Cohomology of configuration spaces of Riemann surfaces

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Abstract

The Križ model for configuration spaces is studied under the action of S_n in order to simplify computations for cohomology. Betti numbers are presented for the ordered and unordered configuration spaces of of 2,3 and 4 points on the torus and surfaces of higher genus.

The Križ model $E(X,n)$

The Križ model [K] is a rational model for the cohomology of configuration spaces coming from the Fulton-MacPherson compactification [FM] of configuration spaces.
For a smooth complex projective variety X (of complex dimension m):

$E(X,n)$ is isomorphic to $H^*(X)^{\otimes n}\otimes\Lambda(G_{ij})$ modulo the ideal of relations:

$(i)\quad G_{ji}=G_{ij}$
$(ii)\quad p_j^*(x)G_{ij}=p_i^*(x)G_{ij},\qquad 1\le i<j\le n,\ x\in H^*(X)$
$(iii)\quad G_{ik}G_{jk}=G_{ij}G_{jk}-G_{ij}G_{ik},\ 1\le i<j<k\le n,$ (*Arnold relations*)

here $\Lambda(G_{ij})$ denotes the exterior algebra generated by G_{ij} for $1\le i<j\le n-1$ and p_i^* denotes pullback of the i-th projection $p_i:X^n\to X$.

The Križ model $E(X,n)=E_q^k(X,n)$ has a natural bigrading:

- q= number of exterior generators G_{ij} in a monomial
- k= total degree of the monomial

Example: $(x\otimes1\otimes1\otimes y\otimes1)G_{12}G_{23}\in E_2^{|x|+|y|+2(2m-1)}$ for $x,y\in H^*(X)$ and $|G_{ij}|=2m-1$.
The nonzero bigraded components $E_q^k(X,n)$ lie in the trapezoid

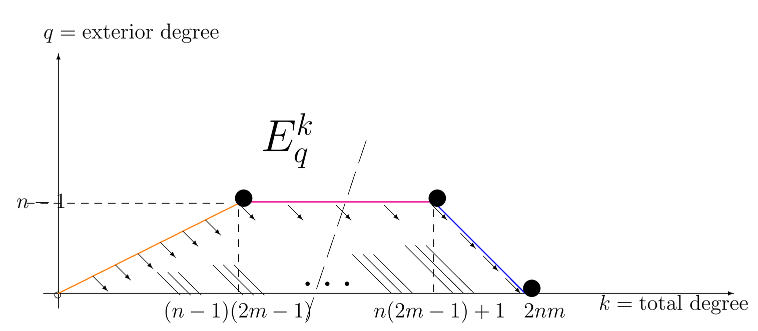


Figure 1: Trapezoid shows the position of bigraded components, arrows show direction of the differentials, whereas the bullets and circles show non-zero components contributing and that do not contribute to cohomology respectively

The differential is defined by $d|_{H^*(X)^{\otimes n}}=0$ and $d(G_{ij})=p_{ij}^*(\Delta)$, where Δ is the class of the diagonal and in our case,

$$\Delta=(1\otimes w+\sum_{i=1}^{g}b_i\otimes a_i-\sum_{i=1}^{g}a_i\otimes b_i+w\otimes 1)\in H^*(\Sigma_g)^{\otimes 2}.$$

S_n acts on $E(X,n)$ by permuting the factors in $H^*(X^n)=H(X)^{*\otimes n}$ and by permuting the indices of the exterior generators G_{ij}

The S_n-equivariance of d and Schur lemma give a splitting of this DGA into subcomplexes corresponding to its decomposition into isotypical components

$$(E_*^*(X,n),d)=\bigoplus_{\lambda\vdash n}(E_*^*(V(\lambda)),d_\lambda)$$

where $E_q^k(V(\lambda))$ denotes the isotypical component corresponding to $\lambda\vdash n$. By the transfer theorem, the S_n-invariant part of the cohomology of ordered configuration space $F(X,n)$ gives cohomology of the unordered configuration space $C(X,n)$.

The *symmetric Poincaré polynomial* for $F(X,n)$ is defined as the formal sum with polynomial coefficients

$$SP_{F(X,n)}(t,s)=\sum_{\lambda\vdash n}(\sum_{k,q}m_{q,\lambda}^k t^k s^q)V(\lambda),$$

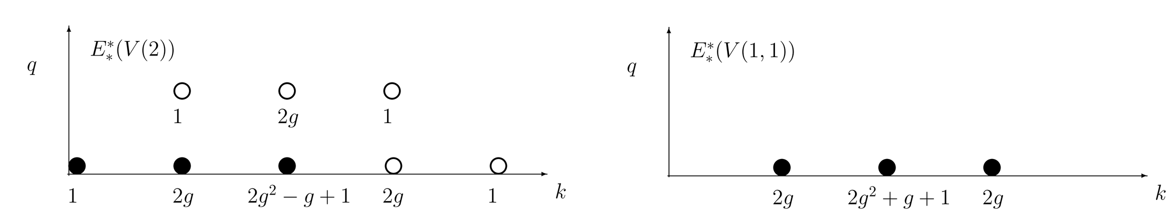
where $m_{q,\lambda}^k$ is the multiplicity of the irreducible representation $V(\lambda)$ in the bigraded component of cohomology $H_q^k=H_q^k(F(X,n))$.

Properties of the differential [AAB]

- The differentials (for $M\neq S^2$) are injective for any $q\in[1,n-1]$:
$$d:E_q^{q(2m-1)}(X,n)\to E_{q-1}^{q(2m-1)+1}(X,n).$$
- The top differentials $d:E_{n-1}^k(X,n)\to E_{n-2}^{k+1}(X,n)$ are injective for any k in the interval $[(n-1)(2m-1),n(2m-1)+1]$.
- All cohomology groups of the following subcomplex
$$0\to E_{n-1}^{n(2m-1)+1}\to E_{n-2}^{n(2m-1)+2}\to\ldots\to E_0^{2nm}\to 0.$$

Configuration space of two points on a Riemann surface

[A] The Križ model $F(\Sigma_g,2)$ for the 2-point configuration space of Riemann surfaces is isomorphic to $H^*(\Sigma_g)^{\otimes 2}\otimes\langle G_{12}\rangle$ modulo the unique relation $p_2^*(x)G_{12}=p_1^*(x)G_{12}$ for any $x\in H^*(\Sigma_g)$.
The bigraded components of the Križ model $E(\Sigma_g,2)$ can be plotted on the trapezoid (considering the isotypical components separately showing multiplicity in each bidegree):



The injectivity of magenta differentials give the symmetric Poincaré polynomial for $F(\Sigma_g,2)$, $g\ge1$

$$SP_{H^*(F(\Sigma_g,2))}(t,s)=(1+2gt+(2g^2-g)t^2)\,V(2)+ +(2gt+(2g^2+g+1)t^2+2gt^3)\,V(1,1).$$

Using the transfer theorem the double Poincaré polynomial of the unordered configuration space $C(\Sigma_g,2)$, $g\ge1$:

$$P_{H^*(C(\Sigma_g,2))}(t,s)=1+2gt+(2g^2-g)t^2.$$

Configuration spaces of three points on a Riemann surface

[A] The symmetric Poincaré polynomial of $F(\Sigma_g,3)$, $g\ge2$, is:

$$\begin{aligned}SP_{H^*(F(\Sigma_g,3))}(s,t)=&\Big[1+2gt+(2g^2-g)t^2+\tfrac{2}{3}(2g^3-3g^2+4g)t^3+\\&+((2g^2+g+1)t^3+2gt^4)s\Big]\,V(3)+\\&+\Big[2gt+4g^2t^2+\tfrac{2}{3}(4g^3-g)t^3\Big]V(2,1)+\\&+\Big[(2g^2+g)t^2+\tfrac{2}{3}(2g^3+3g^2+4g)t^3+\\&+(2g^2+g)t^4\Big]\,V(1,1,1).\end{aligned}$$

For $g=1$:

$$SP_{H^*(F(T^2,3))}(s,t)=(1+t)^2(1+2st^2)\,V(3)+2t(1+t)^2\,V(2,1)+3t^2(1+t)^2\,V(1,1,1).$$

Using the transfer theorem we have the double Poincaré polynomial of the unordered configuration space $C(\Sigma_g,3)$, $g\ge2$:

$$P_{H_*^*(C(\Sigma_g,3))}(s,t)=1+2gt+(2g^2-g)t^2+\frac{2}{3}(2g^3-3g^2-2g)t^3+(2g^2+g+1)t^3s+2gt^4s.$$

For $g=1$ we have

$$P_{H^*(C(T^2,3))}(s,t)=(1+t)^2(1+2st^2).$$

Unordered configuration space of four points on a torus

[A] The double Poincaré polynomial of the 4-point configuration space $C(T^2,4)$:

$$P_{H^*(C(T^2,4))}(s,t)=1+2t+t^2+(2t^2+5t^3+4t^4+t^5)s.$$

Lastly the Poincaré polynomials reads:

$$P_{H^*(C(T^2,4))}(s,t)=(1+t)^2(1+2t^2+t^3).$$

Future work on algebra structure of $H^*(C(T^2,n))$

Here we roughly state some observations about the algebra structure of cohomology of the unordered configuration space of few points on the torus, and conjecture what it would look like for n-points.

- $H^*(C(T^2,1))\cong H^*(T^2)\cong\Lambda(a,b)$
- $H^*(C(T^2,2))\cong\Lambda(\alpha,\beta)$
- $H^*(C(T^2,3))\cong\Lambda(\alpha,\beta)\otimes M(G_a,G_b)$
- $H^*(C(T^2,4))\cong\Lambda(\alpha,\beta)\otimes M(G_a,G_b,C)$
for cohomology generators of the torus a and b: α and β are the S_n-orbits $\sum_{\sigma\in S_n}\sigma\ (a\otimes1\otimes\ldots\otimes1)$, $\sum_{\sigma\in S_n}\sigma\ (b\otimes1\otimes\ldots\otimes1)$ respectively.
The cocycles G_a and G_b are uniquely determined in bidegree $\binom{2}{1}$ and the cocycles $C,C'\ldots$ are the simplest choices. The expression $M(\cdot)$ denotes a module generated over $\Lambda(\alpha,\beta)$.
- More generally, $H^*(C(T^2,n))\cong\Lambda(\alpha,\beta)\otimes M(G_a,G_b,C,C',\ldots)$.

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THANK YOU!