_n\) and describing those simplices whose volume is exactly equal to \(v_n\) is definitely non-trivial.

It is well-known that the area of every \(\text{finite triangle}\) is strictly smaller than \(\pi\), while every \(\text{ideal triangle}\) is regular and has area equal to \(\pi\), so \(v_2 = \pi\). In dimension 3, Milnor proved that \(v_3\) is equal to the volume of the \(\text{regular ideal simplex}\), and that any simplex having volume equal to \(v_3\) is \(\text{regular and ideal}\) [Thu79, Chapter 7]. This result was then extended to any dimension by Haagerup and Munkholm [HM81] (see also [Pey02] for a beautiful alternative proof based on Steiner symmetrization):

**Theorem 9.1 ([HM81, Pey02]).** Let \(\Delta\) be a geodesic \(n\)-simplex in \(\mathbb{H}^n\). Then \(\text{Vol}(\Delta) \leq v_n\), and \(\text{Vol}(\Delta) = v_n\) if and only if \(\Delta\) is \(\text{ideal and regular}\).

Moreover, the volume of geodesic simplices is reasonably continuous with respect to the position of the vertices (see e.g. [Luo06, Proposition 4.1] or [Rat94, Theorem 11.4.2]), so we have

\[
\lim_{\ell \to \infty} \text{Vol}(\tau_\ell) = \text{Vol}(\tau_\infty) = v_n
\]

(see e.g. [FP10, Lemma 3.6] for full details).

**2. The seminorm of the volume form**

Let us now come back to the study of the seminorm of \([\text{Vol}_{\mathbb{H}^n}]^G_c \in H^n(C^*_c(\tilde{M})^G)\). As a corollary of Theorem 9.1, we immediately obtain the inequality

\[
\|\text{Vol}_{\mathbb{H}^n}\|_c^G \leq \|\text{Vol}_{\mathbb{H}^n}\|_\infty = v_n
\]

We will show that this inequality is in fact an equality. To do so, we need to compute

\[
\inf \{ \|\text{Vol}_{\mathbb{H}^n} + \delta \varphi\|_\infty \mid \varphi \in C^{n-1}_c(\mathbb{H}^n)^G \}.
\]

Let us fix \(\varphi \in C^{n-1}_c(\mathbb{H}^n)^G\). Since an affine automorphism of the standard \(n\)-simplex preserves the orientation if and only if the induced permutation of the vertices is even, the cochain \(\text{Vol}\) is alternating. Therefore, by alternating the cochain \(\text{Vol}_{\mathbb{H}^n} + \delta \varphi\) we obtain

\[
\|\text{Vol}_{\mathbb{H}^n} + \delta \text{alt}^{n-1}(\varphi)\|_\infty = \|\text{alt}^n(\text{Vol}_{\mathbb{H}^n} + \delta \varphi)\|_\infty \leq \|\text{Vol}_{\mathbb{H}^n} + \delta \varphi\|_\infty
\]
3. The case of surfaces

Let us denote by \( s_\ell \) an orientation-preserving barycentric parameterization of the geodesic simplex \( \tau_\ell, \ell > 0 \) (see e.g., [FFM12] for the definition of barycentric parameterization of a geodesic simplex). Also let \( \partial_1 s_\ell \) the \( i \)-th face of \( s_\ell \), let \( H \) be the hyperplane of \( \mathbb{H}^n \) containing \( \partial_1 s_\ell \), and let \( v, w \in \mathbb{H}^n \) be two vertices of \( \partial_1 s_\ell \). Let \( h \) be the isometry of \( H \) given by the reflection with respect to the \((n-2)\)-plane containing all the vertices of \( \partial_1 s_\ell \) which are distinct from \( v, w \), and the midpoint of the edge joining \( v \) to \( w \). Then \( h \) may be extended in a unique way to an orientation-preserving isometry \( g \) of \( H \) (such an isometry interchanges the sides of \( H \) in \( \mathbb{H}^n \)). Since \( \partial_1 s_\ell \) is the barycentric parameterization of a regular \((n-1)\)-simplex, this readily implies that

\[
g \circ \partial_1 s_\ell = \partial_1 s_\ell \circ \sigma \ ,
\]

where \( \sigma \) is an affine automorphism of the standard \((n-1)\)-simplex which induces an odd permutation of the vertices. But \( \varphi \) is both alternating and \( G \)-invariant, so we have \( \varphi(\partial_1 s_\ell) = 0 \), whence

\[
\delta \varphi(s_\ell) = 0 .
\]

As a consequence, for every \( \ell > 0 \) we have

\[
\| \text{Vol}_{\mathbb{H}^n} + \delta \varphi \|_\infty \geq \| (\text{Vol}_{\mathbb{H}^n} + \delta \varphi)(s_\ell) \| = \| \text{Vol}_{\mathbb{H}^n}(s_\ell) \| = \text{Vol}(\tau_\ell) .
\]

Taking the limit of the right-hand side as \( \ell \to \infty \) and using (11) we get

\[
\| \text{Vol}_{\mathbb{H}^n} + \delta \varphi \|_\infty \geq v_n ,
\]

whence

\[
\|[\text{Vol}_{\mathbb{H}^n}]_C \|_\infty \geq v_n
\]

by the arbitrariness of \( \varphi \). Putting this estimates together with inequality (12) we finally obtain the following:

**Theorem 9.2 (Gromov and Thurston).** We have \( \|[\text{Vol}_{\mathbb{H}^n}]_C \|_\infty = v_n \), so

\[
\| M \| = \frac{\text{Vol}(M)}{v_n}
\]

for every closed hyperbolic manifold \( M \).

3. The case of surfaces

Let us denote by \( \Sigma_g \) the closed oriented surface of genus \( g \). It is well-known that \( \Sigma_g \) supports a hyperbolic metric if and only if \( g \geq 2 \). In that case, by Gauss-Bonnet Theorem we have

\[
\| \Sigma_g \| = \frac{\text{Area}(\Sigma_g)}{v_2} = \frac{2\pi |H\Sigma_g|}{\pi} = 2|H\sigma_g| = 4g - 4 .
\]

According to the proof of this equality given in the previous section, we may observe that the lower bound \( \| \Sigma_g \| \geq 4g - 4 \) follows from the easy upper bound \( \| \text{Vol}_{\mathbb{H}^2} \| \leq v_2 = \pi \), while the upper bound \( \| \Sigma_g \| \leq 4g - 4 \) follows
from the much less immediate lower bound $\|\text{Vol}_{\mathbb{H}^2}\| \geq \pi$. However, in the case of surfaces the inequality $\|\Sigma_g\| \leq 4g - 4$ may be deduced via a much more direct argument involving triangulations.

In fact, being obtained by gluing the sides of a $4g$-agon, the surface $\Sigma_g$ admits a triangulation by $4g - 2$ triangles (we employ here the word “triangulation” in a loose sense, as is customary in geometric topology: a triangulation of a manifold $M$ is the realization of $M$ as the gluing of finitely many simplices via some simplicial pairing of their facets). Such a triangulation defines a fundamental cycle whose $\ell^1$-norm is bounded above by $4g - 2$, so $\|\Sigma_g\| \leq 2|Hi(\Sigma_g)| + 2$. In order to get the desired estimate, we now need to slightly improve this inequality. To do so we exploit a standard trick, which is based on the fact that the simplicial volume is multiplicative with respect to finite coverings.

Since $\pi_1(\Sigma_g)$ surjects onto $H_1(\Sigma_g, \mathbb{Z}) \cong \mathbb{Z}^{2g}$, for every $d \in \mathbb{N}$ we may construct a subgroup $\pi_1(\Sigma_g')$ of index $d$ (just take the kernel of an epimorphism of $\pi_1(\Sigma_g)$ onto $\mathbb{Z}_d$). Therefore, $\Sigma_g$ admits a $d$-sheeted covering, whose total space is a surface $\Sigma_g'$ such that $|Hi(\Sigma_g')| = d|Hi(\Sigma_g)|$ (recall that the Euler characteristic is multiplicative with respect to finite coverings). By the multiplicativity of the simplicial volume we now get

$$\|\Sigma_g\| = \frac{|\Sigma_g'|}{d} \leq \frac{2|Hi(\Sigma_g')| + 2}{d} = 2|Hi(\Sigma_g)| + \frac{2}{d}.$$ 

Since $d$ is arbitrary, we may conclude that $\|\Sigma_g\| \leq 2|Hi(\Sigma_g)|$, which gives the desired upper bound.

One may wonder whether this strategy could be exploited in general to compute the simplicial volume of manifolds. In order to formalize and discuss this issue we now introduce the notion of stable complexity, which was first introduced by Milnor and Thurston in [MT77].

4. Stable complexity of manifolds

Following Milnor and Thurston [MT77], a numerical invariant $\alpha(M)$ associated to any closed $n$-manifold $M$ is a characteristic number if for every degree-$d$ covering $M \xrightarrow{d} N$ we have $\alpha(M) = d \cdot \alpha(N)$. For example, the simplicial volume and the Euler characteristic provide examples of characteristic numbers.

As mentioned above, a (loose) triangulation of a closed $n$-dimensional manifold $M$ is the realization of $M$ as the gluing of finitely many $n$-simplices via some simplicial pairing of their facets. The $\Delta$-complexity $\sigma(M)$ of $M$ is the minimal number of simplices needed to triangulate $M$. The $\Delta$-complexity is clearly not a characteristic number: for every degree-$d$ covering $M \xrightarrow{d} N$ we have

$$\sigma(M) \leq d \cdot \sigma(N),$$

but such inequality is very often strict, i.e. we typically get $\sigma(M) < d \cdot \sigma(N)$. We can however easily promote $\sigma$ to a characteristic number as follows. We
define the stable $\Delta$-complexity $\sigma_\infty(M)$ of $M$ by setting

$$\sigma_\infty(M) = \inf_{\tilde{M} \xrightarrow{d} M} \left\{ \frac{\sigma(\tilde{M})}{d} \right\}$$

where the infimum is taken over all finite coverings $\tilde{M} \xrightarrow{d} M$ of any finite degree $d$. Stable $\Delta$-complexity is easily seen to be a characteristic number, that is we have

$$\sigma_\infty(M) = d \cdot \sigma_\infty(N)$$

for every finite covering $M \xrightarrow{d} N$. The characteristic number $\sigma_\infty$ was first defined by Milnor and Thurston in [MT77].

**Proposition 9.3.** We have

$$\|M\| \leq \sigma_\infty(M) \leq \sigma(M)$$

**Proof.** The assertion $\sigma_\infty(M) \leq \sigma(M)$ follows from the definitions. In order to prove the other inequality we may suppose that $M$ is oriented. Let $T$ be a triangulation of $M$ with $m = \sigma(M)$ simplices, and let $s_1, \ldots, s_m$ be suitably chosen orientation-preserving parameterizations of the simplices of $T$. We would like to say that $s_1 + \ldots + s_m$ represents the fundamental class in $H_n(M; \mathbb{Z})$, however this singular chain is not necessarily a cycle. We can fix this problem easily by averaging each $s_i$ on all its permutations. That is, we define for any simplex $s$ the chain

$$\text{alt}_n(s) = \frac{1}{(n+1)!} \sum_{\tau \in S_{n+1}} (-1)^{\text{sgn}(\tau)} s \circ \tau,$$

where $\tau$ is the unique affine diffeomorphism of the standard $n$-simplex $\Delta^n$ corresponding to the permutation $\tau$ of the vertices of $\Delta^n$. Now it is immediate to verify that the chain $z = \text{alt}(s_1) + \ldots + \text{alt}(s_m)$ is a cycle which represents the fundamental class of $M$. Moreover, the sum of the absolute values of the coefficients of $z$ is at most $m$, and this implies the inequality $\|M\| \leq \sigma(M)$. The fact that $\|M\| \leq \sigma_\infty(M)$ now follows from the fact that the simplicial volume is multiplicative under finite coverings. \qed

Of course, the equality $\|M\| = \sigma_\infty(M)$ cannot hold in general: for example, if $M$ is an $n$-dimensional sphere, $n \geq 2$, then $\pi_1(M) = 1$, so $M$ does not admit any non-trivial covering and $\sigma_\infty(M) = \sigma(M) > 0$, while $\|M\| = 0$. On the other hand, we have seen that for every $g \geq 2$ we have $\|\Sigma_g\| = \sigma_\infty(\Sigma_g)$ (and the same results holds true also for $g = 1$).

It is tempting to guess that $\|M\| = \sigma_\infty(M)$ at least when $M$ is hyperbolic, because $\pi_1(M)$ is residually finite and hence $M$ has plenty of finite coverings of arbitrarily large injectivity radius. However, this guess is wrong, due to the following result:
Theorem 9.4. [FFM12] In every dimension \( n \geq 4 \) there is a constant \( C_n < 1 \) such that \( \|M\| \leq C_n \sigma_\infty(M) \) for every closed hyperbolic \( n \)-manifold \( M \).

We have seen that Theorem 9.4 does not hold in dimension two (as we have seen above) and three: in fact, there exists a sequence \( M_i \) of closed hyperbolic 3-manifolds such that
\[
\frac{\sigma_\infty(M_i)}{\|M_i\|} \to 1
\]

(see [FFM12]). However, it is not known any closed hyperbolic 3-manifold \( M \) for which \( \sigma_\infty(M) = \|M\| \), neither any closed hyperbolic 3-manifold \( M \) for which \( \sigma_\infty(M) \neq \|M\| \).

The main difference between dimensions two, three, and higher depends on the fact that the regular ideal hyperbolic \( n \)-simplex \( \Delta^n \) can tile \( \mathbb{H}^n \) only in dimensions 2 and 3: the dihedral angle of \( \Delta^n \) does not divide \( 2\pi \) when \( n \geq 4 \).

5. The simplicial volume of negatively curved manifolds

Let now now discuss the more general case when \( M \) is a closed \( n \)-manifold endowed with a Riemannian metric with sectional curvature \( \leq -\varepsilon \), where \( \varepsilon \) is a positive constant. Then it is not difficult to show that there exists a finite constant \( \kappa(n, \varepsilon) \) (which depends only on the dimension and on the upper bound on the sectional curvature) which bounds from above the volume of any straight simplex in \( \tilde{M} \) (see e.g. [IY82]). If \( G \) is the group of the orientation-preserving isometries of \( \tilde{M} \), then
\[
\|\text{Vol}_{\tilde{M}}^G\|_{\infty} \leq \|\text{Vol}_{\tilde{M}}\|_{\infty} \leq \kappa(n, \varepsilon),
\]
so
\[
\|M\| = \frac{\text{Vol}(M)}{\|\text{Vol}_{\tilde{M}}^G\|_{\infty}} > 0.
\]
In particular, any closed manifold supporting a negatively curved Riemannian metric has non-vanishing simplicial volume.

6. The simplicial volume of flat manifolds

Let \( M \) be a closed manifold. We have already mentioned the fact closed manifolds with non-negative Ricci tensor is null [Gro82], so we have \( \|M\| = 0 \). Let us give a proof of this fact which is based on the techniques developed above.

Since \( M \) is flat, the universal covering of \( M \) is isometric to the Euclidean space \( \mathbb{R}^n \). If we denote by \( G \) the group of the orientation-preserving isometries of \( \mathbb{R}^n \), then we are left to show that
\[
\|\text{Vol}_{\mathbb{R}^n}^G\|_{\infty} = \infty.
\]
Let \( \tau_\ell \) be the Euclidean geodesic simplex of edgelength \( \ell \), and observe that
\[
\lim_{\ell \to \infty} \text{Vol}(\tau_\ell) = \infty.
\]
The computations carried out in Section 2 in the
hyperbolic case apply verbatim to the Euclidean case, thus showing that \( \| [\text{Vol}_{\mathbb{R}^n}]^G_c \|_\infty \geq \text{Vol}(\tau_\ell) \) for every \( \ell > 0 \). By taking the limit as \( \ell \) tends to infinity we obtain (13), which in turn implies \( \| M \| = 0 \).

Another proof of the vanishing of the simplicial volume of flat manifolds follows from Bieberbach Theorem: if \( M \) is closed and flat, then it is finitely covered by an \( n \)-torus, so \( \| M \| = 0 \) by the multiplicativity of the simplicial volume with respect to finite coverings.
Additivity of the simplicial volume

This chapter is devoted to the proof of Theorem 7.5. Our approach makes an essential use of the duality between singular homology and bounded cohomology. In fact, the additivity of the simplicial volume for gluings along boundary components with amenable fundamental group will be deduced from a suitable “dual” statement about bounded cohomology (see Theorem 10.2). It is maybe worth mentioning that, as far as the author knows, there exists no proof of Gromov Additivity Theorem which avoids the use of bounded cohomology: namely, no procedure is known which allows to split a fundamental cycle for a manifold into efficient fundamental cycles for the pieces appearing in the decomposition of the manifold arising from cutting along hypersurfaces with amenable fundamental groups.

Let us now fix some notation. Let $M_1, \ldots, M_k$ be oriented $n$-manifolds, $n \geq 2$, such that the fundamental group of every component of $\partial M_j$ is amenable. We fix a pairing $(S^+_1, S^-_1), \ldots, (S^+_h, S^-_h)$ of some boundary components of $\bigsqcup_{j=1}^k M_j$, and for every $i = 1, \ldots, h$ we fix an orientation-reversing homeomorphism $f_i: S^+_i \to S^-_i$. We denote by $M$ the oriented manifold obtained by gluing $M_1, \ldots, M_k$ along $f_1, \ldots, f_h$, and we suppose that $M$ is connected. We also denote by $i_j: M_j \to M$ the obvious quotient map (so $i_j$ is an embedding provided that no boundary component of $M_j$ is paired with another boundary component of $M_j$).

For every $i = 1, \ldots, h$ we denote by $j^\pm(i)$ the index such that $S^\pm_i \subseteq M_{j^\pm(i)}$, and by $K^\pm_i$ the kernel of the map $\pi_1(S^\pm_i) \to \pi_1(M_{j^\pm(i)})$ induced by the inclusion. We recall that the gluings $f_1, \ldots, f_h$ are compatible if the equality

$$(f_i)_*(K^+_i) = K^-_i$$

holds for every $i = 1, \ldots, h$. Gromov Additivity Theorem states that

(14) $\|M, \partial M\| \leq \|M_1, \partial M_1\| + \ldots + \|M_k, \partial M_k\|$, 

and that, if the gluings defining $M$ are compatible, then

(15) $\|M, \partial M\| = \|M_1, \partial M_1\| + \ldots + \|M_k, \partial M_k\|$. 

1. A cohomological proof of subadditivity

If $S_i$ is a component of $\partial M_j$, we denote by $\overline{S}_i \subseteq M$ the image of $S_i$ via $i_j$, and we set $S = \bigcup_{i=1}^h \overline{S}_i \subseteq M$. Then the map $i_j$ is also a map of pairs
$i_j: (M_j, \partial M_j) \to (M, S \cup \partial M)$, and we denote by

$$H^b_0(i_j): H^b_0(M, S \cup \partial M) \to H^b_0(M_j, \partial M_j)$$

the induced map in bounded cohomology. For every compact manifold $N$, the pair $(N, \partial N)$ has the homotopy type of a finite CW-complex [KS69], so the inclusions of relative cochains into absolute cochains induce isometric isomorphisms $H^b_0(M, S \cup \partial M) \cong H^b_0(M)$, $H^b_0(M, \partial M) \cong H^b_0(M)$ (see Theorem 5.14). As a consequence, also the inclusion $C^b_0(M, S \cup \partial M) \to C^b_0(M, \partial M)$ induces an isometric isomorphism

$$\zeta^n: H^b_0(M, S \cup \partial M) \to H^b_0(M, \partial M).$$

For every $j = 1, \ldots, k$, we finally define the map

$$\zeta^n_j = H^b_0(i_j) \circ (\zeta^n)^{-1}: H^b_0(M, \partial M) \to H^b_0(M_j, \partial M_j).$$

**Lemma 10.1.** For every $\varphi \in H^b_0(M, \partial M)$ we have

$$\langle \varphi, [M, \partial M] \rangle = \sum_{j=1}^k \langle \zeta^n_j(\varphi), [M_j, \partial M_j] \rangle.$$

**Proof.** Let $c_j \in H_n(M_j)$ be a real chain representing the fundamental class of $M_j$. With an abuse, we identify any chain in $M_j$ (resp. in $S_i^\pm$) with the corresponding chain in $M$ (resp. in $S_i$), and we set $c = \sum_{j=1}^k c_j \in C_n(M)$. We now suitably modify $c$ in order to obtain a relative fundamental cycle for $M$. It is readily seen that $\partial c_j$ is the sum of real fundamental cycles of the boundary components of $M_j$. Therefore, since the gluing maps defining $M$ are orientation-reversing, we may choose a chain $c' \in \oplus_{i=1}^N C_n(S_i)$ such that $\partial c - \partial c' \in C_{n-1}(\partial M)$. We set $c'' = c - c'$. By construction $c''$ is a relative cycle in $C_n(M, \partial M)$, and it is immediate to check that it is in fact a relative fundamental cycle for $M$. Let now $\psi \in C^b_0(M, S \cup \partial M)$ be a representative of $(\zeta^n)^{-1}(\varphi)$. By definition we have

$$\psi(c) = \sum_{j=1}^k \psi(c_j) = \sum_{j=1}^k \langle \zeta^n_j(\varphi), [M_j, \partial M_j] \rangle.$$

On the other hand, since $\psi$ vanishes on chains supported on $S$, we also have

$$\psi(c) = \psi(c'' + c') = \psi(c'') = \langle \varphi, [M, \partial M] \rangle,$$

and this concludes the proof. \qed

We are now ready to exploit duality to prove that the simplicial volume is subadditive with respect to gluings along boundary components having an amenable fundamental group. By Proposition 7.10 we may choose an element $\varphi \in H^b_0(M, \partial M)$ such that

$$\|M, \partial M\| = \langle \varphi, [M, \partial M] \rangle, \quad \|\varphi\|_{\infty} \leq 1.$$
Observe that \( \| \zeta^n_j(\varphi) \|_\infty \leq \| \varphi \|_\infty \leq 1 \), so by Lemma 10.1
\[
\| M, \partial M \| = \langle \varphi, [M, \partial M] \rangle = \sum_{j=1}^k \langle \zeta^n_j(\varphi), [M_j, \partial M_j] \rangle \leq \sum_{j=1}^k \| M_j, \partial M_j \| .
\]
This concludes the proof of inequality (14).

2. A cohomological proof of Gromov Additivity Theorem

The proof that equality (15) is based on the following extension property for bounded coclasses:

**Theorem 10.2.** Suppose that the gluings defining \( M \) are compatible, let \( \varepsilon > 0 \) and take an element \( \varphi_j \in H^n_b(M_j, \partial M_j) \) for every \( j = 1, \ldots, k \). Then there exists a coclass \( \varphi \in H^n_b(M, \partial M) \) such that \( \zeta^n_j(\varphi) = \varphi_j \) for every \( j = 1, \ldots, k \), and
\[
\| \varphi \|_\infty \leq \max \{ \| \varphi_j \|, j = 1, \ldots, k \} + \varepsilon .
\]

Henceforth we assume that the gluings defining \( M \) are compatible. We first show how Theorem 10.2 may be used to conclude the proof of Gromov Additivity Theorem. By Proposition 7.10, for every \( j = 1, \ldots, k \), we may choose an element \( \varphi_j \in H^n_b(M_j, \partial M_j) \) such that
\[
\| M_j, \partial M_j \| = \langle \varphi_j, [M_j, \partial M_j] \rangle, \quad \| \varphi_j \|_\infty \leq 1 .
\]
By Theorem 10.2, for every \( \varepsilon > 0 \) there exists \( \varphi \in H^n_b(M, \partial M) \) such that
\[
\| \varphi \|_\infty \leq 1 + \varepsilon, \quad \zeta^n_j(\varphi) = \varphi_j, \quad j = 1, \ldots, k .
\]
Using Lemma 10.1 we get
\[
\sum_{j=1}^k \| M_j, \partial M_j \| = \sum_{j=1}^k \langle \varphi_j, [M_j, \partial M_j] \rangle = \langle \varphi, [M, \partial M] \rangle \leq (1 + \varepsilon) \cdot \| M, \partial M \| .
\]
Since \( \varepsilon \) is arbitrary, this implies that \( \| M, \partial M \| \) cannot be strictly smaller than the sum of the \( \| M_j, \partial M_j \| \). Together with inequality (14), this implies that
\[
\| M, \partial M \| = \| M_1, \partial M_1 \| + \ldots + \| M_k, \partial M_k \| .
\]
Therefore, in order to conclude the proof of Gromov Additivity Theorem we are left to prove Theorem 10.2.

We denote by \( p: \widetilde{M} \to M \) the universal covering of \( M \). We first describe the structure of \( \widetilde{M} \) as a tree of spaces. We construct a tree \( T \) as follows. Let us set \( N_j = M_j \setminus \partial M_j \) for every \( j = 1, \ldots, k \). We pick a vertex for every connected component of \( p^{-1}(N_j) \), \( j = 1, \ldots, k \), and we join two vertices if the closures of the corresponding components intersect (along a common boundary component). Therefore, edges of \( T \) bijectively correspond to the connected components of \( p^{-1}(S) \). It is easy to realize \( T \) as a retract of \( \widetilde{M} \), so \( T \) is simply connected, i.e. it is a tree. We denote by \( V(T) \) the set of
vertices of $T$, and for every $v \in V(T)$ we denote by $\widetilde{M}_v$ the closure of the component of $\widetilde{M}$ corresponding to $v$.

The following lemma exploits the assumption that the gluings defining $M$ are compatible.

**Lemma 10.3.** For every $v \in V(T)$ there exists $j(v) \in \{1, \ldots, k\}$ and a universal covering map $p_v: \widetilde{M}_v \rightarrow M_{j(v)}$ such that the following diagram commutes:

\[
\begin{array}{ccc}
\widetilde{M}_v & \xrightarrow{p_v} & M_{j(v)} \\
\downarrow & & \downarrow i_j \\
\widetilde{M}_v & \xrightarrow{p_v|_{\widetilde{M}_v}} & M_{j(M)}
\end{array}
\]

**Proof.** An application of Seifert-Van Kampen Theorem shows that, under the assumption that the gluings defining $M$ are compatible, for every $j$ the map $\pi_1(M_j) \rightarrow \pi_1(M)$ induced by $i_j$ is injective. This implies in turn that $p$ restricts to a universal covering $\widetilde{M}_v \setminus \partial\widetilde{M}_v \rightarrow N_{j(v)}$. It is now easy to check that this restriction extends to the required map $p_v: \widetilde{M}_v \rightarrow M_{j(v)}$. \qed

Let us now proceed with the proof of Theorem 10.2. For every $j = 1, \ldots, k$, we are given a coclass $\varphi_j \in H^n_b(M_j, \partial M_j)$. Recall that, since the fundamental group of every component of $\partial M_j$ is amenable, the inclusion of relative cochains into absolute ones induces an isometric isomorphism in bounded cohomology. Therefore, Corollary 5.16 implies that $\varphi_j$ may be represented by a special cocycle $f_j \in Z^n_{bs}(M_j, \partial M_j)$ such that $\|f_j\|_\infty < \|\varphi\|_\infty + \varepsilon$ (we refer the reader to Section 8 for the definition of the module $C^n_{bs}(X,Y)$ of special cochains, whenever every component of $Y$ has an amenable fundamental group).

We identify the group of the covering automorphisms of $p: \widetilde{M} \rightarrow M$ with the fundamental group $\Gamma = \pi_1(M)$ of $M$. Moreover, for every vertex $v \in V(T)$ we denote by $\Gamma_v$ the stabilizer of $\widetilde{M}_v$ in $\Gamma$. Observe that $\Gamma_v$ is canonically isomorphic (up to conjugacy) to $\pi_1(M_{j(v)})$. For every $v \in V(T)$, we denote by $f_v \in Z^n_{bs}(\widetilde{M}_v, \partial \widetilde{M}_v)^{\Gamma_v}$ the pull-back of $f_j$ via the covering $\widetilde{M}_v \rightarrow M_{j(v)}$. In order to prove Theorem 10.2 it is sufficient to show that there exists a bounded cocycle $f \in Z^n_{bs}(\widetilde{M}, \partial \widetilde{M})^\Gamma$ which restricts to $f_v$ on each $C_n(\widetilde{M}_v)$, $v \in V(T)$.

Let $s: \Delta^n \rightarrow \widetilde{M}$ be a singular simplex, $n \geq 2$, and let $q_0, \ldots, q_n$ be the vertices of $a$ (i.e. the images via $a$ of the vertices of the standard simplex). We say that the vertex $v \in V(T)$ is a barycenter of $s$ the following condition holds:

- if $i, j \in \{0, \ldots, n\}$, $i \neq j$, then every path in $\widetilde{M}$ joining $q_i$ and $q_j$ intersects $\widetilde{M}_v \setminus \partial \widetilde{M}_v$.  


Lemma 10.4. The simplex $s$ has at most one barycenter. If $s$ is supported in $M_v$ for some $v \in V(T)$, then $v$ is the barycenter of $s$ if and only if the vertices of $S$ lie in pairwise distinct components of $\partial M_v$. If this is not the case, then $s$ does not have any barycenter.

Proof. Suppose by contradiction that $v_1$ and $v_2$ are distinct barycenters of $s$. Let $A$ be the connected component of $\tilde{M} \setminus \tilde{M}_{v_1}$ that contains $\tilde{M}_{v_2} \setminus \partial \tilde{M}_{v_2}$. Since $v_1$ is a barycenter of $s$, at most one vertex of $s$ can be contained in $A$. Moreover, the set $\tilde{M} \setminus A$ is path-connected and disjoint from $M_{v_2} \setminus \partial \tilde{M}_{v_2}$. Since $v_2$ is a barycenter of $s$, this implies that at most one vertex of $s$ can belong to $\tilde{M} \setminus A$. As a consequence, $s$ cannot have more than two vertices, and this contradicts the assumption that $s$ is a singular $n$-simplex, $n \geq 2$.

The second and the third statements of the lemma are obvious. □

Let now $v \in V(T)$ be fixed. We associate (quite arbitrarily) to $s$ a singular simplex $s_v$ with vertices $q'_0, \ldots, q'_n$ in such a way that the following conditions hold:

- $s_v$ is supported in $\tilde{M}_v$;
- if $q_i \in M_v$, then $q'_i = q_i$;
- if $q_i \notin M_v$, then there exists a unique component $B$ of $\partial \tilde{M}_v$ such that every path joining $q_i$ with $M_v$ intersects $B$; in this case, we choose $q'_i$ to be any point of $B$.

The simplex $s_v$ may be thought as a “projection” of $s$ onto $\tilde{M}_v$. We may now set, for every $v \in V(T)$,

$$\hat{f}_v(s) = f_v(s_v) .$$

Observe that, even if $s_v$ may be chosen somewhat arbitrarily, the vertices of $s_v$ only depend on $s$, up to considering equivalent points which lie on the same connected component of $\partial \tilde{M}_v$. As a consequence, the fact that $f_v$ is special implies that $\hat{f}_v(s)$ is indeed well-defined (i.e. it does not depend on the choice of $s_v$).

Lemma 10.5. Suppose that $v$ is not a barycenter of $s$. Then $\hat{f}_v(s) = 0$.

Proof. If $v$ is not a barycenter of $s$ then there exists a component of $\partial \tilde{M}_v$ containing at least two vertices of $s_v$, so $\hat{f}_v(s) = f_v(s_v) = 0$ since $f_v$ is special. □

We are now ready to define the cochain $f$ as follows: for every singular $n$-simplex $s$ with values in $\tilde{M}$ we set

$$f(s) = \sum_{v \in V(T)} \hat{f}_v(s) .$$

By Lemmas 10.4 and 10.5, the sum on the right-hand site either is empty or it consists of a single term. This already implies that

$$\|f\|_\infty \subseteq V(T) \|f_v\|_\infty = \{\|f_j\|_\infty, j = 1, \ldots, k\} .$$
Moreover, it is clear from the construction that $f$ is $\Gamma$-invariant, so $f \in C^n_b(\tilde{M}, \partial \tilde{M})^\Gamma$. Let us now suppose that the singular $n$-simplex $s$ is supported in $\tilde{M}_v$ for some $v \in V(T)$. If $v$ is the barycenter of $s$, then we have $f(s) = \hat{f}_v(s) = f_v(s_v) = f_v(s)$. Otherwise, by Lemma 10.4 we know that $s$ has no barycenters, so $f(s) = 0$, and two vertices of $s$ lie on the same component of $\partial \tilde{M}_v$, so $f_v(s) = 0$, and again $f(s) = f_v(s)$. We have thus shown that $f$ indeed coincides with $f_v$ on simplices supported in $\tilde{M}_v$, so we are now left to prove that $f$ is a cocycle. So, let $s'$ be a singular $(n+1)$-simplex with values in $\tilde{M}$, and denote by $s'_v$ a projection of $s'$ on $\tilde{M}_v$, according to the procedure described above in the case of $n$-simplices. It readily follows from the definitions that $\partial_i(s'_v)$ is a projection of $\partial_i s'$ on $\tilde{M}_v$, so

$$\hat{f}_v(\partial s') = \sum_{i=0}^{n+1} (-1)^i f_v((\partial_i s')_v) = \sum_{i=0}^{n+1} (-1)^i f_v(\partial_i s'_v) = f_v(\partial s'_v) = 0,$$

where the last equality is due to the fact that $f_v$ is a cocycle. As a consequence we get

$$f(\partial s') = \sum_{v \in V(T)} \hat{f}_v(\partial s') = 0,$$

so $f$ is also a cocycle. This concludes the proof of Theorem 10.2, whence of Gromov Additivity Theorem.

**Further readings**
CHAPTER 11

Group actions on the circle

Bounded cohomology has been successfully exploited in the study of the
dynamics of homeomorphisms of the circle. Here we review some fundamental
results mainly due to Ghys [Ghy87, Ghy99, Ghy01], who proved that
semi-conjugacy classes of representations into the group of homeomorphisms
of the circle are completely classified by their bounded Euler class. Then,
following Burger, Iozzi and Wienhard [BIW10, BIW11] we concentrate
our attention on the study of representations of the fundamental group of
surfaces.

Probabilmente l’ultima cosa non verrà fatta

1. Homeomorphisms of the circle and the Euler class

Henceforth we identify $S^1$ with the quotient $\mathbb{R}/\mathbb{Z}$ of the real line by
the subgroup of the integers, and we fix the corresponding quotient (universal covering) map $p: \mathbb{R} \to S^1$. We denote by Homeo$^+(S^1)$ the group
of the orientation-preserving homeomorphisms of $S^1$. We also denote by
\(\widetilde{\text{Homeo}}_+(S^1)\) the group of the homeomorphisms of the real line obtained by
lifting elements of Homeo$^+(S^1)$. In other words, Homeo$^+(S^1)$ is the set of
increasing homeomorphisms of $\mathbb{R}$ that commute with every integral trans-
lation. Since $p$ is a universal covering, every $f \in \text{Homeo}^+(S^1)$ lifts to a
map $\widetilde{f} \in \widetilde{\text{Homeo}}_+(S^1)$. On the other hand, every map $\widetilde{f} \in \widetilde{\text{Homeo}}_+(S^1)$
descends to a map $p_*(\widetilde{f}) \in \text{Homeo}^+(S^1)$ such that $p_*(\widetilde{f}) = \widetilde{f}$, so we have
a well-defined surjective homomorphism $p_*: \text{Homeo}^+_+(S^1) \to \text{Homeo}^+(S^1)$,
whose kernel consists of the group of integral translations. Therefore, we
get an extension

\[
1 \to \mathbb{Z} \xrightarrow{\iota} \widetilde{\text{Homeo}}_+(S^1) \xrightarrow{p_*} \text{Homeo}^+(S^1) \to 1,
\]

where $\iota(n) = \tau_n$ is the translation $x \mapsto x + n$ (henceforth we will often
identify $\mathbb{Z}$ with its image in $\widetilde{\text{Homeo}}_+(S^1)$ via $\iota$). By construction, the above
extension is central.

**Definition 11.1.** The coclass

\[ e \in H^2(\widetilde{\text{Homeo}}_+(S^1), \mathbb{Z}) \]
associated to the central extension 16 (see Section 2) is the Euler class of \( \text{Homeo}^+(S^1) \).

The Euler class plays an important role in the study of the group of the homeomorphisms of the circle. Before going on, we point out the following easy result, which will be often exploited later.

**Lemma 11.2.** Let \( \tilde{\psi}_1, \tilde{\psi}_2 : \mathbb{R} \to \mathbb{R} \) be (not necessarily continuous) maps such that \( \tilde{\psi}_i(x + 1) = \tilde{\psi}_i(x) + 1 \) for \( i = 1, 2 \) for every \( x \in \mathbb{R} \). Also suppose that \( \tilde{\psi}_i \) is strictly increasing on the set \( \{y_0\} \cup (x_0 + \mathbb{Z}) \). Then

\[
|\tilde{\psi}_1(y_0) - \tilde{\psi}_1(x_0) - (\tilde{\psi}_2(y_0) - \tilde{\psi}_2(x_0))| = |\tilde{\psi}_1(y_0) - \tilde{\psi}_2(y_0) - (\tilde{\psi}_1(x_0) - \tilde{\psi}_2(x_0))| < 1.
\]

In particular, if \( \tilde{f} \in \text{Homeo}_+(S^1) \), then for every \( x, y \in \mathbb{R} \) we have

\[
|\tilde{f}(y) - y - (\tilde{f}(x) - x)| < 1.
\]

**Proof.** Since \( \tilde{\psi}_i \) commutes with integral translations, we may assume that \( x_0 \leq y_0 < x_0 + 1 \). The condition above implies that \( \tilde{\psi}_1(x_0) \leq \tilde{\psi}_1(y_0) < \tilde{\psi}_1(x_0) + 1 \), so \( 0 \leq \tilde{\psi}_1(y_0) - \tilde{\psi}_1(x_0) < 1 \). This concludes the proof. \( \square \)

### 2. The bounded Euler class

Let us fix a point \( x_0 \in \mathbb{R} \). Then we may define a set-theoretic section \( s_{x_0} : \text{Homeo}^+(S^1) \to \text{Homeo}_+(S^1) \) such that \( p_* \circ s_{x_0} = \text{Id}_{\text{Homeo}^+(S^1)} \) by setting \( s_{x_0}(f) = \tilde{f}_{x_0} \), where \( \tilde{f}_{x_0} \) is the unique lifting of \( f \) such that \( \tilde{f}_{x_0}(x_0) - x_0 \in [0, 1) \). Then the Euler class \( e \in H^2(\text{Homeo}^+(S^1), \mathbb{Z}) \) is represented by the cocycle

\[
c_{x_0} : \text{Homeo}^+(S^1) \times \text{Homeo}^+(S^1) \to \mathbb{Z}, \quad (f, g) \mapsto \left( \tilde{f} \circ \tilde{g} \right)^{-1}_{x_0} \tilde{f}_{x_0} \tilde{g}_{x_0} \in \mathbb{Z}
\]

(where we identify as usual \( \mathbb{Z} \) with \( \iota(\mathbb{Z}) \in \text{Homeo}_+(S^1) \)). Equivalently, \( c_{x_0} \) is defined by the condition

\[
(\tilde{f} \circ \tilde{g})_{x_0} \circ \tau_{c_{x_0}(f,g)} = \tilde{f}_{x_0} \tilde{g}_{x_0}.
\]

By Lemma 11.2, for every \( f \in \text{Homeo}^+(S^1), x \in \mathbb{R} \) we have

\[
|\tilde{f}_{x_0}(x) - x| < |\tilde{f}_{x_0}(x) - x - (\tilde{f}_{x_0}(x_0) - x_0)| + |\tilde{f}_{x_0}(x_0) - x_0| < 2,
\]

whence also

\[
|\tilde{f}_{x_0}^{-1}(x) - x| < 2. \tag{17}
\]

**Lemma 11.3.** The cocycle \( c_{x_0} \) takes values into the set \( \{0, 1\} \) (in particular, it is bounded). Moreover, if \( x_0, x_1 \in \mathbb{R} \), then \( c_{x_0} - c_{x_1} \) is the coboundary of a bounded integral 1-cochain.
But in order to conclude, it is sufficient to show that every element of setting \( q \) is bounded. Moreover, since \( \delta q \) is bounded, it is sufficient to show that \( q \) is bounded and that \( c_{x_1} - c_{x_0} = \delta q \). Using (17) we get

\[
q(fg)^{-1} = \left( (f \circ g)_{x_0} (f \circ g)_{x_1} \right)^{-1} = (f \circ g)_{x_1} (f \circ g)_{x_0} \]

\[
= \left( \bar{g}_{x_1}^{-1} \bar{f}_{x_1}^{-1} \tau_{c_{x_1}}(f,g) \right) \left( \bar{f}_{x_0} \bar{g}_{x_0} \tau_{-c_{x_1}(f,g)} \right) \]

\[
= \tau_{c_{x_1}(f,g) - c_{x_0}(f,g)} \bar{g}_{x_1}^{-1} \bar{f}_{x_1}^{-1} \bar{f}_{x_0} \bar{g}_{x_0} = \tau_{c_{x_1}(f,g) - c_{x_0}(f,g)} q(g)^{-1} q(f)^{-1},
\]

so

\[
\delta q(f,g) = q(fg)^{-1} q(f) q(g) = \tau_{c_{x_1}(f,g) - c_{x_0}(f,g)}.
\]

We have thus shown that \( c_{x_1} - c_{x_0} = \delta q \), and this concludes the proof. \( \square \)

**Definition 11.4.** Let \( x_0 \in \mathbb{R} \), and let \( c_{x_0} \) be the bounded cocycle introduced above. Then, the *bounded Euler class* \( e_b \in H^2_b(\text{Homeo}^+(S^1), \mathbb{Z}) \) is defined by setting \( e_b = [c_{x_0}] \) for some \( x_0 \in \mathbb{R} \). The previous lemma shows that \( e_b \) is indeed well-defined, and by construction the comparison map \( H^2_b(\text{Homeo}^+(S^1), \mathbb{Z}) \to H^2(\text{Homeo}^+(S^1), \mathbb{Z}) \) sends \( e_b \) to \( e \).

Henceforth, for every \( f \in \text{Homeo}^+(S^1) \) we simply denote by \( \bar{f} \) the lifting \( \bar{f}_0 \) of \( f \) such that \( \bar{f}(0) \in [0,1) \), and by \( c \) the cocycle \( c_0 \).

## 3. The (bounded) Euler class of a representation

Let \( \Gamma \) be a group, and consider a representation \( \rho: \Gamma \to \text{Homeo}^+(S^1) \). Then the Euler class \( e(\rho) \) and the bounded Euler class of \( e_b(\rho) \) of \( \rho \) are the elements of \( H^2(\Gamma, \mathbb{Z}) \) and \( H^2_b(\Gamma, \mathbb{Z}) \) defined by

\[
e(\rho) = \rho^*(e), \quad e_b(\rho) = \rho^*(e_b),
\]
where, with a slight abuse, we denote by $\rho^*$ both the morphisms $H^2(\rho)$, $H^2_b(\rho)$ induced by $\rho$ in cohomology and in bounded cohomology.

The (bounded) Euler class of a representation captures several interesting features of the dynamics of the corresponding action. It is worth mentioning that the bounded Euler class often carries much more information than the usual Euler class. For example, if $\Gamma = \mathbb{Z}$, then $H^2(\Gamma, \mathbb{Z}) = 0$, so $e(\rho) = 0$ for every $\rho: \Gamma \to \text{Homeo}^+(S^1)$. On the other hand, we have seen in Proposition 2.9 that $H^2_b(\mathbb{Z}, \mathbb{Z})$ is canonically isomorphic to $\mathbb{R}/\mathbb{Z}$, so one may hope that $e_b(\rho)$ does not vanish at least for some representation $\rho: \mathbb{Z} \to \text{Homeo}^+(S^1)$. And this is indeed the case, as we will show in the following section.

4. The rotation number of a homeomorphism

Let $f$ be a fixed element of $\text{Homeo}^+(S^1)$, and let us consider the representation $\rho_f: \mathbb{Z} \to \text{Homeo}^+(S^1)$, $\rho_f(n) = f^n$. Then we define the rotation number of $f$ by setting

$$\text{rot}(f) = e_b(\rho_f) \in \mathbb{R}/\mathbb{Z} \cong H^2_b(\mathbb{Z}, \mathbb{Z}).$$

The definition of rotation number for a homeomorphism of the circle dates back to Poincaré [Poi81]. Of course, the traditional definition of $\text{rot}$ did not involve bounded cohomology, so equation (18) is usually introduced as a theorem establishing a relationship between bounded cohomology and more traditional dynamical invariants. However, in this monograph we go the other way round, i.e. we recover the classical definition of the rotation number from the approach via bounded cohomology.

In fact, if $\tilde{f}$ is the preferred lift of $f$, then a representative for $\rho^*_f(e_b)$ is given by

$$\rho^*_f(c)(n,m) = c(f^n, f^m) = \tilde{f}^{n+m-1}\tilde{f}^n\tilde{f}^m.$$

Let us define $u: \mathbb{Z} \to \mathbb{Z}$ by requiring that

$$\tilde{f}^n = \tau_{u(n)}\tilde{f}^n$$

for every $n \in \mathbb{N}$. Then

$$\rho^*_f(c)(n,m) = \tilde{f}^{n-m}\tau_{u(n+m)}\tilde{f}^n\tau_{-u(n)}\tilde{f}^m\tau_{-u(m)} = \tau_{u(n+m) - u(n) - u(m)},$$

so $\rho^*_f(c) = -\delta u$. Since we already know that $\rho^*_f(c)$ is bounded, this implies that $u$ is a quasimorphism, so Proposition 2.6 implies that the limit

$$\lim_{n \to \infty} \frac{u(n)}{n} \in \mathbb{R}$$

exists. Moreover, by Proposition 2.9 (see also Remark 2.10) we have

$$\text{rot}(f) = \rho^*_f(e_b) = \left[ \lim_{n \to \infty} \frac{u(n)}{n} \right] \in \mathbb{R}/\mathbb{Z}.$$
On the other hand, by evaluating Equation (20) in 0 we get
\[ u(n) = \tilde{f}^n(0) - \hat{f}^n(0) = \lfloor \hat{f}^n(0) \rfloor \]
(henceforth for every \( x \in \mathbb{R} \) we denote by \( \lfloor x \rfloor \) the largest integer which does not exceed \( x \)), so
\[
\text{rot}(f) = \left[ \lim_{n \to \infty} \frac{u(n)}{n} \right] = \left[ \lim_{n \to \infty} \frac{\lfloor \tilde{f}^n(0) \rfloor}{n} \right] = \left[ \lim_{n \to \infty} \frac{\hat{f}^n(0)}{n} \right] \in \mathbb{R}/\mathbb{Z}.
\]

Therefore, we have the following:

**Proposition 11.5.** Let \( f \in \text{Homeo}^+(S^1) \), let \( \tilde{f} \) be any lift of \( f \) and \( x_0 \in \mathbb{R} \) be any point. Then
\[
\text{rot}(f) = \left[ \lim_{n \to \infty} \frac{\tilde{f}^n(x_0)}{n} \right] \in \mathbb{R}/\mathbb{Z}
\]
(in particular, the above limit exists).

**Proof.** We have already proved the statement in the case when \( \tilde{f} = \tilde{f}_0 \) and \( x_0 = 0 \). However, by Lemma 11.2 the limit \( \lim_{n \to \infty} \tilde{f}^n(x_0)/n \) does not depend on the choice of \( x_0 \). Moreover, if \( \tilde{f}' = \tau_k \tilde{f} \), then \( \lim_{n \to \infty} (\tilde{f}')^n(x_0)/n = k + \lim_{n \to \infty} \tilde{f}^n(x_0)/n \), and this concludes the proof. \( \square \)

**Corollary 11.6.** For every \( f \in \text{Homeo}^+(S^1) \), \( n \in \mathbb{Z} \) we have
\[
\text{rot}(f^n) = n \text{rot}(f).
\]

Roughly speaking, Proposition 11.5 shows that \( \text{rot}(f) \) indeed measures the average rotation angle of \( f \). For example, if \( f \) is a genuine rotation of angle \( 0 \leq \alpha < 1 \) (recall that \( S^1 = \mathbb{R}/\mathbb{Z} \) has length one in our setting), then \( \tilde{f}(x) = x + \alpha \), so \( \tilde{f}^n(0) = n\alpha \) and \( \text{rot}(f) = \lfloor \alpha \rfloor \). On the other hand, if \( f \) admits a fixed point, then we may choose a lift \( \tilde{f} \) of \( f \) admitting a fixed point, so Proposition 11.5 implies that \( \text{rot}(f) = 0 \). We will see in Corollary 11.23 that also the converse implication holds: if \( \text{rot}(f) = 0 \), then \( f \) fixes a point in \( S^1 \).

### 5. Increasing degree one map of the circle

Let \( \Gamma \) be a group. Two representations \( \rho_1, \rho_2 \) of \( \Gamma \) into \( \text{Homeo}^+(S^1) \) are conjugated if there exists \( \varphi \in \text{Homeo}^+(S^1) \) such that \( \rho_1(g) = \varphi \rho_2(g) \varphi^{-1} \) for every \( g \in \Gamma \). It is easy to show that the bounded Euler classes (whence the usual Euler classes) of conjugated representations coincide (this fact also follows from Theorem 11.22 below). Unfortunately it is not true that representations sharing the same bounded Euler class are necessarily conjugated. However, it turns out that the bounded Euler class completely determines the **semi-conjugacy** class of a representation. The subject we are going to describe was first investigated by Ghys [Ghy87], who first noticed the relationship between quasi-conjugation and the bounded Euler class. However,
Glyks himself proposed different definitions of quasi-conjugation in his papers [Ghy87, Ghy99, Ghy01] (see Remark 11.14 below). Here we adopt the definition of semi-conjugation given in [Buc08].

**Definition 11.7.** Let us consider an ordered $k$-tuple $(x_1, \ldots, x_k) \in (S^1)^k$. We say that such a $k$-tuple is

- **positively oriented** if there exists an orientation-preserving embedding $\gamma: [0, 1] \to S^1$ such that $x_i = \gamma(t_i)$, where $t_i < t_{i+1}$ for every $i = 1, \ldots, k-1$.
- **weakly positively oriented** if there exists an orientation-preserving immersion $\gamma: [0, 1] \to S^1$ which restricts to an embedding on $[0, 1)$, and is such that $x_i = \gamma(t_i)$, where $t_i \leq t_{i+1}$ for every $i = 1, \ldots, k-1$.

Observe that cyclic permutations leave the property of being (weakly) positively oriented invariant.

**Definition 11.8.** A (not necessarily continuous) map $\varphi: S^1 \to S^1$ is increasing of degree one if the following condition holds: if $(x_1, \ldots, x_k) \in (S^1)^k$ is weakly positively oriented, then $(\varphi(x_1), \ldots, \varphi(x_k))$ is weakly positively oriented.

For example, any constant map $\varphi: S^1 \to S^1$ is increasing of degree one. The following lemma provides an alternative description of increasing maps of degree one. Observe that the composition of increasing maps of degree one is increasing of degree one. We stress that, in our terminology, a (not necessarily continuous) map $\varphi: \mathbb{R} \to \mathbb{R}$ is increasing if it is weakly increasing, i.e. if $\varphi(x) \leq \varphi(y)$ whenever $x \leq y$.

**Lemma 11.9.** Let $\varphi: S^1 \to S^1$ be any map. Then the following conditions are equivalent:

1. $\varphi$ is increasing of degree one;
2. if $(x_1, \ldots, x_4) \in (S^1)^4$ is positively oriented, then $(\varphi(x_1), \ldots, \varphi(x_4))$ is weakly positively oriented;
3. there exists a (set-theoretical) increasing lift $\tilde{\varphi}: \mathbb{R} \to \mathbb{R}$ of $\varphi$ such that $\tilde{\varphi}(x + 1) = \tilde{\varphi}(x) + 1$ for every $x \in \mathbb{R}$.

**Proof.** (1) $\Rightarrow$ (2) is obvious, so we begin by showing that (2) $\Rightarrow$ (3). It is immediate to check that $\varphi$ satisfies (2) (resp. (3)) if and only if $r \circ \varphi$ does, where $r: S^1 \to S^1$ is any rotation. Therefore, we may assume that $\varphi([0]) = [0]$. For $t \in [0, 1)$, we define $\tilde{\varphi}(t) \in [0, 1]$ as follows. If $\varphi([t]) = [0]$, then we distinguish two cases: if $\varphi([s]) = [0]$ for every $0 \leq s \leq t$, then we set $\tilde{\varphi}(t) = 0$; otherwise we set $\tilde{\varphi}(t) = 1$. If $\varphi([t]) \neq [0]$, then $\tilde{\varphi}(t)$ is the unique point in $(0, 1]$ such that $\tilde{\varphi}(t) = \varphi([t])$.

We have thus defined a map $\tilde{\varphi}: [0, 1) \to [0, 1]$ such that $\tilde{\varphi}(0) = 0$. This map uniquely extends to a map (still denoted by $\tilde{\varphi}$) such that $\tilde{\varphi}(x + 1) = \tilde{\varphi}(x) + 1$. In order to conclude we need to show that $\tilde{\varphi}$ is increasing. By construction, this boils down to show that $\tilde{\varphi}|_{[0,1)}$ is increasing, i.e. that
\( \bar{\varphi}(t_0) \leq \bar{\varphi}(t_1) \) whenever \( 0 \leq t_0 < t_1 < 1 \). Since \( \bar{\varphi}(0) = 0 \) we may suppose that \( 0 < t_0 \).

Assume first that \( \varphi([t_0]) = [0] \). If \( \varphi([s]) = [0] \) for every \( 0 \leq s \leq t_0 \), then \( \bar{\varphi}(t_0) = 0 \leq \bar{\varphi}(t_1) \), and we are done. Otherwise \( \bar{\varphi}(t_0) = 1 \), and there exists \( 0 < s < t_0 \) such that \( \varphi([s]) \neq [0] \). Observe that the quadruple \( ([0], [s], [t_0], [t_1]) \) is positively oriented. As a consequence, the quadruple \( \bar{\varphi}([0]), \bar{\varphi}([s]), \bar{\varphi}([t_0]), \bar{\varphi}([t_1]) \) is weakly positively oriented. Since \( \varphi([s]) \neq [0] \), this implies in turn that \( \bar{\varphi}([t_1]) = 0 \), so by definition \( \bar{\varphi}(t_1) = 1 \), and again \( \bar{\varphi}(t_0) \leq \bar{\varphi}(t_1) \).

Assume now that \( \varphi([t_0]) \neq [0] \). If \( \varphi([t_1]) = [0] \), then \( \bar{\varphi}(t_1) = 1 \), and we are done. Otherwise, observe that, since \( \varphi \) takes positively oriented quadruples into weakly positively oriented quadruples, \( \varphi \) also takes positively oriented triples into weakly positively oriented triples. Therefore, the triple \( ([0], \varphi([t_0]), \varphi([t_1])) \) is weakly positively oriented. Since \( \varphi([t_i]) \neq [0] \) for \( i = 0, 1 \), this is equivalent to the fact that \( \bar{\varphi}(t_0) \leq \bar{\varphi}(t_1) \).

Let us now prove that (3) implies (1). Let \( (x_1, \ldots, x_k) \in (S^1)^k \) be positively oriented. We \( t_i \in \mathbb{R} \) such that \( [t_i] = x_i \) and \( t_1 \leq t_i < t_1 + 1 \) for \( i = 1, \ldots, k \). Since \( (x_1, \ldots, x_k) \in (S^1)^k \) is positively oriented we have \( t_i \leq t_{i+1} \) for every \( i = 1, \ldots, k - 1 \), so

\[
\bar{\varphi}(t_1) \leq \bar{\varphi}(t_2) \leq \ldots \leq \bar{\varphi}(t_k) \leq \bar{\varphi}(t_1) + 1,
\]

which readily implies that \( (\varphi(x_1), \ldots, \varphi(x_k)) \) is weakly positively oriented.

If \( \varphi \) is increasing of degree one, then a lift \( \bar{\varphi} \) as in point (3) of the previous lemma is called a good lift of \( \varphi \). For example, the constant map \( \varphi : S^1 \to S^1 \) mapping every point to \( [0] \) admits \( \bar{\varphi}(x) = [x] \) as a good lift (in fact, for every \( \alpha \in \mathbb{R} \) the maps \( x \mapsto [x + \alpha] \) and \( x \mapsto [x + \alpha] \) are good lifts of \( \varphi \)).

**Remark 11.10.** Condition (3) of the previous lemma (i.e. the existence of an increasing lift commuting with integral translations) is usually described as the condition defining the notion of increasing map of degree one. This condition is not equivalent to the request that positively oriented triples are taken into weakly positively oriented triples. For example, if \( \varphi([t]) = [0] \) for every \( t \in \mathbb{Q} \) and \( \varphi([t]) = [1/2] \) for every \( t \in \mathbb{R} \setminus \mathbb{Q} \), then \( \varphi \) takes any triple into a weakly positively oriented one, but it does not admit any increasing lift. On the other hand, the quadruple \( ([0], [\sqrt{2}/4], 1/2, [\sqrt{2}/2]) \) is taken by \( \varphi \) into \( ([0], [1/2], [0], [1/2]) \), which is not weakly positively oriented.

The following definition will prove useful later:

**Definition 11.11.** Let \( \varphi : S^1 \to S^1 \) be an increasing map of degree one. Then \( \varphi \) is upper semicontinuous if it admits an upper semicontinuous good lift \( \bar{\varphi} : \mathbb{R} \to \mathbb{R} \).
6. Semi-conjugation

The following definition plays an essential role in the study of the dynamics of group actions on the circle.

DEFINITION 11.12. Let $\rho_i : \Gamma \to \text{Homeo}^+(S^1)$ be group representations, $i = 1, 2$. We say that $\rho_1$ is semi-conjugated to $\rho_2$ if the following conditions hold:

1. There exists an increasing map $\varphi$ of degree one such that
   $$\rho_1(g)\varphi = \varphi\rho_2(g)$$
   for every $g \in \Gamma$.
2. There exists an increasing map $\varphi^*$ of degree one such that
   $$\rho_2(g)\varphi = \varphi\rho_1(g)$$
   for every $g \in \Gamma$.

The main result of this section is Theorem 11.22, which shows that the bounded Euler class is a complete invariant of semi-conjugation. We begin by observing that semi-conjugation is an equivalence relation:

LEMMA 11.13. Semi-conjugation is an equivalence relation.

PROOF. Reflexivity and symmetry are obvious, while transitivity readily follows from the fact that the composition of increasing maps of degree one is an increasing map of degree one. \qed

Our definition of semi-conjugation is slightly different from the ones proposed by Ghys in [Ghy87, Ghy99, Ghy01] (see Remark 11.14 below) and is in fact equivalent to the one given in [Buc08] (Remark 11.26).

REMARK 11.14. One may wonder whether condition (2) in the definition of semi-conjugacy is a consequence of condition (1). We will see in Proposition 11.24 that this is often the case. Nevertheless, the following example, which is due to Bucher [Buc08], shows that, in general, condition (2) cannot be dropped from the definition of semi-conjugacy.

Let $\rho_1$ be the trivial representation, and suppose that there exists $\bar{g}$ such that $\rho_2(\bar{g})$ does not have any fixed point in $S^1$ (this is the case, for example, if $\Gamma = \mathbb{Z}$, $\bar{g} = 1$ and $\rho_2(1)$ is a non-trivial rotation). If $\varphi$ is the constant map with image $x_0 \in S^1$, then for every $g \in \Gamma$ the maps $\rho_1(g)\varphi$ and $\varphi\rho_2(g)$ both coincide with the constant map $\varphi$, so

$$\rho_1(g)\varphi = \varphi\rho_2(g)$$

for every $g \in \Gamma$. On the other hand, if $\varphi : S^1 \to S^1$ were any map such that $\rho_2(\bar{g})\varphi = \varphi\rho_1(\bar{g})$, then for every $x \in S^1$ we would have

$$\rho_2(\bar{g})(\varphi(x)) = \varphi(\rho_1(\bar{g})(x)) = \varphi(x),$$

a contradiction since $\rho_2(\bar{g})$ acts freely on $S^1$.

Therefore, condition (2) in the definition of semi-conjugation is necessary in order to ensure that semi-conjugation is an equivalence relation. Indeed,
this is false for the notion of semi-conjugation defined in [Ghy99, Ghy01], where condition (2) of Definition 11.12 is replaced with the request that \( \varphi \) be continuous (but not necessarily injective), or for the definition given in [Ghy87], where condition (2) is dropped.

It readily follows from the definitions that conjugated representations are semi-conjugated. However, semi-conjugation is a much weaker condition than conjugation: for example, any representation admitting a fixed point is semi-conjugated to the trivial representation (see Proposition 11.18). On the other hand, semi-conjugation implies conjugation for minimal representations (a representation \( \rho \) is minimal if every \( \rho(\Gamma) \)-orbit is dense):

**Proposition 11.15.** Let \( \rho_1, \rho_2 : \Gamma \to \text{Homeo}^+(S^1) \) be semi-conjugated representations, and suppose that every orbit of \( \rho_i(\Gamma) \) is dense for \( i = 1, 2 \). Then \( \rho_1 \) is conjugated to \( \rho_2 \).

**Proof.** Let \( \varphi \) be an increasing map of degree one such that

\[
\rho_1(g)\varphi = \varphi\rho_2(g)
\]

for every \( g \in \Gamma \),

and denote by \( \tilde{\varphi} \) a good lift of \( \varphi \). The image of \( \varphi \) is obviously \( \rho_1(\Gamma) \)-invariant, so our assumptions imply that \( \text{Im} \varphi \) is dense in \( S^1 \). This implies in turn that the image of \( \tilde{\varphi} \) is dense in \( \mathbb{R} \). So the map \( \tilde{\varphi} \), being increasing, is continuous and surjective. Therefore, the same is true for \( \varphi \), and we are left to show that \( \varphi \) is also injective.

Suppose by contradiction that there exist points \( x, y \in S^1 \) such that \( \varphi(x) = \varphi(y) \), and choose lifts \( \tilde{x}, \tilde{y} \) of \( x, y \) in \( \mathbb{R} \) such that \( \tilde{x} < \tilde{y} < \tilde{x} + 1 \). Since \( \tilde{\varphi} \) is increasing and commutes with integral translations, we have either \( \tilde{\varphi}(\tilde{y}) = \tilde{\varphi}(\tilde{x}) \) or \( \tilde{\varphi}(\tilde{y}) = \tilde{\varphi}(\tilde{x} + 1) \). In any case, \( \tilde{\varphi} \) is constant on a non-trivial interval, so there exists an open subset \( U \subseteq S^1 \) such that \( \varphi|_U \) is constant. Let now \( x \) be any point of \( S^1 \). Our assumptions imply that there exists \( g \in \Gamma \) such that \( \rho_2(g)^{-1}(x) \in U \), so \( x \in V = \rho_2(g)(U) \), where \( V \subseteq S^1 \) is open. Observe that

\[
\varphi|_V = (\varphi\rho_2(g))|_U \circ \rho_2(g)^{-1}|_V = (\rho_1(g)\varphi)|_U \circ \rho_2(g)^{-1}|_V
\]

is constant. We have thus proved that \( \varphi \) is locally constant, whence constant, and this contradicts the fact that \( \varphi \) is surjective. \( \square \)

There are examples where the maps \( \varphi, \varphi' \) described in the definition of semi-conjugation may not be chosen to be continuous. Things get better if we replace continuity with the less demanding notion of upper semicontinuity:

**Lemma 11.16.** Let \( \rho_i : \Gamma \to \text{Homeo}^+(S^1) \) be group representations, \( i = 1, 2 \), and let \( \varphi : S^1 \to S^1 \) be an increasing map of degree one such that

\[
\rho_1(g)\varphi = \varphi\rho_2(g)
\]
for every $g \in \Gamma$. Then, there exists an upper semicontinuous increasing map of degree one $\varphi': S^1 \to S^1$ such that

$$\rho_1(g)\varphi' = \varphi'\rho_2(g)$$

for every $g \in \Gamma$.

**Proof.** Let $\tilde{\varphi}$ be a good lift of $\varphi$, and set

$$\tilde{\varphi}': \mathbb{R} \to \mathbb{R}, \quad \tilde{\varphi}'(x) = \sup \{ \tilde{\varphi}(y) \mid y < x \}.$$  

It is immediate to check that $\tilde{\varphi}'$ is increasing and commutes with integral translations. Therefore, there exists a unique increasing map $\varphi': S^1 \to S^1$ admitting $\tilde{\varphi}'$ as a good lift. By definition, such a map is upper semicontinuous.

Let now $g \in \Gamma$ and fix the preferred lifts $\tilde{\rho}_1(g)$, $\tilde{\rho}_2(g)$ in $\text{Homeo}_+(S^1)$. Also fix $a \in S^1$, and choose $x \in p^{-1}(a) \subseteq \mathbb{R}$. We have

$$\tilde{\rho}_1(g)(\tilde{\varphi}'(x)) = \tilde{\rho}_1(g)(\sup \{ \tilde{\varphi}(y) \mid y < x \}) = \sup \{ \tilde{\rho}_1(g)(\tilde{\varphi}(y)) \mid y < x \}.$$  

Since $\rho_1(g)\varphi = \varphi\rho_2(g)$, for every $y \in \mathbb{R}$ there exists $k_y \in \mathbb{Z}$ such that

$$\tilde{\rho}_1(g)(\tilde{\varphi}(y)) = \varphi(\tilde{\rho}_2(g)(y)) + k_y$$

(beware: $k_y$ may indeed depend on $y$, see Remark 11.20). Moreover, since $\tilde{\varphi}$ is locally bounded and commutes with integral translations, the set $\{k_y, y \in \mathbb{R}\}$ is finite, so putting together (21) and (22) we may conclude that there exists an integer $k$ such that

$$\tilde{\rho}_1(g)(\tilde{\varphi}'(x)) = \tilde{\varphi}(\tilde{\rho}_2(g)(x)) + k.$$  

This implies that $\rho_1(g)\varphi' = \varphi'\rho_2(g)$, and concludes the proof. \hfill $\square$

As mentioned above, in the following paragraphs we will show that two representations $\rho_1, \rho_2$ are semi-conjugated if and only if $e_b(\rho_1) = e_b(\rho_2)$. In the following proposition we prove one implication:

**Proposition 11.17.** Let $\rho_1, \rho_2$ be representations of $\Gamma$ into $\text{Homeo}^+(S^1)$ such that $e_b(\rho_1) = e_b(\rho_2)$. Then $\rho_1$ is semi-conjugated to $\rho_2$.

**Proof.** By symmetry, it is sufficient to exhibit an increasing map of degree one $\varphi$ such that

$$\rho_1(g)\varphi = \varphi\rho_2(g)$$

for every $g \in \Gamma$.

Since $e_b(\rho_1) = e_b(\rho_2)$ we have a fortiori $e(\rho_1) = e(\rho_2)$. Let

$$1 \longrightarrow \mathbb{Z} \overset{\iota}{\longrightarrow} \Gamma \overset{\pi}{\longrightarrow} \Gamma \longrightarrow 1$$

be the central extension associated to $e(\rho_1) = e(\rho_2)$. Since $e_b(\rho_1) = e_b(\rho_2)$, if $c_\ast = \rho'_1(c)$ is the pull-back of the usual representative of the Euler class of $\text{Homeo}^+(S^1)$, then there exists a bounded cochain $u: \Gamma \to \mathbb{Z}$ such that $\delta u = c_1 - c_2$. Let $s_1: \Gamma \to \Gamma$ be a section of $\pi$ such that

$$i(c_1(g_1, g_2)) = s_1(g_1g_2)^{-1}s_1(g_1)s_1(g_2).$$
If we set \( s_2(g) = s_1(g) \iota(u(g)) \), then we get

\[
i(c_2(g_1, g_2)) = s_2(g_1 g_2)^{-1} s_2(g_1) s_2(g_2) .
\]

Of course, once \( i = 1, 2 \) is fixed, every element of \( \mathcal{G} \in \Gamma \) admits a unique expression \( \mathcal{G} = \iota(n) s_i(g) \). For \( i = 1, 2 \) we define the map

\[
\overline{\rho}_i : \Gamma \to \text{Homeo}_+(S^1), \quad \overline{\rho}_i(\iota(n) s_i(g)) = \tau_n \rho_i(g) .
\]

Let us prove that \( \overline{\rho}_i \) is a homomorphism. Take elements \( \mathcal{G}_j = i(n_j) s_i(g_j) \), \( j = 1, 2 \). Then

\[
\mathcal{G}_1 \mathcal{G}_2 = \iota(n_1) s_i(g_1) \iota(n_2) s_i(g_2) = \iota(n_1 + n_2) \iota(c_1(g_1, g_2)) s_i(g_1 g_2)
\]

so

\[
\overline{\rho}_i(\mathcal{G}_1 \mathcal{G}_2) = \tau_{n_1+n_2} \tau_{c_1(g_1, g_2)} \rho_i(\mathcal{G}_1 \mathcal{G}_2) = \tau_{n_1} \rho_i(\mathcal{G}_1) \tau_{n_2} \rho_i(\mathcal{G}_2) = \overline{\rho}_i(\mathcal{G}_1) \overline{\rho}_i(\mathcal{G}_2),
\]

and the map \( \overline{\rho}_i \) is a homomorphism.

Let now \( \mathcal{G} = \iota(n_1) s_1(g) = \iota(n_1 - u(g)) s_2(g) \). Then

\[
\overline{\rho}_1(\mathcal{G})^{-1} \overline{\rho}_2(\mathcal{G}) = \left( \tau_{n_1} \rho_1(\mathcal{G}) \right)^{-1} \tau_{n_1-u(g)} \rho_2(\mathcal{G}) = \tau_{-u(g)} \left( \rho_1(\mathcal{G}) \right)^{-1} \rho_2(\mathcal{G}) .
\]

Recall now that there exists \( M \geq 0 \) such that \( |u(g)| \leq M \) for every \( g \in \Gamma \). Moreover, using that \( \rho_i(\mathcal{G})(0) \in [0,1) \) for every \( g \in \Gamma \), \( i = 1, 2 \), it is immediate to realize that

\[
x - 2 \leq \left( \rho_1(\mathcal{G}) \right)^{-1} (\rho_2(\mathcal{G})(x)) \leq x + 2 \quad \text{for every } x \in \mathbb{R}.
\]

As a consequence, for every \( x \in \mathbb{R} \) we have

\[
x - M - 2 \leq \overline{\rho}_1(\mathcal{G})^{-1} (\overline{\rho}_2(\mathcal{G})(x)) \leq x + M + 2 ,
\]

so we may define the value

\[
\tilde{\varphi}(x) = \sup_{\mathcal{G} \in \Gamma} \overline{\rho}_1(\mathcal{G})^{-1} (\overline{\rho}_2(\mathcal{G})(x)) \in [x - M - 2, x + M + 2] .
\]

We will now check that \( \tilde{\varphi} \) realizes the desired semi-conjugation between \( \rho_1 \) and \( \rho_2 \).

Being the supremum of (strictly) increasing maps which commute with integral translations, the map \( \tilde{\varphi} : \mathbb{R} \to \mathbb{R} \) is (possibly non-strictly) increasing and commutes with integral translations, so it is a good lift of an increasing map of degree one \( \varphi : S^1 \to S^1 \).
For every $g_0 \in \Gamma$, $x \in \mathbb{R}$, we have
\[
\tilde{\varphi}(\rho_2(g_0)(x)) = \sup_{g \in \Gamma} \rho_1(g)^{-1}(\rho_2(g)(\rho_2(g_0)(x)))
\]
\[
= \sup_{g \in \Gamma} \rho_1(g)^{-1}(\rho_2(g \cdot g_0)(x))
\]
\[
= \sup_{g \in \Gamma} \rho_1(g)^{-1}(\rho_2(g)(\rho_2(g_0)(x)))
\]
\[
= \rho_1(g_0) \left( \sup_{g \in \Gamma} \rho_1(g)^{-1}(\rho_2(g)(x)) \right)
\]
\[
= \rho_1(g_0)(\tilde{\varphi}(x)) ,
\]
and this implies that
\[
\varphi \rho_2(g_0) = \rho_1(g_0) \varphi
\]
for every $g_0 \in \Gamma$.

We are now ready to characterize the set of representations having vanishing bounded Euler class as the semi-conjugacy class of the trivial representation:

**Proposition 11.18.** Let $\rho: \Gamma \to \text{Homeo}^+(S^1)$ be a representation. Then the following conditions are equivalent:

1. $\rho$ fixes a point, i.e., there exists a point $x_0 \in S^1$ such that $\rho(g)(x_0) = x_0$ for every $g \in \Gamma$;
2. $\rho$ is semi-conjugated to the trivial representation;
3. $e_b(\rho) = 0$.

**Proof.** (1) $\implies$ (3): Suppose that $\rho$ fixes the point $x_0$, and consider the cocycle $c_{x_0}$ representing the bounded Euler class $e_b \in H^2(\text{Homeo}^+(S^1), \mathbb{Z})$. By definition, the pull-back $\rho^*(c_{x_0})$ vanishes identically, and this implies that $e_b(\rho) = \rho^*(e_b) = 0$.

The implication (3) $\implies$ (2) is an immediate consequence of Proposition 11.17.

(2) $\implies$ (1): If $\rho$ is semi-conjugated to the trivial representation $\rho_0$, then there exists an increasing map $\varphi: S^1 \to S^1$ of degree one such that $\rho(g) \varphi = \varphi \rho_0(g) = \varphi$ for every $g \in \Gamma$. Let now $x_0 = \varphi(y_0)$ be a point in the image of $\varphi$. By evaluating the above equality in $y_0$, we get
\[
\rho(g)(x_0) = \rho(g) \varphi(y_0) = \varphi(y_0) = x_0
\]
for every $g \in \Gamma$. Therefore, $\rho$ fixes $x_0$, and we are done.

We go on in our analysis of semi-conjugation with the following:
Lemma 11.19. Let $\rho_1, \rho_2 : \Gamma \to \text{Homeo}^+(S^1)$ be representations, and let $
abla : S^1 \to S^1$ be an increasing map of degree one such that

$$\rho_1(g) \nabla = \nabla \rho_2(g)$$

for every $g \in \Gamma$.

Let also $\tilde{\nabla}$ be a good lift of $\nabla$, and denote by $\tilde{\rho}_i(g)$ the preferred lift of $\rho_i(g)$, $g \in \Gamma$, $i = 1, 2$ (recall that such a lift is determined by the request that $\tilde{\rho}_i(0) \in [0, 1]$). We assume that (at least) one of the following conditions hold:

1. $\rho_1(\Gamma)$ does not fix a point, or
2. there exists a $\rho_2(\Gamma)$-invariant subset $K \subseteq S^1$ containing at least two points such that $\nabla|_K$ is injective.

Then, there exists a bounded map $u : \Gamma \to \mathbb{Z}$ such that

$$\tilde{\rho}_1(g) \tilde{\nabla} = \tilde{\nabla} \rho_2(g)\tau_u(g)$$

for every $g \in \Gamma$.

Proof. Let us fix $g \in \Gamma$, and set $f_i = \rho_i(g)$ and $\tilde{f}_i = \tilde{\rho}_i(g)$. Since $\tilde{f}_1 \tilde{\nabla}$ and $\tilde{\nabla} \tilde{f}_2$ are both lifts of the same map $f_1 \nabla = \nabla f_2$, for every $x \in \mathbb{R}$ the quantity $\tau_g(x) = \tilde{f}_1(\tilde{\nabla}(x) - \tilde{\nabla} f_2(x))$ is an integer. Moreover, of course we have $\tau_g(x + k) = \tau_g(x)$ for every $x \in \mathbb{R}$, $k \in \mathbb{Z}$. We need to show that $\tau_g : \mathbb{R} \to \mathbb{Z}$ is constant, and bounded independently of $g$.

We define a subset $\tilde{K} \subseteq \mathbb{R}$ as follows. If condition (1) holds, then $\nabla$ cannot be constant, because otherwise $\rho_1(\Gamma)$ would fix the point in the image of $\nabla$. Therefore, we may choose two points $x, y \in S^1$ such that $\nabla(x) \neq \nabla(y)$, and we set $\tilde{K} = p^{-1}(\{x, y\})$. If condition (2) holds, then we simply set $\tilde{K} = p^{-1}(K)$. Observe that, with any of our choices, the restriction of $\tilde{\nabla}$ to $\tilde{K}$ is injective. Since $\tilde{f}_1$ is injective, this implies at once that the composition $\tilde{f}_1 \tilde{\nabla}$ is also injective on $\tilde{K}$. We claim that also $\tilde{\nabla} \tilde{f}_2|_{\tilde{K}}$ is injective. This is clear when condition (2) holds, because in this case $\tilde{K}$ is $\tilde{f}_2$-invariant. On the other hand, if condition (1) holds, then $\nabla$ is injective on $\{x, y\}$, so $f_1(\nabla(x)) \neq f_1(\nabla(y))$ and $\nabla(f_2(x)) \neq \nabla(f_2(y))$, whence the conclusion again.

We first show that $\tau_g|_{\tilde{K}}$ is constant. Let $\tilde{k}, \tilde{k}'$ lie in $\tilde{K}$. Since $\tau$ is $\mathbb{Z}$-periodic, we may assume that $\tilde{k} < \tilde{k}' < \tilde{k} + 1$. Since both $\tilde{\psi}_1 = \tilde{f}_1 \tilde{\nabla}$ and $\tilde{\psi}_2 = \tilde{\nabla} \tilde{f}_2$ are strictly increasing on $\tilde{K}$, we may apply Lemma 11.2, thus getting

$$|\tau_g(\tilde{k}') - \tau_g(\tilde{k})| = |\tilde{\psi}_2(\tilde{k}') - \tilde{\psi}_1(\tilde{k}) - (\tilde{\psi}_2(\tilde{k}') - \tilde{\psi}_1(\tilde{k}))| < 1.$$

Since $\tau_g$ has integral values, this implies that $\tau_g(\tilde{k}') = \tau_g(\tilde{k})$.

Let now $\tilde{z} \in \mathbb{R} \setminus \tilde{K}$ be fixed. In order to show that $\tau_g$ is constant it is sufficient to show that there exists $\tilde{k} \in \tilde{K}$ such that $\tau_g(\tilde{z}) = \tau_g(\tilde{k})$. Our assumptions imply that $\tilde{K} \cap (\tilde{z}, \tilde{z} + 1) \indices{s$ contains at least two elements. Since $\tilde{\phi}$ is strictly monotone on $\tilde{K}$, this implies that there exists $\tilde{k} \in \tilde{K}$ such that $\tilde{\phi}$ is strictly monotone on $\{\tilde{z}\} \cup (\tilde{k} + \mathbb{Z})$. Therefore, we may apply again
Lemma 11.2 and conclude that $|\tau_g(\tilde{z}) - \tau_g(\tilde{k})| < 1$, so $\tau_g(\tilde{z}) = \tau_g(\tilde{k})$. We have thus shown that $\tau_g$ is constant.

We are left to show that the value taken by $\tau_g$ is bounded independently of $g$. Let $a \in \mathbb{R}$ be such that $\varphi([0, 1]) \subseteq [a, a + 1]$. For every $g \in \Gamma$, $i = 1, 2$ we have that $\rho_i(g)(0) \in [0, 1)$, so

$$|\varphi(\rho_2(g)(0))| \leq |a| + 1$$

and

$$|\rho_1(g)(\varphi(0))| \leq |\varphi(0)| + |\rho_1(g)(\varphi(0)) - \varphi(0) - (\rho_1(g)(0) - 0)| + |\rho_1(g)(0) - 0|$$

$$< |\varphi(0)| + 1 + 1 \leq |a| + 3$$

(see Lemma 11.2). Therefore, we may conclude that

$$|u(g)| = |\rho_1(g)(\varphi(0)) - \varphi(\rho_2(g)(0))| < |\rho_1(g)(\varphi(0))| + |\varphi(\rho_2(g)(0))| < 2|a| + 4$$

and we are done.

\textbf{Remark 11.20.} The following example, which is due to Elia Fioravanti, shows that, even in the case when $\rho_1$ is semi-conjugated to $\rho_2$, the previous lemma does not hold if we remove the requests (1) and (2). With notation as in the proof of Lemma 11.19, let $f_2 \in \text{Homeo}^+(S^1)$ admit the lift $\tilde{f}_2(x + n) = x^2 + n$ for every $x \in [0, 1)$, $n \in \mathbb{Z}$, and let $\varphi(x) = |x + 1/2|$ be a good lift of the increasing map of degree one $\varphi$ which sends every point of $S^1$ to $[0]$. Then we have

$$\varphi(f_2(x)) = [0] = \varphi(x)$$

for every $x \in S^1$. Therefore, if $\rho_1 : \mathbb{Z} \to \text{Homeo}^+(S^1)$ is the trivial representation and $\rho_2 = \rho_{f_2}$ sends 1 to $f_2$, then $\varphi$ satisfies the hypotheses of Lemma 11.19 (and $\rho_1$ is semi-conjugated to $\rho_2$ by Proposition 11.18). As a lift of $\rho_1(1) = \text{Id}_{S^1}$ we choose the identity $\tilde{f}_1$ of $\mathbb{R}$. Then we have

$$\tilde{f}_1(\varphi(0)) - \varphi(\tilde{f}_2(0)) - \varphi(0) = 0$$

while

$$\tilde{f}_1(\varphi(1/2)) - \varphi(\tilde{f}_2(1/2)) = \tilde{f}_1(1) - \varphi(1/4) = 1$$

\textbf{Proposition 11.21.} Let $\rho_1, \rho_2 : \Gamma \to \text{Homeo}^+(S^1)$ be representations, and let $\varphi : S^1 \to S^1$ be an increasing map of degree one such that

$$\rho_1(g) \varphi = \varphi \rho_2(g)$$

for every $g \in \Gamma$.

We assume that (at least) one of the following conditions hold:

1. $\rho_1(\Gamma)$ does not fix a point, or
2. there exists a $\rho_2(\Gamma)$-invariant subset $K \subseteq S^1$ containing at least two points such that $\varphi|_K$ is injective.

Then $e_b(\rho_1) = e_b(\rho_2)$. 

6. SEMI-CONJUGATION

Proof. We denote by \( \tilde{\varphi} \) a good lift of \( \varphi \), and by \( u : \Gamma \to \mathbb{Z} \) the bounded function defined in point (2) of Lemma 11.19. In order to conclude that \( e_b(\rho_1) = e_b(\rho_2) \) it is sufficient to show that, if \( c \) is the canonical representative of the bounded Euler class of \( \text{Homeo}^+(S^1) \), then

\[
\rho_1^*(c) - \rho_2^*(c) = \delta u.
\]

Let us fix \( g_1, g_2 \in \Gamma \). By the definition of \( u \) we have

\[
\tilde{\varphi} = \rho_1(g_1) \circ \tilde{\varphi} \circ \left( \rho_2(g_1) \right)^{-1} \circ \tau_{-u(g_1)},
\]

\[
\tilde{\varphi} = \rho_1(g_2) \circ \tilde{\varphi} \circ \left( \rho_2(g_2) \right)^{-1} \circ \tau_{-u(g_2)},
\]

\[
\tilde{\varphi} = \left( \rho_1(g_1 g_2) \right)^{-1} \circ \tilde{\varphi} \circ \rho_2(g_1 g_2) \circ \tau_{u(g_1 g_2)}.
\]

Now we substitute the first equation in the right-hand side of the second, and the third equation in the right-hand side of the obtained equation. Using that

\[
\tau_{c(\rho_1, \rho_2)} = \rho_1(g_1) \rho_2(g_2) \left( \rho_1(g_1 g_2) \right)^{-1}
\]

we finally get

\[
\tilde{\varphi} = \tilde{\varphi} \circ \tau_{c(\rho_1, \rho_2)}(g_1, g_2) - \rho_2^*(c)(g_1, g_2) - u(g_1) - u(g_2) + u(g_1 g_2),
\]

which is equivalent to Equation (23).

We are now ready to prove the main result of this section.

Theorem 11.22. Let \( \rho_1, \rho_2 \) be representations of \( \Gamma \) into \( \text{Homeo}^+(S^1) \). Then \( \rho_1 \) is semi-conjugated to \( \rho_2 \) if and only if \( e_b(\rho_1) = e_b(\rho_2) \).

Proof. We already know from Proposition 11.17 that representations sharing the same bounded Euler class are semi-conjugated.

Let us suppose that \( \rho_1 \) is semi-conjugated to \( \rho_2 \). If \( \rho_1(\Gamma) \) fixes a point of \( S^1 \), then Proposition 11.18 implies that \( e_b(\rho_1) = e_b(\rho_2) = 0 \), and we are done. On the other hand, if \( \rho_1(\Gamma) \) does not fix a point in \( S^1 \), then \( e_b(\rho_1) = e_b(\rho_2) \) by Proposition 11.21.

In the case of a single homeomorphism we obtain the following:

Corollary 11.23. Let \( f \in \text{Homeo}^+(S^1) \). Then:

1. \( \text{rot}(f) = 0 \) if and only if \( f \) has a fixed point.
2. \( \text{rot}(f) \in \mathbb{Q}/\mathbb{Z} \) if and only if \( f \) has a finite orbit.
3. If \( \text{rot}(f) = [\alpha] \notin \mathbb{Q}/\mathbb{Z} \) and every orbit of \( f \) is dense, then \( f \) is conjugated to the rotation of angle \( \alpha \) (the converse being obvious).

Proof. Point (1) follows from Proposition 11.18, and point (2) follows from point (1) and the fact that \( \text{rot}(f^n) = n \text{rot}(f) \) for every \( n \in \mathbb{Z} \) (see Corollary 11.6).

If \( f \) is as in (3), then Theorem 11.22 implies that \( f \) is semi-conjugated to the rotation of angle \( \text{rot}(f) \), so the conclusion follows from Proposition 11.15.

□
The following proposition is a direct consequence of Proposition 11.21 and Theorem 11.22, and shows that Ghys’ original definition of quasi-conjugation differs from ours only in the case when one representation fixes a point:

**Proposition 11.24.** Let \( \rho_1, \rho_2 : \Gamma \to \text{Homeo}^+(S^1) \) be representations, and let \( \varphi : S^1 \to S^1 \) be an increasing map of degree one such that
\[
\rho_1(g)\varphi = \varphi \rho_2(g) \quad \text{for every } g \in \Gamma.
\]
Also suppose that \( \rho_1(\Gamma) \) does not fix a point. Then, \( \rho_1 \) is semi-conjugated to \( \rho_2 \).

\[
□
\]

We conclude the chapter by establishing some alternative characterizations of semi-conjugation:

**Proposition 11.25.** Let \( \rho_1, \rho_2 : \Gamma \to \text{Homeo}^+(S^1) \) be representations, and let \( \varphi : S^1 \to S^1 \) be an upper semicontinuous increasing map of degree one such that
\[
\rho_1(g)\varphi = \varphi \rho_2(g) \quad \text{for every } g \in \Gamma.
\]
Then, the following conditions are equivalent:

1. \( \rho_1 \) is semi-conjugated to \( \rho_2 \);
2. There exist \( \rho(\Gamma_i) \)-invariant subsets \( K_i \subseteq S^1 \) such that \( \varphi \) restricts to a bijection between \( K_2 \) and \( K_1 \);
3. There exist a \( \rho_1(\Gamma) \)-orbit \( O_1 \subseteq S^1 \) and a \( \rho_2(\Gamma) \)-orbit \( O_2 \subseteq S^1 \) such that \( \varphi \) restricts to a bijection between \( O_2 \) and \( O_1 \);
4. There exists a \( \rho_2(\Gamma) \)-invariant subset \( K \subseteq S^1 \) such that \( \varphi|_K \) is injective.

**Proof.** We first show that conditions (2), (3) and (4) are pairwise equivalent. Then, we will show that (1) holds if and only if (4) does.

(2) \( \iff \) (3): If \( x_2 \in K_2 \), then the \( \rho_2(\Gamma) \)-orbit \( O_2 \) of \( x_2 \) is contained in \( K_2 \), so it is sufficient to observe that \( O_1 = \varphi(O_2) \) coincides with the \( \rho_1(\Gamma) \)-orbit of \( \varphi(x_2) \).

(3) \( \iff \) (4): It is sufficient to set \( K = O_2 \).

(4) \( \iff \) (2): It is sufficient to set \( K_2 = K \) and \( K_1 = \varphi(K) \).

Let us now come to the more interesting implications between conditions (1) and (4).

(1) \( \iff \) (4): Suppose first that \( \rho_2(\Gamma) \) fixes a point \( x_2 \). Then, for every \( g \in \Gamma \) we have
\[
\rho_1(g)(\varphi(x_2)) = \varphi(\rho_2(g)(x_2)) = \varphi(x_2)
\]
so \( x_1 = \varphi(x_2) \) is fixed by \( \rho_1(\Gamma) \), and it is sufficient to set \( K_i = \{x_i\} \). Suppose now that \( \rho_2(\Gamma) \) does not fix a point. We define the map \( \bar{\varphi}^* : \mathbb{R} \to \mathbb{R} \) as follows. Since \( \bar{\varphi} \) is increasing and commutes with integral translations, for every \( x \in \mathbb{R} \) the set \( \{y \in \mathbb{R} \mid \bar{\varphi}(y) \leq x\} \) is non-empty and bounded above, so it makes sense to set
\[
\bar{\varphi}^*(x) = \sup\{y \in \mathbb{R} \mid \bar{\varphi}(y) \leq x\}.
\]
By construction, \( \bar{\varphi}^* \) commutes with integral translations, so its image \( \bar{K} \subseteq \mathbb{R} \) is left invariant by integral translations. We define \( K \subseteq S^1 \) as the unique set such that \( p^{-1}(K) = \bar{K} \). In order to conclude, it is sufficient to show that \( K \) is \( \rho_2(\Gamma) \)-invariant, and that \( \bar{\varphi} \) is strictly increasing on \( \bar{K} \).

Since we are assuming that \( \rho_1(\Gamma) \) does not fix a point, Proposition 11.18 implies that also \( \rho_1(\Gamma) \) does not fix a point, so by Lemma 11.19 there exists a bounded map \( u : \Gamma \to \mathbb{Z} \) such that

\[
\bar{\rho}_1(g)\bar{\varphi} = \bar{\varphi}\bar{\rho}_2(g)\tau_u(g)
\]

for every \( g \in \Gamma \). Therefore, for \( g \in \Gamma, x \in \mathbb{R} \), we have

\[
\bar{\rho}_2(g)(\bar{\varphi}^*(x)) = \sup\{\bar{\rho}_2(g)(y) \mid \bar{\varphi}(y) \leq x\}
= \sup\{y \mid \bar{\varphi}(\bar{\rho}_2^{-1}(g)(y)) \leq x\}
= \sup\{y \mid \bar{\rho}_1(g)^{-1}(\bar{\varphi}(\tau_{-u(g)}(y))) \leq x\}
= \sup\{y \mid \bar{\rho}_1(g)^{-1}(\bar{\varphi}(y)) \leq x + u(g)\}
= \sup\{y \mid \bar{\rho}_1(g)^{-1}(\bar{\varphi}(y)) \leq x\} + u(g)
= \sup\{y \mid \bar{\varphi}(y) \leq \bar{\rho}_1(g)(x)\} + u(g)
= \bar{\varphi}^*(\rho_1(g)(x)) + u(g).
\]

As a consequence, the image \( \bar{K} \) of \( \bar{\varphi}^* \) is left invariant by the action of \( \bar{\rho}_2(g) \) for every \( g \in \Gamma \), and this shows that \( K \) is \( \rho_2(\Gamma) \)-invariant. Let us now prove that \( \bar{\varphi} \) is strictly increasing on \( \bar{K} \). Observe that, if \( z > \bar{\varphi}^*(x_1) \), then \( z > \sup\{y \in \mathbb{R} \mid \bar{\varphi}(y) \leq x_1\} \), so \( \bar{\varphi}(z) > x_1 \). Setting \( z = \bar{\varphi}^*(x_2) \), we get that

\[
\bar{\varphi}^*(x_2) > \bar{\varphi}^*(x_1) \implies \bar{\varphi}(\bar{\varphi}^*(x_2)) > x_1.
\]

Now observe that

\[
\bar{\varphi}(\bar{\varphi}^*(x_1)) = \bar{\varphi}(\sup\{y \mid \bar{\varphi}(y) \leq x_1\}) = \sup\{\bar{\varphi}(y) \mid \bar{\varphi}(y) \leq x_1\} \leq x_1,
\]

where the second inequality is due to the fact that \( \bar{\varphi} \) is upper semicontinuous. Putting together the inequalities (24), (25) we conclude that \( \bar{\varphi} \) is strictly increasing on \( \bar{K} \).

(4) \( \implies \) (1): If \( K \) consists of a point \( x_0 \), then \( \rho_2(\Gamma) \) fixes \( x_0 \), while \( \rho_1(\Gamma) \) fixes \( \varphi(x_0) \), so the conclusion follows from Proposition 11.18. If \( K \) contains at least two points, then Proposition 11.21 implies that \( e_b(\rho_1) = e_b(\rho_2) \), so \( \rho_1 \) and \( \rho_2 \) are semi-conjugated by Theorem 11.22. \( \square \)

**Remark 11.26.** In [Buc08], two representations \( \rho_1, \rho_2 : \Gamma \to \text{Homeo}^+(S^1) \) are defined to be semi-conjugated if and only if the following conditions hold:

1. There exists an increasing map \( \varphi : S^1 \to S^1 \) of degree one such that

\[
\rho_1(g)\varphi = \varphi\rho_2(g) \quad \text{for every } g \in \Gamma,
\]

2. There exist \( \rho(\Gamma) \)-invariant subsets \( K_i \subseteq S^1 \) such that \( \varphi \) restricts to a bijection between \( K_2 \) and \( K_1 \).
Our definition of semi-conjugation is indeed equivalent to Bucher’s one. In fact, if \( \rho_1 \) and \( \rho_2 \) are semi-conjugated (according to our definition), then condition (1) above holds by definition, and condition (2) is implied by Lemma 11.16 and Proposition 11.25. On the other hand, if condition (1) holds, then \( \rho_1 \) and \( \rho_2 \) are semi-conjugated by the implication (2) \( \Rightarrow \) (1) in Proposition 11.25, which holds without any assumption on the upper semicontinuity of \( \varphi \).
1. Topological, smooth and linear sphere bundles

If $X, Y$ are topological spaces, we denote by $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$ the natural projections. Let $M$ be a topological space, and fix $n \geq 1$. An oriented $n$-sphere bundle over $M$ is a map $\pi : E \to M$ such that the following conditions hold:

- For every $x \in M$, the subspace $E_x = \pi^{-1}(x) \subseteq E$ is homeomorphic to the $n$-dimensional sphere $S^n$;
- Each $E_x$ is endowed with an orientation;
- For every $x \in M$ there exist a neighbourhood $U$ and a homeomorphism $\psi : \pi^{-1}(U) \to U \times S^n$ such that the following diagram commutes

\[
\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\psi} & U \times S^n \\
\downarrow{\pi} & & \downarrow{\pi_U} \\
U & & \\
\end{array}
\]

and the homeomorphism $\pi_{S^n} \circ \psi|_{E_x} : E_x \to S^n$ is orientation-preserving for every $x \in U$. Such a diagram provides a local trivialization of $E$ (over $U$).

Henceforth we refer to oriented $n$-sphere bundles simply as to $n$-sphere bundles, or to sphere bundles when the dimension is understood. A 1-sphere bundle will be called circle bundle. An isomorphism between two sphere bundles $\pi : E \to M$, $\pi' : E' \to M$ is a homeomorphism $h : E \to E'$ such that $\pi = \pi' \circ h$.

The definition above fits into the more general context of (sphere) bundles with structure group. In fact, when $G$ is a subgroup of $\text{Homeo}^+(S^n)$, we say that the map $\pi : E \to M$ is a sphere bundle with structure group $G$ (or, simply, a sphere $G$-bundle) if $M$ admits an open cover $\mathcal{U} = \{U_i, i \in I\}$ such that the following conditions hold: every $U_i \in \mathcal{U}$ is the base of a local
trivialization
\[
\pi^{-1}(U_i) \xrightarrow{\psi_i} U_i \times S^n \xrightarrow{p} U_i
\]
and for every \(i, j \in I\) there exists a map \(\lambda_{ij}: U_i \cap U_j \to G\) such that
\[
\psi_i(\psi_j^{-1}(x,v)) = (x, \lambda_{ij}(x)(v))
\]
for every \(x \in U_i \cap U_j\) and \(v \in S^n\). Such a collection of local trivializations is called a \(G\)-atlas for \(E\), and the \(\lambda_{ij}\)'s are called the transition functions of the atlas. In the general theory of bundles with structure group \(G\), usually \(G\) is a topological group, and transition maps are required to be continuous. In this monograph, unless otherwise stated, we endow every subgroup \(G < \text{Homeo}^+(S^n)\) with the compact-open topology, i.e. the topology of the uniform convergence with respect to any metric inducing the usual topology of \(S^n\) (but see also Remark 12.19). If this is the case, then the transition maps are automatically continuous. As usual, we understand that two atlases define the same structure of \(G\)-bundle if their union is still a \(G\)-atlas. Equivalently, \(G\)-bundle structures on \(M\) correspond to maximal \(G\)-atlases for \(M\).

An isomorphism of sphere \(G\)-bundles \(\pi: E \to M\), \(\pi': E' \to M\) is an isomorphism of sphere bundles \(h: E \to E'\) such that, for every \(x \in M\), there exist a neighbourhood \(U\) of \(x\) and trivializations \(\psi: \pi^{-1}(U) \to U \times S^n\), \(\psi': (\pi')^{-1}(U) \to U \times S^n\) in the maximal \(G\)-atlases of \(E\) and \(E'\) such that
\[
\psi'(h(\psi^{-1}(x,v))) = (x, f(x)(v))
\]
for some function \(f: U \to G\).

In this monograph we will be interested just in the following cases:

1. \(G = \text{Homeo}^+(S^n)\): in this case, standard properties of the compact-open topology on \(\text{Homeo}^+(S^n)\) (in fact, on any space of homeomorphisms of a compact metric space) imply that \(E\) is just what we called a sphere bundle. Moreover, two sphere bundles are isomorphic as \(\text{Homeo}^+(S^n)\)-bundles if and only if they are isomorphic according to the definition given at the beginning of the section. Henceforth, we will sometimes stress these facts by saying that \(E\) is a topological sphere bundle.

2. \(G = \text{Diffeo}^+(S^n)\): in this case we say that \(E\) is a smooth sphere bundle; we also say that two smooth sphere bundles are smoothly isomorphic if they are isomorphic as sphere \(\text{Diffeo}^+(S^n)\)-bundles.

3. \(G = \text{Gl}^+(n+1, \mathbb{R})\), where the action of \(G\) is defined via the identification of \(S^n\) with the space of rays in \(\mathbb{R}^{n+1}\). In this case we say that \(E\) is a linear sphere bundle; we also say that two linear sphere bundles are linearly isomorphic if they are isomorphic as sphere \(\text{Gl}^+(n+1, \mathbb{R})\)-bundles.
Remark 12.1. Since $\text{GL}^+(n+1, \mathbb{R}) \subseteq \text{Diffeo}_+(S^n)$, every linear sphere bundle admits a unique compatible structure of smooth sphere bundle. On the other hand, there could exist sphere bundles which are not isomorphic to smooth sphere bundles, and smooth sphere bundles that are not smoothly isomorphic (or even topologically isomorphic) to linear bundles.

Add references for the facts above

Also observe that two smooth fiber bundles could be topologically isomorphic without being smoothly isomorphic, and two linear fiber bundles could be isomorphic or even smoothly isomorphic without being linearly isomorphic.

If $\pi: E \to M$ is a vector bundle, i.e. a locally trivial bundle with fiber $\mathbb{R}^{n+1}$ and structure group $\text{GL}^+(n+1, \mathbb{R})$, then we may associate to $E$ the $n$-sphere bundle $S(E)$ as follows. Let us denote by $E_0 \subseteq E$ the image of the zero section $\sigma_0: M \to E$. As a topological space, $S(E)$ is the quotient of $E \setminus E_0$ by the equivalence relation given by $v \equiv w$ if $v, w$ belong to the same fiber $E_x$ of $E$ and $v = \lambda \cdot w$ for some $\lambda \in \mathbb{R}^+$, where the product is defined according to the natural structure of real vector space on $E_x$. It is immediate to check that the bundle map $\pi: E \to M$ induces a continuous map $S(E) \to M$ which is an $n$-sphere bundle, called the sphere bundle associated to $E$. In fact, by construction $S(E)$ admits a $\text{GL}^+(n+1, \mathbb{R})$-atlas, so it admits a natural structure of linear sphere bundle. Conversely, every linear sphere bundle $\pi: S \to M$ admits an atlas with transition functions with values in $\text{GL}^+(n+1, \mathbb{R})$. Such an atlas also defines a vector bundle $\pi: E \to M$ with fiber $\mathbb{R}^{n+1}$, and by construction we have $S(E) = S$. In other words, there is a natural correspondence between linear $n$-sphere bundles and vector bundles of rank $n+1$ over the same base $M$.

In the following sections we will be interested in particular in the study of circle bundles. As a consequence, an important role will be played by the topology of the group $\widetilde{\text{Homeo}}_+(S^1)$. Recall that, unless otherwise stated, we understand that $\text{Homeo}^+(S^1)$, $\text{Homeo}_+(S^1)$ and $\text{Homeo}^+(S^n)$ are endowed with the compact-open topology (we refer the reader to Section 1 for the definition of $\widetilde{\text{Homeo}}_+(S^1)$). We record the following result for later reference:

Lemma 12.2. \(1\) The topological group $\widetilde{\text{Homeo}}_+(S^1)$ is contractible.

\(2\) The map $p_*: \widetilde{\text{Homeo}}_+(S^1) \to \text{Homeo}^+(S^1)$ induced by the projection $p: \mathbb{R} \to S^1$ is a universal covering.

\(3\) If $x$ is any point in $S^1$, the map

$$v_x: \text{Homeo}^+(S^1) \to S^1, \quad v_x(f) = f(x)$$

is a homotopy equivalence.
PROOF. (1): The map \( \widetilde{\text{Homeo}}_+(S^1) \times [0, 1] \to \widetilde{\text{Homeo}}_+(S^1) \) defined by \((f, t) \mapsto tf + (1 - t)\text{Id}_R\) defines a contraction of \(\text{Homeo}_+(S^1)\), which, therefore, is contractible.

(2): The fact that \(p_*\) is a covering is an easy exercise, so \(p_*\) is a universal covering by point (1).

(3): Let \(r_x : S^1 \to \text{Homeo}^+(S^1)\) be defined as follows: for every \(y \in S^1\), \(r_x(y)\) is the unique rotation of \(S^1\) taking \(x\) into \(y\). We obviously have \(v_x \circ r_x = \text{Id}_{S^1}\), so we only have to check that \(r_x \circ v_x\) is homotopic to the identity of \(\text{Homeo}^+(S^1)\). Let us fix a point \(\bar{x} \in p^{-1}(x) \subseteq \mathbb{R}\). For every \(f \in \text{Homeo}^+(S^1)\) we fix an arbitrary lift \(\widetilde{f}\) in \(\text{Homeo}^+(S^1)\) of \(f\). Then, we define \(\widetilde{H} : \text{Homeo}^+(S^1) \times [0, 1] \to \text{Homeo}^+(S^1), \quad \widetilde{H}(f, t) = (1 - t)\widetilde{f} + t \cdot \tau_{\widetilde{f}(\bar{x})-\bar{x}}\), and we set \(H : \text{Homeo}^+(S^1) \times [0, 1] \to \text{Homeo}^+(S^1), \quad H = p_* \circ \widetilde{H}\).

It is immediate to check that the definition of \(H(f, t)\) does not depend on the arbitrary choice of the lift \(\widetilde{f}\). Using this, one easily checks that \(H\) is indeed continuous. The conclusion follows from the fact that \(H(f, 0) = f\) and \(H(f, 1) = r_x(v_x(f))\) for every \(f \in \text{Homeo}^+(S^1)\).

\[\text{Corollary 12.3.} \quad \text{Let } \sigma_1 : S^1 \to S^1 \text{ be a continuous map, let } b : S^1 \to \text{Homeo}^+(S^1) \text{ be a loop, and let } \sigma_2 : S^1 \to S^1 \text{ be defined by } \sigma_2(x) = b(x)(\sigma_1(x)). \text{ Then } b \text{ is null-homotopic if and only if } \deg \sigma_1 = \deg \sigma_2.\]

\[\text{Proof.} \quad \text{Let } \widetilde{\sigma}_1 : [0, 1] \to \mathbb{R} \text{ be a lift of the composition } [0, 1] \xrightarrow{\sigma_1} S^1 \xrightarrow{\sigma_2} S^1. \text{ Let also } \widetilde{b} : [0, 1] \to \text{Homeo}_+(S^1) \text{ be a lift of the map } [0, 1] \to S^1 \xrightarrow{b} \text{Homeo}^+(S^1) \text{ such that } \widetilde{b}(0)(\widetilde{\sigma}_1(0)) = \widetilde{\sigma}_2(0). \text{ Being lifts of the map } [0, 1] \to S^1 \xrightarrow{\sigma_2} S^1 \text{ which coincide in } 0, \text{ the paths } t \mapsto \widetilde{b}(t)(\widetilde{\sigma}_1(t)) \text{ and } \widetilde{\sigma}_2 \text{ coincide for every } t \in [0, 1]. \text{ In particular, we have } \widetilde{b}(1)(\widetilde{\sigma}_1(1)) = \widetilde{\sigma}_2(1), \text{ so}\]
\[
\widetilde{b}(1)(\widetilde{\sigma}_1(0)) + \deg \sigma_1 = \widetilde{b}(1)(\widetilde{\sigma}_1(0) + \deg \sigma_1) = \widetilde{b}(1)(\widetilde{\sigma}_1(1)) = \widetilde{\sigma}_2(1) = \widetilde{\sigma}_2(0) + \deg \sigma_2 = \widetilde{b}(0)(\widetilde{\sigma}_1(0)) + \deg \sigma_2.
\]

Therefore, \(\deg \sigma_1 = \deg \sigma_2\) if and only if \(\widetilde{b}(1)(\widetilde{\sigma}_1(0)) = \widetilde{b}(0)(\widetilde{\sigma}_1(0))\). Since two lifts of \(b\) coincide if and only if they coincide on a point, we conclude that \(\deg \sigma_1 = \deg \sigma_2\) if and only if \(b\) is a loop. But \(\widetilde{b}\) is a loop if and only if \(b\) is homotopically trivial, and this concludes the proof.

\[\text{2. The Euler class of a sphere bundle}\]

Let us briefly recall the definition of the Euler class \(eu(E) \in H^{n+1}(M, \mathbb{Z})\) associated to an \(n\)-sphere bundle \(\pi : E \to M\). We follow the approach via obstruction theory, i.e. we describe \(eu(E)\) as the obstruction to the existence of a section of \(\pi\) (and we refer the reader e.g. to [Ste51] for more details).
Let \( s: \Delta^k \to M \) be a singular \( k \)-simplex. A section \( \sigma \) over \( s \) is a continuous map \( \sigma: \Delta^k \to E \) such that \( \sigma(x) \in E_{s(x)} \) for every \( x \in \Delta^k \). Equivalently, if we denote by \( E_s \) the sphere bundle over \( s \) given by the pull-back of \( E \) via \( \sigma \), then \( \sigma \) may be seen as a genuine section of \( E_s \). Observe that the sphere bundle \( E_s \) is always trivial, since the standard simplex is contractible.

Let us now fix the dimension \( n \) of the fibers of our sphere bundle \( \pi: E \to M \). We will inductively construct compatible sections over all the singular \( k \)-simplices with values in \( M \), for every \( k \leq n \). More precisely, for every \( k \leq n \), for every singular \( k \)-simplex \( s: \Delta^k \to M \), we will define a section \( \sigma_s: \Delta^k \to E \) over \( s \) in such a way that the restriction of \( \sigma_s \) to the \( i \)-th face of \( \Delta^k \) coincides with the section \( \sigma_{\partial_i s}: \Delta^{k-1} \to E \) over the \( i \)-th face \( \partial_i s: \Delta^{k-1} \to M \) of \( s \).

We first choose, for every 0-simplex (i.e. point) \( x \in M \), a section \( \sigma_x \) over \( x \), i.e. an element \( \sigma_x \) of the fiber \( E_x \). Let \( s: [0,1] \to M \) be a singular 1-simplex such that \( \partial s = y - x \). Then, since \( E_s \) is trivial and \( S^n \) is path connected, it is possible to choose a section \( \sigma_s: [0,1] \to E \) over \( s \) such that \( \sigma_s(1) = \sigma_y \) and \( \sigma_s(0) = \sigma_x \), and we fix such a choice for every 1-simplex \( s \). If \( n = 1 \) we are done. Otherwise, we assume that the desired sections have been constructed for every singular \((k-1)\)-simplex, and, for any given singular \( k \)-simplex \( s: \Delta^k \to M \), we define \( \sigma_s \) as follows.

We fix a trivialization

\[
E_s \xrightarrow{\psi_s} \Delta^k \times S^n
\]

The inductive hypothesis ensures that the sections defined on the \((k-1)\)-faces of \( s \) coincide on the \((k-2)\)-faces of \( s \), so they may be glued together into a section \( \sigma_{\partial s}: \partial \Delta^k \to E_s \) of the restriction of \( E_s \) to \( \partial \Delta^k \). Since \( k \leq n \), we have \( \pi_{k-1}(S^n) = 0 \), so the composition

\[
\pi_{S^n} \circ \psi_s \circ \sigma_{\partial s}: \partial \Delta^k \to S^n
\]

extends to a continuous \( B: \Delta^k \to S^n \). Then, an extension of \( \sigma_{\partial s} \) over \( \Delta^k \) is provided by the section

\[
\sigma_s: \Delta^k \to E_s, \quad \sigma_s(x) = \psi_s^{-1}(x, B(x)).
\]

Let us now consider a singular \((n+1)\)-simplex \( s: \Delta^{n+1} \to M \). Just as we did above, we may fix a trivialization

\[
E_s \xrightarrow{\psi_s} \Delta^{n+1} \times S^n
\]
of $E_s$, and observe that the sections defined on the faces of $s$ glue into a section $\sigma_s$: $\partial \Delta^{n+1} \to E_s$. The obstruction to extend this section over $\Delta^{n+1}$ is measured by the degree $z(s)$ of the map

$$\partial \Delta^{n+1} \xrightarrow{\sigma_s} E_s \xrightarrow{\psi_s} \Delta^{n+1} \times S^n \xrightarrow{\pi S^n} S^n$$

(this degree is well-defined since $\partial \Delta^{n+1}$ inherits an orientation from the one of $\Delta^{n+1}$). The following result shows that $z$ represents a well-defined cohomology class:

**Proposition 12.4.** The singular cochain $z \in C^{n+1}(M, \mathbb{Z})$ defined above satisfies the following properties:

1. $z(s)$ does not depend on the choice of the (orientation-preserving) trivialization of $E_s$;
2. $\delta z = 0$, i.e. $z$ is a cycle;
3. Different choices for the sections over $k$-simplices, $k \leq n$, lead to another cocycle $z' \in C^{n+1}(M, \mathbb{Z})$ such that $[z] = [z']$ in $H^{n+1}(M, \mathbb{Z})$.
4. If $z' \in C^{n+1}(M, \mathbb{Z})$ is a cycle such that $[z] = [z']$ in $H^{n+1}(M, \mathbb{Z})$, then $z'$ can be realized as the cocycle corresponding to suitable choices for the sections over $k$-simplices, $k \leq n$.

**Proof.** (1): It is readily seen that, if $\psi_s, \psi'_s$ are distinct trivializations of $E_s$ and $\alpha, \alpha': \partial \Delta^{n+1} \to S^n$ are given by $\alpha = \pi_{S^n} \circ \psi_s \circ \sigma_s$, $\alpha' = \pi_{S^n} \circ \psi'_s \circ \sigma_s$, then $\alpha'(x) = h(x)(\alpha(x))$ for every $x \in \partial \Delta^{n+1}$, where $h$ is the restriction to $\partial \Delta^{n+1}$ of a continuous map $\Delta^{n+1} \rightarrow \text{Homeo}^+(S^n)$. Being extendable to $\Delta^{n+1}$, the map $h$ is homotopic to a constant map. Therefore, $\alpha'$ is homotopic to the composition of $\alpha$ with an element of $\text{Homeo}^+(S^n)$, so deg $\alpha = \deg \alpha'$, i.e. $z(s) = z'(s)$, and this concludes the proof.

(2): Let $s \in S_{n+2}(M)$, and fix a trivialization $\psi_s: E_s \rightarrow \Delta^{n+2} \times S^n$. The fixed collection of compatible sections induces a section $\sigma'_s$ of $E_s$ over the $n$-skeleton $T$ of $\Delta^{n+2}$. For every $i = 0, \ldots, n+2$, we denote by $F_i$ the geometric $i$-th face of $\Delta^{n+2}$, and we fix an integral fundamental cycle $c_i = C_n(\partial F_i, \mathbb{Z}) \subseteq C_n(T, \mathbb{Z})$ for $\partial F_i$ (here we endow $\partial F_i$ with the orientation inherited from $F_i$, which in turn is oriented as the boundary of $\Delta^{n+2}$). Then we have $\sum_{i=0}^{n+2} [c_i] = 0$ in $H_n(T, \mathbb{Z})$. Observe that $\psi_s$ restricts to a trivialization $\psi_i: E_{\partial_i s} \rightarrow \Delta^{n+1} \times S^n$ for any single face $\partial_i s$ of $s$. By (1), we may exploit these trivializations to compute $z(\partial_i s)$ for every $i$, thus getting

$$z(\partial_i s) = H_n(\pi_{S^n} \circ \psi_s \circ \sigma'_i)((-1)^i [c_i]) \in \mathbb{Z} = H_n(S^n, \mathbb{Z})$$

(the sign $(-1)^i$ is due to the fact that the orientations on $\Delta^{n+1}$ and on $F_i$ agree if and only if $n$ is even). Since the composition $\pi_{S^n} \circ \psi_s \circ \sigma'_i$ is defined over $T$, this implies that

$$z(\sigma s) = \sum_{i=0}^{n+2} (-1)^i z(\partial_i s) = H_n(\pi_{S^n} \circ \psi_s \circ \sigma'_s)\left(\sum_{i=0}^{n+2} (-1)^i [c_i]\right) = 0 .$$

Since this holds for every $s \in S_{n+2}(M)$, we conclude that $z$ is a cycle.
(3): Let us fix two compatible families of sections \( \{ \sigma_s | s \in S_n(M) \} \), \( \{ \sigma'_s | s \in S_n(M) \} \), and observe that, being compatible, these families also define families of sections \( \{ \sigma_s | s \in S_j(M), j \leq n \} \), \( \{ \sigma'_s | s \in S_j(M), j \leq n \} \).

Using that \( \pi_k(S^n) = 0 \) for every \( k < n \), it is not difficult to show that these families are homotopic up to dimension \( n - 1 \), i.e., that there exists a collection of compatible homotopies \( \{ H_s | s \in S_j(M), j \leq n - 1 \} \) between the \( \sigma_s \)'s and the \( \sigma'_s \)'s, where \( s \) ranges over all the singular simplices of dimension at most \( n - 1 \). Here the compatibility condition requires that the restriction of \( H_s \) to the face \( \partial_i s \) is equal to \( H_{s_0} \). This implies in turn that we may modify the compatible family \( \{ \sigma'_s | s \in S_n(M) \} \) without altering the induced Euler cocycle, in such a way that the restrictions of \( \sigma'_s \) and of \( \sigma_s \) to \( \partial \Delta^n \) coincide for every \( s \in S_n(M) \).

Let us now fix a singular simplex \( s \in S_n(M) \), and a trivialization \( \psi_s : E_s \to \Delta^n \times S^n \). Let \( \Delta^n_1, \Delta^n_2 \) be copies of the standard simplex \( \Delta^n \), and endow \( \Delta^n_1 \) (resp. \( \Delta^n_2 \)) with the same (resp. opposite) orientation of \( \Delta^n \). Also denote by \( S \) the sphere obtained by gluing \( \Delta^n_1 \) with \( \Delta^n_2 \) along their boundaries via the identification \( \partial \Delta^n_1 \equiv \partial \Delta^n \equiv \partial \Delta^n_2 \). Observe that \( S \) admits an orientation which extends the orientations of \( \Delta^n_1, i = 1, 2 \). As discussed above, we may assume that the restrictions of \( \sigma'_s \) and of \( \sigma_s \) to \( \partial \Delta^n \) coincide for every \( s \in S_n(M) \), so we may define a continuous map \( \varphi : S \to E_s \) which coincides with \( \sigma_s \) on \( \Delta^n_1 \) and with \( \sigma'_s \) on \( \Delta^n_2 \). After composing with \( \psi_s \) and projecting to \( S^n \), this map gives rise to a map \( S \to S^n \). We denote its degree by \( \varphi(s) \in \mathbb{Z} \). Now it is easy to show that, if \( z \) and \( z' \) are the Euler cocycles associated respectively with the \( \sigma_s \)'s and the \( \sigma'_s \)'s, then \( z - z' = \delta \varphi \).

(4): Suppose that \( z \in C^{n+1}(M, \mathbb{Z}) \) is the representative of \( \text{eu}(E) \) corresponding to the compatible collection \( \{ \sigma_s, s \in S_n(M) \} \) of sections over the elements of \( S_n(M) \). Our assumption implies that \( z' = z - \delta \varphi \) for some \( \varphi \in C^n(M, \mathbb{Z}) \). For every \( s \in S_n(M) \), let us modify \( \sigma_s \) as follows. Let \( \Delta^n_1, \Delta^n_2 \) and \( S \) be as in the previous point. We consider a trivialization

\[
E_s \xrightarrow{\psi_s} \Delta^n \times S^n \xrightarrow{\pi} S^n \xrightarrow{\pi_{\Delta^n}} \Delta^n
\]

and denote by \( g : \Delta^n_1 \to S^n \) the map \( g = \pi_{\Delta^n} \circ \sigma_s \), where we are identifying \( \Delta^n_1 \) with \( \Delta^n \). We extend \( g \) to a map \( G : S \to S^n \) such that \( \deg G = \varphi(s) \). If \( \overline{g} : \Delta^n \equiv \Delta^n_2 \to S^n \) is the restriction of \( G \) to \( \Delta^n_2 \), then we set

\[
\sigma'_s : \Delta^n \to E_s, \quad \sigma'_s(x) = \psi_s^{-1}(x, \overline{g}(x)).
\]

With this choice, \( \sigma'_s \) and \( \sigma_s \) coincide on \( \partial \Delta^n \), so the collection of sections \( \{ \sigma'_s, s \in S_n(M) \} \) is still compatible. Moreover, it is easily checked that the Euler cocycle corresponding to the new collection of sections is equal to \( z' \). \( \square \)
The previous proposition implies that the following definition is well-posed:

**Definition 12.5.** The *Euler class* $\text{eu}(E) \in H^{n+1}(M, \mathbb{Z})$ of $\pi: E \to M$ is the coclass represented by the cocycle $z$ described above.

**Remark 12.6.** In the case when $M$ is a CW-complex, the construction above may be simplified a bit. In fact, using the fact that $\pi_k(S^n) = 0$ for $k < n$ one may choose a global section of $E$ over the $n$-skeleton $M^{(n)}$ of $M$. Then, working with $(n+1)$-cells rather than with singular $(n+1)$-simplices, one may associate an integer to every $(n+1)$-cell of $M$, thus getting a cellular $(n+1)$-cochain, which is in fact a cocycle whose cohomology class does not depend on the choice of the section on $M^{(n)}$. This approach is explained in detail e.g. in [Ste51]. Our approach via singular homology has two advantages:

1. It allows a neater discussion of the boundedness of the Euler class.
2. It shows directly that, even when $M$ admits the structure of a CW-complex, the Euler class $\text{eu}(E)$ does not depend on the realization of $M$ as a CW-complex.

The Euler class is natural, in the following sense:

**Lemma 12.7.** Let $\pi: E \to M$ be a sphere $n$-bundle, and let $f: N \to M$ be any continuous map. Then, if $f^*E$ denotes the pull-back of $E$ via $f$, then

$$e(f^*E) = H^{n+1}(f)(e(E)),$$

where $H^{n+1}(f): H^{n+1}(M, \mathbb{Z}) \to H^{n+1}(N, \mathbb{Z})$ is the map induced by $f$ in cohomology.

**Proof.** Via the map $f$, a collection of compatible sections of $E$ over the simplices in $M$ gives rise to a collection of compatible sections of $f^*E$ over the simplices in $N$. The conclusion follows at once. \qed

By construction, the Euler class provides an obstruction for $E$ to admit a section:

**Proposition 12.8.** If the sphere bundle $\pi: E \to M$ admits a section, then $\text{eu}(E) = 0$.

**Proof.** Let $\sigma: M \to E$ be a global section. If $s: \Delta^k \to M$ is any singular simplex, then we may define a section $\sigma_s$ of $E_s$ simply as the pull-back of $\sigma$ via $s$. With this choice, for every singular $n$-simplex $s$, the partial section $\sigma_{\partial s}$ of $E_s$ extends from $\partial \Delta^n$ to $\Delta^n$. As a consequence, the Euler class admits an identically vanishing representative, so $\text{eu}(E) = 0$. \qed

It is well-known that the vanishing of the Euler class does not ensure the existence of a section in general (see e.g. [BT82, Example 23.16]). However, we have the following positive result in this direction:
Proposition 12.9. Let \( \pi: E \to M \) be an \( n \)-sphere bundle over a simplicial complex. Then \( \text{eu}(E) = 0 \) if and only if the restriction of \( E \) to the \((n+1)\)-skeleton of \( M \) admits a section.

Proof. By Proposition 12.8, we need to construct a section over \( M^{(n+1)} \) under the assumption that \( \text{eu}(E) = 0 \). Let \( S_n(M) \) be the set of singular \( n \)-simplices with values in \( M \). By Proposition 12.4–(4), there exists a compatible collection \( \{ \sigma_s, s \in S_n(M) \} \) of sections over the elements of \( S_n(M) \) such that the corresponding Euler cocycle is identically equal to zero. These sections may be exploited to define a global section \( \sigma: M^{(n+1)} \to E \) over the \( n \)-skeleton of \( M \). Using that the corresponding Euler cocycle vanishes, it is easy to show that this section extends over \( M^{(n+1)} \), and we are done. \( \Box \)

It is not known to the author whether the following result holds for any topological sphere bundle:

Proposition 12.10. Let \( \pi: E \to M \) be a linear \( n \)-sphere bundle, where \( n \) is even. Then \( 2\text{eu}(E) = 0 \in H^{n+1}(M, \mathbb{Z}) \).

Proof. Since \( E \) is linear, the structure group of \( E \) commutes with the antipodal map of \( S^n \). As a consequence, there exists a well-defined involutive bundle map \( i: E \to E \) such that the restriction of \( i \) to the fiber \( E_x \) has degree \((-1)^{n+1} \) for every \( x \in M \).

Let now \( S_n(M) \) be the set of singular \( n \)-simplices with values in \( M \), take a collection of compatible sections \( \{ \sigma_s, s \in S_n(M) \} \) over the elements of \( S_n(M) \), and let \( z \) be the corresponding Euler cocycle. By composing each \( \sigma_s \) with \( i \), we obtain a new collection of compatible sections \( \{ \sigma'_s, s \in S_n(M) \} \), which define the Euler cocycle \( z' = -(1)^{n+1}z = -z \). Therefore, we have \( \text{eu}(E) = -\text{eu}(E) \), whence the conclusion. \( \Box \)

The notion of Euler class may be extended also to the context of orientable vector bundles: namely, if \( \pi: E \to M \) is such a bundle, then we simply set

\[
\text{eu}(E) = \text{eu}(S(E)) .
\]

If the rank of \( E \) is equal to 2, then we may associate to \( E \) another (smooth) sphere bundle, the projective bundle \( \mathbb{P}(E) \to M \). Such a bundle is defined as follows: the fiber of \( \mathbb{P}(E) \) over \( x \in M \) is just the projective space \( \mathbb{P}(E_x) \), i.e. the space of 1-dimensional subspaces of \( E_x \). Using the fact that the one-dimensional real projective space \( \mathbb{P}^1(\mathbb{R}) \) is homeomorphic to the circle, it is not difficult to check that \( \mathbb{P}(E) \to M \) is a smooth circle bundle, according to our definitions (in fact, a \( \text{GL}^+(2, \mathbb{R}) \)-atlas for \( E \) induces a \( \text{PGl}^+(2, \mathbb{R}) \)-atlas for \( \mathbb{P}(E) \), which, therefore, is endowed with the structure of a smooth circle bundle). Then, using that the natural projection \( \mathbb{R}^2 \supseteq S^1 \to \mathbb{P}^1(\mathbb{R}) \) has degree 2, it is easy to show that

\[
\text{eu}(\mathbb{P}(E)) = 2\text{eu}(S(E)) = 2\text{eu}(E) .
\]
**Definition 12.11.** Let $\pi: E \to M$ be an $n$-sphere bundle, and suppose that $M$ is a closed oriented $(n+1)$-manifold. Then $H^n(M, \mathbb{Z})$ is canonically isomorphic to $\mathbb{Z}$, so $\text{eu}(E)$ may be identified with an integer, still denoted by $\text{eu}(E)$. We call such an integer the **Euler number** of the fiber bundle $E$.

**Proposition 12.12.** Let $\pi: E \to M$ be an $n$-sphere bundle over a smooth closed oriented $(n+1)$-manifold. Then $\text{eu}(E) = 0$ if and only if $E$ admits a section.

**Proof.** Since $M$ can be smoothly triangulated, the conclusion follows from Proposition 12.9. □

**Proposition 12.13.** Let $\pi: E \to M$ be a linear $n$-sphere bundle over a closed oriented $(n+1)$-manifold, where $n$ is even. Then $\text{eu}(E) = 0$, so $E$ admits a section.

**Proof.** Since $H^{n+1}(M, \mathbb{Z})$ is torsion-free, the conclusion follows from Propositions 12.10 and 12.12. □

The Euler number of the tangent bundle of a closed manifold is just equal to the Euler characteristic of $M$:

**Proposition 12.14.** Let $M$ be a closed oriented smooth manifold, and denote by $TM$ the tangent bundle of $M$. Then

$$\text{eu}(TM) = \chi(M).$$

**Proof.** Let $\tau$ be a smooth triangulation of $M$. We first exploit $\tau$ to construct a continuous vector field $X: M \to TM$ having only isolated zeroes. Let $n = \dim M$. For every $k$-simplex $\Delta$ of $\tau$, $k \leq n$, we choose a point $p_\Delta$ in the interior of $\Delta$ (where we understand that, if dim $\Delta = 0$, i.e. if $\Delta$ is a vertex of $\tau$, then $p_\Delta = \Delta$). Then we build our vector field $X$ in such a way that the following conditions hold:

1. $X$ is null exactly in the $p_\Delta$’s;
2. for every $\Delta$ with dim $\Delta = k$, there exist local coordinates $(x_1, \ldots, x_n)$ on a neighbourhood $U$ of $p_\Delta$ such that $U \cap \Delta = \{x_{k+1} = \ldots = x_n = 0\}$ and

$$X|_U = -\left(x_1 \frac{\partial}{\partial x_1} + \ldots + x_k \frac{\partial}{\partial x_k}\right) + x_{k+1} \frac{\partial}{\partial x_{k+1}} + \ldots x_n \frac{\partial}{\partial x_n}$$

(so $p_\Delta$ is a sink for the restriction of $X$ to $\Delta$ and a source for the restriction of $X$ to a transversal section of $\Delta$). See Figure 1 for a picture in the 2-dimensional case.
Let us now choose another triangulation $\tau' = \{\Delta'_1, \ldots, \Delta'_m\}$ of $M$ such that no zero of $X$ lies on the $(n-1)$-skeleton of $\tau'$, then every $\Delta'_i$ contains at most one zero of $X$ and, if $\Delta'_i$ contains a zero of $X$, then $\Delta'_i$ is so small that the restriction of $X$ to $\Delta_i$ admits a local description as in point (2) above. For every $i = 1, \ldots, m$, we would like to choose an orientation-preserving parameterization $s_i : \Delta^i \to M$ of $\Delta'_i$ in such a way that the chain $c = s_1 + \ldots + s_m$ represents a generator of $H_n(M, \mathbb{Z})$. This is not possible in general. However, as observed in the proof of Proposition 9.3, the $s_i$’s may be chosen so that the rational chain $\text{alt}_n(c)$ obtained by alternating $c$ represents the image of the integral fundamental class of $M$ in $H_n(M, \mathbb{Q})$ via the change of coefficients homomorphism.

The vector field $X$ defines a nowhere vanishing section on the $(n-1)$-skeleton $(\tau')^{n-2}$ of $\tau'$, which induces in turn a section from $(\tau')^{n-1}$ to $S(TM)$. We may use this section to define $z(s_i)$ for every $i = 1, \ldots, m$, where $z$ is the representative of the Euler class constructed above in this section. If $\Delta'_i$ does not contain any singularity of $X$, then $X$ may be extended over $\Delta'_i$, so $z(s_i) = 0$. On the other hand, if $\Delta'_i$ contains the singularity $p_\Delta$, then $\dim \Delta = k$, then $z(s_i)$ is equal to the degree of the map

$$S^{n-1} \to S^{n-1}, \ (x_1, \ldots, x_n) \mapsto (-x_1, \ldots, -x_k, x_{k+1}, \ldots, x_n),$$

so $z(s_i) = (-1)^k$. Observe now that from the very definitions it follows that $z(\text{alt}_n(s_i)) = z(s_i)$ for every $i = 1, \ldots, m$. Moreover, the Euler number of $S(TM)$ may be computed by evaluating $z$ over any rational cycle which is homologous over $\mathbb{Q}$ to an integral fundamental cycle for $M$. Putting together these facts we finally get

$$\text{eu}(TM) = \sum_{i=1}^{m} z(\text{alt}_n(s_i)) = \sum_{i=1}^{m} z(s_i) = \sum_{j=1}^{n} (-1)^k \cdot \# \{\Delta \in \tau \mid \dim \Delta = k\} = \chi(M).$$

The problem of which integral cohomology classes may be realized as Euler classes of some sphere bundle is open in general: for example, a classical result by Milnor, Atiyah and Hirzebruch implies that, if $m \neq 1, 2, 4$, then the Euler class of any linear $n$-sphere bundle over $S^{2m}$ is of the form $2 \cdot \alpha$ for some $\alpha \in H^{n+1}(S^{2m}, \mathbb{Z})$ [Mil58b, MFA61], while, if $k$ is even, there is a number $N(k,n)$ such that for every $n$-dimensional CW-complex $X$ and every cohomology class $a \in H^k(X, \mathbb{Z})$, there exists a linear $(k-1)$-sphere bundle $E$ over $X$ such that $\text{eu}(E) = 2N(k,n)a$ [LG02].

However, in the case of closed surfaces it is easily seen that every integer is the Euler number of some circle bundle:

**Proposition 12.15.** For every $n \in \mathbb{Z}$ there exists a circle bundle $E_n$ over $\Sigma_g$ such that $\text{eu}(E_n) = n$. 

Proof. Let $\overline{D} \subseteq \Sigma_g$ be an embedded closed disk, let $D$ be the internal part of $\overline{D}$, and consider the surface with boundary $\Sigma'_g = \Sigma_g \setminus D$. Take the products $E_1 = \Sigma'_g \times S^1$ and $E_2 = \overline{D} \times S^1$, and fix an identification between $\partial \Sigma'_g = \partial \overline{D}$ and $S^1$, which is orientation-preserving on $\partial \Sigma'_g$ and orientation-reversing on $\partial \overline{D}$. Let now $n \in \mathbb{Z}$ be fixed, and glue $\partial E_1 = (\partial \Sigma'_g) \times S^1 \cong S^1 \times S^1$ to $\partial E_2 = (\partial \overline{D}) \times S^1 \cong S^1 \times S^1$ via the attaching map

$$
\partial E_1 \to \partial E_2, \quad (\theta, \varphi) \mapsto (\theta, \varphi - n\theta).
$$

If we denote by $E$ the space obtained via this gluing, the projections $\Sigma'_g \times S^1 \to \Sigma'_g, \overline{D} \times S^1 \to \overline{D}$ induce a well-defined circle bundle $\pi: E \to \Sigma_g$.

Let us now fix a triangulation $\mathcal{T}$ of $\Sigma_g$ such that $\overline{D}$ coincides with one triangle $T_0$ of $\mathcal{T}$, and consider the fundamental cycle $c$ for $\Sigma_g$ given by the (alternating average) of the triangles of $\mathcal{T}$. Our construction provides preferred trivializations of $E_1$ and $E_2$. If $\sigma_i, i = 1, 2,$ are the “constant” sections of $E_1$ over $\Sigma'_g$ and of $E_2$ over $\overline{D}$, then $\sigma_1, \sigma_2$ induce sections over all the triangles of $\mathcal{T}$. We may exploit such sections to compute $z(T)$ for every $T \in \mathcal{T}$. By construction, we get $z(T) = n$ and $z(T) = 0$ for every $T \neq T_0$.

This implies that $\text{eu}(E) = n$. \hfill \Box

In fact, the Euler number completely classifies the isomorphism type of topological circle bundles on surfaces. This is a consequence of Theorem 12.17 below. We first need the following:

Lemma 12.16. Let $\sigma, \sigma': [0, 1] \to S^1$ be continuous paths, and let $h_0, h_1 \in \text{Homeo}^+(S^1)$ be such that $h_i(\sigma(i)) = \sigma'(i)$, $i = 0, 1$. Then there exists a path $\psi: [0, 1] \to \text{Homeo}^+(S^1)$ such that $\psi(t) = h_t$ for $i = 0, 1,$ and $\psi(t)(\sigma(t)) = \sigma'(t)$ for every $t \in [0, 1]$.

Proof. Let $\tilde{\sigma}, \tilde{\sigma}': [0, 1] \to \mathbb{R}$ be continuous lifts of $\sigma, \sigma'$, and choose $\tilde{h}_t \in \text{Homeo}_+(S^1)$ which lifts $h_t$ and is such that $\tilde{h}_i(\tilde{\sigma}(i)) = \tilde{\sigma}'(i)$, $i = 0, 1$. For $t \in [0, 1]$ we now define

$$
\tilde{\nu}_t \in \text{Homeo}_+(S^1), \quad \tilde{\nu}_t(x) = (1 - t)\tilde{h}_0(x) + t\tilde{h}_1(x),
$$

$$
\tilde{\psi}_t \in \text{Homeo}_+(S^1), \quad \tilde{\psi}_t(x) = \tilde{\nu}_t(x) + \tilde{\sigma}'(t) - \tilde{\nu}_t(\tilde{\sigma}(t)).
$$

Then the required path is obtained by projecting the path $t \mapsto \tilde{\psi}_t$ from $\text{Homeo}_+(S^1)$ to $\text{Homeo}^+(S^1)$. \hfill \Box

Theorem 12.17. Let $M$ be a simplicial complex, and let $\pi: E \to M$, $\pi': E' \to M$ be circle bundles such that $\text{eu}(E) = \text{eu}(E')$ in $H^2(M, \mathbb{Z})$. Then $E$ is (topologically) isomorphic to $E'$.

Proof. Since $\text{eu}(E) = \text{eu}(E')$, we may choose compatible collections of sections $\{\sigma_s\}_{s \in S_L(M)}$ and $\{\sigma'_s\}_{s \in S_L(M)}$ respectively for $E$ and $E'$ in such a way that the corresponding Euler cocycles $z, z' \in C^2(M, \mathbb{Z})$ coincide (see Proposition 12.4–(4)). Such sections glue up into partial sections $\sigma: M^{(1)} \to$
$E_1, \sigma': M^{(1)} \to E'_1$, where $M^{(1)}$ is the 1-skeleton of $M$, and $E_1$ (resp. $E'_1$) is the restriction of $E$ (resp. $E'$) to $M^{(1)}$.

For every vertex $v$ of $M$ we choose an orientation-preserving homeomorphism $h_v: E_v \to E'_v$ such that $h_v(\sigma(v)) = \sigma'(v)$. The collections of these maps is just a bundle map between the restrictions of $E$ and of $E'$ to the 0-skeleton of $M$. Using Lemma 12.16, we may extend this map to a bundle map $h: E_1 \to E'_1$ such that $h(\sigma(x)) = \sigma'(x)$ for every $x \in M$.

We will show that $h$ may be extended to a bundle map on the whole 2-skeleton of $M$. Let $\Delta$ be a 2-dimensional simplex of $M$, and choose trivializations $\psi_\Delta: E_{\Delta} \to \Delta \times S^1$, $\psi'_\Delta: E'_{\Delta} \to \Delta \times S^1$ of the restrictions of $E$ and $E'$ over $\Delta$. For every $x \in \partial\Delta$, the restriction of the composition $\psi'_\Delta \circ h \circ \psi^{-1}_\Delta$ to $\{x\} \times S^1$ defines an orientation-preserving homeomorphism of $\{x\} \times S^1$, which gives in turn an element $b(x) \in \text{Homeo}^+(S^1)$. Moreover, the map $b: \partial \Delta^2 \to \text{Homeo}^+(S^1)$ is continuous.

Putting together Proposition 12.15 and Theorem 12.17 we get the following:

**Corollary 12.18.** Let $\Sigma_g$ be a closed oriented surface. Then, the Euler number establishes a bijection between the set of isomorphism classes of topological circle bundles on $\Sigma_g$ and the set of integers.

### 3. Flat sphere bundles

We now concentrate our attention on a special class of fiber bundles. As usual, let $G$ be a subgroup of $\text{Homeo}^+(S^n)$.

A flat sphere bundle with structure group $G$ (or, simply, a flat $G$-bundle) is a sphere bundle $\pi: E \to M$ endowed with an open cover $U = \{U_i, i \in I\}$
of the base $M$ and a collection of local trivializations

$$
\pi^{-1}(U_i) \xrightarrow{\psi_i} U_i \times S^n
$$

such that, for every $i, j \in I$, there exists a \emph{locally constant} map $\lambda_{ij}: U_i \cap U_j \to G$ such that

$$
\psi_i(\psi_j^{-1}(x, v)) = (x, \lambda_{ij}(x)(v))
$$

for every $x \in U_i \cap U_j$, $v \in S^n$. Such a collection of local trivializations is a \emph{flat $G$-atlas} for $E$, and the $\lambda_{ij}$'s are called the \emph{transition functions} of the atlas. Two flat $G$-atlases define the same structure of flat $G$-bundle if their union is still a flat $G$-atlas.

An isomorphism of flat sphere $G$-bundles $\pi: E \to M$, $\pi': E' \to M$ is an isomorphism of sphere bundles $h: E \to E'$ such that, for every $x \in M$, there exist trivializations $\psi_i: \pi^{-1}(U_i) \to U_i \times S^n$, $\psi_{i'}: \pi'^{-1}(U_{i'}) \to U_{i'} \times S^n$ in the flat $G$-atlases of $E$ and $E'$ such that

$$
\psi_{i'}(h(\psi_i^{-1}(x, v))) = (x, f(x)(v))
$$

for some \emph{locally constant} function $f: U_i \cap U_{i'} \to G$.

\textbf{Remark 12.19.} If $G < \text{Homeo}^+(S^n)$ is any group, we denote by $G^\delta$ the topological group given by the abstract group $G$ endowed with the discrete topology. Then, a flat $G$-structure on a sphere bundle is just a $G^\delta$-structure, and an isomorphism between flat $G$-structures corresponds to an isomorphism between $G^\delta$-structures. Therefore, flat $G$-bundles may be inscribed in the general theory of $G$-bundles, provided that the group $G$ is not necessarily endowed with the compact-open topology.

We will call a flat sphere $G$-bundle \emph{flat topological sphere bundle} (resp. \emph{flat smooth sphere bundle}, \emph{flat linear sphere bundle}) if $G = \text{Homeo}^+(S^n)$ (resp. $G = \text{Diffeo}_+(S^n)$, $G = \text{Gl}^+(n + 1, \mathbb{R})$). Of course, a structure of flat $G$-bundle determines a unique structure of $G$-bundle. However, there exist sphere $G$-bundles which do not admit any flat $G$-structure (in fact, this chapter is mainly devoted to describe an important and effective obstruction on topological sphere bundles to be topologically flat). Moreover, the same $G$-structure may be induced by non-isomorphic flat $G$-structures on a sphere bundle. Finally, whenever $G' \subseteq G$, a flat $G'$-structure obviously define a unique flat $G'$-structure, but there could exist flat $G$-structures that cannot be induced by any $G'$-flat structure: for example, in Remark 12.28 we exhibit a flat circle bundle (i.e. a flat $\text{Homeo}^+(S^1)$-bundle) which does not admit any flat $\text{Gl}^+(2, \mathbb{R})$-structure.

In order to avoid pathologies, henceforth we assume that the base $M$ of every sphere bundle satisfies the following properties:

- $M$ is locally path-connected;
\[ S \text{ for some countable subset } \Lambda \text{ of } \\mathcal{U}, \text{ such that, for every leaf } E \text{ condition holds: there exists a } G \text{ holonomy of the flat bundle } (\text{see } \text{[Ste51, §13]} \text{ for the details). This homomorphism is usually called the holonomy of the flat bundle } E. \text{ By choosing a possibly different basepoint } x_0 \in M \text{ and/or a possibly different identification of the corresponding fiber with } S^n, \text{ one is lead to a possibly different homomorphism, which, however, is conjugated to the original one by an element of } G. \text{ Therefore, to any flat } G\text{-bundle on } M \text{ there is associated a holonomy representation } \rho: \Gamma \to G, \text{ which is well-defined up to conjugacy by elements of } G. \]

Let us now fix a basepoint \( x_0 \in M \), and denote by \( h: S^n \to E_{x_0} \) the homeomorphism induced by a local trivialization of \( E \) over a neighbourhood of \( x_0 \) belonging to the atlas which defines the flat \( G \)-structure of \( E \). If \( \alpha: [0, 1] \to M \) is a loop based at \( x_0 \), then for every \( q \in E_{x_0} \) we consider the horizontal lift \( \pi_q \) of \( \alpha \) starting at \( q \), and we set \( t_\alpha(q) = \pi(1) \in E_{x_0} \). Then \( t_\alpha \) is a homeomorphism which only depends on the homotopy class of \( \alpha \) in \( \pi_1(M, x_0) = \Gamma \). Now, for every \( g \in \Gamma \) we take a representative \( \alpha \) of \( g^{-1} \), and we set \( \rho(g) = h^{-1} \circ t_\alpha \circ h: S^n \to S^n \). It is not difficult to check that \( \rho(g) \) belongs to \( G \), and that the map \( \rho: \Gamma \to G \) is a homomorphism (see [Ste51, §13] for the details). This homomorphism is usually called the holonomy of the flat bundle \( E \).

Let us now come back to our fixed flat \( G \)-bundle \( E \). If we define the equivalence relation \( \equiv \) on \( E \) such that \( x \equiv y \) if and only if there exists a horizontal path starting at \( x \) and ending in \( y \), then each equivalence class of \( \equiv \) is a leaf of a foliation \( \mathcal{F} \) of \( E \). Observe that, if \( F \) is a leaf of this foliation and \( x \) is a point of \( M \), then the set \( E_x \cap F \) may be identified with an orbit of a holonomy representation for \( E \), so it is countable. It readily follows that the foliation \( \mathcal{F} \) is transverse to the fibers, according to the following:

**Definition 12.20.** Let \( \pi: E \to M \) be a (not necessarily flat) sphere \( G \)-bundle. A foliation \( \mathcal{F} \) of \( E \) is transverse (to the fibers) if the following condition holds: there exists a \( G \)-atlas \( \{(U_i, \psi_i)\}_{i \in I} \) for the \( G \)-structure of \( E \) such that, for every leaf \( F \) of \( \mathcal{F} \) and every \( i \in I \), we have

\[
\psi_i(F \cap \pi^{-1}(U_i)) = U_i \times \Lambda
\]

for some countable subset \( \Lambda \) of \( S^n \).
We have shown that any flat $G$-structure on a sphere bundle $E$ determines a foliation transverse to the fibers. In fact, this foliation completely determines the flat structure on $E$:

**Theorem 12.21.** Let $\pi: E \to M$ be a sphere $G$-bundle. Then there exists a bijective correspondence between flat $G$-structures on $E$ compatible with the given $G$-structure on $E$, and transverse foliations of $E$. Moreover, if $\pi: E \to M$, $\pi': E' \to M$ are flat $G$-bundles with corresponding foliations $\mathcal{F}, \mathcal{F}'$, then an isomorphism of $G$-bundles $h: E \to E'$ is an isomorphism of flat $G$-bundles if and only if it carries any leaf of $\mathcal{F}$ onto a leaf of $\mathcal{F}'$.

**Proof.** Suppose that $E$ admits a transverse foliation $\mathcal{F}$, and let $U$ be the maximal atlas with the properties described in Definition 12.20. Since countable subsets of $S^n$ are totally disconnected, it is easily seen that the transition functions of $U$ are locally constant, so $U$ is a maximal flat $G$-atlas, and $E$ may be endowed with a flat $G$-structure canonically associated to $\mathcal{F}$. The theorem readily follows. □

Flat $G$-bundles may be characterized also in terms of their holonomy. Set $\Gamma = \pi_1(M)$ and let us first construct, for any given representation $\rho: \Gamma \to G$, a flat $G$-bundle $E_\rho$ with holonomy $\rho$. We fix a point $\tilde{x}_0 \in \tilde{M}$, we set $x_0 = p(\tilde{x}_0) \in M$, and we fix the corresponding identification of $\Gamma$ with the group of the covering automorphisms of $\tilde{M}$, so that the automorphism $\psi$ gets identified with the element of $\pi_1(M, p(\tilde{x}_0))$ represented by the projection of a path joining $\tilde{x}_0$ with $\psi(\tilde{x}_0)$. We set $\tilde{E} = \tilde{M} \times S^n$, and we define $E_\rho$ as the quotient of $\tilde{E}$ by the diagonal action of $\Gamma$, where $\Gamma$ is understood to act on $S^n$ via $\rho$. The composition of the projections $\tilde{E} \to \tilde{M} \to M$ induces a quotient map $\pi: E_\rho \to M$. If the open subset $U \subseteq M$ is evenly covered and $\tilde{U} \subseteq \tilde{M}$ is an open subset such that $p|_{\tilde{U}}: \tilde{U} \to U$ is a homeomorphism, then the composition

$$U \times S^n \xrightarrow{(p^{-1}, \text{id})} \tilde{U} \times S^n \xrightarrow{p|_{\tilde{U}}} \tilde{E} \xrightarrow{E_\rho}$$

induces a homeomorphism $\psi: U \times S^n \to \pi^{-1}(U)$ which provides a local trivialization of $E_\rho$ over $U$. It is easily checked that, when $U$ runs over the set of evenly covered open subsets of $M$, the transition functions corresponding to these local trivializations take values in $G$, and are locally constant. Indeed, the subsets of $\tilde{E}$ of the form $\tilde{M} \times \{p\}$ project onto the leaves of a transverse foliation of $E_\rho$. As a consequence, $E_\rho$ is endowed with the structure of a flat $G$-bundle. Moreover, by construction the action of $\Gamma$ on $E_{\tilde{x}_0}$ via the holonomy representation is given exactly by $\rho$. The following proposition implies that, up to isomorphism, every flat $G$-bundle may be obtained via the construction we have just described.
Proposition 12.22. The map
\[
\{\text{Representations of } \Gamma \text{ in } G\} \to \{\text{Flat } G \text{- bundles}\}
\]
\[
\rho \mapsto E_\rho
\]
induces a bijective correspondence between conjugacy classes of representations of \( \Gamma \) into \( G \) and isomorphism classes of flat \( G \)-bundles over \( M \). In particular, every representation of \( \Gamma \) in \( G \) arises as the holonomy of a flat \( G \)-bundle over \( M \), and flat \( G \)-bundles with conjugated holonomies are isomorphic as flat \( G \)-bundles.

Proof. Let us first show that every flat sphere \( G \)-bundle is isomorphic to a bundle \( E_\rho \) for some \( \rho : \Gamma \to G \). To this aim, it is sufficient to prove that the holonomy completely determines the isomorphism type of the bundle. So, let \( E \) be a flat sphere \( G \)-bundle, fix points \( x_0 \in M \) and \( \tilde{x}_0 \in p^{-1}(x_0) \) and fix the identification between \( \Gamma = \pi_1(M, x_0) \) and the group of covering automorphisms of \( M \) such that the projection on \( M \) of any path in \( \tilde{M} \) starting at \( \tilde{x}_0 \) and ending at \( g(\tilde{x}_0) \) lies in the homotopy class corresponding to \( g \). We also denote by \( h : S^n \to E_{x_0} \) the homeomorphism induced by a local trivialization of \( E \) over a neighbourhood of \( x_0 \) belonging to the atlas which defines the flat \( G \)-structure of \( E \), and we denote by \( \rho : \Gamma \to G \) the holonomy representation associated to these data.

We will now show that \( E_\rho \) and \( E \) are isomorphic as flat \( G \)-bundles. To this aim we define a map \( \tilde{\eta} : \tilde{M} \times S^n \to E \) as follows: for every \( \tilde{x} \in \tilde{M} \) we fix a path \( \tilde{\beta}_\tilde{x} \) joining \( \tilde{x}_0 \) to \( \tilde{x} \) in \( \tilde{M} \), we set \( \beta_\tilde{x} = p \circ \tilde{\beta}_\tilde{x} : [0, 1] \to M \), and we define \( \tilde{\eta}(\tilde{x}, q) \) as the endpoint of the horizontal lift of \( \beta_\tilde{x} \) starting at \( q \). The map \( \tilde{\eta} \) is well-defined (i.e. it is independent of the choice of \( \tilde{\beta}_\tilde{x} \)), and induces the quotient map \( \eta : E_\rho \to E \), which is easily seen to be an isomorphism of \( G \)-bundles. Moreover, by construction \( \eta \) carries the transverse foliation of \( E_\rho \) into the transverse foliation of \( E \), so \( \eta \) is indeed an isomorphism of flat \( G \)-bundles.

In order to conclude we are left to show that two representations \( \rho, \rho' : \Gamma \to G \) define isomorphic flat \( G \)-bundles if and only if they are conjugated. Suppose first that there exists \( g_0 \in G \) such that \( \rho'(g) = g_0 \rho(g) g_0^{-1} \) for every \( g \in \Gamma \). Then the map \( \tilde{E}_\rho \to \tilde{E}_{\rho'} \) given by \( (x, v) \mapsto (g_0x, g_0v) \) induces an isomorphism of flat \( G \)-bundles \( E_\rho \to E_{\rho'} \). On the other hand, if \( \rho, \rho' : \Gamma \to G \) \( h : E_\rho \to E_{\rho'} \) is an isomorphism of flat \( G \)-bundles, then we may lift \( h \) to a map \( \tilde{h} : \tilde{E}_\rho \to \tilde{E}_{\rho'} \) such that \( \tilde{h}(\tilde{x}_0, v) = (\tilde{x}_0, g_0v) \) for some \( g_0 \in G \). Since \( h \) preserves both the fibers and the transverse foliations, this easily implies that \( \tilde{h}(\tilde{x}, v) = (\tilde{x}, g_0v) \) for every \( \tilde{x} \in \tilde{M}, v \in S^n \). Moreover, for every \( g \in \Gamma \) there must exist \( g' \in \Gamma \) such that
\[
(g\tilde{x}_0, g_0\rho(g)v) = \tilde{h}(g \cdot (\tilde{x}_0, v)) = g' \cdot \tilde{h}(\tilde{x}_0, v) = (g'\tilde{x}_0, \rho'(g')g_0v).
\]
This equality implies \( g' = g \), whence \( g_0\rho(g) = \rho'(g)g_0 \) for every \( g \in \Gamma \). We have thus shown that \( \rho' \) is conjugated to \( \rho \), and this concludes the proof. \( \square \)
4. The bounded Euler class of a flat circle bundle

Let \( \pi: E \to M \) be a flat circle bundle. We fix points \( x_0 \in M \) and \( \bar{x}_0 \in \rho^{-1}(x_0) \), and we identify \( \Gamma = \pi_1(M,x_0) \) with the group of the covering automorphisms of \( \bar{M} \) so that the projection on \( M \) of any path in \( \bar{M} \) starting at \( \bar{x}_0 \) and ending at \( g(\bar{x}_0) \) lies in the homotopy class corresponding to \( g \). We also choose a set of representatives \( R \) for the action of \( \Gamma = \pi_1(M) \) on \( \bar{M} \) containing \( \bar{x}_0 \).

By exploiting the flat structure on \( E \), we are going to construct a bounded representative \( z_b \in C^2_b(M,\mathbb{Z}) \) for the Euler class \( eu(E) \). In fact, we will see that such a cocycle represents a well-defined bounded class in \( H^2_b(M,\mathbb{Z}) \). By Proposition 12.22 we may suppose \( E = E_\rho \) for some representation \( \rho: \Gamma \to \text{Homeo}^+(S^1) \), and we denote by \( j: \tilde{M} \times S^1 \to E_\rho \) the quotient map with respect to the diagonal action of \( \Gamma \) on \( \tilde{M} \times S^1 \). We pick a point \( q \in E_{x_0} \) and we denote by \( \theta_0 \in S^1 \) the unique element such that \( j(\bar{x}_0,\theta_0) = q \).

For every \( x \in M \), we denote by \( \bar{x} \) the unique preimage of \( x \) in \( R \subseteq \bar{M} \), and we choose the section \( \sigma_x \in E_x \) defined by \( \sigma_x = j(\bar{x},\theta_0) \). Let now \( s: [0,1] \to M \) be a singular simplex. We denote by \( \bar{s}: [0,1] \to \bar{M} \) the unique lift of \( s \) starting at a point in \( R \), and by \( g_s \) the unique element of \( \Gamma \) such that \( \bar{s}(1) = g_s(R) \). We also choose a path \( \tilde{h}_s(t): [0,1] \to \text{Homeo}^+(S^1) \) such that \( \tilde{h}_s(0) = \text{Id} \) and \( \tilde{h}_s(1) = \rho(g_s) \), where as usual \( \rho(g_s) \) denotes the lift of \( \rho(g_s) \) taking \( 0 \in \mathbb{R} \) into \( [0,1] \subseteq \mathbb{R} \), and we denote by \( h_s(t) \) the projection of \( \tilde{h}_s(t) \) in \( \text{Homeo}^+(S^1) \). We finally define the section \( \sigma_s: [0,1] \to E \) by setting

\[
\sigma_s(t) = j(\bar{s}(t),h_s(t)(\theta_0))
\]

Our choices imply that \( \sigma_s \) is indeed a section of \( E \) over \( s \) such that \( \sigma_s(0) = \sigma_s(0) \) and \( \sigma_s(1) = \sigma_s(1) \).

Let us now evaluate the Euler cocycle \( z_b \) corresponding to the chosen sections on a singular 2-simplex \( s: \Delta^2 \to M \). We denote by \( e_0,e_1,e_2 \) the vertices of \( \Delta^2 \), by \( \bar{s}: \Delta^2 \to \bar{M} \) the unique lift of \( s \) such that \( \bar{s}(e_0) \in R \), and by \( g_1,g_2 \) the elements of \( \Gamma \) such that \( \bar{s}(e_1) \in g_1(R), \bar{s}(e_2) \in g_1g_2(R) \). The pull-back \( E_s \) of \( E \) over \( s \) admits the trivialization

\[
\psi_s: E_s \to \Delta^2 \times S^1, \quad \psi_s^{-1}(x,\theta) = j(\bar{s}(x),\theta).
\]

Let \( \sigma_{\partial s}: \partial \Delta^2 \to E_s \) be the section obtained by concatenating \( \sigma_{\partial s_i}, i = 1,2,3 \). In order to compute \( z_b(s) \) we need to write down an expression for the map \( \pi_{S^1} \circ \psi_s \circ \sigma_{\partial s}: \partial \Delta^2 \to S^1 \). For convenience, we set \( \varepsilon_i = (-1)^i \), and for any path \( \alpha \) defined over \( [0,1] \) we denote by \( \alpha^{-1} \) the inverse path (i.e. the path such that \( \alpha^{-1}(t) = \alpha(1-t) \)). If \( \alpha \) is a path with values in \( M \), we also denote by \( \tilde{\alpha} \) the lift of \( \alpha \) with initial point in \( R \). Now we have

\[
\partial_2 \bar{s} = \partial_2 \bar{s}, \quad \partial_0 \bar{s} = g_1 \circ \partial_0 \bar{s}, \quad (\partial_1 \bar{s})^{-1} = g_1g_2 \circ (\partial_1 \bar{s})^{-1}.
\]
Recall that, to every 1-simplex $\partial s$, there is associated a path $h_i = h_{(\partial_i s)i} : [0,1] \to \text{Homeo}^+(S^1)$ such that $\sigma_{(\partial_i s)i}(t) = j((\partial_i s)i(t), h_i(t)(\theta_0))$. Therefore, if $\alpha_i : [0, 1] \to S^1$ is defined by
\[
\alpha_2(t) = h_2(t)(\theta_0), \quad \alpha_0(t) = \rho(g_1) h_0(t)(\theta_0), \quad \alpha_1(t) = \rho(g_1 g_2) h_1(t)(\theta_0)
\]
and $\beta_i(t) = ((\partial_i s)i(t), \alpha_i(t)) \in \tilde{M} \times S^1$, then the desired section of $E_s$ over $\partial \Delta^2$ is obtained by projecting onto $E$ the concatenation $\beta_2 \ast \beta_0 \ast \beta_1$. As a consequence, the value of $z_b(s)$ is equal to the element of $\pi_1(S^1) \cong \mathbb{Z}$ defined by the map $\gamma = \alpha_2 \ast \alpha_0 \ast \alpha_1 : [0, 3] \to S^1$. In order to compute this element, we observe that the lift $\tilde{\gamma} : [0, 3] \to \mathbb{R}$ of $\gamma$ such that $\tilde{\gamma}(0) = 0$ is given by $\tilde{\gamma}(1) = \tilde{\alpha}_1(1) = \rho(g_1) \rho(g_2) \rho((g_1 g_2)^{-1})(0) = \rho^*(c)(g_1, g_2)$, where $c \in C^2_b(\text{Homeo}^+(S^1), \mathbb{Z})$ is the canonical representative of the bounded Euler class $e_b \in H^2_b(\text{Homeo}^+(S^1), \mathbb{Z})$.

We have thus shown that the representative $z_b \in C^2(M, \mathbb{Z})$ of the Euler class of $E$ is bounded. Moreover, if $s : \Delta^2 \to M$ is any singular simplex whose vertices lie respectively in $R, g_1(R)$ and $g_1 g_2(R)$, then $z_b(s) = \rho^*(c)(g_1, g_2)$. This fact may be restated by saying that $z_b = r^2(\rho^*(c))$, where
\[
r^2 : C^2_b(\Gamma, \mathbb{Z}) \to C^2(M, \mathbb{Z})
\]

is the map described in Lemma 5.2. In other words, the natural map
\[
H^2_b(\rho^*) : H^2_b(\Gamma, \mathbb{Z}) \to H^2_b(M, \mathbb{Z})
\]
described in Corollary 5.3 takes the bounded Euler class $e_b(\rho)$ into the class $[z_b] \in H^2_b(M, \mathbb{R})$. Therefore, it makes sense to define the bounded Euler class $e_{b}(E)$ of the flat circle bundle $E$ by setting
\[
e_{b}(E) = [z_b] \in H^2_b(M, \mathbb{R})
\]
Recall that, if $E, E'$ are isomorphic flat circle bundles, then $E = E_{\rho}$ and $E' = E_{\rho'}$ for conjugated representations $\rho, \rho' : \Gamma \to \text{Homeo}^+(S^1)$. Since conjugated representations share the same bounded Euler class, we have that $e_{b}(E)$ is a well-defined invariant of the isomorphism type of $E$ as a flat circle bundle. We may summarize this discussion in the following:

**Theorem 12.23.** To every flat topological circle bundle $\pi : E \to M$ it is possible to associate a bounded class $e_{b}(E) \in H^2_b(M, \mathbb{Z})$, in such a way that:

- $\|e_{b}(E)\|_{\infty} \leq 1$;
- If $E = E_{\rho}$, then $e_{b}(E) = H^2_b(\rho^*)(e_{b}(\rho))$;
• If $\pi': E' \to M$ is isomorphic to $\pi: E \to M$ as a topological flat circle bundle, then $e_{b}(E') = e_{b}(E)$.

• The comparison map $H^2_b(M, \mathbb{Z}) \to H^2(M, \mathbb{Z})$ takes $e_{b}(E)$ into $eu(E)$.

Theorem 12.23 already provides an obstruction for a circle bundle to admit a flat structure:

**Corollary 12.24.** Let $E \to M$ be a circle bundle. If $E$ admits a flat structure, then $eu(E)$ lies in the image of the unit ball via the comparison map $H^2_b(M, \mathbb{Z}) \to H^2(M, \mathbb{Z})$.

By passing to real coefficient, the above estimate on the seminorm of $e_{b}(E)$ may be improved. A celebrated result by Milnor and Wood implies that, in the case of closed surfaces, this fact may be exploited to prove sharp estimates on the Euler number of flat circle bundles. In the following section we describe a proof of Milnor-Wood inequalities which makes use of the machinery introduced so far.

### 5. Milnor-Wood inequalities

Let us now concentrate our attention on circle bundles on surfaces. We recall that, for every $g \in \mathbb{N}$, the symbol $\Sigma_g$ denotes the closed orientable surface of genus $g$. In this section we describe classical results by Milnor [Mil58a] and Wood [Woo71], which provide sharp estimates on the Euler number of flat circle bundles on $\Sigma_g$. To this aim, it is useful to consider the real (bounded) Euler class of bundles and representations, which are defined as follows.

**Definition 12.25.** The real Euler class $e^{R} \in H^2(\text{Homeo}^{+}(S^1), \mathbb{R})$ (resp. bounded real Euler class $e_{b}^{R} \in H^2_b(\text{Homeo}^{+}(S^1), \mathbb{R})$) is the image of $e \in H^2(\text{Homeo}^{+}(S^1), \mathbb{Z})$ (resp. of $e_{b} \in H^2_b(\text{Homeo}^{+}(S^1), \mathbb{Z})$) under the obvious change of coefficients homomorphism. If $\rho: \Gamma \to \text{Homeo}^{+}(S^1)$ is any representation, then we also set $e^{R}(\rho) = \rho^{*}(e^{R})$ and $e_{b}^{R}(\rho) = \rho^{*}(e_{b}^{R})$.

**Definition 12.26.** Let $\pi: E \to M$ be a topological $n$-sphere bundle. Then the real Euler class $e^{R}(E) \in H^{n+1}(M, \mathbb{R})$ of $E$ is the image of $e(E) \in H^{n+1}(M, \mathbb{Z})$ under the change of coefficients homomorphism. If $n = 1$ and $E$ is topologically flat, then we denote by $e_{b}^{R}(E) \in H^2_b(M, \mathbb{R})$ the image of $e_{b}(E) \in H^2_b(M, \mathbb{Z})$ under the change of coefficients homomorphism.

Let us summarize what we can already deduce from the preceding sections:

1. If $g = 0$, i.e. $\Sigma_g = S^2$, then $\pi_1(\Sigma_g)$ is trivial, so for every $G < \text{Homeo}^{+}(S^1)$ the space of representations of $\pi_1(\Sigma_g)$ into $G$ is trivial. As a consequence of Proposition 12.22, the unique flat $G$-bundle over $S^2$ is the trivial one. More precisely, the only circle bundle over $S^2$ supporting a flat $G$-structure is the topologically trivial
one, and the unique flat $G$-structure supported by this bundle is the trivial one.

(2) If $g = 1$, i.e. $\Sigma_g = S^1 \times S^1$, then $\pi_1(\Sigma_g)$ is amenable, so $H^2_b(\Sigma_g, \mathbb{R}) = 0$. Therefore, if $\pi: E \to \Sigma_g$ is a flat circle bundle, then $e^b_\rho(E) = 0$, whence $e^\mathbb{R}(E) = 0$. But the change of coefficients map $H^2(\Sigma_g, \mathbb{Z}) \to H^2(\Sigma_g, \mathbb{R})$ is injective, so the Euler number of $E$ vanishes. By Theorem 12.17, this implies that $E$ is topologically trivial. Therefore, the unique circle bundle over $S^1 \times S^1$ supporting a flat structure is the trivial one. However, there are infinitely many non-conjugate representations of $\pi_1(S^1 \times S^1)$ into $\text{Homeo}^+(S^1)$. Therefore, Proposition 12.22 implies that the trivial circle bundle over $S^1 \times S^1$ admits infinitely many pairwise non-isomorphic flat structures.

The following proposition provides interesting examples of flat circle bundle over $\Sigma_g$, when $g \geq 2$.

**Proposition 12.27.** For every $g \geq 2$, there exists a flat circle bundle $\pi: E \to \Sigma_g$ such that

$$|\text{eu}(E)| = 2g - 2.$$

**Proof.** Let $\Gamma_g$ be the fundamental group of $\Sigma_g$. It is well-known that the surface $\Sigma_g$ supports a hyperbolic metric. As a consequence, we may identify the Riemannian universal covering of $\Sigma_g$ with the hyperbolic plane $\mathbb{H}^2$, and $\Sigma_g$ is realized as the quotient of $\mathbb{H}^2$ by the action of $\rho(\Gamma_g)$, where $\rho: \Gamma_g \to \text{Isom}^+(\mathbb{H}^2)$ is a faithful representation with discrete image. The hyperbolic plane admits a natural compactification $\mathbb{H}^2 = \mathbb{H}^2 \cup \partial \mathbb{H}^2$ which can be roughly described as follows (see e.g. [BP92] for the details): points of $\partial \mathbb{H}^2$ are equivalence classes of geodesic rays of $\mathbb{H}^2$, where two such rays are equivalent if their images lie at finite Hausdorff distance one from the other (recall that a geodesic ray in the hyperbolic plane admits a natural compactification which can be roughly described as follows (see e.g. [BP92] for the details)): points of $\partial \mathbb{H}^2$ are equivalence classes of geodesic rays of $\mathbb{H}^2$, where two such rays are equivalent if their images lie at finite Hausdorff distance one from the other. A metrizable topology on $\partial \mathbb{H}^2$ may be defined by requiring that a sequence $(p_i)_{i \in \mathbb{N}}$ in $\partial \mathbb{H}^2$ converges to $p \in \partial \mathbb{H}^2$ if and only if there exists a choice of representatives $\gamma_i \in p_i$, $\gamma \in p$ such that $\gamma_i \to \gamma$ with respect to the compact-open topology. Let us denote by $\pi: T_1 \mathbb{H}^2 \to \mathbb{H}^2$ the unit tangent bundle of $\mathbb{H}^2$, and by $\gamma_v$, $v \in T_1 \mathbb{H}^2$, the geodesic ray in $\mathbb{H}^2$ with initial speed $v$. Then the map

$$\Psi: T_1 \mathbb{H}^2 \to \mathbb{H}^2 \times \partial \mathbb{H}^2, \quad \Psi(v) = (\pi(v), [\gamma_v])$$

is a homeomorphism. In particular, $\partial \mathbb{H}^2$ is homeomorphic to $S^1$. Moreover, any isometry of $\mathbb{H}^2$ extends to a homeomorphism of $\partial \mathbb{H}^2$, so $\rho(\Gamma_g)$ acts on $\partial \mathbb{H}^2$. Of course, $\rho(\Gamma_g)$ also acts on $T_1 \mathbb{H}^2$, and it follows from the definitions that the map $\Psi$ introduced above is $\rho(\Gamma_g)$-equivariant with respect to these actions. Observe now that the quotient of $T_1 \mathbb{H}^2$ by the action of $\rho(\Gamma_g)$ may be canonically identified with the unit tangent bundle $T_1 \Sigma_g$, so $|\text{eu}(T_1 \Sigma_g)| = 2g - 2$ by Proposition 12.14. On the other hand, since $\partial \mathbb{H}^2$ is homeomorphic to $S^1$, the quotient of $\mathbb{H}^2 \times \partial \mathbb{H}^2$ by the diagonal action of $\rho(\Gamma_g)$ is a flat circle bundle on $\Sigma_g$. □
Remark 12.28. The previous result holds also if we require $E$ to be flat as a smooth circle bundle. In fact, $\partial \mathbb{H}^2$ admits a natural smooth structure. With respect to this structure, the map $\Psi$ introduced in the proof of Proposition 12.27 is a diffeomorphism, and $\text{Isom}^+(\mathbb{H}^2)$ acts on $\partial \mathbb{H}^2$ via diffeomorphisms. As a consequence, $T_1 \Sigma_g$ admits a structure of flat smooth circle bundle.

On the other hand, as a consequence of a classical result by Milnor (see Theorem 12.32 below), the bundle $T_1 \Sigma_g$ does not admit any flat linear structure for $g \geq 2$.

Proposition 12.27 shows that, when $g \geq 2$, in order to get bounds on the Euler number of flat bundles over $\Sigma_g$ a more refined analysis is needed. We first provide an estimate on the norm of $e^R_b \in H^2_b(\text{Homeo}^+(S^1), \mathbb{R})$.

Lemma 12.29. We have

$$\|e^R_b\|_\infty \leq \frac{1}{2}.$$  

Proof. Recall from Lemma 11.3 that $e_b$ admits a representative $c_{x_0}$ taking values in the set $\{0, 1\}$. If $\varphi: C^1(\text{Homeo}^+(S^1), \mathbb{R})^{\text{Homeo}^+(S^1)}$ is the constant cochain taking the values $-1/2$ on every pair $(g_0, g_1) \in \text{Homeo}^+(S^1)^2$, then the cocycle $c + \delta \varphi$ still represents $e_b$, and takes values in $\{-1/2, 1/2\}$. The conclusion follows. □

Corollary 12.30. Let $\pi: E \to M$ be a flat circle bundle. Then

$$\|e^R_b(E)\|_\infty \leq \frac{1}{2}.$$  

Proof. The conclusion follows from Lemma 12.29 and Theorem 12.23. □

We are now ready the prove Wood’s estimate of the Euler number of flat circle bundles. For convenience, for every $g$ we set $\chi_-(\Sigma_g) = \min\{\chi(\Sigma_g), 0\}$, and we recall from Section 3 that the simplicial volume of $\Sigma_g$ is given by $\|\Sigma_g\| = 2|\chi_-(\Sigma_g)|$.

Theorem 12.31 ([Woo71]). Let $\pi: E \to \Sigma_g$ be a flat circle bundle. Then

$$|e^R_b(E)| \leq |\chi_-(\Sigma_g)|.$$  

Moreover, this inequality is sharp, i.e. for every $g \in \mathbb{N}$ there exists a flat circle bundle over $\Sigma_g$ with Euler number equal to $|\chi_-(\Sigma_g)|$.

Proof. Let us denote by $\langle \cdot, \cdot \rangle: H^2_b(\Sigma_g, \mathbb{R}) \times H_2(\Sigma_g, \mathbb{R}) \to \mathbb{R}$ the Kronecker product (see Chapter 6). The Euler number of $E$ may be computed by evaluating $e(E) \in H^2(\Sigma_g, \mathbb{Z})$ on the integral fundamental class $[\Sigma_g]$ of $\Sigma_g$, or, equivalently, by evaluating the class $e^R(E) \in H^2(\Sigma_g, \mathbb{R})$ on the image $[\Sigma_g]_\mathbb{R} \in H_2(\Sigma_g, \mathbb{R})$ of the integral fundamental class of $\Sigma_g$ under the change of coefficients homomorphism. As a consequence, we have

$$|e^R_b(E)| = \langle e^R_b(E), [\Sigma_g]_\mathbb{R} \rangle \leq \|e^R_b(E)\|_\infty \|\Sigma_g\| \leq \frac{1}{2}|2\chi_-(\Sigma_g)|.$$  


The fact that Wood’s inequality is sharp is a direct consequence of Proposition 12.27.

As a consequence, we obtain the following corollary, which is due to Milnor:

**Theorem 12.32 ([Mil58a]).** Let \( \pi: E \to \Sigma_g \) be a flat linear circle bundle. Then

\[
|\text{eu}(E)| \leq \frac{\chi(\Sigma_g) - 2}{2}.
\]

**Proof.** If \( \pi: E \to \Sigma_g \) is a flat linear circle bundle, then the associated projective bundle \( \mathbb{P}(E) \) is a flat topological circle bundle, and \( 2|\text{eu}(E)| = |\text{eu}(\mathbb{P}(E))| \leq |\chi(\Sigma_g)| \), whence the conclusion.

In fact, the bounds in the last two theorems are sharp in the following strict sense: every Euler number which is not forbidden is realized by some flat topological (resp. linear) circle bundle.

Theorem 12.31 allow us to compute the exact norm of the real Euler class:

**Theorem 12.33.** The norm of the Euler class \( e^R_b \in H^2_b(\text{Homeo}^+(S^1), \mathbb{R}) \) is given by

\[
\|e^R_b\|_\infty = \frac{1}{2}.
\]

**Proof.** The inequality \( \leq \) was proved in Lemma 12.29. Now let \( g = 2 \), and consider the flat circle bundle on \( \pi: E \to \Sigma_2 \) described in Proposition 12.27. We have

\[
2 = 2g - 2 = |\text{eu}(E)| \leq \|e^R_b\|_\infty \|\Sigma_2\| = 4\|e^R_b\|_\infty,
\]

so \( \|e^R_b\|_\infty \geq 1/2 \), and we are done.

**Remark 12.34.** One could exploit the estimate described in the proof of Theorem 12.33 also to compute a (sharp) lower bound on the simplicial volume of \( \Sigma_g \), \( g \geq 2 \). In fact, once one knows that \( \|e^R_b\|_\infty \leq 1/2 \) and that there exists a flat circle bundle \( E \) over \( \Sigma_g \) with Euler number \( 2g - 2 \), the estimate

\[
2g - 2 = |\text{eu}(E)| \leq \|e^R_b\|_\infty \|\Sigma_g\| \leq \|\Sigma_g\|/2
\]

implies that \( \|\Sigma_g\| \geq 2|\chi(\Sigma_g)| \).
CHAPTER 13

The Euler class in higher dimension and the Chern Conjecture

In the previous chapter we have discussed some applications of the boundedness of the Euler class to the study of flat topological circle bundles. We now analyze the case of flat $n$-sphere bundles for $n \geq 2$. We concentrate here on the case of linear sphere bundles, which is better understood (see Conjecture 13.12 below for a discussion of the general case).

The first description of a bounded representative for the Euler class of a flat linear sphere bundle $E$ is due to Sullivan. Since $E$ is linear, we have that $E = S(V)$ for some flat rank-$(n + 1)$ vector bundle (a flat vector bundle of rank $(n + 1)$ is a vector bundle endowed with an $\text{Gl}^+(n + 1, \mathbb{R})$-atlas with locally constant transition functions). In order to compute the Euler class of $E$, one may analyze just affine sections of $V$ over singular simplices, where the word affine will make sense exactly because $V$ is a flat vector bundle.

We follow an argument due to Sullivan [Sul76]. In fact, Sullivan’s original argument works for a simplicial complex $M$ and provides a simplicial cocycle of norm at most one. Of course, any simplicial cocycle on a compact simplicial complex is bounded. However, the fact that the norm of Sullivan’s cocycle is bounded by 1 already implies that, if $M$ is a triangulated oriented manifold of dimension $(n + 1)$, then the Euler number of $E$ is bounded by the number of top-dimensional simplices in a triangulation of $M$. Just as we did in the case of Milnor-Wood’s inequality, we would like to promote the bounded simplicial Euler cocycle to a bounded singular cocycle. This will also allow to replace the number of top-dimensional simplices in a triangulation of $M$ with the simplicial volume of $M$ in the above upper bound for the Euler number of any flat linear sphere bundle on $M$. This can be done essentially in two ways: one could either invoke a quite technical result by Gromov [Gro82, §3.2], which ensures that the bounded cohomology (with real coefficients) of $M$ is isometrically isomorphic to the simplicial bounded cohomology of a suitable multicomplex $K(M)$, thus reducing computations in singular cohomology to computations in simplicial cohomology, or explicitly describe a bounded singular cocycle representing the Euler class.

Here we described the approach to the second strategy developed by Ivanov and Turaev in [IT82] (we refer the reader also to [BM12] for more recent developments).
1. Ivanov-Turaev cocycle

Let $\pi: E \to M$ be a flat linear $n$-sphere bundle, and recall that $E = S(V)$ for some flat rank-$(n+1)$ vector bundle (a flat vector bundle of rank $(n+1)$ is a vector bundle endowed with an $\text{GL}^+(n + 1, \mathbb{R})$-atlas with locally constant transition functions). Sullivan’s strategy is based on the following simple idea: in order to compute the Euler class of $E$, one may analyze just affine sections of $V$ over singular simplices, where the word affine will make sense exactly because $V$ is a flat vector bundle.

We begin with the following lemma. Point (6) will prove useful to get representatives of the Euler class of small norm, as suggested in [Smii]. Recall that a subset $\Omega \subseteq \mathbb{R}^{n+1}$ is in general position if, for every subset $\Omega_k$ of $\Omega$ with exactly $k$ elements, $k \leq n+2$, the dimension of the smallest affine subspace of $\mathbb{R}^{n+1}$ containing $\Omega_k$ is equal to $k - 1$.

**Definition 13.1.** Let $N$ be an integer (in fact, we will be interested only in the case $N \leq n+1$, and mainly in the case $N = n+1$). An $(N+1)$-tuple $(v_0, \ldots, v_N) \in (\mathbb{R}^{n+1})^{N+1}$ is generic if the following condition holds: for every $(\varepsilon_0, \ldots, \varepsilon_N) \in \{\pm 1\}^{N+1}$, the set $\{0, \varepsilon_0v_0, \ldots, \varepsilon_Nv_N\}$ is in general position (in particular, the set $\{0, \varepsilon_0v_0, \ldots, \varepsilon_Nv_N\}$ consists of $N+2$ distinct points).

**Lemma 13.2.** Let $(v_0, \ldots, v_{n+1})$ be a generic $(n+2)$-tuple. For every $I = (\varepsilon_0, \ldots, \varepsilon_{n+1}) \in \{\pm 1\}^{n+2}$, let $\sigma_I: \Delta^{n+1} \to \mathbb{R}^{n+1}$ be the affine map defined by

$$\sigma_I(t_0, \ldots, t_{n+1}) = \sum_{i=0}^{n+1} t_i \varepsilon_i v_i,$$

and let $S \cong S^n$ be the sphere of rays of $\mathbb{R}^{n+1}$. Then:

1. The restriction of $\sigma_I$ to $\partial \Delta^{n+1}$ takes values in $\mathbb{R}^{n+1} \setminus \{0\}$, thus defining a map $\tilde{\sigma}_I: \partial \Delta^{n+1} \to S$.
2. The map $\sigma_I$ is a smooth embedding.
3. If the image of $\sigma_I$ does not contain 0, then $\deg \tilde{\sigma}_I = 0$.
4. If the image of $\sigma_I$ contains 0 and $\sigma_I$ is orientation-preserving, then $\deg \tilde{\sigma}_I = 1$.
5. If the image of $\sigma_I$ contains 0 and $\sigma_I$ is orientation-reversing, then $\deg \tilde{\sigma}_I = -1$.
6. There exist exactly two elements $I_1, I_2 \in \{\pm 1\}^{n+2}$ such that the image of $\sigma_{I_j}$ contains 0 for $j = 1, 2$. For such elements we have $\sigma_{I_1}(x) = -\sigma_{I_2}(x)$ for every $x \in \partial \Delta^{n+1}$.

**Proof.** (1) and (2) follow from the fact that $\{0, \varepsilon_0v_0, \ldots, \varepsilon_{n+1}v_{n+1}\}$ is in general position. If the image of $\sigma_I$ does not contain 0, then $\tilde{\sigma}_I$ continuously extends to a map from $\Delta^{n+1}$ to $S$, and this implies (3). Suppose now that the image of $\sigma_I$ contains 0. Since $\{0, \varepsilon_0v_0, \ldots, \varepsilon_{n+1}v_{n+1}\}$ is in general position, 0 cannot belong to the image of any face of $\Delta^{n+1}$, so the image of $\sigma_I$ contains a neighbourhood of 0 in $\mathbb{R}^n$, and $\tilde{\sigma}_I: \partial \Delta^{n+1} \to S$ is surjective. Since the
image of \( \sigma_I \) is star-shaped with respect to 0, the map \( \tilde{\sigma}_I: \partial \Delta^{n+1} \to S \) is also injective, so it is a homeomorphism. Moreover, \( \tilde{\sigma}_I \) is orientation-preserving if and only if \( \sigma_I \) is, and this concludes the proof of (4) and (5).

Let us prove (6). Since \( v_0, \ldots, v_{n+1} \) are linearly dependent in \( \mathbb{R}^{n+1} \), we have \( \sum_{i=0}^{n+1} \alpha_i v_i = 0 \) for some \( (\alpha_0, \ldots, \alpha_{n+1}) \in \mathbb{R}^{n+2} \setminus \{0\} \). Since the set \( \{0, v_0, \ldots, v_{n+1}\} \) is in general position, we have \( \alpha_i \neq 0 \) for every \( i \), and, if \( \sum_{i=0}^{n+1} \lambda_i v_i = 0 \), then \( (\lambda_0, \ldots, \lambda_{n+1}) = \mu (\alpha_0, \ldots, \alpha_{n+1}) \) for some \( \mu \in \mathbb{R} \). We set

\[
\ell_i = \frac{|\alpha_i|}{\sum_{i=0}^{n+1} |\alpha_i|} \in [0, 1], \quad \varepsilon_1 = \text{sign}(\alpha_i).
\]

Since \( \sum_{i=0}^{n+1} t_i \varepsilon_i v_i = 0 \) and \( \sum_{i=0}^{n+1} \ell_i = 1 \), if we set \( I_1 = (\varepsilon_0, \ldots, \varepsilon_{n+1}), I_2 = (-\varepsilon_0, \ldots, -\varepsilon_{n+1}) \), then 0 belongs to the image of \( \sigma_{I_1} \) and \( \sigma_{I_2} \), and \( \sigma_{I_1}(x) = -\sigma_{I_2}(x) \) for every \( x \in \partial \Delta^{n+1} \).

Suppose now that \( \sum_{i=0}^{n+1} t_i \varepsilon_i v_i = 0 \), where \( t_i \in [0, 1], \sum_{i=0}^{n+1} t_i = 1 \). Then there exists \( \mu \in \mathbb{R} \) such that \( t_i \varepsilon_i = \mu \ell_i \varepsilon_1 \) for every \( i \). This readily implies that \( \varepsilon_i = \varepsilon_1 \) for every \( i \) (if \( \mu > 0 \), or \( \varepsilon_i = -\varepsilon_1 \) (if \( \mu < 0 \); observe that \( \mu = 0 \) is not possible since \( t_i \neq 0 \) for some \( i \)). In other words, \( (\varepsilon_0, \ldots, \varepsilon_{n+1}) \) is equal either to \( I_1 \) or to \( I_2 \), and this concludes the proof. \( \square \)

Until the end of the section we simply denote by \( G \) the group \( \text{GL}^+(n+1, \mathbb{R}) \), and by \( D \) the closed unit ball in \( \mathbb{R}^{n+1} \). We will understand that \( D \) is endowed with the standard Lebesgue measure (and any product \( D^k \) is endowed with the product of the Lebesgue measures of the factors).

**Definition 13.3.** Take \( \overline{g} = (g_0, \ldots, g_{n+1}) \in G^{n+2} \). We say that an \((n+2)\)-tuple \((v_0, \ldots, v_{n+1}) \in D^{n+2} \) is \( \overline{g} \)-generic if \((g_0v_0, \ldots, g_{n+1}v_{n+1}) \) is generic.

**Lemma 13.4.** For every \( \overline{g} \in G^{n+2} \), the set of \( \overline{g} \)-generic \((n+2)\)-tuples has full measure in \( D^{n+2} \).

**Proof.** Let \( \overline{g} = (g_0, \ldots, g_{n+1}) \). For every fixed \((\varepsilon_0, \ldots, \varepsilon_{n+1}) \in \{\pm 1\}^{n+2} \), the subset of \( \mathbb{R}^{n+1})^{n+2} \) of elements \((v_0, \ldots, v_{n+1}) \) such that

\[
\{0, \varepsilon_0g_0(v_0), \ldots, \varepsilon_{n+1}g_{n+1}(v_{n+1})\}
\]

consists of \( n+3 \) distinct points in general position is the non-empty complement of a real algebraic subvariety of \( \mathbb{R}^{(n+1)(n+2)} \), so it has full measure in \( \mathbb{R}^{(n+1)(n+2)} \). Therefore, being the intersection of a finite number of full measure sets, the set of \( \overline{g} \)-generic \((n+2)\)-tuples has itself full measure. \( \square \)

For convenience, in this section we work mainly with cohomology with real coefficients (but see Corollary 13.11).

We are now ready to define the Euler cochain \( eul \in C^{n+1}_b(G, \mathbb{R}) \). First of all, for every \((n+2)\)-tuple \( \overline{v} = (v_0, \ldots, v_{n+1}) \in \mathbb{R}^{n+1})^{n+2} \) we define a value \( t(\overline{v}) \in \{-1, 0, 1\} \) as follows. If the set \( \{0, v_0, \ldots, v_{n+1}\} \) is not in
general position, then \( t(\tau) = 0 \). Otherwise, if \( \sigma_\tau \colon \Delta^{n+1} \to \mathbb{R}^{n+1} \) is the affine embedding with vertices \( v_0, \ldots, v_{n+1} \), then:

\[
t(\tau) = \begin{cases} 
1 & \text{if } 0 \in \text{Im } \sigma_\tau \text{ and } \sigma_\tau \text{ is positively oriented} \\
-1 & \text{if } 0 \in \text{Im } \sigma_\tau \text{ and } \sigma_\tau \text{ is negatively oriented} \\
0 & \text{otherwise} .
\end{cases}
\]

Then, for every \( (g_0, \ldots, g_{n+1}) \in G^{n+2} \) we set

\[
eul(g_0, \ldots, g_n) = \int_{D^{n+2}} t(g_0 v_0, \ldots, g_{n+1} v_{n+1}) \, dv_0 \ldots dv_{n+1} .
\]

**Lemma 13.5.** The element \( \text{eul} \in C^{n+1}_b(G, \mathbb{R}) \) is a \( G \)-invariant alternating cocycle. Moreover:

1. \( \|\text{eul}\|_\infty \leq 2^{n-1} \), and \( \text{eul} = 0 \) if \( n \) is even.
2. \( \text{eul}(g_0, \ldots, g_{n+1}) = 0 \) if there exist \( i \neq j \) such that \( g_i, g_j \in SO(n+1) \).

**Proof.** The fact that \( \text{eul} \) is \( G \)-invariant follows from the fact that, if \( g \cdot \bar{\tau} \) is the \((n+2)\)-tuple obtained by translating every coordinate of \( \bar{\tau} \in (\mathbb{R}^{n+1})^{n+2} \) by \( g \in G \), then \( \sigma_{g \cdot \bar{\tau}} = g \circ \sigma_{\bar{\tau}} \), so \( t(g \cdot \bar{\tau}) = t(\bar{\tau}) \). Moreover, \( \text{eul} \) is alternating, since \( t \) is. Let us prove that \( \text{eul} \) is a cocycle. So, let \( \bar{\gamma} = (g_0, \ldots, g_{n+2}) \in G^{n+3} \), and set \( \partial_i \bar{\gamma} = (g_0, \ldots, \hat{g}_i, \ldots, g_{n+2}) \). We need to show that

\[
\delta \text{eul}(\bar{\gamma}) = \sum_{i=0}^{n+2} \text{eul}(\partial_i \bar{\gamma}) = 0 .
\]

Let us denote by \( \Omega \in D^{n+2} \) the set of \((n+2)\)-tuples which are generic for every \( \partial_i \bar{\gamma} \). Then, \( \Omega \) has full measure in \( D^{n+2} \), so

\[
\text{eul}(\partial_i \bar{\gamma}) = \int_{\Omega} t(g_0 v_0, \ldots, \hat{g}_i v_i, \ldots, g_{n+2} v_{n+2}) \, dv_0 \ldots \hat{d}v_i \ldots dv_{n+2} .
\]

Therefore, in order to conclude it is sufficient to show that

\[
T(v_0, \ldots, v_{n+2}) = \sum_{i=0}^{n+2} (-1)^i t(v_0, \ldots, \hat{v}_i, \ldots, v_{n+2}) = 0
\]

for every \((n+3)\)-tuple \((v_0, \ldots, v_{n+2})\) such that \((v_0, \ldots, \hat{v}_i, \ldots, v_{n+2})\) is generic for every \( i \). Let \( \lambda \colon \partial \Delta^{n+2} \to \mathbb{R}^{n+1} \) be the map which sends the \( i \)-th vertex of \( \Delta^{n+2} \) to \( v_i \), and is affine on each face of \( \Delta^{n+2} \). Then, it follows from the very definitions that \( T(v_0, \ldots, v_{n+2}) \) is equal to the degree of \( \lambda \), which is obviously null. This concludes the proof that \( \text{eul} \) is a \( G \)-invariant alternating cocycle.

Let us now prove (1). We fix \( \bar{\gamma} = (g_0, \ldots, g_{n+1}) \in G^{n+2} \), and denote by \( \Omega \subset D^{n+2} \) the set of \( \bar{\gamma} \)-generic \((n+2)\)-tuples, so that

\[
\text{eul}(g_0, \ldots, g_{n+1}) = \int_{\Omega} t(g_0 v_0, \ldots, g_{n+1} v_{n+1}) \, dv_0 \ldots dv_{n+1} .
\]
For every \( I = (\varepsilon_0, \ldots, \varepsilon_{n+1}) \in \{\pm 1\}^{n+2} \) we set
\[
 t_I(w_0, \ldots, w_{n+1}) = t(\varepsilon_0 w_0, \ldots, \varepsilon_{n+1} w_{n+1}) ,
\]
\[
eul_I(g_0, \ldots, g_{n+1}) = \int_{\Omega} t_I(g_0 v_0, \ldots, g_{n+1} v_{n+1}) \, dv_0 \ldots dv_{n+1} .
\]
Since the map
\[
 (v_0, \ldots, v_{n+1}) \mapsto (\varepsilon_0 v_0, \ldots, \varepsilon_{n+1} v_{n+1})
\]
is a measure-preserving automorphism of \( \Omega \) and each \( g_i \) is linear, we have \( \text{eul} = \text{eul}_I \) for every \( I \), so
\[
eul(g_0, \ldots, g_{n+1}) = 2^{-n-2} \sum_I \text{eul}_I(g_0, \ldots, g_{n+1}) =
\]
\[
 2^{-n-2} \int_{\Omega} \left( \sum_I t_I(g_0 v_0, \ldots, g_{n+1} v_{n+1}) \right) \, dv_0 \ldots dv_{n+1} .
\]
But point (6) of Lemma 13.2 implies that, for every \((v_0, \ldots, v_{n+1}) \in \Omega\), there exist exactly two multiindices \( I_1, I_2 \) such that \( t_I(g_0 v_0, \ldots, g_{n+1} v_{n+1}) \) does not vanish. Moreover, since \( v \mapsto -v \) is orientation-preserving (resp. reversing) if \( n \) is odd (resp. even), we have
\[
 t_{I_1}(g_0 v_0, \ldots, g_{n+1} v_{n+1}) = t_{I_2}(g_0 v_0, \ldots, g_{n+1} v_{n+1}) = \pm 1 \quad \text{if } n \text{ is odd},
\]
\[
 t_{I_1}(g_0 v_0, \ldots, g_{n+1} v_{n+1}) = -t_{I_2}(g_0 v_0, \ldots, g_{n+1} v_{n+1}) \quad \text{if } n \text{ is even} .
\]
This concludes the proof of (1).

Suppose now that there exist \( g_i, g_j \) such that \( g_i, g_j \in SO(n+1) \), \( i \neq j \). The map \( \psi: D^{n+2} \to D^{n+2} \) which acts on \( D^{n+2} \) as \( g_i^{-1} \) (resp. \( g_j^{-1} \)) on the \( i \)-th (resp. \( j \)-th) factor of \( D^{n+2} \), and as the identity on the other factors, is a measure-preserving automorphism of \( D^{n+2} \). Therefore, if \( g'_i = g_j = 1 \) and \( g'_k = g_k \) for every \( k \notin \{i, j\} \), then
\[
eul(g_0, \ldots, g_{n+1}) = \text{eul}(g'_0, \ldots, g'_{n+1}) = 0 ,
\]
where the last equality is due to the fact that \( \text{eul} \) is alternating, and \( g'_i = g_j \).

**Definition 13.6.** The \((n+1)\)-dimensional bounded Euler class is the element
\[
 [\text{eul}]_b \in H_{b}^{n+1}(\text{GL}(n+1, \mathbb{R}), \mathbb{R}) .
\]

2. Representing cycles via simplicial cycles

As mentioned above, in order to prove that the Euler class of a linear sphere bundle is bounded, it is convenient to make simplicial chains come into play. To this aim we introduce a machinery which is very well-suited to describe singular cycles as push-forwards of simplicial cycles.

Let \( M \) be a topological space, and let \( z \) be an \( n \)-dimensional cycle in \( C_n(M, \mathbb{R}) \). For technical reasons that will be clear later, we first take the double barycentric subdivision \( \overline{z} = \sum_{i=1}^{k} a_i \sigma_i \) of \( z \), where \( \sigma_i \) is a singular
n-simplex on $M$, and $a_i \in \mathbb{R}$ for every $i$. Also assume that $\sigma_i \neq \sigma_j$ for $i \neq j$.

We now construct a simplicial complex $P$ associated to $\pi$ (whence to $z$). Let us consider $k$ distinct copies $\Delta^n_1, \ldots, \Delta^n_k$ of the standard $n$-simplex $\Delta^n$. For every $i$ we fix an identification between $\Delta^n_i$ and $\Delta^n$, so that we may consider $\sigma_i$ as defined on $\Delta^n_i$. For every $i = 1, \ldots, k$, $j = 0, \ldots, n$, we denote by $F^i_j$ the $j$-th face of $\Delta^n_i$, and by $\partial^i_j: \Delta^{n-1} \to F^i_j \subseteq \Delta^n$ the usual face inclusion.

We say that the faces $F^i_j$ and $F^{i'}_{j'}$ are equivalent if $\sigma_i|_{F^i_j} = \sigma_{i'}|_{F^{i'}_{j'}}$. Since $z$, whence $\pi$, is a cycle, equivalence classes of faces contain an even number of elements. Therefore, it is possible to construct a pairing between the faces in such a way that paired faces are equivalent.

Let us define a simplicial complex $P$ as follows. The simplices of $P$ are $\Delta^n_1, \ldots, \Delta^n_k$, and, if $F^i_j$, $F^{i'}_{j'}$ are paired, then we identify them via the affine diffeomorphism $\partial^i_j \circ (\partial^{i'}_{j'})^{-1}: F^i_j \to F^{i'}_{j'}$. The fact that $P$ is indeed a simplicial complex is a consequence of the fact that $\pi$ is the double barycentric subdivision of a cycle.

By construction, the maps $\sigma_1, \ldots, \sigma_k$ glue up to a well-defined continuous map $f: |P| \to M$, where $|P|$ denotes the topological realization of $P$. For every $i = 1, \ldots, k$, let $\tilde{\sigma}_i: \Delta^n \to |P|$ be the simplicial simplex obtained by composing the identification $\Delta^n \cong \Delta^n_i$ with the quotient map with values in $|P|$, and let us consider the simplicial chain $z_P = \sum_{i=1}^k a_i \tilde{\sigma}_i$. By construction, $z_P$ is a cycle, and the push-forward of $z_P$ via $f$ is equal to $\pi$, whence homologous to $z$. We have thus proved the following:

**Lemma 13.7.** Take an element $\alpha \in H_n(M, \mathbb{R})$. Then, there exist a finite simplicial complex $P$, a real simplicial $n$-cycle $z_P$ on $P$ and a continuous map $f: |P| \to M$ such that $H_n(f)(|z_P|) = \alpha$ in $H_n(M, \mathbb{R})$.

### 3. The bounded Euler class of a linear sphere bundle

Let now $\pi: E \to M$ be a flat linear $n$-sphere bundle, and set $\Gamma = \pi_1(M)$. We denote by

$$r^M_\bullet: C_\bullet(\tilde{M}, \mathbb{R}) \to C_\bullet(\Gamma, \mathbb{R})$$

$$r^*_M: C^b_\bullet(\Gamma, \mathbb{R})^\Gamma \to C^b_\bullet(\tilde{M}, \mathbb{R})^\Gamma = C^b_\bullet(M, \mathbb{R})$$

the classifying maps defined in Lemma 5.2. We know from Proposition 12.22 that $E$ is (linearly) isomorphic to $E_{\rho}$ for a representation $\rho: \pi_1(M) \to \text{GL}^+(n+1, \mathbb{R})$. We define the real bounded Euler class $c^b_\rho(E) \in H^{n+1}_b(M, \mathbb{R})$ of $E$ as the pull-back of $[\text{eul}]_{\rho}$ via $\rho$ (and the classifying map), i.e. we set

$$c^b_\rho(E) = H^{n+1}_b(r^*_M) \circ H^{n+1}_b(\rho^*)([\text{eul}]_{\rho}) \in H^{n+1}_b(M, \mathbb{R}).$$

We are now ready to show that the real bounded Euler class of $E$ is mapped by the comparison map onto the Euler class of $E$. We first deal with the simplicial case, and then we reduce the general case to the simplicial one.
Proposition 13.8. Suppose that $M$ is a simplicial complex, and let $z$ be an $(n+1)$-dimensional simplicial cycle on $M$. Then
\[
 r^{n+1}_M(p^{n+1}(\text{eul}))(z) = \langle e^\mathbb{R}(E), [z] \rangle .
\]

Proof. Let $q_1, \ldots, q_N$ be the vertices of $M$, and for every $i = 1, \ldots, N$ let us take an element $v_i \in S^n$. After choosing a suitable trivialization of $E$ over the $q_i$’s, we would like to exploit the $v_i$ to define a section over the $q_i$’s. Then, we will affinely extend such a section over the $n$-skeleton of $M$, and use the resulting section to compute the Euler class of $E$. In order to do so, we need to be sure that the resulting section does not vanish on the $n$-skeleton of $M$, and to this aim we have to carefully choose the $v_i$’s we start with.

We fix points $x_0 \in M$ and $\tilde{x}_0 \in p^{-1}(x_0)$, and we identify $\Gamma = \pi_1(M, x_0)$ with the group of the covering automorphisms of $\tilde{M}$ so that the projection on $M$ of any path in $\tilde{M}$ starting at $\tilde{x}_0$ and ending at $g(\tilde{x}_0)$ lies in the homotopy class corresponding to $g$. We also choose a set of representatives $R$ for the action of $\Gamma = \pi_1(M)$ on $\tilde{M}$ containing $\tilde{x}_0$, and we denote by by $j: \tilde{M} \times S^n \to E_\rho = E$ the quotient map with respect to the diagonal action of $\Gamma$ on $\tilde{M} \times S^n$.

Let us now fix an $N$-tuple $(v_1, \ldots, v_N) \in D^N$. If $v_i \neq 0$ for every $i$, such an $N$-tuple gives rise to a section $s^{(0)}$ of $E$ over the $0$-skeleton $P^{(0)}$ which is defined as follows: if $\tilde{q}_i$ is the unique lift of $q_i$ in $R$, then $s^{(0)}(q_i) = j(\tilde{q}_i, v_i/\|v_i\|)$. We are now going to affinely extend such a section over the $n$-skeleton of $P$. However, as mentioned above, this can be done only under some additional hypothesis on $(v_1, \ldots, v_N)$, which we are now going to describe.

For every $k$-simplex $\tau$ of $M$, $k \leq n + 1$, we denote by $\tilde{\tau} \subseteq \tilde{M}$ the lift of $\tau$ having the first vertex in $R$ (recall that, by definition of simplex in a simplicial complex, the vertices of simplices are ordered). Let now $q_{\tilde{j}}$ be the $i$-th vertex of $\tau$. Then there exists $q_i \in \Gamma$ such that the $i$-th vertex $\tilde{v}_i$ of $\tilde{\tau}$ is equal to $g_i(q_{\tilde{j}})$. Moreover, we may choose the classifying map $r^k_M$ in such a way that
\[
 r^k_M(\tilde{\tau}) = (g_0, \ldots, g_k)
\]
(see Lemma 5.2). We now say that $(v_1, \ldots, v_N)$ is $\tau$-generic if the $(k+1)$-tuple
\[
 (\rho(g_0)(v_{j_0}), \rho(g_1)(v_{j_1}), \ldots, \rho(g_n)(v_{j_k})) \in D^{k+1}
\]
is generic. If this is the case and $k \leq n$, then the composition
\[
 \tau \xrightarrow{l} \tilde{\tau} \xrightarrow{\text{Id} \times a} \tilde{M} \times S^n \xrightarrow{j} E ,
\]
where $l$ is just the lifting map, and
\[
 a(t_0\tilde{v}_0 + \ldots + t_k\tilde{v}_k) = \frac{t_0\rho(g_0)(v_{j_0}) + \ldots + t_k\rho(g_k)(v_{j_k})}{\|t_0\rho(g_0)(v_{j_0}) + \ldots + t_k\rho(g_k)(v_{j_k})\|}
\]
is a well-defined section of $E$ over $\tau$, which extends $s^{(0)}$. 

We say that the $N$-tuple $(v_1, \ldots, v_N)$ is generic if it is $\tau$-generic for every $k$-simplex $\tau$ of $M$, $k \leq n+1$, and we denote by $\Omega \subseteq D^N$ the subset of generic $N$-tuples. A similar argument to the proof of Lemma 13.4 implies that there exist Borel subsets $\Omega_i \subseteq D$, $i = 1, \ldots, N$, such that each $\Omega_i$ has full measure in $D$ and

$$\Omega_1 \times \ldots \times \Omega_N \subseteq \Omega$$

(in particular, $\Omega$ has full measure in $D^N$). In fact, one can set $\Omega_1 = D$, and define inductively $\Omega_{i+1}$ by imposing that $v \in \Omega_{i+1}$ if and only if the following condition holds: for every $(v_1, \ldots, v_i) \in \Omega_1 \times \ldots \times \Omega_i$, the $(i+1)$-tuple $(v_1, \ldots, v_i, v)$ is $\tau$-generic for every $k$-simplex $\tau$ of $M$, $k \leq n+1$, with vertices in in $\{q_1, \ldots, q_i+1\}$ (observe that it makes sense to require that an $(i+1)$-tuple is $\tau$-generic, provided that the vertices of $\tau$ are contained in $\{q_1, \ldots, q_i+1\}$). It is not difficult to check that indeed $\Omega_i$ has full measure in $D$ for every $i$.

Let us now pick an element $v = (v_1, \ldots, v_N) \in \Omega$. Such an element defines a compatible family of sections over all the simplicial $k$-simplices of $M$, $k \leq n$, which can be extended in turn to a compatible family of sections over all the singular $n$-simplices in $M$. If we denote by $\varphi_v$ the Euler cocycle associated to this compatible family of sections, then Lemma 13.2 implies that, for every $(n+1)$-dimensional simplex $\tau$ of $M$, we have

$$\varphi_v(\tau) = t(g_0 v_j_0, \ldots, g_{n+1} v_{j_{n+1}}),$$

where $q_j_i$ is the $i$-th vertex of $\tau$, and

$$(g_0, \ldots, g_{n+1}) = r^M_{n+1}(\tau).$$

Therefore, if $z$ is a fixed real simplicial $(n+1)$-cycle, then

$$\langle e^{R}(E), [z] \rangle = \varphi_v(z),$$

so

$$(27) \quad \langle e^{R}(E), [z] \rangle = \int_{\tau \in \Omega_1 \times \ldots \times \Omega_N} \varphi_v(z),$$

(wherwe used that the product of the $\Omega_i$’s has full, i.e. unitary, measure in $D^N$).

On the other hand, for every $(n+1)$-simplex $\tau$ we have

$$\int_{\tau \in \Omega_1 \times \ldots \times \Omega_N} \varphi_v(\tau) = \int_{\tau \in \Omega_1 \times \ldots \times \Omega_N} t(g_0 v_{j_{0}}, \ldots, g_{n+1} v_{j_{n+1}}) \, d\nu$$

$$= \int_{v_j \in \Omega_j} t(g_0 v_{j_0}, \ldots, g_{n+1} v_{j_{n+1}}) \, dv_{j_0} \ldots dv_{j_{n+1}}$$

$$= \int_{D^{n+2}} t(g_0 v_0, \ldots, g_{n+1} v_{n+1}) \, dv_0 \ldots dv_{n+1}$$

$$= r^M_{n+1}(\text{eul})(\tau),$$
(where we used again that \( \Omega_{\ell} \) has full (whence unitary) measure in \( D \), so by linearity
\[
(28) \quad \int_{\sigma \in \Omega_{1} \times \cdots \times \Omega_{N}} \varphi_\pi(z) = r_{M}^{n+1}(\text{eul})(z) .
\]
Putting together Equations (27) and (28) we obtain that \( \langle e^\mathbb{R}(E), [z] \rangle = \varphi_\pi(z) = r_{M}^{n+1}(\text{eul})(z) \), whence the conclusion. \( \square \)

We are now ready to prove that, via the classifying map, the group cochain \( \text{eul} \) indeed provides a representative of the Euler class of \( E \), even in the case when \( M \) is not assumed to be a simplicial complex:

**Proposition 13.9.** The cochain \( r_{M}^{n+1}(\rho^{n+1}(\text{eul})) \in C_{b}^{n+1}(M, \mathbb{R}) \) is a representative of the real Euler class of \( E \).

**Proof.** By the Universal Coefficient Theorem, it is sufficient to show that
\[
(29) \quad r_{M}^{n+1}(\rho^{n+1}(\text{eul}))(z) = \langle e^\mathbb{R}(E), [z] \rangle
\]
for every singular cycle \( z \in C_{n+1}(M, \mathbb{R}) \). So, let us fix such a cycle, and take a finite simplicial complex \( P \), a real simplicial \( (n+1) \)-cycle \( z_{P} \) on \( P \) and a continuous map \( f : |P| \to M \) such that \( \{H_{n+1}(f)([z_{P}]) = [z] \text{ in } H_{n+1}(M, \mathbb{R})\} \) (see Lemma 13.7). Since \( z \) is homologous to \( C_{n+1}(f)(z_{P}) \), we have
\[
(30) \quad r_{M}^{n+1}(\rho^{n+1}(\text{eul}))(z) = C^{n+1}(f)(r_{M}^{n+1}(\rho^{n+1}(\text{eul})))((z_{P}) .
\]
Moreover, one may choose a classifying map
\[
r_{\ast} : C^{\ast}(\pi_{1}(|P|, \mathbb{R})^{\pi_{1}(|P|)} \to C^{\ast}(|\widetilde{P}|, \mathbb{R})^{\pi_{1}(|P|)} = C^{\ast}(|P|, \mathbb{R})
\]
in such a way that the diagram
\[
\begin{array}{ccc}
C^{\ast}(\Gamma, \mathbb{R}) & \xrightarrow{f_{\ast}} & C^{\ast}(\pi_{1}(|P|, \mathbb{R})^{\pi_{1}(|P|)} \\
\downarrow r_{M}^{\ast} & & \downarrow r_{\ast}^{\ast} \\
C^{\ast}(|M|, \mathbb{R}) & \xrightarrow{C^{\ast}(f)} & C^{\ast}(|P|, \mathbb{R})
\end{array}
\]
commutes, where we denote by \( f_{\ast} \) the map induced by \( f \) on fundamental groups. Therefore, if we denote by \( \rho' : \pi_{1}(|P|) \to G \) the composition \( \rho' = \rho \circ f_{\ast} \), then
\[
C^{n+1}(f)(r_{M}^{n+1}(\rho^{n+1}(\text{eul}))) = r_{P}^{n+1}(f_{\ast}^{n+1}(\rho^{n+1}(\text{eul}))) = r_{P}^{n+1}((\rho')^{n+1}(\text{eul})).
\]
Putting this equality together with (30) we obtain that
\[
(31) \quad r_{M}^{n+1}(\rho^{n+1}(\text{eul}))(z) = r_{P}^{n+1}((\rho')^{n+1}(\text{eul}))(z_{P}).
\]
On the other hand, the bundle \( E \) pulls back to a flat linear \( S^{n} \)-bundle \( f^{\ast}E \) on \( |P| \), and it readily follows from the definitions that \( f^{\ast}E \) is isomorphic to the sphere bundle associated to the representation \( \rho' : \pi_{1}(|P|) \to G \) just
introduced. Therefore, Proposition 13.8 (applied to the case \( M = P \)) implies that
\[
(32) \quad r_P^{n+1}(\rho')^{n+1}(\text{eul})(z_P) = \langle e^R(f^*E), [z_P] \rangle,
\]
while Lemma 12.7 (the statement with integral coefficients implies the one with real coefficients) gives
\[
(33) \quad \langle e^R(f^*E), [z_P] \rangle = \langle H^{n+1}(f)(e^R(E)), [z_P] \rangle = \langle c_R(E), [z] \rangle.
\]
Putting together (31), (32) and (33) we finally obtain the desired equality (29), whence the conclusion. \qed

Putting together the previous proposition and Lemma 13.5 we obtain the following result, which generalizes to higher dimensions Corollary 12.30:

**Theorem 13.10.** Let \( \pi: E \rightarrow M \) be a flat linear \( n \)-sphere bundle, and let \( \rho: \Gamma \rightarrow \text{GL}^+(n+1, \mathbb{R}) \) be the associated representation, where \( \Gamma = \pi_1(M) \). Then the real Euler class of \( E \) is given by
\[
e^R(E) = c(e^R_b(E)),
\]
where \( c: H^{n+1}_b(\Gamma, \mathbb{R}) \rightarrow H^{n+1}(\Gamma, \mathbb{R}) \) is the comparison map. Therefore,
\[
\|e^R(E)\|_\infty \leq 2^{-n-1}.
\]

We have shown that the Euler class of a flat linear sphere bundle is bounded. By Proposition 2.11, this implies that such a class is bounded even as an integral class:

**Corollary 13.11.** Let \( \pi: E \rightarrow M \) be a flat linear \( n \)-sphere bundle. Then the integral Euler class \( e(E) \) admits a bounded representative.

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**Commento sulla canonicita della classe limitata**

**Conjecture 13.12.** Let \( \pi: E \rightarrow M \) be a topologically flat sphere bundle. Then \( eu(E) \) admits a bounded representative.
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