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Title: Volumes of Polyhedra and their applications to Knot Theory.

Abstract: Volume calculations for polyhedra in the spaces of constant curvature is very old and difficult problem. The first result in this direction was obtained by Tartaglia (1494) who found a formula for volume of Euclidean tetrahedron. Now this formula is known as Cayley-Menger formula. The volumes for non-Euclidean biorthogonal tetrahedra were given by Lobachevsky (1836) and Schläfli (1858). Vinberg (1991) determined the volume for tetrahedron with at least one vertex on the infinity. The volume formula for the Lambert cube was obtained by Ruth Kellerhals (1989) and Mednykh and Derevnin (2001) in hyperbolic and spherical spaces, respectively.

Despite of this partial results, a formula for the volume of an arbitrary hyperbolic tetrahedron has been unknown until very recently. The general algorithm for obtaining such a formula was indicated by Hsiang (1988) and the complete solution of the problem was given by Yu. Cho and H. Kim (1999), J. Murakami and M. Yano (2001), A. Ushijima (2002).

In these papers the volume is expressed as an analytic formula involving 16 Dilogarithm or Lobachevsky functions whose arguments depend on the dihedral angles of the tetrahedron and on some additional parameter which is a root of some complicated quadratic equation with complex coefficients.

Just recently, it became known that very simple and beautiful formula for hyperbolic tetrahedron was discovered by Italian Duke Gaetano Sforza (1906). More precisely, the volume for hyperbolic tetrahedron is given by

$$\mathrm{Vol}\,(\mathrm{T}) = \frac{1}{4}\int_{A_0}^{A} \log \frac{c_A - \sqrt{-\det G}\ \sin A}{c_A + \sqrt{-\det G}\ \sin A}\,\mathrm{d}A,$$

where G is Gram matrix of the tetrahedron, c_A is algebraic compliment of G with respect to dihedral angle A, and A_0 is a root of the equation $\det G = 0$.

We discuss similar results for a wide class of polyhedra in the spaces of constant curvature. As application, we give explicit formulae for the Figure eight knot, Whitehead link, Borromean rings and others cone-manifolds.